ON CERTAIN INEQUALITIES FOR NORMAL DISTRIBUTIONS AND THEIR APPLICATIONS TO SIMULTANEOUS CONFIDENCE BOUNDS[1]

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1. Introduction and summary. In practical situations, one is generally faced with multivariate problems in the form of testing the hypotheses or obtaining a set of simultaneous confidence bounds on certain parameters of interest. We shall consider here the variates under study to be normally distributed. A lot of work on the univariate and multivariate normal populations for the simultaneous confidence bounds on the location and scale parameters has been done, (see references, not necessarily exhaustive). Establishing certain inequalities for normal variates, we try to give shorter confidence bounds on variances and on a given set of linear functions of location parameters when this set is previously chosen for study. For the univariate case, Dunn [6], [8] using the Bonferroni inequality, obtained shorter confidence bounds when the number of linear functions is not too large. We may note that Nair [12], David [5], Dunn [6], [7], [8] and Siotani [22], [24] have studied the closeness of the Bonferroni inequality while deriving the percentage points of certain statistics in univariate and multivariate normal cases. In this paper, we improve the Bonferroni inequality in all the situations considered by Siotani [22], [23], [24] and Dunn [6], [7], [8], and point out various uses of these results in obtaining simultaneous confidence bounds on variances and on linear functions of means (or location parameters) with confidence greater than or equal to $1 - \alpha$ where α is the size of the test. We mention our main results in Section 2 for those who are interested in results and not in proofs. Since our results are extensions of Dunn [6], [8], Siotani [22], [24] and Banerjee [2], [3], their comments on the shortness of the confidence bounds apply to our cases too.

2. Notations and main results.

2.1. *Notations.* As far as possible, a column vector (or a matrix) will be denoted by a small (or capital) bold face letter. $\mathbf{A}: n \times m$ means a matrix with n rows and m columns. An identity matrix will be denoted by $\mathbf{I}$ (or $\mathbf{I}_p$, p being the order of $\mathbf{I}$, wherever there is confusion), and a null matrix will be denoted by $\mathbf{0}$ without any discrimination. $\mathbf{A}'$ means a transpose of $\mathbf{A}$. $|\mathbf{A}|$ means the determinant of $\mathbf{A}$ while $|a|$ means the modulus of a scaler quantity a. If $\mathbf{A}$ is a symmetric positive (semi-)definite (s.p.sd.) matrix, then there exists a symmetric matrix denoted by $\mathbf{A}^{\frac{1}{2}}$ such that $(\mathbf{A}^{\frac{1}{2}})^2 = \mathbf{A}$. A matrix $\mathbf{A}: p \times p$ will be said to have a

Received 26 July 1965; revised 14 November 1966.

[1] This research was supported by the Mathematics Division of the Air Force Office of Scientific Research Contract No. AF-AFOSR-84-63.

structure l if $a_{ii'} = \alpha_i\alpha_{i'}(a_{ii}\, a_{i'i'})^{\frac{1}{2}}$ for $|\alpha_i| \leqq 1, i \neq i' = 1, 2, \cdots, p$, and $a_{ii} > 0$ for all i.

A region $D = D(\mathbf{x})$ will be said to be *symmetric in* $\mathbf{x}$ *about the origin if* $\mathbf{x} \varepsilon D$ implies $-\mathbf{x} \varepsilon D$, while a region $D = D(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n)$ will be said to be *separately symmetric in* $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ *about the origin* if $(\mathbf{x}_1, \cdots, \mathbf{x}_n) \varepsilon D$ implies $(\epsilon_1\mathbf{x}_1, \cdots, \epsilon_n\mathbf{x}_n) \varepsilon D$ for all $\epsilon_1, \cdots, \epsilon_n$ such that $\epsilon_j = +1$ or -1 for $j = 1, 2, \cdots, n$. If $f(\mathbf{x})$ is the density function of $\mathbf{x}$ and $D = D(\mathbf{x})$ is any region, then $P(D)$ is given by

$$P(D) = \int_D f(\mathbf{x})\, d\mathbf{x}. \tag{1}$$

Throughout the paper, E will stand for expectation over the random variables involved. If $\mathbf{x}: p \times 1$ and $\mathbf{y}: q \times 1$ are random vectors, then $\text{Cov}(\mathbf{x}, \mathbf{y}) = E(\mathbf{x} - E(\mathbf{x}))(\mathbf{y} - E(\mathbf{y}))'$ is a matrix of p rows and q columns. Note that $\text{Cov}(\mathbf{y}, \mathbf{x}) = [\text{Cov}(\mathbf{x}, \mathbf{y})]'$. We shall denote $V(\mathbf{x}) = \text{Cov}(\mathbf{x}, \mathbf{x})$. $\chi^2(n)$ will be denoted by a χ^2-statistic with n degrees of freedom. For $x > 0$, we shall write

$$\beta_2(a, b; x) = [B(a, b)]^{-1} \int_0^x y^{a-1}(1 + y)^{-a-b}\, dy \quad \text{and} \tag{2}$$
$$\beta_2(b, a; x^{-1}) = 1 - \beta_2(a, b; x).$$

If $\mathbf{x}$ is a random vector with density function $f(\mathbf{x})$, it is denoted by $\mathbf{x} \sim f(\mathbf{x})$. If $\mathbf{x}_j, j = 1, 2, \cdots, n$, are independent random vectors with the respective density function $f_j(\mathbf{x}_j)$, it is denoted by $\mathbf{x}_j \sim If_j(\mathbf{x}_j), j = 1, \cdots, n$. $\mathbf{x}: p \times 1 \sim N(\boldsymbol{\mu}, V(\mathbf{x}); \mathbf{x})$ provided $\mathbf{x}$ is distributed as normal with mean $\boldsymbol{\mu}$ and variance matrix $V(\mathbf{x})$, and if $V(\mathbf{x})$ is nonsingular, it is written as

$$N(\boldsymbol{\mu}, V(\mathbf{x}); \mathbf{x}) = (2\pi)^{-\frac{1}{2}p}|V(\mathbf{x})|^{-\frac{1}{2}} \exp[-\tfrac{1}{2}(\mathbf{x} - \boldsymbol{\mu})'\{V(\mathbf{x})\}^{-1}(\mathbf{x} - \boldsymbol{\mu})] \quad \text{for all } |x_i| \leqq \infty. \tag{3}$$

A random s.p.d. matrix $\mathbf{S}: p \times p \sim W(p, m, \boldsymbol{\Sigma}; \mathbf{S})$, $m \geqq p$, provided

$$W(p, m, \boldsymbol{\Sigma}; \mathbf{S}) = \{2^{\frac{1}{2}pm}\pi^{p(p-1)/4}\textstyle\prod_{i=1}^p \Gamma(\tfrac{1}{2}(m - i + 1))|\boldsymbol{\Sigma}|^{\frac{1}{2}m}\}^{-1} \cdot |\mathbf{S}|^{\frac{1}{2}(m-p-1)} \exp(-\tfrac{1}{2} \operatorname{tr} \boldsymbol{\Sigma}^{-1}\mathbf{S}). \tag{4}$$

Let $\Gamma_p(m) = \pi^{p(p-1)/4} \prod_{i=1}^p \Gamma(\frac{1}{2}(m - i + 1))$.

A random matrix $\mathbf{Y}: p \times n \sim \Delta(p, n, m; \mathbf{Y})$, $m \geqq p$, provided

$$\Delta(p, n, m; \mathbf{Y}) = \pi^{-\frac{1}{2}pn} \textstyle\prod_{i=1}^p \{\Gamma(\tfrac{1}{2}(m + n - i + 1))/\Gamma(\tfrac{1}{2}(m - i + 1)\} \cdot |\mathbf{I}_p + \mathbf{Y}\mathbf{Y}'|^{-\frac{1}{2}(m+n)}$$

(5) or

$$\Delta(p, n, m; \mathbf{Y}) = \pi^{-\frac{1}{2}pn} \textstyle\prod_{j=1}^n \{\Gamma(\tfrac{1}{2}(m + j))/\Gamma(\tfrac{1}{2}(m - p + j))\} |\mathbf{I}_n + \mathbf{Y}'\mathbf{Y}|^{-\frac{1}{2}(m+n)}.$$

2.2. *Main results.* We give below some of the important results with certain applications without actual proofs.

THEOREM 1. *Let* $D_1 = D_1(\mathbf{x}^{(1)})$ *and* $D_2 = D_2(\mathbf{x}^{(2)})$ *be two convex regions re-*

spectively symmetric in $\mathbf{x}^{(1)}$ *and* $\mathbf{x}^{(2)}$ *about the origin and let* $\mathbf{x} = (\mathbf{x}^{(1)'}, \mathbf{x}^{(2)'})' \sim N(\mathbf{0}, V(\mathbf{x}); \mathbf{x})$. *Then* $P(D_1D_2) \geqq P(D_1)P(D_2)$ *provided the rank of* Cov $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$ *is at the most one.*

THEOREM 2. $P(|x_i| \geqq c_i, i = 1, 2, \cdots, p) \geqq \prod_{i=1}^{p} P(|x_i| \geqq c_i)$ *provided* $\mathbf{x} = (x_1, \cdots, x_p)' \sim N(\mathbf{0}, V(\mathbf{x}); \mathbf{x})$ *and* $V(\mathbf{x})$ *has the structure* l.

