CYLINDRICALLY ROTATABLE DESIGNS OF TYPES 1, 2 AND 3[1]

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1. Introduction. In Herzberg (1966a), we defined cylindrically rotatable designs as designs such that the variances of the estimated responses at points on the same $(k - 1)$-dimensional hyper-sphere centred on a specified axis are equal. Here we enlarge this definition to include all designs such that the variances of the estimated responses at points on the same s-dimensional cylinder in k-dimensional space with certain characteristics are equal. It turns out that there are three possible subclasses of such designs. We shall call such designs *cylindrically rotatable designs of types 1, 2 and 3.* In particular, cylindrically rotatable designs of type 1 are such that the variances of the estimated responses at all points on the same s-dimensional cylinder with the same remaining $(k - s)$ co-ordinates are equal; designs of type 2 are such that the variances of the estimated responses at all points on the same s-dimensional cylinder with the same remaining squares of the $(k - s)$ co-ordinates are equal; designs of type 3 are such that the variances of the estimated responses at all points on the same s-dimensional cylinder with the same remaining sum of squares of the $(k - s)$ co-ordinates are equal (geometrically this means that the variances of the estimated responses at all points at the intersection of two cylinders are equal). If a cylindrically rotatable design is rotated in a certain manner, the variances and covariances of the estimated coefficients of the response function remain unchanged. Cylindrically rotatable designs are identical to rotatable designs of the same order except in the required levels of $(k - s)$ factors. Therefore, as in the case of rotatable designs, if the experimenter has some prior knowledge about the shape of the response surface, the design may be rotated to reduce possible bias. If he has no previous knowledge of the orientation of the surface, the requirement of cylindrical rotatability is a reasonable one since the orientation of the design may be chosen at random. The cylindrically rotatable designs mentioned in Herzberg (1966a) are a special case of cylindrically rotatable designs of type 1.

Without loss of generality, we can relabel the axes in such a way that the first s axes refer to the dimensions of the s-dimensional cylinder.

2. Cylindrically rotatable designs of type 1. We wish to find conditions for designs such that the variances of the estimated responses at all points on the same s-dimensional cylinder with the same remaining $(k - s)$ co-ordinates are equal. We shall call these designs cylindrically rotatable designs of type 1.

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The following method for developing the conditions that designs of order d be cylindrically rotatable of type 1 is similar to the method used by Box and Hunter (1957) to develop the conditions that designs of order d be rotatable.

Let $\hat{y}(\mathbf{x})$ denote the value of the polynomial fitted by the method of least squares to the response at $\mathbf{x}$ and $V(\hat{y}(\mathbf{x}))$ be the variance of the estimated response at $\mathbf{x}$, where $\mathbf{x} = (x_1, x_2, \cdots, x_k)$.

Suppose

$$(2.1)\quad \hat{y}(\mathbf{x}) = b_0 + b_1x_1 + b_2x_2 + \cdots + b_kx_k + b_{11}x_1^2 + \cdots + b_{12}x_1x_2 + \cdots + b_{111}x_1^3 + \cdots \text{ of degree } d$$

or, in matrix notation,

$$(2.2)\quad \hat{y}(\mathbf{x}) = \mathbf{x}'^{[d]}\mathbf{b},$$

where $\mathbf{x}' = (1, x_1, x_2, \cdots, x_k)$, $\mathbf{x}'^{[d]}$ is such that $\mathbf{x}'^{[d]}\mathbf{x}^{[d]} = (\mathbf{x}'\mathbf{x})^d$, and $\mathbf{b}$ contains all the $b_{1^\alpha 2^\gamma \cdots k^\tau}$'s with suitable multipliers attached in order that (2.1) will equal (2.2).

From Box and Hunter (1957), equation 26, we know that

$$(2.3)\quad V(\hat{y}(\mathbf{x})) = \mathbf{x}'^{[d]}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{x}^{[d]}\sigma^2,$$

where $\mathbf{X}$ is the $N \times L$ matrix of independent variables. (N is the number of design points and L is the number of terms in (2.1).)

