LOCAL ASYMPTOTIC POWER AND EFFICIENCY OF TESTS OF KOLMOGOROV-SMIRNOV TYPE

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0. Summary. This paper is devoted to some problems related to asymptotic power and efficiency of tests of Kolmogorov-Smirnov type. The results are derived by means of the theory of stochastic processes. The concepts of local asymptotic power and local asymptotic efficiency of tests are introduced according to [7]. Section 4 contains the evaluation of the local asymptotic power of one-sided tests of Kolmogorov-Smirnov type and of Rényi's one-sided test. The asymptotic power of these tests may be expressed by series expansion. Some theoretical results related to terms of the series are given and used for approximations and bounds of the asymptotic power.

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1. Introduction. Let $X_1, \cdots, X_N$ be random variables. We shall consider the general regression model

$$X_i = \alpha + \beta c_i + Y_i \qquad (i = 1, \cdots, N),$$

where α and β are unknown parameters, $c_1, \cdots, c_N$ are some known constants and $Y_1, \cdots, Y_N$ are independent random variables with a common (but unknown) continuous distribution. In the special case when

$$\begin{aligned} c_i &= 0 \quad \text{for} \quad i = 1, \cdots, n, \\ &= 1 \quad \text{for} \quad i = n+1, \cdots, N, \end{aligned}$$

we have the classical two-sample problem.

The tests of Kolmogorov-Smirnov type may also be defined for regression alternatives. (See [6], [7] and Section 2.) In this paper we shall investigate the one-sided Kolmogorov-Smirnov test and the Rényi test only. The two-sided versions of these tests as well as the Cramér-von Mises test will not be considered here.

The local asymptotic efficiency is a new concept introduced first in [7]. If the test statistic is asymptotically normal under both the hypothesis and the alternative and if we have a two-sample problem, it coincides with the Pitman asymptotic efficiency. The asymptotic power of tests of Kolmogorov-Smirnov type may be expressed as the probabilities of events defined by a Brownian bridge process. It is possible to evaluate these probabilities by means of the heat conduction equation with rather complicated boundary conditions. We use a different method and get an approximate solution with the help of the theory of stochastic proc-

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esses. Our considerations correspond to the small-parameter method for the solution of the differential equation. The relation between both methods is not analysed here.

2. Preliminaries. A real stochastic process $z(t, \omega)$, $0 \leqq t \leqq 1$, will be called a Brownian bridge, if it is Gaussian with

$$(2.1) \qquad Ez(t) = 0, \qquad 0 \leqq t \leqq 1,$$

$$(2.2) \qquad B(s, t) = Ez(s)z(t) = s(1 - t), \qquad 0 \leqq s \leqq t \leqq 1,$$

and if all paths $z(\cdot)$ are continuous functions on [0, 1].

It may be proved from a well-known theorem of Kolmogorov (see [3], p. 576) that a separable Gaussian process with parameters (2.1) and (2.2) is continuous with probability 1. Then, of course, the process may be adjusted on ω-subsets having probability 0, so that all paths are continuous.

Let Ω be the space of continuous functions $\gamma(t)$, $0 \leqq t \leqq 1$, with $\gamma(0) = \gamma(1) = 0$. Let $\mathfrak{A}$ be the σ-field generated by the system of open sets of continuous functions; the distance ρ is $\rho(\gamma_1, \gamma_2) = \max_{0 \leqq t \leqq 1} |\gamma_1(t) - \gamma_2(t)|$. Obviously $z(t)$ may be realized on $(\Omega, \mathfrak{A}, P)$, when we put $z(t, \gamma(\cdot)) = \gamma(t)$.

We denote by L_2 the set of functions $\{\varphi\}$ on [0, 1] such that $\int_0^1 |\varphi(t)|^2 \, dt < \infty$; further we denote by L_2^0 the subset of L_2 such that $\int_0^1 \varphi(t) \, dt = 0$.

We define the integral $\int_0^1 \varphi(t) \, dz(t)$, where $\varphi \in L_2$ and $z(t)$ is the Brownian bridge. This can be done for step functions by the standard procedure (see [3], Chapter IX). We may derive the relation

$$(2.3) \qquad E\left[\int_0^1 \varphi(t) \, dz(t)\right]^2 = \int_0^1 [\varphi(t) - \bar{\varphi}]^2 \, dt \leqq \int_0^1 [\varphi(t)]^2 \, dt$$

for any step function $\varphi(t)$; we have denoted $\bar{\varphi} = \int_0^1 \varphi(t) \, dt$. Using (2.3), the integral $\int_0^1 \varphi(t) \, dz(t)$ may be defined for any $\varphi \in L_2$ as the limit in quadratic mean (with respect to P) of integrals $\int_0^1 \varphi_n(t) \, dz(t)$, where $\{\varphi_n\}$ is a sequence of step functions such that $\varphi_n \to \varphi$ in quadratic mean. It is obvious that this limit does not depend on the choice of the particular sequence $\{\varphi_n\}$. Note the validity of (2.3) for any $\varphi \in L_2$. Moreover, it is easy to obtain the relation

$$(2.4) \qquad \operatorname{cov}\left(\int_0^1 \psi(t) \, dz(t), \int_0^1 \varphi(t) \, dz(t)\right) = \int_0^1 [\psi(t) - \bar{\psi}][\varphi(t) - \bar{\varphi}] \, dt, \quad \varphi, \psi \in L_2.$$

Put

$$\psi(t) = 1 - s \quad \text{for} \quad 0 \leqq t \leqq s,$$
$$= -s \quad \text{for} \quad s < t \leqq 1.$$

Then we get

$$(2.5) \qquad z(s) = \int_0^1 \psi(t) \, dz(t).$$

Suppose $\bar{\varphi} = 0$. Then (2.4) implies

$$(2.6) \qquad \operatorname{cov}\left(z(s), \int_0^1 \varphi(t) \, dz(t)\right) = \int_0^s \varphi(t) \, dt.$$

