THE CONDITIONAL LEVEL OF STUDENT'S t TEST[1]

BY L. BROWN

Cornell University

1. Introduction. Buehler and Fedderson (1963) considered the conditional significance level of Student's two-sided t-test and the coverage of the related confidence intervals. They conditioned on a subset of the form $|\bar{x}|/s \leqq c$ and found in one special case ($n = 2$, $\alpha = .5$) that for any values of μ_0 and σ^2 the conditional level of the t-test that the population mean is μ_0 is smaller than the unconditional level. In fact it is strictly smaller than a constant $a < \alpha = .5$. (For $c = \frac{2}{3}$ they were able to choose $a = .482$). Hence the conditional confidence coefficient of the confidence interval procedure is greater than $1 - a > .5$.

In this note we will show that similar results are valid for Student's two sided t-test at all levels and for all sample sizes, $n \geqq 2$. Also we show that the disparity between the conditional and unconditional levels is larger than was previously assumed. For example, in the case $n = 2$, $\alpha = .5$ we show that the conditional probability of acceptance given $|\bar{x}|/s \leqq \tan \pi/8 = 2^{\frac{1}{2}} + 1$ is bounded below by $\frac{2}{3}$.

In view of the well known optimum properties of the t-test it is not clear that the results of this note can possibly lead to any practically useful new procedures. (It is not even clear that any remotely reasonable test procedures exist for this problem which do not have conditional properties similar to those described here.)

We hope that these results about the t-test will help add to the general knowledge concerning its characteristics. In particular, let us point out that these results are somewhat related to the fact that the usual invariant estimator of σ is inadmissible (see Brown (to appear)). However it would appear that, if anything, these results concerning tests depend more strongly on normality than do the results for estimation.

2. Statement and proof of the main theorem. Let $X_1, X_2, \cdots, X_n$, $n \geqq 2$, be independent normal random variables with mean μ and variance σ^2. Let $\bar{x} = n^{-1} \sum_{i=1}^{n} x_i$ and $s^2 = n^{-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$. Let the (unconditional) rejection region for testing $\mu = \mu_0$ be of the form $K = \{\bar{x}, s : |\bar{x} - \mu_0|/s > k\}$. Then the level of significance, $\alpha = \Pr(K \mid \mu_0, \sigma)$, is independent of σ^2. Let the "conditioning" set be $C = \{\bar{x}, s : |\bar{x}|/s \leqq c\}$.

THEOREM. *Suppose* $c > k/[(1 + k^2)^{\frac{1}{2}} - 1]$. *Then there is a constant* $a < \alpha$ *such that* $\Pr\{K \mid (\bar{x}, s) \varepsilon C, \mu_0, \sigma^2\} \leqq a < \alpha$ *for all* μ_0, σ^2.

PROOF. Since K and C depend only on the ratios $\bar{x}/s$ and μ/s, and μ/σ, $\Pr\{K \mid (\bar{x}, s) \varepsilon C, \mu_0, \sigma\}$ is a function of the parameters only through the ratio

Received 1 December 1966.

[1] This research was supported in part by the Office of Naval Research Contract number Nonr 401(50) and in part by the National Science Foundation grant GP4867. Reproduction in whole or in part is permitted for any purpose of the United States Government.

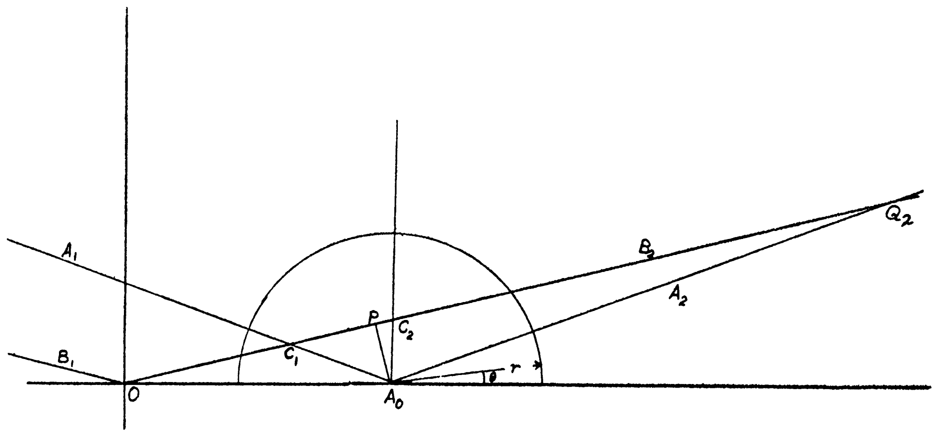

FIG. 1

μ_0/σ. There is therefore no loss of generality in assuming $\sigma^2 = 1$, and $\mu = \mu_0$ and we shall do so for the remainder of the proof. The theorem is clearly true if $\mu_0 = 0$. Using symmetry there is then no loss of generality if we assume $\mu_0 > 0$.