We now wish to consider the variance of $\hat{y}(\mathbf{z})$, where $\mathbf{z}$ is such that

$$(2.4)\quad \sum_{j=1}^{s} x_j^2 = \sum_{j=1}^{s} z_j^2,\ x_{s+1} = z_{s+1}, \cdots, x_k = z_k, \qquad (s < k).$$

Suppose

$$(2.5)\quad \mathbf{z} = \mathbf{M}\mathbf{x},$$

where $\mathbf{x}' = (1, x_1, x_2, \cdots, x_k)$ and $\mathbf{M}$ is a matrix of the following form:

$$(2.6)\quad \mathbf{M} = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ k \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & \cdots & s & s+1 & & \cdots & k \end{array} \\ \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & & \cdots & 0 \\ 0 & m_{11} & \cdots & m_{1s} & 0 & & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & & \vdots \\ 0 & m_{s1} & \cdots & m_{ss} & 0 & & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \end{array},$$

where

$$\begin{bmatrix} m_{11} & \cdots & m_{1s} \\ \vdots & & \vdots \\ m_{s1} & \cdots & m_{ss} \end{bmatrix}$$

is an orthogonal $s \times s$ matrix. Then

$$(2.7) \qquad V(\hat{y}(\mathbf{z})) = \mathbf{z}'^{[d]}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{z}^{[d]}\sigma^2 = \mathbf{x}'^{[d]}\mathbf{M}'^{[d]}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{M}^{[d]}\mathbf{x}^{[d]}\sigma^2,$$

where $\mathbf{z}' = (1, z_1, z_2, \cdots, z_k)$. ($\mathbf{M}^{[d]}$ is defined in such a way that $\mathbf{z}^{[d]} = \mathbf{M}^{[d]}\mathbf{x}^{[d]}$.)

For the variance to be constant at points satisfying (2.4), we require that $V(\hat{y}(\mathbf{x})) = V(\hat{y}(\mathbf{z}))$. In order that $V(\hat{y}(\mathbf{x})) = V(\hat{y}(\mathbf{z}))$ we have, from (2.3) and (2.7),

$$(2.8) \qquad (\mathbf{X}'\mathbf{X})^{-1} = \mathbf{M}'^{[d]}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{M}^{[d]}$$

for every matrix $\mathbf{M}$ of the form (2.6).

Box and Hunter (1957) show that

$$(2.9) \qquad Q = N^{-1}\mathbf{t}'^{[d]}\mathbf{X}'\mathbf{X}\mathbf{t}^{[d]}, \quad \text{where} \quad \mathbf{t}' = (1, t_1, t_2, \cdots, t_k), \\ = N^{-1}\textstyle\sum_{u=1}^{N} (1 + t_1x_{1u} + t_2x_{2u} + \cdots + t_kx_{ku})^{2d}$$

is the generating function of moments of order $2d$ or less of a design. If we let $[1^{\alpha_1}, 2^{\alpha_2}, \cdots, k^{\alpha_k}] = N^{-1}\sum_{u=1}^{N} x_{1u}^{\alpha_1}x_{2u}^{\alpha_2}\cdots x_{ku}^{\alpha_k}$, then the coefficient of $t_1^{\alpha_1}t_2^{\alpha_2}\cdots t_k^{\alpha_k}$ in Q is

$$(2.10) \qquad [(2d)!/\textstyle\prod_{j=1}^{k}\alpha_j!(2d-\alpha)!][1^{\alpha_1}, 2^{\alpha_2}, \cdots, k^{\alpha_k}],$$

where $\alpha = \sum_{j=1}^{k}\alpha_j$ = order of the moment and $0 \leqq \alpha \leqq 2d$.