The main consequences of the above theorems are Corollaries 5, 7 and 8 of which 5 and 7 are given in particular forms as under:

COROLLARY 5$'$. *If* $\mathbf{x}_j = (x_{1j}, x_{2j}, \cdots, x_{pj})' \sim IN(\mathbf{0}, \mathbf{\Sigma}; \mathbf{x}_j), j = 1, 2, \cdots, n$, and $\mathbf{A}$ is s.p.sd., *then*

$$P((x_{i1}, \cdots, x_{in})\mathbf{A}(x_{i1}, \cdots, x_{in})' \leqq c_i, i = 1, \cdots, p) \geqq \prod_{i=1}^{p} P((x_{i1}, \cdots, x_{in})\mathbf{A}(x_{i1}, \cdots, x_{in})' \leqq c_i).$$

COROLLARY 7$'$. *If* $\mathbf{x}_j = (x_{1j}, x_{2j}, \cdots, x_{pj})' \sim IN(\mathbf{0}, \mathbf{\Sigma}; \mathbf{x}_j), j = 1, 2, \cdots, n$, $\mathbf{\Sigma}$ *has the structure* l, *and* $\mathbf{A}$ *is* s.p.sd., *then*

$$P((x_{i1}, \cdots, x_{in})\mathbf{A}(x_{i1}, \cdots, x_{in})' \geqq c_i, i = 1, \cdots, p) \geqq \prod_{i=1}^{p} P((x_{i1}, \cdots, x_{in})\mathbf{A}(x_{i1}, \cdots, x_{in})' \geqq c_i).$$

The immediate consequences of the above results in simultaneous confidence bounds on variances, which do not require any proofs, are the following (6), (7) and (8):

Let $\mathbf{x}_j \sim IN(\boldsymbol{\mu}, \mathbf{\Sigma}; \mathbf{x}_j)$, $j = 1, 2, \cdots, n$. Let $\bar{\mathbf{x}} = \sum_{j=1}^{n} \mathbf{x}_j/n$ and $s_{ii} = \sum_{j=1}^{n} (x_{ij} - \bar{x}_i)^2$, $i = 1, 2, \cdots, p$. Then, with confidence coefficient greater than or equal to $(1 - \alpha)$, we have the lower limit on σ_{ii}, $i = 1, 2, \cdots, p$, as

$$\sigma_{ii} \geqq c_{i,1}^{-1} s_{ii}, i = 1, 2, \cdots, p, \tag{6}$$

where $c_{i,1}$'s are to be calculated from $(1 - \alpha) = \prod_{i=1}^{p} P(\chi^2(n - 1) \leqq c_{i,1})$, while if $\mathbf{\Sigma}$ has the structure l, the simultaneous confidence bounds on the upper limit of σ_{ii}, $i = 1, \cdots, p$, with confidence greater than or equal to $(1 - \alpha)$ is given by

$$\sigma_{ii} \leqq c_{i,2}^{-1} s_{ii}, i = 1, 2, \cdots, p, \tag{7}$$

where $c_{i,2}$'s are to be calculated from $(1 - \alpha) = \prod_{i=1}^{p} P(\chi^2(n - 1) \geqq c_{i,2})$.

We give one more application on variances. Let $\mathbf{x}_j : p \times 1$ and $\mathbf{y}_{j'} : q \times 1$, $j = 1, 2, \cdots, n$ and $j' = 1, 2, \cdots, m$, be independent random vectors such that $\mathbf{x}_j \sim IN(\boldsymbol{\mu}, \mathbf{\Sigma}_1; \mathbf{x}_j)$ and $\mathbf{y}_{j'} \sim IN(\boldsymbol{\nu}, \mathbf{\Sigma}_2; \mathbf{y}_{j'})$ in which $\mathbf{\Sigma}_2$ has the structure l. Let $s_{ii,1} = \sum_{j=1}^{n} (x_{ij} - \bar{x}_i)^2$ and $s_{i'i',2} = \sum_{j'=1}^{m} (y_{i'j'} - y_{i'})^2$ for $i = 1, 2, \cdots, p$ and $i' = 1, 2, \cdots, q$. Then, with confidence coefficient greater than or equal to $(1 - \alpha)$, we have the simultaneous lower limit on $\sigma_{ii,1}/\sigma_{i'i',2}$ for $i = 1, \cdots, p$ and $i' = 1, \cdots, q$ as

$$\sigma_{ii,1}/\sigma_{i'i',2} \geqq s_{ii,1}/cs_{i'i',2} \qquad \text{for all } i \text{ and } i' \tag{8}$$

where c is a constant to be determined from $(1 - \alpha) = P(\chi^2_{\max,1}(n - 1) \leqq c\chi^2_{\min,2}(m - 1))$, $\chi^2_{\max,1}(n - 1) = \max_i \chi^2_{i,1}(n - 1)$, $\chi^2_{\min,2}(m - 1) =$

$\min_{i'} \chi^2_{i',2}(m-1)$ and $\chi^2_{i,1}(n-1)$ and $\chi^2_{i',2}(m-1)$, $i = 1, \cdots, p$ and $i' = 1, \cdots, q$, are independent χ^2-variates. Note that the distribution of the above statistic in general is not known. Hence, we can calculate c from $(1-\alpha) = \prod_{i'=1}^{q} P(\chi^2_{\max,1}(n-1) \leqq c\chi^2_{i'}(m-1))$. The value of c can be calculated by using the tables of Pillai and Ramachandran [13] and Krishnaiah [11]. Even if these tables are not available, we can calculate c from $(1-\alpha) = \prod_{i=1}^{p} \prod_{i'=1}^{q} \cdot P(\chi^2_{i,1}(n-1) \leqq c\chi^2_{i',2}(m-1))$. Similarly, we can consider the simultaneous upper limit of $\sigma_{ii,1}/\sigma_{i'i',2}$ for all i and i' by interchanging the structure l from $\mathbf{\Sigma}_2$ to $\mathbf{\Sigma}_1$. The proofs of (6), (7) and (8) are not given in the text and they are left to the reader.

The following two theorems are connected with maximum (or minimum) Hotelling T^2 as defined by Siotani [22], [23], [24] when the tables are not available.

THEOREM 3. *Let* $\mathbf{x}_j : p \times 1, j = 1, 2, \cdots, n$, *be normally distributed with zero means and* Cov $(\mathbf{x}_j, \mathbf{x}_{j'}) = b_{jj'}\mathbf{\Sigma}$ for $j, j' = 1, 2, \cdots, n$, *and* $\mathbf{\Sigma}$ *is any* s.p.d. *Let* $\mathbf{S}: p \times p \sim W(p, m, \mathbf{\Sigma}; \mathbf{S})$ *and be independent of* $(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n)$. *Then*,

$$(9)\quad P(\mathbf{x}_j'\mathbf{S}^{-1}\mathbf{x}_j \leqq c_j b_{jj}, j = 1, \cdots, n) \geqq \prod_{j=1}^{n} P(\mathbf{y}_j'\mathbf{y}_j \leqq c_j, j = 1, \cdots, n)$$

where $\mathbf{Y}: p \times n = (\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n) \sim \Delta(p, n, m; \mathbf{Y})$. *When the tables on the right side of* (9) *are not available, we can use the inequalities*

$$(10)\qquad P(\mathbf{y}_j'\mathbf{y}_j \leqq c, j = 1, 2, \cdots, n) \geqq \{\beta_2(\tfrac{1}{2}p, \tfrac{1}{2}(m+1-p); c\}^n$$

and

$$(11)\quad \prod_{j=1}^{n} \beta_2(\tfrac{1}{2}p, \tfrac{1}{2}(m+j-p); c_{(j)}) \geqq P(\mathbf{y}_j'\mathbf{y}_j \leqq c_j, j = 1, \cdots, n) \geqq w \prod_{j=1}^{n} \beta_2(\tfrac{1}{2}p, \tfrac{1}{2}(m+n-p); c_j)$$

where $c_{(1)} \leqq c_{(2)} \leqq \cdots \leqq c_{(n)}$ *are ordered values of* c_j's, *and*

$$(12)\quad w = [\prod_{j=1}^{n-1} \{\Gamma(\tfrac{1}{2}(m+j))/\Gamma(\tfrac{1}{2}(m-p+j))\}] \cdot [\Gamma(\tfrac{1}{2}(m+n-p))/\Gamma(\tfrac{1}{2}(m+n))]^{n-1}.$$

THEOREM 4. *Let* $\mathbf{x}_j : p \times 1, j = 1, 2, \cdots, n$, *be normally distributed with zero means and* Cov $(\mathbf{x}_j, \mathbf{x}_{j'}) = b_{jj'}\mathbf{\Sigma}$ *for* $j, j' = 1, 2, \cdots, n$, $\mathbf{\Sigma}$ *is any* s.p.d. *and* $(b_{jj'})$ *has the structure* l. *Let* $\mathbf{S}: p \times p \sim W(p, m, \mathbf{\Sigma}; \mathbf{S})$ *and be independent of* $(\mathbf{x}_1, \cdots, \mathbf{x}_n)$. *Then*,

$$(13)\quad P(\mathbf{x}_j'\mathbf{S}^{-1}\mathbf{x}_j \geqq c_j b_{jj}, j = 1, \cdots, n) \geqq \prod_{j=1}^{n} P(\mathbf{y}_j'\mathbf{y}_j \geqq c_j, j = 1, 2, \cdots, n)$$

where $\mathbf{Y}: p \times n = (\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_n) \sim \Delta(p, n, m; \mathbf{Y})$. *If the tables on the right side of* (13) *are not available, we can use the inequalities*

$$(14')\quad P(\mathbf{y}_j'\mathbf{y}_j \geqq c, j = 1, 2, \cdots, n) \geqq [\beta_2(\tfrac{1}{2}(m+1-p), \tfrac{1}{2}p; c^{-1})]^n$$

and

$$(14)\quad P(\mathbf{y}_j'\mathbf{y}_j \geqq c_j, j = 1, 2, \cdots, n) \geqq \prod_{j=1}^{n} \beta_2(\tfrac{1}{2}(m+j-p), \tfrac{1}{2}p; c^{-1}_{(n-j+1)})$$

where $c_{(1)} \leqq \cdots \leqq c_{(n)}$ *are ordered values of* c_j's. *Note that* (14′) *is better than* (14) *when all* c_j's *are equal.*

No applications of Theorem 4 is given. We give below some results on simultaneous confidence bounds on the location parameters.