THEOREM 2.1. *Let P_Δ, $\Delta \neq 0$, denote the distribution of the process $v(t) = z(t) + \Delta\psi(t)$, where $z(t)$ is a Brownian bridge and ψ is a given function on* [0, 1]. *Put $P = P_0$.*

Then P_Δ is absolutely continuous with respect to P, if and only if $\psi(0) = \psi(1) = 0$ and $\psi(t)$ is absolutely continuous with square integrable $\psi'(t)$. In this case

$$dP_\Delta/dP = \exp\{\Delta\xi - \tfrac{1}{2}\Delta^2 \operatorname{var} \xi\},$$

where

$$\xi = \int_0^1 \psi'(t)\, dz(t), \qquad \operatorname{var} \xi = \int_0^1 |\psi'(t)|^2 dt = \sigma^2.$$

The random variable ξ is a sufficient statistic for the system $\{P_\Delta\}$ and ξ has distribution $N(0, \sigma^2)$ under P_0.

PROOF. Consider the system $\{\int_0^1 \varphi(t)\, dz(t)\}$, $\varphi \in L_2^0$. This system is closed with respect to passage to the limit in mean for [P] and it is the minimal closed linear manifold, which contains the random variables $z(s)$ (cf. (2.5)). The assertions of Theorem 2.1 follow then from (2.6) and [5]. Q.E.D.

We call a function $f(x)$ absolutely continuous, if it is finite and if for given $\epsilon > 0$ such $\delta > 0$ exists, that for any finite system of disjoint open intervals $(a_1, b_1), \cdots, (a_n, b_n)$, where $\sum_{k=1}^n (b_k - a_k) < \delta$, the relation

$$\left| \sum_{k=1}^n [f(b_k) - f(a_k)] \right| < \epsilon$$

holds. (See [10].)

Let X be a random variable with distribution function $F(x)$ and density $f(x)$. Suppose that $f(x)$ is absolutely continuous and

$$I(f) = \int_{-\infty}^{\infty} |f'(x)/f(x)|^2 f(x)\, dx < \infty.$$

$I(f)$ is called Fisher's information for $f(x)$. Define the φ-function

$$\varphi(u, f) = -f'(F^{-1}(u))/f(F^{-1}(u)), \qquad 0 < u < 1,$$

where F^{-1} is the inverse of F. From the assumption $I(f) < \infty$ it follows that $\varphi(u, f) \in L_2^0$ and $I(f) = \int_0^1 \varphi^2(u, f)\, du$.

Let $x = (x_1, \cdots, x_N)$ be a real vector such that $x_i \neq x_j$ for $i \neq j$ $(i, j = 1, \cdots, N)$. Denote by $r_i(x)$ the number of components of x whose values are not greater than x_i. If $X = (X_1, \cdots, X_N)$ is a random vector, the statistic $R_i = r_i(X)$ is called the rank of X_i.

Introduce the scores $a_N(i, t)$ for $0 \leqq t \leqq 1$ and for $N \geqq 1$ in this way:

$$\begin{aligned} a_N(i, t) &= 0 && \text{if } i \leqq tN, \\ &= i - tN && \text{if } tN \leqq i \leqq tN + 1, \\ &= 1 && \text{if } tN + 1 \leqq i. \end{aligned}$$

Let $c = (c_1, \cdots, c_N)$ be a vector with real components. Put

$$T_c(t) = [\sum_{i=1}^N (c_i - \bar{c})^2]^{-\frac{1}{2}} \sum_{i=1}^N (c_i - \bar{c}) a_N(R_i, t),$$

where $\bar{c} = N^{-1} \sum_{i=1}^{N} c_i$. Let

$$\sum_{i=1}^{N} (c_i - \bar{c})^2 / \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2 \to \infty \quad \text{for} \quad N \to \infty. \tag{2.7}$$

We have the hypothesis H_0 that the simultaneous density of $X_1, \cdots, X_N$ is $\prod_{i=1}^{N} f(x_i)$. Then under H_0 $T_c(\cdot)$ converges in distribution in the measurable space of continuous functions to the Brownian bridge $z(\cdot)$ (see [6]).

Consider the statistic $K_c^+ = \max_{0 \leqq t \leqq 1} T_c(t)$. In the special case when

$$c_i = 0 \quad (i = 1, \cdots, n), \qquad c_i = 1 \quad (i = n+1, \cdots, N), \tag{2.8}$$

the statistic K_c^+ coincides with the well-known one-sided Kolmogorov-Smirnov statistic. It is proved (see [7]) that under H_0 and (2.7)

$$P\{K_c^+ \geqq (-\tfrac{1}{2} \ln \alpha)^{\frac{1}{2}}\} \to \alpha, \qquad \alpha \,\varepsilon\, (0, 1).$$

The statistic

$$R_{ac}^+ = \max_{0 < a \leqq t \leqq 1} t^{-1} T_c(t)$$

coincides in the special case (2.8) with the usual one-sided Rényi statistic. Let $\Phi(x)$ denote the distribution function of $N(0, 1)$ and $\Phi(u_\alpha) = \alpha$. Then under H_0 and (2.7)

$$P\{R_{ac}^+ \geqq [(1 - a)/a]^{\frac{1}{2}} u_{1-\alpha/2}\} \to \alpha, \qquad \alpha \,\varepsilon\, (0, 1).$$

