TIMID PLAY IS OPTIMAL[1]

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You are in a gambling house Γ which is infinitely wealthy and offers all subfair bets, except

(a) it transacts business only in integer multiples of a dollar,

(b) it allows no credit, and

(c) money must be won or lost on each bet.

You start with a finite number of dollars, and keep betting until you go broke. How should you gamble so as to delay this fate as long as possible? Provided you have money, you should gamble next time so as to win or lose a dollar with probability $\frac{1}{2}$ each; that is, you should play timidly.

This theorem will be stated formally as Theorem 1, and proved below. A very similar result was obtained independently by Molenaar and van der Velde (1967). I first learned of the gambling house Γ while reading a draft of Leo Breiman's book for Addison-Wesley. Breiman showed that you do go broke.

Theorem 2 partially extends Theorem 1 to the continuous case. Theorem 3 is an analog of Theorem 1, with all bets uniformly subfair in a certain sense. Of course, Theorem 3 can be extended to the continuous case.

Let $X_1, X_2, \cdots$ be integer-valued random variables, $S_0 = 0$, $S_n = X_1 + \cdots + X_n$. Let j be a nonnegative integer. Say $j + S_n : n = 0, 1, \cdots$ is a *j-process* iff for all $n \geqq 0$: (i) $j + S_n \geqq 0$; (ii) $E(X_{n+1} \mid X_1, \cdots, X_n) \leqq 0$; (iii) $X_{n+1} \neq 0$ on $j + S_n \neq 0$. From (i) and (ii), $X_{n+1} = 0$ on $j + S_n = 0$. Informally, $j + S_n : n = 0, 1, \cdots$ is a possible process of fortunes if you gamble in Γ, starting with j (dollars). Let N_j be the least $n \geqq 0$ if any with $j + S_n = 0$; if none, $N_j = \infty$.

Theorem 1. *For nonnegative integer j and k, among all j-processes, $P(N_j > k)$ is maximized when: given $X_1, \cdots, X_n$, on $j + S_n > 0$, X_{n+1} is ± 1 with conditional probability $\frac{1}{2}$ each, for $0 \leqq n \leqq k - 1$.*

Corollary (Breiman). $P(N_j < \infty) = 1$.

The proof is easy, with the help of Lemmas 1, 2 and 3. A slightly more careful argument shows the maximum is strict. To state Lemma 1, let $u = \{u(n) : n = 0, 1, \cdots\}$ be a sequence of real numbers. Define Tu, another sequence, as follows: $(Tu)(0) = 0$ and $(Tu)(n) = \frac{1}{2}u(n + 1) + \frac{1}{2}u(n - 1)$ for $n = 1, 2, \cdots$. Say u is *nice* iff $u(0) = 0$, $u(n)$ is nondecreasing with n, and $u(n + 1) - u(n)$ is nonincreasing with n.

Lemma 1. *If u is nice, so is Tu.*

Proof. Easy. □

To state Lemma 2, let $v(0) = 0$ and $v(n) = 1$ for $n \geqq 1$. Plainly, v is nice.

Received 9 December 1966.

[1] Prepared with the partial support of the U.S. Army Research Office (Durham), Grant DA-31-124-ARO-D-548.

Let $Y_1, Y_2, \cdots$ be independent and identically distributed, Y_n being ± 1 with probability $\frac{1}{2}$ each. Let $Z_0 = 0$ and $Z_n = Y_1 + \cdots + Y_n$.

LEMMA 2. $(T^k v)(j) = P(j + Z_n > 0$ *for all* $n \leqq k)$.

PROOF. Induction on k. □

For Lemma 3, let u be nice, and suppose

(1) m is a nonnegative integer, X_1 is an integer-valued random variable

and either

$$(2) \qquad m = 0, \qquad X_1 = 0 \quad \text{a.e.}$$

or

$$(3) \quad m > 0, \qquad m + X_1 \geqq 0 \quad \text{a.e.}, \qquad |X_1| \geqq 1 \quad \text{a.e.}, \qquad E(X_1) = 0.$$

LEMMA 3. $E\{u(m + X_1)\} \leqq (Tu)(m)$.

PROOF. Convexity argument. □

PROOF OF THEOREM 1. The theorem is trivial for $k = 0$, and easy for $k = 1$. Suppose the theorem holds for some k. That is, by Lemma 2, the probability that a j-process does not hit zero on or before time k is at most $(T^k v)(j) = u(j)$. Now consider the probability π that an m-process $\{m, m + X_1, m + X_1 + X_2, \cdots\}$ does not hit zero on or before time $k + 1$. Plainly, m and X_1 satisfy (1) and either (2) or (3). Given X_1, the process $\{m + X_1, m + X_1 + X_2, m + X_1 + X_2 + X_3, \cdots\}$ is conditionally an $(m + X_1)$-process. By the inductive assumption, $\pi \leqq E\{u(m + X_1)\}$. By Lemmas 1 and 3, $E\{u(m + X_1)\} \leqq (Tu)(m) = (T^{k+1}v)(m)$. □

The idea behind this induction is, of course, familiar to dynamic programmers.