From (2.8), we see that a design will be cylindrically rotatable of type 1 if and only if

$$(2.11) \qquad \begin{aligned} Q &= N^{-1}\mathbf{t}'^{[d]}\mathbf{X}'\mathbf{X}\mathbf{t}^{[d]} \\ &= N^{-1}\mathbf{t}'^{[d]}(\mathbf{M}'^{[d]}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{M}^{[d]})^{-1}\mathbf{t}^{[d]} \\ &= N^{-1}\mathbf{t}'^{[d]}(\mathbf{M}^{[d]})^{-1}\mathbf{X}'\mathbf{X}(\mathbf{M}'^{[d]})^{-1}\mathbf{t}^{[d]} \\ &= N^{-1}(\mathbf{t}'\mathbf{M}')^{[d]}\mathbf{X}'\mathbf{X}(\mathbf{M}\mathbf{t})^{[d]}. \end{aligned}$$

Therefore, Q is a function of $\sum_{j=1}^{s}t_j^2, t_{s+1}, \cdots, t_k$. Since Q is a polynomial in the t_j's, it must be of the form

$$(2.12) \quad Q = \textstyle\sum_{p=0}^{d}\sum_{\alpha_{s+1}=0}^{2d}\cdots\sum_{\alpha_k=0}^{2d} a_{2p,\alpha_{s+1},\cdots,\alpha_k}(\sum_{j=1}^{s}t_j^2)^p t_{s+1}^{\alpha_{s+1}}\cdots t_k^{\alpha_k},$$

where $2p + \alpha_{s+1} + \cdots + \alpha_k \leqq 2d$. The coefficient of $t_1^{\alpha_1}t_2^{\alpha_2}\cdots t_k^{\alpha_k}$ in Q is zero if any of the α_j, $j = 1, \cdots, s$, is odd and is

$$(2.13) \qquad a_{\Sigma_{j=1}^{s}\alpha_j,\alpha_{s+1},\cdots,\alpha_k}\cdot[(\tfrac{1}{2}(\textstyle\sum_{j=1}^{s}\alpha_j))!/\prod_{j=1}^{s}(\tfrac{1}{2}\alpha_j)!]$$

if all the α_j, $j = 1, \cdots, s$, are even integers.

Equating (2.10) and (2.13), we obtain

$$(2.14) \quad [1^{\alpha_1}, 2^{\alpha_2}, \cdots, k^{\alpha_k}] = a_{\Sigma_{j=1}^{s}\alpha_j,\alpha_{s+1},\cdots,\alpha_k} \\ \cdot(\tfrac{1}{2}(\textstyle\sum_{j=1}^{s}\alpha_j))!\prod_{j=1}^{k}\alpha_j!(2d-\alpha)!/\prod_{j=1}^{s}(\tfrac{1}{2}\alpha_j)!(2d)!.$$

Then letting

$$\lambda_{\Sigma_{j=1}^{s}\alpha_j,\alpha_{s+1},\cdots,\alpha_k} = a_{\Sigma_{j=1}^{s}\alpha_j,\alpha_{s+1},\cdots,\alpha_k} \cdot 2^{\alpha/2}\left(\tfrac{1}{2}\left(\textstyle\sum_{j=1}^{s}\alpha_j\right)\right)!\prod_{j=s+1}^{k}\alpha_j!(2d-\alpha)!/(2d)!, \tag{2.15}$$

we see that

$$\begin{aligned}[1^{\alpha_1}, 2^{\alpha_2}, \cdots, k^{\alpha_k}] &= 0, \qquad \text{any } \alpha_j \text{ odd}, \qquad j = 1, \cdots, s,\\ &= \lambda_{\Sigma_{j=1}^{s}\alpha_j,\alpha_{s+1},\cdots,\alpha_k}\cdot\left[\textstyle\prod_{j=1}^{s}\alpha_j!/2^{\alpha/2}\prod_{j=1}^{s}(\tfrac{1}{2}\alpha_j)!\right],\\ &\qquad \text{all } \alpha_j \text{ even}, j = 1, \cdots, s.\end{aligned} \tag{2.16}$$

The design points are chosen in such a manner that the moments are of the form given in (2.16) and such that the variance-covariance matrix is non-singular.