Consider the family of alternatives that the simultaneous density of $X_1, \cdots, X_N$ is

$$q_d = \prod_{i=1}^{N} f_0(x_i - d_i),$$

where f_0 is a given absolutely continuous density with $I(f_0) < \infty$. Put $\bar{d} = N^{-1} \sum_{i=1}^{N} d_i$. Suppose (2.7) and let

$$[\sum_{i=1}^{N} (c_i - \bar{c})(d_i - \bar{d})][\sum_{i=1}^{N} (c_i - \bar{c})^2 \sum_{i=1}^{N} (d_i - \bar{d})^2]^{-\frac{1}{2}} \to \rho_2, \tag{2.9}$$

$$\max_{1 \leqq i \leqq N} (d_i - \bar{d})^2 \to 0, \tag{2.10}$$

$$I(f_0) \sum_{i=1}^{N} (d_i - \bar{d})^2 \to b^2, \qquad 0 < b < \infty. \tag{2.11}$$

If Q_d is the probability corresponding to q_d, then

$$Q_d\{K_c^+ \geqq (-\tfrac{1}{2} \ln \alpha)^{\frac{1}{2}}\}$$
$$\to P\{\max_{0 \leqq t \leqq 1} \{z(t) + b\rho_2[I(f_0)]^{-\frac{1}{2}} f_0(F_0^{-1}(t))\} \geqq (-\tfrac{1}{2} \ln \alpha)^{\frac{1}{2}}\},$$
$$Q_d\{R_{ac}^+ \geqq [(1 - a)/a]^{\frac{1}{2}} u_{1-\alpha/2}\}$$
$$\to P\{\max_{a \leqq t \leqq 1} \{z(t) + b\rho_2[I(f_0)]^{-\frac{1}{2}} f_0(F_0^{-1}(t)) - t[(1 - a)/a]^{\frac{1}{2}} u_{1-\alpha/2}\} \geqq 0\}.$$

These probabilities are the asymptotic power of the one-sided test of Kolmogorov-Smirnov type and of the one-sided test of Rényi type respectively.

Thus we are facing the evaluation of probabilities

$$P\{\max_{0 \leqq t \leqq 1} [z(t) + b\psi(t) - a] \geqq 0\}, \qquad a > 0,$$

and

$$P\{\max_{0<a\leqq t\leqq 1}[z(t) + b\psi(t) - \lambda t] \geqq 0\}, \qquad \lambda > 0,$$

where $\psi(t)$ is an absolutely continuous function such that $\psi' \varepsilon L_2^0$.

Let $B(\alpha, f_0, b)$ be the asymptotic power of a test. Let us assume

$$B(\alpha, f_0, b) = \alpha + bB_1(\alpha, f_0) + o(b). \tag{2.12}$$

Then $B_1(\alpha, f_0)$ is called the local asymptotic power of the given test. The evaluation of $B_1(\alpha, f_0)$ is often relatively simpler than that of $B(\alpha, f_0, b)$. The local asymptotic power was introduced in [7].

The asymptotic power of the asymptotically most powerful one-sided test on the level α is equal to $1 - \Phi(u_{1-\alpha} - b)$. Define b' by

$$B(\alpha, f_0, b) = 1 - \Phi(u_{1-\alpha} - b').$$

The asymptotic efficiency of the given test is defined by

$$e(\alpha, b) = (b'/b)^2.$$

But $e(\alpha, b)$ depends on both α and b. In accordance with [7] we introduce the local asymptotic efficiency

$$e(\alpha) = \lim_{b\to 0^+} e(\alpha, b).$$

In the case of one-sided tests of Kolmogorov-Smirnov type $e(\alpha)$ was first derived in [7] by Hájek. We may obtain that

$$\begin{aligned} e(\alpha) &= \lim_{b\to 0^+}[(B(\alpha, f_0, b) - \alpha)/(1 - \Phi(u_{1-\alpha} - b) - \alpha)]^2 \\ &= 2\pi \exp(u_{1-\alpha}^2) B_1^2(\alpha, f_0). \end{aligned}$$

Approximately (see Table 4, columns $Q(\Delta)$ and $\Phi(\rho_0 + \rho_1\Delta)$) one has that

$$B(\alpha, f_0, b) \sim 1 - \Phi(u_{1-\alpha} - [e(\alpha)]^{\frac{1}{2}} b).$$

3. Series expansion for $P_\Delta(A)$. We shall consider the events $A \varepsilon \mathfrak{A}$ only. In order to calculate $P_\Delta(A)$ we deduce a theoretical expansion in the form of sums of absolutely convergent power series in Δ. The remainder term is calculated for two special cases. Without loss of generality we assume $\Delta > 0$, since $\Delta\psi(t) = (-\Delta)(-\psi(t))$.

LEMMA 3.1. *For any $A \varepsilon \mathfrak{A}$*

$$\int_A e^{\Delta\xi}\, dP = \sum_{j=0}^\infty (\Delta^j/j!) \int_A \xi^j\, dP$$

holds, where the series on the right side is absolutely convergent.

PROOF. Using a well-known theorem (see [9], p. 134) we have

$$\begin{aligned} \sum_{j=0}^\infty (\Delta^j/j!)\left|\int_A \xi^j\, dP\right| &\leqq \sum_{j=0}^\infty (\Delta^j/j!) \int_\Omega |\xi^j|\, dP = \int_\Omega e^{|\Delta\xi|}\, dP \\ &= (2\pi)^{-\frac{1}{2}}\sigma^{-1} \int_{-\infty}^\infty \exp\{|\Delta x| - x^2/2\sigma^2\}\, dx < \infty. \end{aligned}$$

Put $f_n(\omega) = \sum_{j=0}^n (\Delta^j/j!)\xi^j(\omega)$. We see that $f_n(\omega) \to e^{\Delta\xi(\omega)}$ everywhere. Further,

$e^{|\Delta\xi(\omega)|}$ is an integrable majorant for $\{\chi_A(\omega)\cdot f_n(\omega)\}$, hence by the Lebesgue theorem

$$\int_A f_n\,dP = \sum_{j=0}^{n} (\Delta^j/j!) \int_A \xi^j\,dP \rightarrow \int_A e^{\Delta\xi}\,dP$$

for $n \rightarrow \infty$ (χ_A is the characteristic function of the set A).