Let $\mathbf{x}_{jt} : p \times 1 \sim IN(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t ; \mathbf{x}_{jt})$ for $j = 1, 2, \cdots, n_t$ and $t = 1, 2, \cdots, k$ such that $\boldsymbol{\Sigma}_t$, $t = 1, 2, \cdots, k$, have the structures l. Let $\bar{\mathbf{x}}_t = \sum_{j=1}^{n_t} \mathbf{x}_{jt}/n_t$, $(s_{ii'}^{(t)}) = \mathbf{S}_t = \sum_{j=1}^{n_t} (\mathbf{x}_{jt} - \bar{\mathbf{x}}_t)(\mathbf{x}_{jt} - \bar{\mathbf{x}}_t)' : p \times p$, $\mathbf{S} = \sum_{t=1}^{n} \mathbf{S}_t$. Then, if $\boldsymbol{\Sigma}_t$'s are unequal, we have the simultaneous confidence bounds on $\nu_{i\gamma} = \sum_{t=1}^{k} \mu_{it} a_{\gamma t}$, $i = 1, 2, \cdots, p$, and $\gamma = 1, 2, \cdots, r$, with confidence greater than or equal to $(1 - \alpha)$ as

$$(15) \quad z_{i\gamma} - \left(\textstyle\sum_{t=1}^{k} f_t a_{\gamma t}^2 s_{ii}^{(t)}/n_t\right)^{\frac{1}{2}} \leqq \nu_{i\gamma} \leqq z_{i\gamma} + \left(\textstyle\sum_{t=1}^{k} f_t a_{\gamma t}^2 s_{ii}^{(t)}/n_t\right)^{\frac{1}{2}} \text{ for all } i \text{ and } \gamma,$$

where $a_{\gamma t}$'s are given real numbers, $z_{i\gamma} = \sum_{t=1}^{k} a_{\gamma t} \bar{x}_{it}$ and f_t's are to be determined from $\beta_2(\frac{1}{2}, \frac{1}{2}(n_t - 1); f_t) = (1 - \alpha)^{1/pr}$.

In the above problem, let us assume that $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_2 = \cdots = \boldsymbol{\Sigma}_k = \boldsymbol{\Sigma}$(say) and $\boldsymbol{\Sigma}$ is any s.p.d. matrix. Then, simultaneous confidence bounds on $\mathbf{b}'\boldsymbol{\nu}_\gamma = \sum_{t=1}^{k} (\mathbf{b}'\boldsymbol{\mu}_t) a_{\gamma t}$ for $\gamma = 1, \cdots, r$ and for all non-null vector $\mathbf{b}: p \times 1$ with confidence greater than or equal $(1 - \alpha)$ as

$$(16) \quad \mathbf{b}'\mathbf{z}_\gamma - \{c_\gamma(\mathbf{b}'\mathbf{S}\mathbf{b})(\textstyle\sum_{t=1}^{k} a_{\gamma t}^2/n_t)\}^{\frac{1}{2}} \leqq \mathbf{b}'\boldsymbol{\nu}_\gamma \leqq \mathbf{b}'\mathbf{z}_\gamma + \{c_\gamma(\mathbf{b}'\mathbf{S}\mathbf{b})(\textstyle\sum_{t=1}^{k} a_{\gamma t}^2/n_t)\}^{\frac{1}{2}}$$

where $\mathbf{z}_\gamma = \sum_{t=1}^{k} a_{\gamma t} \bar{\mathbf{x}}_t$, and c_γ's are to be determined from

$$(17) \qquad 1 - \alpha = P(\mathbf{y}_\gamma'\mathbf{y}_\gamma \leqq c_\gamma,\ \gamma = 1, 2, \cdots, r),$$
$$\mathbf{Y}: p \times r = (\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_r) \sim \Delta(p, r, \textstyle\sum_{t=1}^{k} n_t - k; \mathbf{Y}).$$

We use the inequalities given in Theorem 3 when the tables for (17) are not known.

In the general MANOVA model, the columns of $\mathbf{X}: q \times n$ are independently distributed as normals with common covariance matrix $\boldsymbol{\Sigma}_1$, and $E(\mathbf{X}) = \mathbf{Q}\boldsymbol{\xi}\mathbf{L}$ where $\mathbf{Q}: q \times s$ and $\mathbf{L}: k \times n$ are known matrices of rank s and k respectively. If $\mathbf{C}: p \times q$ and $\mathbf{A}: k \times r$ are known matrices of respective ranks p and r, let us write

$$\boldsymbol{\nu} = (\nu_{i\gamma}): p \times r = \mathbf{C}\boldsymbol{\xi}\mathbf{A},$$
$$\mathbf{Z} = (z_{i\gamma}): p \times r = \mathbf{C}(\mathbf{Q}'\mathbf{Q})^{-1}\mathbf{Q}'\mathbf{X}\mathbf{L}'(\mathbf{L}\mathbf{L}')^{-1}\mathbf{A},$$
$$\mathbf{B} = (b_{\gamma\gamma'}): r \times r = \mathbf{A}'(\mathbf{L}\mathbf{L}')^{-1}\mathbf{A},$$
$$\mathbf{S}_0 = (s_{ii',0}): q \times q = \mathbf{C}(\mathbf{Q}'\mathbf{Q})^{-1}\mathbf{Q}'\mathbf{X}[\mathbf{I} - \mathbf{B}]\mathbf{X}'\mathbf{Q}(\mathbf{Q}'\mathbf{Q})^{-1}\mathbf{C}'$$
$$\text{and } \boldsymbol{\Sigma}_0: q \times q = \mathbf{C}(\mathbf{Q}'\mathbf{Q})^{-1}\mathbf{Q}'\boldsymbol{\Sigma}_1\mathbf{Q}(\mathbf{Q}'\mathbf{Q})^{-1}\mathbf{C}'.$$

Then, $(\mathbf{z}_\gamma - \boldsymbol{\nu}_\gamma)$, $\gamma = 1, 2, \cdots, r$, are normally distributed with zero means and Cov $(\mathbf{z}_\gamma, \mathbf{z}_{\gamma'}) = b_{\gamma\gamma'}\boldsymbol{\Sigma}_0$ and independently distributed of $\mathbf{S}_0 \sim W(p, n - k, \boldsymbol{\Sigma}_0 ; \mathbf{S}_0)$. Then, simultaneous confidence bounds on $\mathbf{d}'\boldsymbol{\nu}_\gamma$ for all γ's and for all non-null vector $\mathbf{d}: p \times 1$ with confidence greater than or equal to $(1 - \alpha)$ as

$$(18) \qquad \mathbf{d}'\mathbf{z}_\gamma - \{c_\gamma(\mathbf{d}'\mathbf{S}_0\mathbf{d})b_{\gamma\gamma}\}^{\frac{1}{2}} \leqq \mathbf{d}'\boldsymbol{\nu}_\gamma \leqq \mathbf{d}'\mathbf{z}_\gamma + \{c_\gamma(\mathbf{d}'\mathbf{S}_0\mathbf{d})b_{\gamma\gamma}\}^{\frac{1}{2}}$$

where c_γ's are given by (17), and $n = \sum_{t=1}^{k} n_t$. If $\mathbf{\Sigma}_0$ has the structure l, then simultaneous confidence bounds on $\mathbf{v}_{i\gamma}$ for all i and γ with confidence greater than or equal to $(1 - \alpha)$ are given by

$$(19) \qquad z_{i\gamma} - (u_{i\gamma} b_{\gamma\gamma} s_{ii,0})^{\frac{1}{2}} \leqq \nu_{i\gamma} \leqq z_{i\gamma} + (u_{i\gamma} b_{\gamma\gamma} s_{ii,0})^{\frac{1}{2}} \qquad \text{for } i = 1, \cdots, p; \gamma = 1, \cdots, r$$

where $u_{i\gamma}$'s are to be determined from $\prod_{i,\gamma} \beta_2(\frac{1}{2}, \frac{1}{2}(n - k); u_{i\gamma}) = 1 - \alpha$.

We may note that in all the above cases, we require the percentage points of t, χ^2 or F distributions, in some cases maximum studentized t-statistic [13] and maximum Hotelling T^2 [22], [23], [24] when their tables are available.

3. Some inequalities for multivariate normal distributions.

Lemma 1. *Let $D = D(\mathbf{x})$ be a convex set symmetric in $\mathbf{x}$ about the origin. Then*

$$\int_D \exp\left[-\tfrac{1}{2}(\mathbf{x} - y_0\mathbf{b})'\mathbf{\Sigma}^{-1}(\mathbf{x} - y_0\mathbf{b})\right] d\mathbf{x}$$

is a monotonic non-increasing function of $|y_0|$ if $\mathbf{\Sigma}: p \times p$ is s.p.d. *and $\mathbf{b}: p \times 1$.*

This follows from the following result proved by Anderson [1], p. 170–1.

Let $D = D(\mathbf{x})$ be a convex set symmetric in $\mathbf{x}$ about the origin and let $f(\mathbf{x})$ be a function of $\mathbf{x}: p \times 1$ such that (i) $f(\mathbf{x}) \geqq 0$, (ii) $f(\mathbf{x}) = f(-\mathbf{x})$, (iii) $\{\mathbf{x} \mid f(\mathbf{x}) \geqq u\}$ is a convex set for every u, and (iv) $\int_D f(\mathbf{x})\, d\mathbf{x} < \infty$. Then

$$\int_D f(\mathbf{x} + y_0\mathbf{b})\, d\mathbf{x} \geqq \int_D f(\mathbf{x} + \mathbf{b})\, d\mathbf{x} \qquad \text{for } 0 \leqq y_0 \leqq 1.$$

For Lemma 1, $f(\mathbf{x}) = \exp(-\frac{1}{2}\mathbf{x}'\mathbf{\Sigma}^{-1}\mathbf{x})$, and note that all the conditions of the Anderson's result are satisfied.