EXAMPLE. Consider the following point sets:

$$\begin{gathered}(\pm a, \pm a, c, e)\\ (\pm a, \pm a, -c, -e),\\ (\pm 2^{3/4}, 0, 0, 0),\\ (0, \pm 2^{3/4}a, 0, 0),\\ (0, 0, \pm d, 0),\\ (0, 0, 0, \pm f).\end{gathered} \tag{2.17}$$

For all values of a, c, d, e and f except zero, these point sets will form a second order cylindrically rotatable design of type 1 in four dimensions since the moments satisfy (2.16) and the variance-covariance matrix is non-singular. The number of points involved is sixteen. The moments of the design are, when multiplied by N,

$$\begin{aligned}\sum x_{1u}^2 = \sum x_{2u}^2 &= 4(2 + 2^{1/2})a^2,\\ \sum x_{3u}^2 &= 8c^2 + 2d^2,\\ \sum x_{4u}^2 &= 8e^2 + 2f^2,\\ \sum x_{1u}^4 = \sum x_{2u}^4 = 3\sum x_{1u}^2x_{2u}^2 &= 24a^4,\\ \sum x_{3u}^4 &= 8c^4 + 2d^4,\\ \sum x_{4u}^4 &= 8e^4 + 2f^4,\\ \sum x_{1u}^2x_{3u}^2 = \sum x_{2u}^2x_{3u}^2 &= 8a^2c^2,\\ \sum x_{1u}^2x_{4u}^2 = \sum x_{2u}^2x_{4u}^2 &= 8a^2e^2,\\ \sum x_{1u}^2x_{3u}x_{4u} = \sum x_{2u}^2x_{3u}x_{4u} &= 8a^2ce,\end{aligned} \tag{2.18}$$

$$\sum x_{3u}^2 x_{4u}^2 = 8c^2e^2,$$

$$\sum x_{3u}^3 x_{4u} = 8c^3e,$$

$$\sum x_{3u} x_{4u}^3 = 8ce^3,$$

$$\sum x_{3u} x_{4u} = 8ce,$$

and all other sums of powers and products up to and including order four are zero. The summation is taken over all design points.

3. Cylindrically rotatable designs of type 2. We wish to find conditions for designs such that the variances of the estimated responses at all points on the same s-dimensional cylinder with the same remaining squares of the $(k - s)$ co-ordinates are equal, that is, the estimated responses at two points $\mathbf{x} = (x_1, x_2, \cdots, x_k)$ and $\mathbf{z} = (z_1, z_2, \cdots, z_k)$ will have the same variance if

$$(3.1) \qquad \sum_{j=1}^{s} x_j^2 = \sum_{j=1}^{s} z_j^2, \; x_{s+1}^2 = z_{s+1}^2, \cdots, x_k^2 = z_k^2, \qquad (s < k)$$

These designs will be called cylindrically rotatable designs of type 2. The moment conditions for designs of type 2 are found in a manner similar to that used for designs of type 1. The details may be found in Herzberg (1966b).

A design will be cylindrically rotatable of type 2 if and only if the moments satisfy the following conditions and the variance-covariance matrix is non-singular:

$$[1^{\alpha_1}, 2^{\alpha_2}, \cdots, k^{\alpha_k}] = 0, \qquad \text{any } \alpha_j \text{ odd},$$

$$(3.2) \qquad = \lambda_{\Sigma_{j=1}^{s} \alpha_j, \alpha_{s+1}, \cdots, \alpha_k} \cdot \left[\prod_{j=1}^{s} \alpha_j! / 2^{\alpha/2} \prod_{j=1}^{s} (\tfrac{1}{2}\alpha_j)!\right],$$

all α_j even

Here $\lambda_{\Sigma_{j=1}^{s} \alpha_j, \alpha_{s+1}, \cdots, \alpha_k}$ is constant for any design and any $\sum_{j=1}^{s} \alpha_j$ and α_l, $l = s + 1, \cdots, k$.