THEOREM 3.1. *For any $A \varepsilon \mathfrak{A}$ the expansion*

$$P_\Delta(A) = c_0 + c_1\Delta + c_2\Delta^2 + \cdots$$

holds, where the series is a product of the Taylor series for

$$\exp\{-\tfrac{1}{2}\Delta^2\sigma^2\} \quad \textit{and of} \quad \sum_{j=0}^{\infty} (\Delta^j/j!) \int_A \xi^j\,dP.$$

PROOF. According to Theorem 2.1 and the Radon-Nikodym theorem we have

$$P_\Delta(A) = \int_A (dP_\Delta/dP)\,dP = \exp\{-\tfrac{1}{2}\Delta^2\sigma^2\} \int_A \exp\{\Delta\xi\}\,dP.$$

Using Lemma 3.. we obtain the product of two absolutely convergent series and therefore again an absolutely convergent series.

For fixed $A \varepsilon \mathfrak{A}$ we shall denote

$$a_k = \int_A \xi^k\,dP, \qquad k = 0, 1, 2, \cdots.$$

THEOREM 3.2. *One can write*

$$P_\Delta(A) = a_0 + a_1\Delta + Z_1,$$

where

$$|Z_1| \leqq |(\exp\{-\tfrac{1}{2}\Delta^2\sigma^2\} - 1)(a_0 + a_1\Delta)| + \tfrac{1}{4}\Delta^2\sigma^2 \exp\{-\tfrac{1}{2}\Delta^2\sigma^2\} + \tfrac{1}{2}\Delta^2\sigma^2[(1 + \Delta^2\sigma^2)\Phi(\Delta\sigma) + \Delta\sigma(2\pi)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}\Delta^2\sigma^2\}].$$

PROOF.

$$P_\Delta(A) = \exp\{-\tfrac{1}{2}\Delta^2\sigma^2\} \int_A \exp\{\Delta\xi\}\,dP = (1 + \exp\{-\tfrac{1}{2}\Delta^2\sigma^2\} - 1)(a_0 + a_1\Delta + \tfrac{1}{2}\Delta^2 \int_A \xi^2 \exp\{\vartheta\Delta\xi\}\,dP$$

for some $\vartheta \varepsilon (0, 1)$. Hence ($\vartheta = \vartheta_{\Delta\xi}$)

$$P_\Delta(A) = a_0 + a_1\Delta + (\exp\{-\tfrac{1}{2}\Delta^2\sigma^2\} - 1)(a_0 + a_1\Delta) + \tfrac{1}{2}\Delta^2 \exp\{-\tfrac{1}{2}\Delta^2\sigma^2\} \int_A \xi^2 \exp\{\vartheta\Delta\xi\}\,dP.$$

But we have

$$\left|\int_A \xi^2 \exp\{\vartheta\Delta\xi\}\,dP\right| \leqq (2\pi)^{-\frac{1}{2}}\sigma^{-1} \int_{-\infty}^{0} x^2 \exp\{-x^2/2\sigma^2\}\,dx + (2\pi)^{-\frac{1}{2}}\sigma^{-1} \int_0^\infty x^2 \exp\{-x^2/2\sigma^2 + \Delta x\}\,dx$$
$$= \tfrac{1}{2}\sigma^2 + \sigma^2 \exp\{\tfrac{1}{2}\Delta^2\sigma^2\}[(1 + \Delta^2\sigma^2)\Phi(\Delta\sigma) + \Delta\sigma(2\pi)^{-\frac{1}{2}} \exp\{-\tfrac{1}{2}\Delta^2\sigma^2\}].$$

THEOREM 3.3. *Denoting*

$$D = \exp\{-\tfrac{1}{2}\Delta^2\sigma^2\} - 1 + \tfrac{1}{2}\Delta^2\sigma^2,$$
$$R = \sigma^3(2\pi)^{-\frac{1}{2}}(4 + \Delta^2\sigma^2) + \Delta\sigma^4 \exp\{\tfrac{1}{2}\Delta^2\sigma^2\}(3 + \Delta^2\sigma^2)\Phi(\Delta\sigma),$$

we have

$$P_\Delta(A) = a_0 + a_1\Delta + \tfrac{1}{2}(a_2 - a_0\sigma^2)\Delta^2 + Z_2,$$

where

$$|Z_2| \leqq \tfrac{1}{6}\Delta^3 R\,|1 - \tfrac{1}{2}\Delta^2\sigma^2 + D| + |D(a_0 + a_1\Delta + \tfrac{1}{2}a_2\Delta^2) - \tfrac{1}{2}a_1\sigma^2\Delta^3 - \tfrac{1}{4}a_2\sigma^2\Delta^4|.$$

The *proof* is analogous to that of the Theorem 3.2.

4. The computation of the coefficient a_1.

THEOREM 4.1. *Let $G(t) = \int_A z(t)\,dP$ be absolutely continuous on* $[0, 1]$ *with derivative $G'(t) = g(t) \varepsilon L_2$. Then for any $\varphi \varepsilon L_2^0$*

$$\int_A [\int_0^1 \varphi(t)\,dz(t)]\,dP = \int_0^1 \varphi(t)g(t)\,dt. \tag{4.1}$$

PROOF. The relation (4.1) obviously holds when $\varphi(t)$ is a step function. Let $\{\varphi_n(t)\}$ be a sequence of step functions such that $\int_0^1 \varphi_n(t)\,dt = 0$ $(n = 1, 2, \cdots)$ and $\varphi_n \to \varphi \varepsilon L_2^0$ in the quadratic mean. Then

$$|\int_A [\int_0^1 \varphi_n(t)\,dz(t)]\,dP - \int_A [\int_0^1 \varphi(t)\,dz(t)]\,dP| \leqq E\,|\int_0^1 [\varphi_n(t) - \varphi(t)]\,dz(t)| \to 0$$

taking into account that

$$E\,|\int_0^1 [\varphi_n(t) - \varphi(t)]\,dz(t)|^2 \leqq \int_0^1 |\varphi_n(t) - \varphi(t)|^2\,dt \to 0.$$

Furthermore we have

$$\int_0^1 \varphi_n(t)g(t)\,dt \to \int_0^1 \varphi(t)g(t)\,dt$$

(convergence in quadratic mean implies weak convergence in L_2). Q.E.D.