Lemma 2. *If $\mathbf{S}: p \times p \sim W(p, m, \mathbf{\Sigma}; \mathbf{S})$, $m \geqq p$, then $P(|\mathbf{S}| \leqq c)$ is a monotonic non-increasing function of m for any given values of c and p.*

Proof. We note that the density function of $y = |\mathbf{S}|$ can be written as

$$f_1(y) = \{f_2(m)\}^{-1} y^{\frac{1}{2}(m-p-1)} \int_{D_1} \exp(-\tfrac{1}{2} \operatorname{tr} \mathbf{\Sigma}^{-1}\mathbf{S})\, d\mathbf{S}$$

where $D_1 = D_1(\mathbf{S} \mid y = |\mathbf{S}|)$ and $f_2(m) = 2^{\frac{1}{2}pm}\pi^{p(p-1)/4} \prod_{i=1}^{p} \Gamma(\frac{1}{2}(m - i + 1)) |\mathbf{\Sigma}|^{\frac{1}{2}m}$. It is easy to show that $E(\log |\mathbf{S}|) = f_2'(m)/f_2(m)$ with $f_2'(m) = (d/dm) f_2(m)$. We have

$$P(|S| \leqq c) = \int_0^c f_1(y)\, dy,$$

and hence

$$\begin{aligned} (d/dm)P(|\mathbf{S}| \leqq c) &= \int_0^\infty f_1(x)\, dx \int_0^c f_1(y)(\log y)\, dy \\ &\qquad - \int_0^\infty f_1(y)(\log y)\, dy \int_0^c f_1(x)\, dx \\ &= \int_c^\infty f_1(x)\, dx \int_0^c f_1(y)(\log y)\, dy \\ &\qquad - \int_c^\infty f_1(y)(\log y)\, dy \int_0^c f_1(x)\, dx. \end{aligned}$$

Now, since $(\log y)$ is an increasing function of y, we have

$$\begin{aligned} \int_c^\infty f_1(y)(\log y)\, dy \int_0^c f_1(x)\, dx &\geqq \int_c^\infty f_1(y)\, dy \int_0^c f_1(x)\, dx (\log c) \\ &\geqq \int_c^\infty f_1(x)\, dx \int_0^c f_1(y)(\log y)\, dy. \end{aligned}$$

Hence, $(d/dm)\,P(|S| \leqq c)$ is never positive and this proves Lemma 2.

LEMMA 3. *A* s.p.d. *matrix* $\mathbf{N}: p \times p \sim MB(p, \frac{1}{2}a_1, \frac{1}{2}a_2; \mathbf{N})$ for $(a_1, a_2) \geqq p$ where

$$(20)\quad MB(p, \tfrac{1}{2}a_1, \tfrac{1}{2}a_2; \mathbf{N}) = \Gamma_p(a_1 + a_2)\{\Gamma_p(a_1)\Gamma_p(a_2)\}^{-1}|\mathbf{N}|^{\frac{1}{2}(a_1-p-1)}|\mathbf{I} + \mathbf{N}|^{-\frac{1}{2}(a_1+a_2)}.$$

Then, $M\beta_2(p, \frac{1}{2}a_1, \frac{1}{2}a_2; c) = P(|N| \leqq c)$ *is* (i) *a monotonic non-increasing function of* a_1 *for given values of* p *and* a_2, *while* (ii) *a monotonic non-decreasing function of* a_2 *for given values of* p *and* a_1.

PROOF. Since

$$2^{\frac{1}{2}pa_1}\Gamma_p(a_1 + a_2)\{\Gamma_p(a_2)\}^{-1}|\mathbf{I} + \mathbf{N}|^{-\frac{1}{2}a_1} = E|\mathbf{S}|^{\frac{1}{2}a_1} \quad \text{if} \quad \mathbf{S} \sim \dot{W}(p, a_2, (\mathbf{I} + \mathbf{N})^{-1}; \mathbf{S}),$$

we can write $P(|\mathbf{N}| \leqq c)$ after some transformations as

$$(21)\qquad P(|\mathbf{N}| \leqq c) = \int_D W(p, a_2, \mathbf{I}; \mathbf{S})W(p, a_1, \mathbf{I}; \mathbf{N})\, d\mathbf{S}\, d\mathbf{N}$$

where $D = D\{\mathbf{N}, \mathbf{S} \mid |\mathbf{N}| \leqq c|\mathbf{S}|, \mathbf{N}$ and $\mathbf{S}$ are s.p.d.$\}$. Using Lemma 2 in (21), we get Lemma 3.

LEMMA 4. $\beta_2(a, a_1; c_1)\beta_2(a, a_2; c_2) \geqq \beta_2(a, a_1; c_2)\beta_2(a, a_2; c_1)$ *if* $a_1 \geqq a_2$ *and* $c_2 \geqq c_1$.

Proof follows from the equality

$$\beta_2(a, a_1; c_1)\beta_2(a, a_2; c_2) - \beta_2(a, a_1; c_2)\beta_2(a, a_2; c_1) = \text{constant} \int_{x=0}^{x=c_1}\int_{y=c_1}^{y=c_2} (xy)^{a-1}(1 + x + y + xy)^{-a-a_2} \cdot\{(1 + x)^{-(a_1-a_2)} - (1 + y)^{-(a_1-a_2)}\}\, dx\, dy$$

and the expression in the curly bracket is non-negative for any value of x and y under consideration.

LEMMA 5. *Let* $g(\mathbf{x})$ *and* $h(\mathbf{x})$ *be two functions of real random vector* $\mathbf{x}$. *Then*

$$Eg(\mathbf{x})h(\mathbf{x}) \geqq Eg(\mathbf{x})Eh(\mathbf{x})$$

provided for any two points $\mathbf{x}_1$ *and* $\mathbf{x}_2$, *either* $g(\mathbf{x}_1) \geqq g(\mathbf{x}_2)$ *and* $h(\mathbf{x}_1) \geqq h(\mathbf{x}_2)$ *or* $g(\mathbf{x}_1) \leqq g(\mathbf{x}_2)$ *and* $h(\mathbf{x}_1) \leqq h(\mathbf{x}_2)$, *while* $Eg(\mathbf{x})h(\mathbf{x}) \leqq Eg(\mathbf{x})Eh(\mathbf{x})$ *provided for any two points* $\mathbf{x}_1$ *and* $\mathbf{x}_2$, *either* $g(\mathbf{x}_1) \geqq g(\mathbf{x}_2)$ *and* $h(\mathbf{x}_1) \leqq h(\mathbf{x}_2)$ *or* $g(\mathbf{x}_1) \leqq g(\mathbf{x}_2)$ *and* $h(\mathbf{x}_1) \geqq h(\mathbf{x}_2)$.

PROOF. Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be any two independent and identical random vectors having the same distribution as that of $\mathbf{x}$. Then $\{g(\mathbf{y}_1) - g(\mathbf{y}_2)\}\{h(\mathbf{y}_1) - h(\mathbf{y}_2)\} \geqq 0$ for all $\mathbf{y}_1$ and $\mathbf{y}_2$ for the first part, while $\{g(\mathbf{y}_1) - g(\mathbf{y}_2)\}\{h(\mathbf{y}_1) - h(\mathbf{y}_2)\} \leqq 0$ for all $\mathbf{y}_1$ and $\mathbf{y}_2$ for the second part. Hence, taking the expectations we get Lemma 5 as required.

COROLLARY 1. *Let* $f = f(\mathbf{x})$ *be a function of a random vector* $\mathbf{x}$. *Then,*

(i) $Ef^{2r} \geqq (Ef^{2s})(Ef^{2t})$ *for any non-negative integers* s *and* t *such that* $s + t = r$.

(ii) $Ef^r \geqq (Ef^s)(Ef^t)$ *provided* $f \geqq 0$ *and* s *and* t *are any non-negative numbers such that* $s + t = r$, *and*

(iii) *if* $f > 0$ *and* Ef^{-t} *exists for any* t, *then* $Ef^r \leqq (Ef^s)(Ef^{-t})$ *provided* $s - t = r$ *and* s *and* t *are non-negative.*

Proof of (i) and (ii) follows from the first part of Lemma 5 and proof of (iii) follows from the second part of Lemma 5.

COROLLARY 2. *Let* $\mathbf{x}_j : p \times 1, j = 1, 2, \cdots, n$, *be independent and identically distributed random vectors and be distributed independently of* $\mathbf{y}$: $q \times 1$. *Then,*

$$\text{(i)} \qquad P(f(\mathbf{x}_j, \mathbf{y}) \geqq c, j = 1, \cdots, n) \geqq [P(f(\mathbf{x}, \mathbf{y}) \geqq c]^n \qquad \text{and}$$

$$\text{(ii)} \quad P(f(\mathbf{x}_j, \mathbf{y}) \leqq c, j = 1, 2, \cdots, n) \geqq [P(f(\mathbf{x}, \mathbf{y}) \leqq c]^n$$

where $\mathbf{x}$ *has the same distribution as that of* $\mathbf{x}_j$.

PROOF. Note that when $\mathbf{y}$ is fixed, $f_1(\mathbf{y}) = P(f(\mathbf{x}, \mathbf{y}) \geqq c \mid \mathbf{y}) \geqq 0$ and then the left side of (i) is equal to $E[f_1(\mathbf{y})]^n$. Hence, using the Corollary 1(ii), we get the Corollary 2(i). Similarly, the second part of the corollary can be proved.