Example. Consider the following point sets:

$$(3.3) \qquad \begin{array}{c} (\pm a, \pm a, \pm c, \pm d), \\ (\pm 2a, 0, 0, 0), \\ (0, \pm 2a, 0, 0), \\ (0, 0, \pm e, 0). \end{array}$$

For all values of a, c, d and e except zero, these point sets will form a second order cylindrically rotatable design of type 2 in four dimensions since the moments satisfy (3.2) and the variance-covariance matrix is non-singular. The number of points involved is twenty-two. The moments of the design are, when multiplied by N,

$$
\begin{aligned}
& \sum x_{1u}^2 = \sum x_{2u}^2 = 24a^2, \\
& \sum x_{3u}^2 = 16c^2 + 2e^2, \\
& \sum x_{4u}^2 = 16d^2, \\
(3.4) \qquad & \sum x_{1u}^4 = \sum x_{2u}^4 = 3\sum x_{1u}^2 x_{2u}^2 = 48a^4, \\
& \sum x_{3u}^4 = 16c^4 + 2e^4, \\
& \sum x_{4u}^4 = 16d^4, \\
& \sum x_{1u}^2 x_{3u}^2 = \sum x_{2u}^2 x_{3u}^2 = 16a^2c^2, \\
& \sum x_{1u}^2 x_{4u}^2 = \sum x_{2u}^2 x_{4u}^2 = 16a^2d^2, \\
& \sum x_{3u}^2 x_{4u}^2 = 16a^2d^2
\end{aligned}
$$

and all other sums of powers and products up to and including order four are zero. The summation is taken over all design points.

4. Cylindrically rotatable designs of type 3. We wish to find designs such that the variances of the estimated responses at all points on the same s-dimensional cylinder with the same remaining sum of squares of the $(k - s)$ co-ordinates are equal, that is, the estimated responses at two points $\mathbf{x} = (x_1, x_2, \cdots, x_k)$ and $\mathbf{z} = (z_1, z_2, \cdots, z_k)$ will have the same variance if

$$
(4.1) \qquad \sum_{j=1}^{s} x_j^2 = \sum_{j=1}^{s} z_j^2 \quad \text{and} \quad \sum_{j=s+1}^{k} x_j^2 = \sum_{j=s+1}^{k} z_j^2, \qquad (s < k).
$$

These designs will be called cylindrically rotatable designs of type 3. The moment conditions for designs of type 3 are found in a manner similar to that used for designs of type 1. The details may be found in Herzberg (1966b).

A design will be cylindrically rotatable of type 3 if and only if the moments satisfy the following conditions and the variance-covariance matrix is nonsingular:

$$
\begin{aligned}
& [1^{\alpha_1}, 2^{\alpha_2}, \cdots, k^{\alpha_k}] = 0, \qquad \text{any } \alpha_j \text{ odd}, \\
(4.2) \qquad & \qquad = \lambda_{\Sigma_{j=1}^{s} \alpha_j, \Sigma_{j=s+1}^{k} \alpha_j} \cdot [\textstyle\prod_{j=1}^{k} \alpha_j!/2^{\alpha/2} \prod_{j=1}^{k} (\frac{1}{2}\alpha_j)!], \\
& \qquad \qquad \text{all } \alpha_j \text{ even}.
\end{aligned}
$$

Here $\lambda_{\Sigma_{j=1}^{s} \alpha_j, \Sigma_{j=s+1}^{k} \alpha_j}$ is constant for any design and any $\sum_{j=1}^{s} \alpha_j$ and $\sum_{j=s+1}^{k} \alpha_j$.

Example. Let $U(x_1, x_2, \cdots, x_k)$ be any one of the smallest 2^{-p} fractions of a 2^k factorial design such that

$$
\sum_u x_{iu}^{\alpha_i} x_{ju}^{\alpha_j} x_{lu}^{\alpha_l} x_{mu}^{\alpha_m} = 0,
$$

where

(i) $i, j, l, m = 1, 2, \cdots, k$ and are distinct,
(ii) at least one of $\alpha_i, \alpha_j, \alpha_l, \alpha_m$ is odd and $0 < \alpha_i + \alpha_j + \alpha_l + \alpha_m \leqq 4$, and

(iii) the summation is taken over all the points of $U(x_1, x_2, \cdots, x_k)$.