Consider the events

$$A_0 = \{\sup_{0\leqq t\leqq 1}[z(t) - a - bt] < 0\}, \qquad a > 0, \quad a + b > 0,$$
$$A_1 = \{\sup_{0\leqq t\leqq 1}[z(t) - a] < 0\}, \qquad a > 0,$$
$$A_2 = \{\sup_{a\leqq t\leqq 1}[z(t) - \lambda t] < 0\}, \qquad a \,\varepsilon\, (0, 1), \quad \lambda > 0.$$

From [2] we easily get the following result, given here without proof.

LEMMA 4.1. *Let $w(s)$ be a real separable Gaussian process on $[0, t]$, $t \varepsilon (0, 1)$ with $Ew(s) = 0$, $Ew(s)w(s') = s(1 - s'/t)$ for $0 \leqq s \leqq s' \leqq t$. For $\gamma > 0$, $\gamma + \delta t > 0$ the probability of exceeding the straight line $\gamma + \delta s$ is*

$$\exp\{-2\gamma^2/t - 2\gamma\delta\}.$$

First we shall consider the event A_0. The conditional process $z(s)$ with condition $z(t) = y$ has

$$E[z(s)\,|\,z(t) = y] = (s/t)y$$

and the correlation function $s(1 - s'/t)$ for $0 \leqq s \leqq s' \leqq t$. The probability of exceeding the barrier of $a + bs$ on $[0, t]$ equals the probability that $w(s)$ considered in Lemma 4.1 exceeds the barrier $a + (b - y/t)s$ on $[0, t]$, i.e.

$$\exp\{-(2a/t)(a + bt - y)\}$$

for $y < a + bt$. Analogously the probability of exceeding the barrier $a + bs$ on $[t, 1]$ by our conditional process is

$$\exp\{-[2(a + b)/(1 - t)](a + bt - y)\}.$$

But the subprocesses $\{z(s), 0 \leqq s \leqq t\}$ and $\{z(s), t \leqq s \leqq 1\}$ are conditionally independent given $z(t) = y$. Consequently

$$P(A_0 \mid z(t) = y) = [1 - \exp\{-(2a/t)(a + bt - y)\}] \cdot [1 - \exp\{-[2(a + b)/(1 - t)](a + bt - y)\}]$$

for $y < a + bt$ and 0 for $y > a + bt$.

In view of Lemma 4.1 (for $t = 1$) we have

$$P^* = P(A_0) = 1 - \exp\{-2a(a + b)\}.$$

The probability density of $y = z(t)$ is

$$h_0(y) = [2\pi t(1 - t)]^{-\frac{1}{2}} \exp\{-y^2/2t(1 - t)\}.$$

The conditional density of $y = z(t)$ given A_0 is

$$h_1(y) = (P^*)^{-1}[1 - \exp\{-(2a/t)(a + bt - y)\}] \cdot [1 - \exp\{-2[(a + b)/(1 - t)](a + bt - y)\}]h_0(y)$$

for $y < a + bt$ and 0 for $y > a + bt$. Let $h(y) = P^* h_1(y)$. Then

$$G_0(t) = \int_{A_0} z(t)\, dP = \int_{-\infty}^{a+bt} yh(y)\, dy.$$

After somewhat lengthy computations we get

$$\begin{aligned} G_0(t) &= 2(a + bt)\Phi(-(a + bt)/[t(1 - t)]^{\frac{1}{2}}) \\ &- 2a(1 - t)\exp\{-2a(a + b)\}\Phi((t(2a + b) - a)/[t(1 - t)]^{\frac{1}{2}}) \\ &- 2(a + b)t\exp\{-2a(a + b)\}\Phi(-(t(2a + b) - a)/[t(1 - t)]^{\frac{1}{2}}). \end{aligned} \tag{4.2}$$

Furthermore we have the kernel

$$\begin{aligned} g_0(t) = G_0'(t) &= 2b\Phi(-(a + bt)/[t(1 - t)]^{\frac{1}{2}}) \\ &+ 2a\exp\{-2a(a + b)\}\Phi((t(2a + b) - a)/[t(1 - t)]^{\frac{1}{2}}) \\ &- 2(a + b)\exp\{-2a(a + b)\}\Phi(-(t(2a + b) - a)/[t(1 - t)]^{\frac{1}{2}}). \end{aligned} \tag{4.3}$$

For the event A_1 we put $b = 0$ and hence obtain the kernel

$$g_1(t) = 2a\exp\{-2a^2\}[2\Phi((2at - a)/[t(1 - t)]^{\frac{1}{2}}) - 1]. \tag{4.4}$$

Now we shall derive the kernel $g_2(t)$ for the event A_2. Consider the conditional

Brownian bridge $z(s)$ given $z(a) = y$. Such a process does not exceed the barrier λs, $s \varepsilon (a, 1)$, with probability

$$S_y = 1 - \exp\{-2\lambda(\lambda a - y)/(1 - a)\} \tag{4.5}$$

for $y < \lambda a$. The probability density of $y = z(a)$ is

$$f(y) = [2\pi a(1 - a)]^{-\frac{1}{2}} \exp\{-y^2/2a(1 - a)\}.$$

The unconditional probability S that $z(s)$ does not exceed λs on $[a, 1]$ is

$$S = \int_{-\infty}^{\lambda a} S_y f(y)\, dy = 2\Phi(\lambda[a/(1 - a)]^{\frac{1}{2}}) - 1. \tag{4.6}$$