The theorem can be partially extended to cover real-valued variables. Let $X_1, X_2, \cdots$ be real-valued random variables, $S_0 = 0$, $S_n = X_1 + \cdots + X_n$. Let x be a nonnegative real number. Say $x + S_n : n = 0, 1, \cdots$ is an *x-process* iff for all $n \geqq 0$: (i) $x + S_n \geqq 0$; (ii) $E(X_{n+1} \mid X_1, \cdots, X_n) \leqq 0$, (iii) $|X_{n+1}| \geqq 1$ on $x + S_n \geqq 1$; (iv) $X_{n+1} = 0$ on $x + S_n < 1$. Informally, condition (a) on Γ is dropped, but (c) is modified so that on each bet at least a dollar is won or lost. Let N_x be the least n if any with $x + S_n < 1$, if none $N_x = \infty$. Thus, N_x is the waiting time to a fortune where you cannot bet.

Define random variables $Y_1, Y_2, \cdots$ as follows. For $x = 0$, all vanish. For $x > 0$, Y_1 has mean 0 and $x + Y_1$ is either the least integer greater than x or the greatest integer less than x. On $x + Y_1 + \cdots + Y_n = 0$, $Y_{n+1} = 0$. Given $Y_1, \cdots, Y_n$, on $x + Y_1 + \cdots + Y_n > 0$, Y_{n+1} is ± 1 with conditional probability $\frac{1}{2}$ each. The process $x + Y_1 + \cdots + Y_n : n = 0, 1, \cdots$ is almost an x-process; it misses for noninteger $x > 0$ because $|Y_1| < 1$. Let M_x be the least $n \geqq 0$ with $x + Y_1 + \cdots + Y_n = 0$. Plainly, $x \to P(M_x > k)$ is continuous, takes the value $(T^k v)(j)$ for $x = j$, and is linearly interpolated.

THEOREM 2. $P(N_x > k) \leqq P(M_x > k)$.

PROOF. As for Theorem 1. □

COROLLARY. $P(N_x < \infty) = 1$.

Here is a different generalization of Theorem 1. Let $r > 0$ with $r \neq 1$. There is a unique probability number p, namely $p = r/(r + 1)$, such that $pr^{-1} + (1 - p)r = 1$. (For $1 < r < \infty$, $\frac{1}{2} < p < 1$; for $0 < r < 1, 0 < p < \frac{1}{2}$.) Let $X_1, X_2, \cdots$ be integer-valued random variables, $S_0 = 0$, $S_n = X_1 + \cdots + X_n$. Let j be a nonnegative integer. Say $j + S_n : n = 0, 1, \cdots$ is an *(j, r)-process* if for all $n \geqq 0$: (i) $j + S_n \geqq 0$; (ii) $r > 1$ implies $E(r^{X_{n+1}} \mid X_1, \cdots, X_n) \leqq 1$, while $r < 1$ implies $E(r^{X_{n+1}} \mid X_1, \cdots, X_n) \geqq 1$; (iii) $X_{n+1} \neq 0$ on $j + S_n > 0$.

Plainly, on $j + S_n = 0$, $X_{n+1} = 0$. (Of course, for $r > 1$, $E(r^X) \leqq 1$ implies $E(X) \leqq 0$; for $r < 1$, $E(r^X) \geqq 1$ places no restriction on $E(X)$, but does limit $P(X \geqq n)$ for $n > 0$.) Let N_j be the least n if any with $j + S_n = 0$; if none, $N_j = \infty$.

THEOREM 3. *For nonnegative integer j and k, among all (j, r)-processes, $P(N_j > k)$ is strictly maximized when: given $X_1, \cdots, X_n$, on $j + S_n > 0$, X_{n+1} is -1 with conditional probability p and $+1$ with conditional probability $1 - p$, for $0 \leqq n \leqq k - 1$.*

PROOF. As in Theorem 1, with these modifications. Replace T by T_p, where $(T_p u)(0) = 0$ and $(T_p u)(m) = pu(m - 1) + (1 - p)u(m + 1)$. Replace nice by *$p$-nice*, where u is p-nice iff $u(0) = 0$, $u(n)$ is nondecreasing with n, and $u(n) \geqq pu(n - 1) + (1 - p)u(n + 1)$ for all $n \geqq 1$. The only new problem is to show that for $m > 0$, $X \geqq -m$, $|X| \geqq 1$, $E(r^X) = 1$, and p-nice u, $Eu(m + X) \leqq (T_p u)(m)$. To do that find a and b with $ar^{m-1} + b = u(m - 1)$ and $ar^{m+1} + b = u(m + 1)$. (For $r > 1$, $a > 0$, while for $r < 1$, $a < 0$.) Check that

$$ar^m + b = (T_p u)(m) \tag{4}$$

and

$$ar^n + b \geqq u(n) \quad \text{for} \quad n \neq m, \qquad n = 0, 1, \cdots. \tag{5}$$

So $E\{u(m + X)\} \leqq E(ar^{m+X} + b) = ar^m + b = (T_p u)(m)$. One easy way to check (4) and (5) is this. Introduce a continuous function v (for $r > 1$, defined on $[1, \infty)$, while for $r < 1$, defined on $(0, 1]$), whose value at r^n is $u(n)$, and which is linearly interpolated between. Because u is p-nice, v is concave. □

COROLLARY. *If $r > 1$, $P(N_x < \infty) = 1$.*

The analog of Theorem 2 can also be proved.

REFERENCE

MOLENAAR W. and VAN DER VELDE E. A. (1967). How to survive a fixed number of fair bets. *Ann. Math. Statist.* **38** 1278–1280.