Then consider the following point sets:

$$
(4.3)\qquad
\begin{aligned}
&U(a, a, a, c, c),\\
&(\pm 2a, 0, 0, 0, 0),\\
&(0, \pm 2a, 0, 0, 0),\\
&(0, 0, \pm 2a, 0, 0),\\
&(0, 0, 0, \pm 2c, 0),\\
&(0, 0, 0, 0, \pm 2c).
\end{aligned}
$$

For all values of a and c except zero, these point sets will form a second order cylindrically rotatable design of type 3 in five dimensions since the moments satisfy (4.2) and the variance-covariance matrix is non-singular. The number of points involved is twenty-six. The moments of the design are, when multiplied by N,

$$
(4.4)\qquad
\begin{aligned}
&\sum x_{1u}^2 = \sum x_{2u}^2 = \sum x_{3u}^2 = 24a^2,\\
&\sum x_{4u}^2 = \sum x_{5u}^2 = 24c^2,\\
&\sum x_{1u}^4 = \sum x_{2u}^4 = \sum x_{3u}^4 = 3\sum x_{1u}^2 x_{2u}^2\\
&\qquad = 3\sum x_{1u}^2 x_{3u}^2 = 3\sum x_{2u}^2 x_{3u}^2 = 48a^4,\\
&\sum x_{4u}^4 = \sum x_{5u}^4 = 3\sum x_{4u}^2 x_{5u}^2 = 48c^4,\\
&\sum x_{iu}^2 x_{ju}^2 = 16a^2c^2, \quad i = 1, 2, 3 \text{ and } j = 4, 5,
\end{aligned}
$$

and all other sums of powers and products up to and including order four are zero. The summation is taken over all design points.

5. Blocking. Here we shall discuss the blocking of only those cylindrically rotatable designs for which all the odd moments are zero. (A moment is said to be odd if at least one of the α_i is odd. We shall also limit our discussion to designs comprising only two blocks and in which one block forms a complete first order cylindrically rotatable design and the other block consists of the additional points needed to make the whole a second order cylindrically rotatable design. Therefore, if it is discovered that the first order design gives an adequate fit, then there is no need to perform the experiments of the second block

Box and Hunter (1957) have shown that the required conditions for block effects to be independent of the estimates of the polynomial coefficients are all automatically satisfied when the design is rotatable except

$$
(5.1)\qquad \sum_u^{n_w'} x_{iu}^2 \Big/ \sum_{u=1}^{N} x_{iu}^2 = n_w'/N,
$$

where n_w' denotes the number of points in the wth block, $w = 1, 2; i = 1, 2, \cdots, k$

and the summation in the numerator is for those values of u in the wth block. Equation (5.1) is also the only condition which is not automatically satisfied when cylindrically rotatable designs are used. The details may be found in Herzberg (1966b).

Let

$$n_w' = n_w + n_{0w}, \tag{5.2}$$

where n_w denotes the number of points not at the centre and n_{0w} denotes the number of centre points in the wth block. The number of centre points in each block is chosen so that the blocking conditions will be satisfied.

Equation (5.1) can be rewritten as

$$n_2' \sum_u^{n_1} x_{iu}^2 - n_1' \sum_u^{n_2} x_{iu}^2 = 0,$$

that is,

$$(n_2 + n_{02}) \sum_u^{n_1} x_{iu}^2 - (n_1 + n_{01}) \sum_u^{n_2} x_{iu}^2 = 0, \tag{5.3}$$

where $i = 1, 2, \cdots, k$. In order to determine the blocked design, the following procedure can be used. For $i = 1, 2, \cdots, s$, we have, from (5.3),

$$n_{01} = \{(n_2 + n_{02}) \sum_u^{n_1} x_{iu}^2 - n_1 \sum_u^{n_2} x_{iu}^2\} / \sum_u^{n_2} x_{iu}^2, \tag{5.4}$$

where n_{02} must be such that n_{01} will be non-negative. n_{01} and n_{02} can be chosen to satisfy (5.4), and then the levels of the remaining $(k - s)$ factors are determined so that (5.3) will be satisfied for $i = s + 1, \cdots, k$.

EXAMPLE. Here we consider a second order cyclindrically rotatable design of type 2 in four dimensions, where $s = 2$.