Moreover, the conditional density $h_a(y)$ of $y = z(a)$ with the condition A_2 is

$$h_a(y) = (S_y/S)f(y), \qquad y < \lambda a.$$

Hence

$$\int_{A_2} z(a)\, dP = S \int_{-\infty}^{\lambda a} y h_a(y)\, dy = -2a\lambda\Phi(-\lambda[a/(1 - a)]^{\frac{1}{2}}).$$

Assume $t \varepsilon (0, a)$. Consider the conditional process $z(s)$ given $z(t) = x$. On $[t, 1]$ this is a Gaussian process with expectation $[x/(1 - t)](1 - s)$, $t \leqq s \leqq 1$, and correlation function $(1 - s')(s - t)/(1 - t)$ for $0 < t \leqq s \leqq s' \leqq 1$. Under this condition $y = z(a)$ has the density

$$k(y) = [2\pi(1 - a)(a - t)/(1 - t)]^{-\frac{1}{2}} \cdot \exp\left\{-\frac{(y + ax(1 - t)^{-1} - x(1 - t)^{-1})^2}{2(1 - a)(a - t)/(1 - t)}\right\}.$$

When $y = z(a)$ is fixed, owing to the independence of the subprocesses on $[0, a]$ and $[a, 1]$ our process does not exceed the barrier λs with probability S_y. The probability that the Brownian bridge with $z(t) = x$ does not exceed λs on $[a, 1]$ is

$$\begin{aligned} S_{x,t} &= \int_{-\infty}^{\lambda a} S_y k(y)\, dy \\ &= \Phi((a\lambda - at\lambda + ax - x)/[(1 - t)(1 - a)(a - t)]^{\frac{1}{2}}) \\ &\quad - \exp\{2\lambda(x - \lambda t)/(1 - t)\} \\ &\quad \cdot \Phi((-a\lambda - at\lambda + ax - x + 2t\lambda)/[(1 - t)(1 - a)(a - t)]^{\frac{1}{2}}). \end{aligned}$$

The conditional density of $x = x(t)$ with condition A_2 is

$$h^*(z) = [2\pi t(1 - t)]^{-\frac{1}{2}} \exp\{-x^2/2t(1 - t)\} \cdot S_{x,t}/S$$

and

$$\begin{aligned} \int_{A_2} z(t)\, dP &= S \int_{-\infty}^{\infty} x h^*(x)\, dx \\ &= -2\lambda t(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\{-\tfrac{1}{2}y^2\} \\ &\quad \cdot \Phi([-a\lambda(1 - t)^{\frac{1}{2}} + (1 - a)t^{\frac{1}{2}}y]/[(1 - a)(a - t)]^{\frac{1}{2}})\, dy \\ &= -2\lambda t\Phi(-\lambda[a/(1 - a)]^{\frac{1}{2}}), \qquad \text{for} \quad t \varepsilon (0, a). \end{aligned} \tag{4.7}$$

Assume $t \varepsilon (a, 1)$. Let $z(t) = x$ be fixed. Then $z(s)$ on $[0, t]$ is Gaussian with expectation $(x/t)s$, $0 \leqq s \leqq t$ and correlation function $s(1 - s'/t)$, $0 \leqq s \leqq s' \leqq t$.

Consider $z(s)$ with condition $z(a) = y$, $z(t) = x$. This process does not exceed the barrier λs on $[t, 1]$ with probability

$$1 - \exp\{-2\lambda(\lambda t - x)/(1 - t)\}, \qquad x < \lambda t,$$

and on $[a, t]$ with probability

$$1 - \exp\{-2(\lambda a - y)(\lambda t - x)/(t - a)\}, \quad y < \lambda a, \quad x < \lambda t.$$

Thus the probability that $z(s)$ with condition $z(t) = x$ only does not exceed λs on $[a, t]$ is

$$\begin{aligned} S'_{x,t} &= \int_{-\infty}^{\lambda a} [1 - \exp\{-2(\lambda a - y)(\lambda t - x)/(t - a)\}] \\ &\quad \cdot [2\pi a(1 - a/t)]^{-\frac{1}{2}} \exp\{-(y - ax/t)^2/2a(1 - a/t)\}\, dy \\ &= 1 - 2\Phi((x - \lambda t)[a/t(t - a)]^{\frac{1}{2}}). \end{aligned}$$

The process $z(s)$ conditioned by $z(t) = x$ does not exceed the barrier λs on $[a, 1]$ with probability

$$S''_{x,t} = S'_{x,t}[1 - \exp\{-2\lambda(\lambda t - x)/(1 - t)\}].$$

The conditional density of $x = z(t)$ under the condition A_2 is then

$$\begin{aligned} h^0(x) &= [2\pi t(1 - t)]^{-\frac{1}{2}} \exp\{-x^2/2t(1 - t)\} S''_{x,t}/S, && x < \lambda t, \\ &= 0, && x \geqq \lambda t. \end{aligned}$$

We have, for $t \varepsilon (a, 1)$

$$\begin{aligned} \int_{A_2} z(t)\, dP &= S \int_{-\infty}^{\lambda t} x h^0(x)\, dx \\ &= 2(1 - t)(2\pi)^{-\frac{1}{2}}[a/(1 - a)]^{\frac{1}{2}} \exp\{-a\lambda^2/2(1 - a)\} \\ &\quad - 4(1 - t)(2\pi)^{-\frac{1}{2}}[a/(1 - a)]^{\frac{1}{2}} \exp\{-a\lambda^2/2(1 - a)\} \\ &\qquad \cdot \Phi(\lambda[(t - a)/(1 - t)(1 - a)]^{\frac{1}{2}}) \\ &\quad - 4\lambda t(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{0} \exp\{-\tfrac{1}{2}y^2\} \\ &\qquad \cdot \Phi([y[t(t - a)]^{\frac{1}{2}} - \lambda t a^{\frac{1}{2}}]/[at(1 - t)]^{\frac{1}{2}})\, dy. \end{aligned} \tag{4.8}$$