THEOREM 1. *Let* $D_1 = D_1(\mathbf{x}^{(1)})$ *and* $D_2 = D_2(\mathbf{x}^{(2)})$ *be two convex regions respectively symmetric in* $\mathbf{x}^{(1)}$ *and* $\mathbf{x}^{(2)}$ *about the origin and let* $\mathbf{x} = (\mathbf{x}^{(1)\prime}, \mathbf{x}^{(2)\prime})' \sim N(\mathbf{0}, V(\mathbf{x}); \mathbf{x})$. *Then* $P(D_1D_2) \geqq P(D_1)P(D_2)$ *provided that the rank of* Cov $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$ *is at the most one.*

PROOF. Let $\mathbf{x}^{(1)}: r \times 1$ and $\mathbf{x}^{(2)}: (p - r) \times 1$. We shall first prove the result when $V(\mathbf{x})$ is s.p.d. Let us consider the random variables $(\mathbf{x}'y_0)' \sim N(\mathbf{0}, V(\mathbf{x}'y_0)'; (\mathbf{x}'y_0)')$ where

$$V\begin{pmatrix} \mathbf{x} \\ y_0 \end{pmatrix} = \begin{pmatrix} V(\mathbf{x}^{(1)}) & \text{Cov}\,(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) & \mathbf{a} \\ \text{Cov}\,(\mathbf{x}^{(2)}, \mathbf{x}^{(1)}) & V(\mathbf{x}^{(2)}) & \mathbf{b} \\ \mathbf{a}' & \mathbf{b}' & 1 \end{pmatrix} \text{ is s.p.d.}$$

Then, it is easy to see that $\mathbf{x} \sim N(\mathbf{0}, V(\mathbf{x}); \mathbf{x})$ and $\mathbf{x}$ given $y_0 \sim N(\mathbf{a}y_0, V(\mathbf{x}^{(1)}) - \mathbf{aa}'; \mathbf{x}^{(1)})N(\mathbf{b}y_0, V(\mathbf{x}^{(2)}) - \mathbf{bb}'; \mathbf{x}^{(2)})$ provided Cov $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = \mathbf{ab}'$. Since the rank of Cov $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$ is at the most one, we can always find two vectors $\mathbf{a}$ and $\mathbf{b}$ such that Cov $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)}) = \mathbf{ab}'$. This means that the distributions of $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ given y_0 are independent normal. Using these facts, we can write $P(D_1D_2) = Eg(y_0)h(y_0)$ where

$$g(y_0) = \int_{D_1} N(\mathbf{a}y_0, V(\mathbf{x}^{(1)}) - \mathbf{aa}'; \mathbf{x}^{(1)})\, d\mathbf{x}^{(1)} \qquad \text{and}$$
$$h(y_0) = \int_{D_2} N(\mathbf{b}y_0, V(\mathbf{x}^{(2)}) - \mathbf{bb}'; \mathbf{x}^{(2)})\, d\mathbf{x}^{(2)}.$$

Note that $Eg(y_0) = P(D_1)$ and $Eh(y_0) = P(D_2)$. Then, using Lemma 1 and the first part of Lemma 5, we get

$$P(D_1D_2) \geqq Eg(y_0)Eh(y_0) = P(D_1)P(D_2)$$

which proves the result when $V(\mathbf{x})$ is s.p.d. When $V(\mathbf{x})$ is singular, we consider $\mathbf{w}^{(1)} = \mathbf{x}^{(1)} + \mathbf{y}^{(1)}$ and $\mathbf{w}^{(2)} = \mathbf{x}^{(2)} + \mathbf{y}^{(2)}$ where $\mathbf{x} \sim N(\mathbf{0}, V(\mathbf{x}); \mathbf{x})$, $\mathbf{y} = (\mathbf{y}^{(1)\prime}, \mathbf{y}^{(2)\prime})' \sim N(\mathbf{0}, q\mathbf{I}; \mathbf{y})$, $\mathbf{x}$ and $\mathbf{y}$ are independently distributed and $q > 0$ (however small) such that $V(\mathbf{x} + \mathbf{y}) = V(\mathbf{x}) + q\mathbf{I}$ is s.p.d. Since $V(\mathbf{x} + \mathbf{y})$ is s.p.d., we have

$$(22) \qquad P(D_1(\mathbf{w}^{(1)})D_2(\mathbf{w}^{(2)})) \geqq P(D_1(\mathbf{w}^{(1)}))P(D_2(\mathbf{w}^{(2)})).$$

We note that as $q \to 0^+$, $\mathbf{y} \to \mathbf{0}$ and consequently $\mathbf{w} \to \mathbf{x}$. Hence taking limits in (22) as $q \to 0^+$, we get Theorem 1 for the singular case too. Thus, Theorem 1 is completely proved.

COROLLARY 3. *Let* $\mathbf{x}_j = (\mathbf{x}_j^{(1)\prime}, \mathbf{x}_j^{(2)\prime})' \sim IN(\mathbf{0}, V(\mathbf{x}_j); \mathbf{x}_j), j = 1, 2, \cdots, n,$ *such that the rank of* Cov $(\mathbf{x}_j^{(1)}, \mathbf{x}_j^{(2)})$ *is at the most one for all* j *and* $V(\mathbf{x}_j) \neq V(\mathbf{x}_{j'})$, $j \neq j'$. *Let* $D_3 = D_3(\mathbf{x}_j^{(1)}, j = 1, 2, \cdots, n)$ *and* $D_4 = D_4(\mathbf{x}_j^{(2)}, j = 1, 2, \cdots, n)$ *be convex regions respectively separately symmetric in* $\mathbf{x}_j^{(1)}, j = 1, \cdots, n$, *and in* $\mathbf{x}_j^{(2)}, j = 1, \cdots, n$, *about the origin. Then,* $P(D_3D_4) \geqq P(D_3)P(D_4)$.

PROOF. Let $N_j = N(\mathbf{0}, V(\mathbf{x}_j); \mathbf{x}_j)$, $N_{j,1} = N(\mathbf{0}, V(\mathbf{x}_j^{(1)}); \mathbf{x}_j^{(1)})$ and $N_{j,2} = N(\mathbf{0}, V(\mathbf{x}_j^{(2)}); \mathbf{x}_j^{(2)})$, for $j = 1, 2, \cdots, n$. Then,

$$P(D_3D_4) = \int_{D_3D_4} \prod_{j=1}^{n} (N_j\, d\mathbf{x}_j).$$

Now, since D_3 and D_4 are convex regions respectively separately symmetric in $\mathbf{x}_j^{(1)}$ $(j = 1, \cdots, n)$ and in $\mathbf{x}_j^{(2)}$ $(j = 1, \cdots, n)$ about the origin, we get D_3 and D_4 as convex regions respectively symmetric in $\mathbf{x}_1^{(1)}$ and $\mathbf{x}_1^{(2)}$ about the origin when $\mathbf{x}_2^{(1)}, \cdots, \mathbf{x}_n^{(1)}, \mathbf{x}_2^{(2)}, \cdots, \mathbf{x}_n^{(2)}$ are kept fixed. Hence, using Theorem 1 for integration over $\mathbf{x}_1$, we get

$$P(D_3D_4) \geqq \int_{D_3D_4} N_{1,1}N_{1,2}\, d\mathbf{x}_1^{(1)}\, d\mathbf{x}_1^{(2)} \prod_{j=2}^{n} (N_j\, d\mathbf{x}_j).$$

Continuing this type of arguments for $\mathbf{x}_2$, then $\mathbf{x}_3, \cdots$, and lastly for $\mathbf{x}_n$, we get

$$P(D_3D_4) \geqq \int_{D_3D_4} \prod_{j=1}^{n} (N_{j,1}\, d\mathbf{x}_j^{(1)} N_{j,2}\, d\mathbf{x}_j^{(2)}) = P(D_3)P(D_4).$$

COROLLARY 4. *Let* $D_i = D_i(x_{ij}, j = 1, 2, \cdots, n)$ *be a convex region separately symmetric in* x_{ij}, $j = 1, \cdots, n$, *about the origin for* $i = 1, 2, \cdots, p$, *and let* $\mathbf{x}_j : p \times 1 \sim IN(\mathbf{0}, V(\mathbf{x}_j); \mathbf{x}_j)$, $j = 1, 2, \cdots, n$. *Then* $P(D_1D_2 \cdots D_p) \geqq \prod_{i=1}^{p} \{P(D_i)\}$.

This follows from Corollary 3.

COROLLARY 5. *Let* $\mathbf{x}_j = (x_{1j}, x_{2j}, \cdots, x_{pj})' \sim IN(\mathbf{0}, \mathbf{\Sigma}; \mathbf{x}_j), j = 1, 2, \cdots, n,$ *and let* $\mathbf{A}_i : n \times n$, $i = 1, 2, \cdots, p$, *be given* s.p.sd. *matrices with real elements such that they are reducible to diagonal matrices by a single orthogonal matrix. Then,*

$$P((x_{i1}, \cdots, x_{in})A_i(x_{i1}, \cdots, x_{in})' \leqq c_i, i = 1, \cdots, p)$$
$$\geqq \prod_{i=1}^{p} P((x_{i1}, \cdots, x_{in})A_i(x_{i1}, \cdots, x_{in})' \leqq c_i).$$

PROOF. Since $\mathbf{A}_i$'s are s.p.sd. and are reducible to diagonal matrices by a single orthogonal matrix $\mathbf{Q}: n \times n$, say, then $\mathbf{A}_i = \mathbf{Q}'\mathbf{T}_i\mathbf{Q}$, for $i = 1, 2, \cdots, p$ where $\mathbf{T}_i$'s are diagonal matrices with non-negative diagonal elements. Let $\mathbf{Q}(x_{i1}, \cdots, x_{in})' = \mathbf{w}_i$, $i = 1, 2, \cdots, p$. Then, $(w_{1j}, \cdots, w_{pj})' \sim IN(\mathbf{0}, \mathbf{\Sigma}; (w_{1j}, \cdots, w_{pj})'), j = 1, 2, \cdots, n$. Then, the left side of Corollary 5 is equal to $P(\mathbf{w}_i'\mathbf{T}_i\mathbf{w}_i \leqq c_i, i = 1, 2, \cdots, p)$ and then using Corollary 4, we get

$$P(\mathbf{w}_i'\mathbf{T}_i\mathbf{w}_i \leqq c_i, i = 1, \cdots, p) \geqq \textstyle\prod_{i=1}^{p} P(\mathbf{w}_i'\mathbf{T}_i\mathbf{w}_i \leqq c_i)$$

$$= \text{the right of Corollary 5.}$$

Thus Corollary 5 is proved.