Throughout the proof we will constantly refer to figure one. In this figure the axes are $\bar{x}$ and s, A_0 is the point $(\mu_0, 0)$, the region contained in $A_1A_0A_2$ is the acceptance region K'. C is the region in B_10B_2, $A_0P \perp 0B_2$, and $A_0C_2 \perp 0A_0$.

The condition $c > k/[(1 + k^2)^{\frac{1}{2}} - 1]$ is precisely the condition which implies $\angle C_2A_0C_1 > 2\cdot \angle C_2A_0P$. ("$\angle C_2A_0P$" denotes the radian measure of the angle between C_2A_0 and A_0P.) It follows that $P \varepsilon K$ and $\overline{PC_1} \geqq \overline{PC_2}$ and $\overline{A_0C_1} \geqq \overline{A_0C_2}$. The lines $0B_2$ and A_0A_2 may intersect in a point Q_2 as shown in Figure one or they may not intersect at all for $s \geqq 0$. If they do not intersect for $s \geqq 0$, write $\overline{A_0Q_2} = \infty$. In either case $\overline{A_0Q_2} > \overline{A_0C_1}$. When $\overline{A_0Q_2} = \infty$ the lines A_0A_1 and $0B_1$ may intersect for some $s > 0$. Call the point of intersection Q_1. As above, if there is no such intersection we define $\overline{A_0C_1} = \infty$.

We consider a system of polar coordinates in Figure one with A_0 as its center, i.e. $r^2 = (\bar{x} - \mu_0)^2 + s^2$ and $\tan\theta = s/(\bar{x} - u_0)$.

Since $\bar{x}$, s are values of a random variable, r and θ may also be considered values of random variables whose probability density is

$$f(r, \theta) = ar^{n-1}(\sin\theta)^{n-2}e^{-nr^2/2}, \qquad r \geqq 0, \quad 0 \leqq \theta \leqq \pi.$$

Hence the conditional density of the variables R, Θ given $R = r$ is $b(\sin\theta)^{n-2}$ for $0 \leqq \theta \leqq \pi$, and is independent of the given value of R. It follows that for all $r > 0$, $\Pr(K' \mid R = r) = 1 - \alpha$. (As before, K' denotes the complement of K.)

Let

$$p_1(r) = \Pr(K' \cap C \cap \{\theta:\theta \leqq \pi/2\} \mid R = r);$$

$$p_2(r) = \Pr(C \cap \{\theta:\theta \leqq \pi/2\} \mid R = r);$$

$$p_3(r) = \Pr(K' \cap C \cap \{\theta:\theta > \pi/2\} \mid R = r);$$

$$p_4(r) = \Pr(C \cap \{\theta:\theta > \pi/2\} \mid R = r);$$

$$\rho(r) = [p_1(r) + p_3(r)]/[p_2(r) + p_4(r)] \quad \text{for} \quad r > \overline{A_0P}.$$

(For $r \leqq \overline{A_0P}$, $p_i(r) = 0, i = 1, 2, 3, 4$; hence $\rho(r)$ is undefined.)

We need consider only the case $r > \overline{A_0P}$. For $r \leqq \overline{A_0C_1}$ and $r \geqq \overline{A_0Q_1}p_4(r) = p_3(r)$. For $\overline{A_0C_1} < r < \overline{A_0Q_1}p_3(r) = (1 - \alpha)/2$ and $p_4(r) \leqq \frac{1}{2}$. (Here we use the fact that the conditional density of Θ is symmetric about $\pi/2$.) Hence $p_3(r)/p_4(r) \geqq 1 - \alpha$ and $p_4(r) \leqq \frac{1}{2}$.

Using Figure one and the expression for the conditional density of Θ given $R = r$ we see that for $r \leqq A_0C_2$, $p_1(r) = p_2(r) = 0$; for $\overline{A_0C_2} < r \leqq \overline{A_0Q_2}$, $p_1(r) > 0$; and for $r > \overline{A_0Q_2}$, $p_1(r) = (1 - \alpha)/2$ and

$$p_2(r) \leqq b \int_{\cot^{-1}c}^{\pi/2} (\sin\theta)^{n-2}\, d\theta < \tfrac{1}{2}.$$

It follows that there is an $\epsilon_1 > 0$ such that $p_1(r)/p_2(r) \geqq 1 - \alpha + \epsilon_1$ whenever $p_2(r) \neq 0$.