These relations may be easily obtained:

$$\begin{aligned} &\lim_{t \to a^-} \int_{A_2} z(t)\, dP = \lim_{t \to a^+} \int_{A_2} z(t)\, dP = \int_{A_2} z(a)\, dP, \\ &\lim_{t \to 0^+} \int_{A_2} z(t)\, dP = \lim_{t \to 1^-} \int_{A_2} z(t)\, dP = 0. \end{aligned} \tag{4.9}$$

The derivative is

$$g_2(t) = -2\lambda\Phi(-\lambda[a/(1 - a)]^{\frac{1}{2}}), \qquad 0 < t < a,$$

$$g_2(t) = -4\lambda(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{0} \exp\{-\tfrac{1}{2}y^2\}\Phi\left(\frac{y[t(t - a)]^{\frac{1}{2}} - \lambda t a^{\frac{1}{2}}}{[at(1 - t)]^{\frac{1}{2}}}\right) dy$$

$$+ 4(2\pi)^{-\frac{1}{2}} [a/(1-a)]^{\frac{1}{2}} \exp \{-a\cdot\lambda^2/2(1-a)\}$$
$$\cdot \Phi(\lambda[(t-a)/(1-t)(1-a)]^{\frac{1}{2}})$$
$$- 2(2\pi)^{-\frac{1}{2}} [a/(1-a)]^{\frac{1}{2}} \exp \{-a\lambda^2/2(1-a)\}, \qquad a < t < 1.$$

We see that

$$\lim_{t\to a^+} g_2(t) = -2\lambda\Phi(-\lambda[a/(1-a)]^{\frac{1}{2}}),$$
$$\lim_{t\to 1^-} g_2(t) = 2(2\pi)^{-\frac{1}{2}}[a/(1-a)]^{\frac{1}{2}} \exp\{-a\lambda^2/2(1-a)\}.$$

Let α be fixed, $\alpha \varepsilon (0, \frac{1}{2})$. If we require $P(A_2) = 1 - \alpha$, then it follows from (4.6)

TABLE 1

Table of the kernel $g_2(t)$ for $\alpha = 0.05$, $a = 0.1$

t	$g_2(t)$
0.10	−0.294
.11	− .109
.13	− .023
.15	.010
.17	.025
.20	.034
.30	.039
.40	.039
.60	.039
.80	.039
1.00	.039

$$\lambda = [(1-a)/a]^{\frac{1}{2}} u_{1-\frac{1}{2}\alpha}. \tag{4.10}$$

For $\alpha = 0.05$ and $a = 0.1$ we get $\lambda = 5.88$. Table 1 contains the values of $g_2(t)$ for several fixed $t \varepsilon [0.1; 1]$.

EXAMPLE. Put

$$\varphi(t) = 1, \qquad 0 \leqq t \leqq \tfrac{1}{2},$$
$$= -1, \qquad \tfrac{1}{2} < t \leqq 1.$$

Such a φ-function corresponds to a double exponential distribution. Then

$$\Delta \int_0^1 \varphi(t) g_2(t)\, dt = \Delta[G_2(\tfrac{1}{2}) - G_2(0) - G_2(1) + G_2(\tfrac{1}{2})]$$
$$= 2\Delta G_2(\tfrac{1}{2}) = 2\Delta \int_{A_2} z(\tfrac{1}{2})\, dP.$$

This expression depends of course on a. Defining $H(a) = \int_{A_2} z(\frac{1}{2})\, dP$ we shall study this function of a. Let us look for such a value $a \varepsilon (0, 1)$ so that $H(a)$ should be minimal. Then with regard to Theorem 3.2 and Theorem 4.1

$$P_\Delta(A_2) \sim a_0 + a_1\Delta = a_0 + 2\Delta H(a)$$

is (for sufficiently small Δ) minimal, i.e. the probability of exceeding the barrier

λs on $[a, 1]$ is maximal. For $\frac{1}{2} < a < 1$ according to (4.7) and (4.10) we obtain

$$H(a) = -\beta[(1-a)/a]^{\frac{1}{2}}\Phi(-\beta), \qquad \beta = u_{1-\frac{1}{2}\alpha},$$

i.e. $H(a)$ is increasing on $(\frac{1}{2}, 1)$.

For $a \varepsilon (0, \frac{1}{2})$ we have from (4.8)

$$\begin{aligned} H(a) = & -2\beta(2\pi)^{-\frac{1}{2}}[(1-a)/a]^{\frac{1}{2}} \int_{-\infty}^{0} \exp\{-\tfrac{1}{2}y^2\} \\ & \cdot\Phi([y(1-2a)^{\frac{1}{2}} - \beta(1-a)^{\frac{1}{2}}]a^{-\frac{1}{2}})\, dy \\ & -(2\pi)^{-\frac{1}{2}}\exp\{-\tfrac{1}{2}\beta^2\}[a/(1-a)]^{\frac{1}{2}}[2\Phi(\beta[(1-2a)/a]^{\frac{1}{2}}) - 1]. \end{aligned} \tag{4.11}$$

We see that

$$\lim_{a\to\frac{1}{2}^+} H(a) = \lim_{a\to\frac{1}{2}^-} H(a) = -\beta\Phi(-\beta).$$

TABLE 2

Table of the function H(a)

a	$H(a)$
0.00	0.00000
.10	− .01948
.20	− .02922
.30	− .03813
.40	− .04622
.45	− .04904
.49	− .04950
.50	− .04900

By somewhat lengthy considerations it may be shown that

$$\lim_{a\to 0^+} H(a) = 0.$$

For $\alpha = 0.05$, $H(a)$, $0 \leqq a \leqq \frac{1}{2}$, is tabulated (see Table 2). Let us find the minimum of $H(a)$. By quadratic interpolation and from (4.11) we find the minimum -0.04965 in the point $a = 0.475$.