THEOREM 2. *$P(|x_i| \geqq c_i, i = 1, 2, \cdots, p) \geqq \prod_{i=1}^{p} P(|x_i| \geqq c_i)$ provided $\mathbf{x} = (x_1, x_2, \cdots, x_p)' \sim N(\mathbf{0}, V(\mathbf{x}); \mathbf{x})$ and $V(\mathbf{x})$ has the structure l.*

PROOF. Since $V(\mathbf{x})$ has the structure l, we can write it as $V(\mathbf{x}) = \mathbf{T} + \boldsymbol{\alpha\alpha}'$ where $\mathbf{T}: p \times p$ is a diagonal matrix with diagonal elements $\sigma_i^2(1 - \alpha_i^2)$, $V(x_i) = \sigma_i^2$, $i = 1, 2, \cdots, p$, and $\boldsymbol{\alpha}' = (\sigma_1\alpha_1, \cdots, \sigma_p\alpha_p)$, $|\alpha_i| \leqq 1$. Let us assume that $V(\mathbf{x})$ is s.p.d. Consider the random variables $(\mathbf{x}'y_0)' \sim N(\mathbf{0}, V(\mathbf{x}'y_0)'; (\mathbf{x}'y_0)')$, where

$$V\begin{pmatrix} \mathbf{x} \\ y_0 \end{pmatrix} = \begin{pmatrix} \mathbf{T} + \boldsymbol{\alpha\alpha}' & \boldsymbol{\alpha} \\ \boldsymbol{\alpha}' & 1 \end{pmatrix} \text{ is s.p.d.}$$

Then, it is easy to see that $\mathbf{x} \sim N(\mathbf{0}, V(\mathbf{x}); \mathbf{x})$ and x_i given

$$y_0 \sim IN(\sigma_i\alpha_i y_0, \sigma_i^2(1 - \alpha_i^2); x_i), \; i = 1, 2, \cdots, p.$$

Hence,

$$P(|x_i| \geqq c_i, i = 1, \cdots, p) = E \textstyle\prod_{i=1}^{p} g_i(y_0)$$

where

$$g_i(y_0) = \int_{|x_i| \geqq c_i} N(\sigma_i\alpha_i y_0, \sigma_i^2(1 - \alpha_i^2); x_i)\, dx_i, \qquad i = 1, 2, \cdots, p.$$

Note that $Eg_i(y_0) = P(|x_i| \geqq c_i)$ and by Lemma 1, $g_i(y_0)$ is a monotonic non-decreasing function of $|y_0|$. Hence, using the first part of Lemma 5, we get Theorem 2 when $V(\mathbf{x})$ is s.p.d. When $V(\mathbf{x})$ is singular, we can use the similar arguments as employed in proving Theorem 1 for the singular case of $V(\mathbf{x})$.

COROLLARY 6. *Let $\mathbf{x}_j = (x_{1j}, \cdots, x_{pj})' \sim IN(\mathbf{0}, V(\mathbf{x}_j); \mathbf{x}_j)$, $j = 1, 2, \cdots, n$, such that all the matrices $V(\mathbf{x}_j)$, $j = 1, \cdots, n$, have the structure l. Then,*

$$P(\textstyle\sum_{j=1}^{n} \lambda_{i,j} x_{ij}^2 \geqq c_i, i = 1, \cdots, p) \geqq \prod_{i=1}^{p} P(\sum_{j=1}^{n} \lambda_{i,j} x_{ij}^2 \geqq c_i)$$

where $\lambda_{i,j} \geqq 0$ and $c_i \geqq 0$ for all i and j.

PROOF. Let $N_j = N(\mathbf{0}, V(\mathbf{x}_j); \mathbf{x}_j)$, $N_{j,i} = N(0, V(x_{ij}); x_{ij})$ and $D_i = D_i(\sum_{j=1}^{n} \lambda_{i,j} x_{ij}^2 \geqq c_i)$ for $i = 1, \cdots, p$ and $j = 1, 2, \cdots, n$. Then,

$$P(\textstyle\sum_{j=1}^{n} \lambda_{i,j} x_{ij}^2 \geqq c_i, i = 1, \cdots, p) = \int_{D_1 D_2 \cdots D_p} \prod_{j=1}^{n} (N_j\, d\mathbf{x}_j).$$

With the help of Theorem 2, the integration over $\mathbf{x}_1$ when $\mathbf{x}_2, \cdots, \mathbf{x}_n$ are fixed gives

$$P(\textstyle\sum_{j=1}^{n} \lambda_{i,j} x_{ij}^2 \geqq c_i, i = 1, \cdots, p) \geqq \int_{D_1 D_2 \cdots D_p} (\prod_{i=1}^{p} N_{1,i}\, dx_{i1})(\prod_{j=2}^{n} N_j\, d\mathbf{x}_j).$$

Continuing this type of arguments for $\mathbf{x}_2, \cdots$, and lastly for $\mathbf{x}_n$, we get Corollary 6. Note that the above arguments are similar to those given in the proof of Corollary 3.

COROLLARY 7. *Let* $\mathbf{x}_j = (x_{1j}, x_{2j}, \cdots, x_{pj})' \sim IN(\mathbf{0}, \mathbf{\Sigma}; \mathbf{x}_j)$, $j = 1, 2, \cdots, n$, *and let* $\mathbf{A}_i : n \times n$, $i = 1, 2, \cdots, p$, *be given* s.p.sd. *matrices with real elements such that they are reducible to diagonal matrices by a single orthogonal matrix. Then, if* $\mathbf{\Sigma}$ *has the structure* l,

$$P((x_{i1}, \cdots, x_{in})\mathbf{A}_i(x_{i1}, \cdots, x_{in})' \geqq c_i, i = 1, \cdots, p) \geqq \prod_{i=1}^{p} P((x_{i1}, \cdots, x_{in})\mathbf{A}_i(x_{i1}, \cdots, x_{in})' \geqq c_i).$$

The proof is similar to that given for Corollary 5. We do not repeat what is done in the proof of Corollary 6.

Combining Corollaries 5 and 7, we get the following result:

COROLLARY 8. *Let* $\mathbf{x}_j : p \times 1 \sim IN(\mathbf{0}, \mathbf{\Sigma}_1; \mathbf{x}_j)$, $\mathbf{y}_{j'} : p \times 1 \sim IN(\mathbf{0}, \mathbf{\Sigma}_2; \mathbf{y}_{j'})$ *for* $j = 1, 2, \cdots, n_1$ *and* $j' = 1, 2, \cdots, n_2$ *and let them be mutually independent. Let* $\mathbf{A}_i : n_1 \times n_1$ *and* $\mathbf{B}_i : n_2 \times n_2$, $i = 1, 2, \cdots, p$, *be given* s.p.sd. *matrices such that* $\mathbf{A}_i$'s *are reducible to diagonal matrices by a single orthogonal matrix and* $\mathbf{B}_i$'s *are reducible to diagonal matrices by a single orthogonal matrix. Then, if* $\mathbf{\Sigma}_2$ *has the structure* l, *then*

$$\begin{aligned} &P((x_{i1}, \cdots, x_{in_1})\mathbf{A}_i(x_{i1}, \cdots, x_{in_1})' \\ &\qquad \leqq c_i(y_{i1}, \cdots, y_{in_2})\mathbf{B}_i(y_{i1}, \cdots, y_{in_2})', \quad i = 1, \cdots, p) \\ &\qquad \geqq \prod_{i=1}^{p} P((x_{i1}, \cdots, x_{in_1})\mathbf{A}_i(x_{i1}, \cdots, x_{in_1})' \\ &\qquad \leqq c_i(y_{i1}, \cdots, y_{in_2})\mathbf{B}_i(y_{i1}, \cdots, y_{in_2})).\end{aligned}$$

Theorems 3 and 4 are given in Section 2 and since they are slightly lengthy, we do not rewrite them at this place. Their proofs are given below:

PROOF OF THEOREM 3. Since the statistics $\mathbf{x}_j'\mathbf{S}^{-1}\mathbf{x}_j$, $j = 1, 2, \cdots, n$, are invariant under the transformations $\mathbf{\Sigma}^{-\frac{1}{2}}\mathbf{S}\mathbf{\Sigma}^{-\frac{1}{2}} \to \mathbf{S}$ and $\mathbf{\Sigma}^{-\frac{1}{2}}\mathbf{x}_j \to \mathbf{x}_j$, we assume without loss of generality $\mathbf{\Sigma} = \mathbf{I}$. Hence, when $\mathbf{S}$ is fixed, the use of Corollary 5 gives

$$(23) \quad P(\mathbf{x}_j'\mathbf{S}^{-1}\mathbf{x}_j \leqq b_{jj}c_j, j = 1, \cdots, n \mid \mathbf{S}) \geqq \prod_{j=1}^{n} P(\mathbf{z}_j'\mathbf{S}^{-1}\mathbf{z}_j \leqq c_j \mid \mathbf{S})$$

where $\mathbf{z}_j : p \times 1 \sim IN(\mathbf{0}, \mathbf{I}; \mathbf{z}_j)$, $j = 1, 2, \cdots, n$. Using the transformation $\mathbf{S}^{-\frac{1}{2}}\mathbf{z}_j = \mathbf{y}_j$, $j = 1, 2, \cdots, n$, and writing $D = D(\mathbf{y}_j'\mathbf{y}_j \leqq c_j, j = 1, 2, \cdots, n)$, we get

$$\prod_{j=1}^{n} P(\mathbf{z}_j'\mathbf{S}^{-1}\mathbf{z}_j \leqq c_j \mid \mathbf{S}) = (2\pi)^{-\frac{1}{2}pn} \int_D |\mathbf{S}|^{\frac{1}{2}n} \exp(-\tfrac{1}{2} \operatorname{tr} \mathbf{S}\mathbf{Y}\mathbf{Y}')\, d\mathbf{Y},$$

where $\mathbf{Y} : p \times n = (\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_2)$. Hence, integrating over $\mathbf{S}$, we get

$$(24) \quad E \prod_{j=1}^{n} P(\mathbf{z}_j'\mathbf{S}^{-1}\mathbf{z}_j \leqq c_j \mid \mathbf{S}) = P(\mathbf{y}_j'\mathbf{y}_j \leqq c_j, j = 1, 2, \cdots, n)$$

where $\mathbf{Y} : p \times n \sim \Delta(p, n, m; \mathbf{Y})$. Taking the expectation of (23) and using (24), we get the result (9). When $c_j = c$ for all j, the use of Corollary 2(ii) and (24) in $P(\mathbf{z}_j'\mathbf{S}^{-1}\mathbf{z}_j \leqq c, j = 1, 2, \cdots, n)$ gives the result (10), for $P(\mathbf{z}_j'\mathbf{S}^{-1}\mathbf{z}_j \leqq c) = \beta_2(\frac{1}{2}p, \frac{1}{2}(m + 1 - p); c)$. We note that $|\mathbf{I} + \mathbf{Y}'\mathbf{Y}| \leqq \prod_{j=1}^{n} (1 + \mathbf{y}_j'\mathbf{y}_j)$ and so