Let

$$(\pm a, \pm a, \pm c, \pm c) \tag{5.5}$$

be the points of the first block, and

$$\begin{array}{cccc}(\pm 2a, & 0, & 0, & 0),\\ (\ 0, & \pm 2a, & 0, & 0),\\ (\ 0, & 0, & \pm e, & 0),\\ (\ 0, & 0, & 0, & \pm e)\end{array} \tag{5.6}$$

be the points of the second block, where (5.5) forms a first order cylindrically rotatable design of type 2. Then, for $i = 1, 2$, (5.4) becomes

$$n_{01} = \{(8 + n_{02})16a^2 - 16 \cdot 8a^2\}/8a^2 = 2n_{02}. \tag{5.7}$$

Let $n_{02} = 0$. Then, from (5.7), $n_{01} = 0$. Substituting these values of n_{01} and n_{02} in (5.3) for $i = 3$ and 4, we obtain

$$8 \cdot 16c^2 - 16 \cdot 2e^2 = 0. \tag{5.8}$$

Therefore,

$$e^2 = 4c^2, \qquad (5.9)$$
$$e = 2c.$$

6. Interpretation. From the moment conditions for cylindrically rotatable designs, it can be seen that a k-dimensional cylindrically rotatable design is an extension of an s-dimensional rotatable design of the same order, $s < k$. The variances of the estimated responses continue to be constant at points on s-dimensional cylinders. Therefore, by using a cylindrically rotatable design, it is possible to preserve this property of an s-dimensional rotatable design with the added advantage of enabling the experimenter to estimate the coefficients of the terms of the polynomial involving the $(s + 1)$th to kth factors.

Since the moment conditions for cylindrically rotatable designs are not as restrictive as the moment conditions for rotatable designs of the same dimension and order, the number of points required for a cylindrically rotatable design is usually less than the number required for a rotatable design.

The three types of cylindrically rotatable design are such that the variances of the estimated responses at certain points on a cylinder whose axis is a coordinate axis are equal. As mentioned previously, cylindrically rotatable designs of type 1 are such that the variances of the estimated responses at all points on the same s-dimensional cylinder with the same remaining $(k - s)$ co-ordinates are equal; designs of type 2 are such that the variances of the estimated responses at all points on the same s-dimensional cylinder with the same remaining squares of the $(k - s)$ co-ordinates are equal; designs of type 3 are such that the variances of the estimated responses at all points on the same s-dimensional cylinder with the same remaining sum of squares of the $(k - s)$ co-ordinates are equal (geometrically this means that the variances of the estimated responses at all points at the intersection of two cylinders are equal). The three types of cylindrically rotatable design differ in their moment conditions as can be seen from (2.16), (3.2) and (4.2). The moment restrictions and, therefore, the number of experimental points required increase when the number of points at which the variances of the estimated responses are equal increases. The experimenter must decide whether the use of a larger number of experimental points is balanced by the larger number of estimated responses having equal variances.

When $s = k - 1$, the moment conditions for a cylindrically rotatable design of type 3 can be written in the following way:

$$(6.1) \quad [1^{\alpha_1}, 2^{\alpha_2}, \cdots, k^{\alpha_k}] = 0, \qquad \text{any } \alpha_j \text{ odd},$$
$$= \lambda'_{\alpha-\alpha_k,\alpha_k} \cdot [\textstyle\prod_{j=1}^{k-1} \alpha_j!/2^{\alpha/2} \prod_{j=1}^{k-1} (\frac{1}{2}\alpha_j)!], \qquad \text{all } \alpha_j \text{ even},$$

where $\lambda'_{\alpha-\alpha_k,\alpha_k} = \lambda_{\alpha-\alpha_k,\alpha_k}[\alpha_k!/(\frac{1}{2}\alpha_k)!]$. Therefore, when $s = k - 1$, cylindrically rotatable designs of types 2 and 3 are equivalent. If the moments of a cylindrically rotatable design of type 1 are such that they are zero for any α_j odd, then cylindrically rotatable designs of types 1 and 2 are equivalent. Therefore, if

$s = k - 1$ and all the odd moments of a cylindrically rotatable design of type 1 are zero, cylindrically rotatable designs of types 1, 2 and 3 are equivalent.

For cylindrically rotatable designs of types 1 and 2, the level of the $(s + 1)$th to kth factors can be integers because of the nature of the moment conditions and, therefore, $(k - s)$ factors can be discrete.

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