Note that $\overline{A_0C_2} < \overline{A_0C_1}$ and that $p_1(r)$ is strictly increasing for $\overline{A_0C_2} < r < \overline{A_0Q_2}$ and non-decreasing for all $r > \overline{A_0C_2}$. Hence there is an $\epsilon_2 > 0$ such that $p_3(r)/p_4(r) < 1$ implies $p_2(r) > \epsilon_2$.

Hence, using the above, either

(i) $p_3(r)/p_4(r) > 1 - \alpha + \epsilon_1$, in which case

$$\rho(r) = [p_1(r) + p_3(r)]/[p_2(r) + p_4(r)] > 1 - \alpha + \epsilon_1,$$

or (ii) $1 - \alpha \leqq p_3(r)/p_4(r)$ and $p_2(r) > \epsilon_2$, in which case

$$\begin{aligned} \rho(r) &= [p_1(r) + p_3(r)]/[p_2(r) + p_4(r)] \\ &\geqq [p_1(r)/p_2(r) + (1 - \alpha)p_4(r)/p_2(r)]/[1 + p_4(r)/p_2(r)] \\ &\geqq 1 - \alpha + \epsilon_1/[1 + p_4(r)/p_2(r)] \geqq 1 - \alpha + \epsilon \end{aligned}$$

where $\epsilon = \epsilon_1/(1 + \frac{1}{2}/\epsilon_2) > 0$.

Since $\Pr\{K' \mid C\} = E(\rho(r))$ we have $\Pr\{K' \mid C\} \geqq 1 - \alpha + \epsilon$ which implies $\Pr\{K \mid C\} \leqq \alpha - \epsilon$. This completes the proof of the theorem.

Note that the conclusion of the theorem will clearly remain true even if the restriction $c > k/[(k^2 + 1)^{\frac{1}{2}} - 1]$ is somewhat relaxed, however some restriction on c is necessary, because $c < k^{-1}$ implies $\angle A_0C_1C_2 > \pi/2$ which implies that the conditional significance level tends to one as $\mu_0 \to \infty$.

3. Numerical results for $n = 2$. When $n = 2$ the distribution of $\bar{x}$, s given r is uniform on the arc of radius r. The computation of $\rho(r)$ then becomes a simple geometrical exercise. It can easily be checked that if $c \geqq k^{-1} = \tan \alpha\pi/2$ then $\rho(r)$ is non-increasing for $r \leqq \overline{0A_0}$ and non-decreasing for $r > \overline{0A_0}$. The minimum value of $\rho(r)$ therefore occurs when $r = \overline{0A_0}$. If we then choose c to maximize this minimum value we find that for $k \geqq 3^{-\frac{1}{2}}$ the appropriate choice of c is such that $\overline{A_00} = \overline{A_0Q}$, i.e. $c = k + (1 + k^2)^{\frac{1}{2}} = \cot \alpha\pi/4 > k^{-1}$. In this case $\rho(\overline{A_00}) =$

$\min_r \rho(r) = 1 - \alpha/(2 - \alpha)$. Hence, for example, for $\alpha = \frac{1}{2}$, $k = 1$ and $c = 1 + 2^{\frac{1}{2}}$. The conditional significance level is bounded above by $\frac{1}{3}$. In general, the conditional significance level for a properly chosen c is bounded above by a number which is asymptotic to $\alpha/2$ as $\alpha \to 0$. It is not clear that we have obtained the best possible inequality in the sense that a slightly smaller choice of c may yield a smaller bound on the overall conditional probability (even though min $\rho(r) < 1 - \alpha/(2 - \alpha)$). For our choice of c the conditional level is exactly $\alpha/(2 - \alpha)$, when $\mu_0 = 0$. (When $\mu_0 = 0$, $\rho(r) = 1 - \alpha/(2 - \alpha)$ for all r). In that sense our upper bound on the conditional level is sharp.

In the less interesting case when $k < 3^{-\frac{1}{2}} (\alpha > \frac{2}{3})$ the best choice of c according to the above reasoning is $c = k^{-1} = \tan \alpha\pi/2$. For this value of c, $\min_r \rho(r) = (1 - \alpha)/\alpha$, so that the conditional significance level is bounded above by $(2\alpha - 1)/\alpha < \alpha$ (for $\alpha < 1$).

REFERENCE

BUEHLER, R. J. and FEDDERSON, A. P. (1963). Note on a conditional property of Student's *t*. *Ann. Math. Statist.* **34** 1098–1100.