$$\begin{aligned}(25) \quad &\Delta(p, n, m; \mathbf{Y}) \\ &\geqq w \prod_{j=1}^{n} \{\Gamma(\tfrac{1}{2}(m + n))[\pi^{\frac{1}{2}p}\Gamma(\tfrac{1}{2}(m + n - p))]^{-1}(1 + \mathbf{y}_j'\mathbf{y}_j)^{-\frac{1}{2}(m+n)}\}\end{aligned}$$

for all $\mathbf{Y}: p \times n$ and w being defined in (12). From (25), we get the right side of the inequality (11). For the left side of the inequality (11), we note that if $\mathbf{Y} = (\mathbf{y}_1, \mathbf{Y}_2)$, $|\mathbf{I}_n + \mathbf{Y}'\mathbf{Y}| = |\mathbf{I}_{n-1} + \mathbf{Y}_2'\mathbf{Y}_2|(1 + \mathbf{y}_1'(\mathbf{I}_p + \mathbf{Y}_2\mathbf{Y}_2')^{-1}\mathbf{y}_1)$. Using the transformation $\mathbf{v}_1 = (\mathbf{I}_p + \mathbf{Y}_2\mathbf{Y}_2')^{-\frac{1}{2}}\mathbf{y}_1$, the domain $\mathbf{y}_1'\mathbf{y}_1 \leqq c_1$ is changed to the domain $\mathbf{v}_1'(\mathbf{I}_p + \mathbf{Y}_2\mathbf{Y}_2')\mathbf{v}_1 \leqq c_1$ and this domain implies the domain $\mathbf{v}_1'\mathbf{v}_1 \leqq c_1$. Hence,

$$\beta_2(\tfrac{1}{2}p, \tfrac{1}{2}(m + n - p); c_1)$$
$$= P(\mathbf{v}_1'\mathbf{v}_1 \leqq c_1) \geqq P(\mathbf{y}_1'\mathbf{y}_1 = \mathbf{v}_1'(\mathbf{I}_p + \mathbf{Y}_2\mathbf{Y}_2')\mathbf{v}_1 \leqq c_1 \mid \mathbf{Y}_2).$$

Using this in the right side of (24), we get

$$P(\mathbf{y}_j'\mathbf{y}_j \leqq c_j, j = 1, \cdots, n)$$
$$\leqq \beta_2(\tfrac{1}{2}p, \tfrac{1}{2}(m + n - p); c_1)P(\mathbf{y}_j'\mathbf{y}_j \leqq c_j, j = 2, \cdots, n).$$

If we integrate $\mathbf{y}_{j_1}$ instead of $\mathbf{y}_1$, we shall get c_{j_1} in place of c_1. Proceeding in this way, we get

$$P(\mathbf{y}_j'\mathbf{y}_j \leqq c_j, j = 1, 2, \cdots, n) \leqq \textstyle\prod_{j'=1}^{n} \beta_2(\tfrac{1}{2}p, \tfrac{1}{2}(m + j - p); c_{j_{j'}})$$

where $(j_1, j_2, \cdots, j_n)$ is any permutation of $(1, 2, \cdots, n)$. Using Lemma 4, we get the left side of the inequality (11). This proves Theorem 3 completely.

The proof of Theorem 4 is exactly similar to that given above, but in order to eliminate certain inequalities, we use Lemma 3 for $p = 1$ only.

Remark. For any non-singular $V(\mathbf{x})$ matrix in Theorem 2, it has been shown that $P(|x_i| \geqq c_i, i = 1, 2, \cdots, p)$ is locally minimum at $\operatorname{Cov}(x_i, x_{i'}) = 0$ for $i \neq i'$, $i, i' = 1, 2, \cdots, p$. This gives some hope that Theorem 2 may be true for any $V(\mathbf{x})$.

4. Simultaneous confidence bounds on a set of linear functions of location parameters.

4.1. *On the means of k independent multivariate normal variates with different covariance matrices having the structures l.* Let $\mathbf{x}_{jt}: p \times 1 \sim IN(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t; \mathbf{x}_{jt})$, $j = 1, 2, \cdots, n_t$, and $t = 1, 2, \cdots, k$ such that $\boldsymbol{\Sigma}_t$, $t = 1, 2, \cdots, k$, have the structures l. Let $(\bar{x}_{1t}, \bar{x}_{2t}, \cdots, \bar{x}_{pt})' = \bar{\mathbf{x}}_t = \sum_{j=1}^{n_t} \mathbf{x}_{jt}/n_t$, $(s_{ii'}^{(t)}) = \mathbf{S}_t = \sum_{j=1}^{n_t} (\mathbf{x}_{jt} - \bar{\mathbf{x}}_t)(\mathbf{x}_{jt} - \bar{\mathbf{x}}_t)': p \times p$. Here, we obtain the simultaneous confidence bounds on $\nu_{i\gamma} = \sum_{t=1}^{k} a_{\gamma t}\mu_{it}$, $i = 1, 2, \cdots, p$, and $\gamma = 1, 2, \cdots, r$ in the similar form as that given by Banerjee [2], [3] for $r = p = 1$. Let $z_{i\gamma} = \sum_{t=1}^{k} a_{\gamma t}\bar{x}_{it}$. Then, $z_{i\gamma}$ for all i and γ are jointly distributed as normal with means $\nu_{i\gamma}$ and covariances given by $\operatorname{Cov}(z_{i\gamma}, z_{i'\gamma'}) = \sum_{t=1}^{k} a_{\gamma t}a_{\gamma' t}\sigma_{ii'}^{(t)}/n_t$ for $i, i' = 1, \cdots, p$ and $\gamma, \gamma' = 1, \cdots, r$, and they are independently distributed of $\mathbf{S}_t \sim IW(p, n_t - 1, \boldsymbol{\Sigma}_t; \mathbf{S}_t)$, $t = 1, 2, \cdots, k$. Since $\boldsymbol{\Sigma}_1$ has the structure l, so using Corollary 7 and taking $f_t \geqq 0$ for all t, we get

$$P((z_{i\gamma} - \nu_{i\gamma})^2 \leqq \textstyle\sum_{t=1}^{k} f_t a_{\gamma t}^2 s_{ii}^{(t)}/n_t, \quad \text{for all} \quad i \quad \text{and} \quad \gamma \mid z_{i\gamma}\text{'s}, \mathbf{S}_2, \cdots, \mathbf{S}_k)$$
$$\geqq P((z_{i\gamma} - \nu_{i\gamma})^2 \leqq f_1 a_{\gamma 1}^2 \sigma_{ii}^{(1)} \chi_i^2(n_1 - 1)/n_1$$
$$+ \textstyle\sum_{t=2}^{n} f_t a_{\gamma t}^2 s_{ii}^{(t)}/n_t, \quad \text{for all} \quad i, \gamma \mid z_{i\gamma}\text{'s}, \mathbf{S}_2, \cdots, \mathbf{S}_k).$$

Now, using the same type of arguments for $\mathbf{S}_2, \cdots, \mathbf{S}_k$, we get

$$P((z_{i\gamma} - \nu_{i\gamma})^2 \leqq \textstyle\sum_{t=1}^k f_t a_{\gamma t}^2 s_{ii}^{(t)}/n_t \quad \text{for all} \quad i, \gamma \mid z_{i\gamma}\text{'s})$$
$$\geqq P((z_{i\gamma} - \nu_{i\gamma})^2 \leqq \textstyle\sum_{t=1}^k f_t a_{\gamma t}^2 \sigma_{ii}^{(t)} \chi_i^2(n_t - 1)/n_t \quad \text{for all} \quad i \quad \text{and} \quad \gamma \mid z_{i\gamma}\text{'s}),$$

where $\chi_i^2(n_t - 1)$, $i = 1, 2, \cdots, p$ and $t = 1, 2, \cdots, k$ are independent χ^2-variates with $n_t - 1$ degrees of freedom. Now, using Corollary 5, we get

$$(26) \quad P((z_{i\gamma} - \nu_{i\gamma})^2 \leqq \textstyle\sum_{t=1}^k f_t a_{\gamma t}^2 s_{ii}^{(t)}/n_t \quad \text{for all} \quad i, \gamma)$$
$$\geqq P(v_{i\gamma}^2 \leqq \textstyle\sum_{t=1}^k f_t w_{i\gamma t} \chi_i^2(n_t - 1) \quad \text{for all} \quad i, \gamma)$$

where $w_{i\gamma t} = (a_{\gamma t}^2 \sigma_{ii}^{(t)}/n_t)/(\sum_{t'=1}^k a_{\gamma t'}^2 \sigma_{ii}^{(t')}/n_{t'})$, $\sum_{t=1}^k w_{i\gamma t} = 1$ for all i and γ, $v_{i\gamma} \sim IN(0, 1; v_{i\gamma})$, $i = 1, 2, \cdots, p$ and $\gamma = 1, 2, \cdots, r$, and are independently distributed of all $\chi_i^2(n_t - 1)$. Referring Banerjee [2], p. 357, [3], p. 361, we can write

$$(27) \quad P(|v_{i\gamma}| \leqq (\textstyle\sum_{t=1}^k f_t w_{i\gamma t} \chi_i^2(n_t - 1))^{\frac{1}{2}} \quad \text{for all} \quad i, \gamma \mid \quad \text{all} \quad \chi_i^2(n_t - 1))$$
$$\geqq \textstyle\prod_{i,\gamma} \{\sum_{t=1}^k w_{i\gamma t} P(v_{i\gamma}^2 \leqq f_t \chi_i^2(n_t - 1) \mid \quad \text{all} \quad \chi_i^2(n_t - 1)\}.$$

Combining (26) and (27), we get finally

$$(28) \quad P((z_{i\gamma} - \nu_{i\gamma})^2 \leqq \textstyle\sum_{t=1}^k f_t a_{\gamma t}^2 s_{ii}^{(t)}/n_t \quad \text{for all} \quad i, \gamma)$$
$$\geqq \textstyle\prod_{i,\gamma} \{\sum_{t=1}^k w_{i\gamma t} \beta_2(\frac{1}{2}, \frac{1}{2}(n_t - 1); f_t)\}$$

because $P(v_{i\gamma}^2 \leqq f_t \chi_i^2(n_t - 1)) = \beta_2(\frac{1}{2}, \frac{1}{2}(n_t - 1); f_t)$. The right side of (28) can be made equal to $(1 - \alpha)$ by choosing f_t such that $\beta_2(\frac{1}{2}, \frac{1}{2}(n_t - 1); f_t) = (1 - \alpha)^{1/pr}$. Hence, with confidence greater than or equal to $(1 - \alpha)$, we have simultaneous confidence bounds on $\nu_{i\gamma}$'s as given in (15).

4.2. *On the means of k independent multivariate normal distributions with equal covariance matrix.* With the same notations as in Section 4.1, we have $\mathbf{\Sigma}_1 = \cdots = \mathbf{\Sigma}_k = \mathbf{\Sigma}$ (say) which is any s.p.sd. matrix. We consider the problem of obtaining the simultaneous confidence bounds on $\mathbf{b}'\mathbf{\nu}_\gamma = \sum_{t=1}^k (\mathbf{b}'\mathbf{\mu}_t) a_{\gamma t}$ for all γ and for all non-null vector $\mathbf{b}: p \times 1$. Let $\mathbf{S} = \sum_{t=1}^k \mathbf{S}_t$ and $\mathbf{z}_\gamma = \sum_{t=1}^k a_{\gamma t} \bar{\mathbf{x}}_t$. Then, $\mathbf{z}_\gamma$, $\gamma = 1, 2, \cdots, r$, are jointly distributed as normal with means $\mathbf{\nu}_\gamma$ and $\operatorname{Cov}(\mathbf{z}_\gamma, \mathbf{z}_{\gamma'}) = (\sum_{t=1}^k a_{\gamma t} a_{\gamma' t}/n_t)\mathbf{\Sigma}$ and is independently distributed of $\mathbf{S} \sim W(p, n - k = \sum_{t=1}^k n_t - k, \mathbf{\Sigma}; \mathbf{S})$. Then by Theorem 3, we have

$$(29) \quad P((\mathbf{z}_\gamma - \mathbf{\nu}_\gamma)'\mathbf{S}^{-1}(\mathbf{z}_\gamma - \mathbf{\nu}_\gamma) \leqq c_\gamma(\textstyle\sum_{t=1}^k a_{\gamma t}^2/n_t) \quad \text{for all} \quad \gamma) \geqq (1 - \alpha)$$

where $c_1, c_2, \cdots, c_r$ are to be determined from

$$(30) \quad P(\mathbf{y}_\gamma'\mathbf{y}_\gamma \leqq c_\gamma, \gamma = 1, \cdots, r) = 1 - \alpha, \mathbf{Y}: p \times r$$
$$= (\mathbf{y}_1, \cdots, \mathbf{y}_r) \sim \Delta(p, r, n - k; \mathbf{Y}).$$

From (29), we have simultaneous confidence bounds on $\mathbf{b}'\mathbf{\nu}_\gamma$ for all non-null vector $\mathbf{b}: p \times 1$ and for all γ with confidence greater than or equal to $(1 - \alpha)$ as given in (16). When $p = 1$ and $c_1 = \cdots = c_r$, then the tables are available from Krishnaiah [10], Dunn [7] and Pillai and Ramachandran [12]. When $p > 1$, some tables are available from Siotani [22], [24]. When the tables are not available, we can use the inequality (10) or the right side of the inequality (11) established in Theorem 3.

4.3. *General MANOVA model for the growth curve problems.* Let $\mathbf{X}: q \times n$ be independently distributed as normals with common covariance matrix $\mathbf{\Sigma}_1$ and $E(\mathbf{X}) = \mathbf{Q}\boldsymbol{\xi}\mathbf{L}$ where $\mathbf{Q}: q \times s$ and $\mathbf{L}: k \times n$ are known matrices of rank s and k respectively and $\boldsymbol{\xi}: s \times k$ is an unknown matrix of location parameters, (see Khatri [10], Potthoff and Roy [14]). We are interested in obtaining simultaneous confidence bounds on $(\nu_{i\gamma'}): p \times r = \boldsymbol{\nu} = \mathbf{C}\boldsymbol{\xi}\mathbf{A}$, where $\mathbf{C}: p \times q$ and $\mathbf{A}: k \times r$ are known matrices of respective ranks p and r. Let $\mathbf{A}'(\mathbf{LL}')^{-1}\mathbf{A} = \mathbf{B} = (b_{\gamma\gamma'})$: $r \times r$, $\mathbf{Z}: p \times r = (z_{i\gamma}) = \mathbf{C}(\mathbf{Q}'\mathbf{Q})^{-1}\mathbf{Q}'\mathbf{X}\mathbf{L}'(\mathbf{LL}')^{-1}\mathbf{A}$, $\mathbf{\Sigma}_0 = \mathbf{C}(\mathbf{Q}'\mathbf{Q})^{-1}\cdot \mathbf{Q}'\mathbf{\Sigma}_1\mathbf{Q}(\mathbf{Q}'\mathbf{Q})^{-1}\mathbf{C}'$ and $\mathbf{S}_0: p \times p = (s_{ii',0}) = \mathbf{C}(\mathbf{Q}'\mathbf{Q})^{-1}\mathbf{Q}'\mathbf{X}[\mathbf{I}_n - \mathbf{L}'(\mathbf{LL}')^{-1}\mathbf{L}]\cdot \mathbf{X}'\mathbf{Q}(\mathbf{Q}'\mathbf{Q})^{-1}\mathbf{C}'$. Then, $\mathbf{z}_\gamma = (z_{1\gamma}, \cdots, z_{p\gamma})'$, $\gamma = 1, 2, \cdots, r$, are jointly distributed as normal with means $\boldsymbol{\nu}_\gamma = (\nu_{1\gamma}, \cdots, \nu_{p\gamma})'$, and $\operatorname{Cov}(\mathbf{z}_\gamma, \mathbf{z}_{\gamma'}) = b_{\gamma\gamma'}\mathbf{\Sigma}_0$ for $\gamma, \gamma' = 1, 2, \cdots, r$, and is independently distributed of $\mathbf{S}_0 \sim W(p, n - k, \mathbf{\Sigma}_0; \mathbf{S}_0)$, (see [14]).

CASE (i). If $\mathbf{\Sigma}_0$ is any s.p.sd. matrix, then using the similar arguments as in Section 4.2, we get simultaneous confidence bounds on $\mathbf{d}'\boldsymbol{\nu}_\gamma$ for all non-null vector $\mathbf{d}: p \times 1$ and for all γ with confidence greater than or equal to $(1 - \alpha)$ as given in (18).

CASE (ii). If $\mathbf{\Sigma}_0$ has the structure l, then using Corollary 8, we get

$$(31) \quad P((z_{i\gamma} - \nu_{i\gamma})^2 \leqq u_{i\gamma}b_{\gamma\gamma}s_{ii,0} \text{ for all } i, \gamma) \geqq \textstyle\prod_{i,\gamma} P((z_{i\gamma} - \nu_{i\gamma})^2 \leqq u_{i\gamma}b_{\gamma\gamma}s_{ii,0}).$$

Hence, simultaneous confidence bounds on $\nu_{i\gamma}$ for all i and γ with confidence greater than or equal to $(1 - \alpha)$ are given by (19).

CASE (iii). If $\mathbf{\Sigma}_0$ has the structure l, $\boldsymbol{\nu}_i = (\nu_{i1}, \nu_{i2}, \cdots, \nu_{ir})'$ and $\mathbf{v}_i = (z_{i1}, \cdots, z_{ir})'$ for $i = 1, 2, \cdots, p$, then using Corollary 8, we have

$$(32) \quad P((\mathbf{v}_i - \boldsymbol{\nu}_i)'\mathbf{B}^{-1}(\mathbf{v}_i - \boldsymbol{\nu}_i) \leqq q_i s_{ii,0},\ i = 1, \cdots, p) \geqq \textstyle\prod_{i=1}^{p} P((\mathbf{v}_i - \boldsymbol{\nu}_i)'\mathbf{B}^{-1}(\mathbf{v}_i - \boldsymbol{\nu}_i) \leqq q_i s_{ii,0}).$$

From this, we have simultaneous confidence bounds on $\mathbf{e}'\mathbf{v}_i$ for all non-null vector $\mathbf{e}: r \times 1$ and for all i with confidence greater than or equal to $1 - \alpha$ as

$$(33) \quad \mathbf{e}'\mathbf{v}_i - (q_i s_{ii,0}\mathbf{e}'\mathbf{B}\mathbf{e})^{\frac{1}{2}} \leqq \mathbf{e}'\boldsymbol{\nu}_i \leqq \mathbf{e}'\mathbf{v}_i + (q_i s_{ii,0}\mathbf{e}'\mathbf{B}\mathbf{e})^{\frac{1}{2}} \quad \text{for all } i \text{ and non-null } \mathbf{e}$$

where q_i's are to be determined from $\prod_{i=1}^{p} \beta_2(\frac{1}{2}r, \frac{1}{2}(n - k); q_i) = 1 - \alpha$.

In concluding, we remark that simultaneous confidence bounds obtained in Section 4 will be shorter than the traditional ones when the number of linear functions is not too large and in some cases it may nearly be the shortest. All the results of Section 4 can easily be extended to regression-like parameters (in testing independence of two sets), to testing the multicollinearity of means (or to covariance analysis), and to the step down procedures, but they are not given.

I am thankful to Dr. D. Basu for some suggestions especially for Theorem 2. I am thankful to the referee for the helpful comments.

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Note added in proof. The condition on Cov $(\mathbf{x}^{(1)}, \mathbf{x}^{(2)})$ in Theorem 1 is not necessary and this will appear elsewhere.