5. Higher order coefficients. We derive the expression of $\int_A \xi^k \, dP$ for a set of functions $\{\varphi(t)\}$ with bounded variation.

Suppose that $\varphi \varepsilon L_2^0$ is a function with bounded variation. Then there exists a sequence $\{\varphi_n(t)\}$ of step functions such that $\varphi_n \varepsilon L_2^0$, $\varphi_n(t) \to \varphi(t)$ everywhere; and such that all $\varphi_n(t)$ are uniformly bounded and have uniformly bounded variations. From Lebesgue's theorem we obtain that $\int [\varphi_n(t) - \varphi(t)]^2 \, dt \to 0$, i.e. $\varphi_n \to \varphi$ in quadratic mean.

LEMMA 5.1. *For any given natural numbers* $k \geqq 1$, $r \geqq 0$,

$$E\,|z(t_1) \cdots z(t_k)|^{2r} \leqq K_{k,r}$$

holds, where $z(t)$ *is the Brownian bridge and* $K_{k,r}$ *is a constant not depending on the choice of points* $t_1, \cdots, t_k \,\varepsilon\, [0, 1]$.

PROOF. By immediate mathematical induction.

For a fixed $A \varepsilon \mathfrak{A}$ let us define

$$G(t_1, \cdots, t_k) = \int_A z(t_1) \cdots z(t_k) \, dP.$$

LEMMA 5.2. *The function* $G(t_1, \cdots, t_k)$ *is continuous on*

$$C_k = \times_1^k [0, 1].$$

PROOF. Let $t_{i_n}' \varepsilon [0, 1]$, $t_{i_n}' \to t_i$, $i = 1, 2, \cdots, k$. We drop the sub-index n. We have

$$\begin{aligned} |G(t_1, \cdots, t_k) - G(t_1', \cdots, t_k')| & \\ &\leqq E\,|z(t_1) \cdots z(t_k) - z(t_1') \cdots z(t_k')| \\ &= E\,|\textstyle\sum_{i=1}^k z(t_1') \cdots z(t_{i-1}')[z(t_i) - z(t_i')]z(t_{i+1}) \cdots z(t_k)| \\ &\leqq \textstyle\sum_{i=1}^k [E\,|z(t_i) - z(t_i')|^2 K_{k-1,1}]^{\frac{1}{2}} \to 0 \end{aligned}$$

(the process $z(t)$ is continuous in the mean).

We denote by $\chi_A(t)$ the characteristic function of a set $A \subset [0, 1]$.

LEMMA 5.3. *Let* $[a_i, b_i] \subset [0, 1]$, *put* $\varphi_i(t) = \chi_{[a_i, b_i]}(t)$, $i = 1, 2, \cdots, k - s$. *Let* $d_1, \cdots, d_s \varepsilon [0, 1]$. *Then*

$$\begin{aligned} (-1)^{k-s} \int_{C_{k-s}} G(t_1, \cdots, t_{k-s}, d_1, \cdots, d_s) \, d\varphi_1(t) \cdots d\varphi_{k-s}(t_{k-s}) & \\ = \int_A z(d_1) \cdots z(d_s)[z(b_1) - z(a_1)] \cdots [z(b_{k-s}) - z(a_{k-s})] \, dP. \end{aligned}$$

PROOF. For $s = k$ the assertion holds. The rest follows by induction from s to $s - 1$.

THEOREM 5.1. *Let* $\varphi \varepsilon L_2^0$ *be a function with bounded variation. Then for* $A \varepsilon \mathfrak{A}$ *the following relation holds:*

$$\int_A [\textstyle\int_0^1 \varphi(t) \, dz(t)]^k \, dP = (-1)^k \int_{C_k} G(t_1, \cdots, t_k) \, d\varphi(t_1) \cdots d\varphi(t_k). \tag{5.1}$$

PROOF. When $\varphi(t)$ is a step function, the theorem obviously holds. In the general case we take a uniformly bounded sequence $\{\varphi_n(t)\}$ of step functions such that $\varphi_n \varepsilon L_2^0$ and $\varphi_n(t) \to \varphi(t)$ everywhere. Denote

$$\xi_n = \int_0^1 \varphi_n(t) \, dz(t), \qquad \xi = \int_0^1 \varphi(t) \, dz(t),$$
$$\sigma_n^2 = \int_0^1 |\varphi_n(t)|^2 \, dt, \qquad \sigma^2 = \int_0^1 |\varphi(t)|^2 \, dt.$$

We know that ξ_n has $N(0, \sigma_n^2)$ and ξ has $N(0, \sigma^2)$. For any odd number n let $n!! = n(n - 2) \cdots 3.1$. Let us note that $\sigma_n < R$, where R is a constant. We obtain

$$\begin{aligned} &|\textstyle\int_A \xi_n^k \, dP - \int_A \xi^k \, dP| \\ &\quad \leqq E\,|\xi_n^k - \xi^k| \\ &\quad = E\,|\textstyle\sum_{i=1}^k \xi_n^{i-1}(\xi - \xi_n)\xi^{k-i}| \\ &\quad \leqq \textstyle\sum_{i=1}^k [E\,|\xi - \xi_n|^2 \, E\xi_n^{2i-2}\xi^{2k-2i}]^{\frac{1}{2}} \\ &\quad \leqq \textstyle\sum_{i=1}^k [\int_0^1 |\varphi(t) - \varphi_n(t)|^2 \, dt]^{\frac{1}{2}}[(4i - 5)!!(4k - 4i - 1)!!R^{4i-4}\sigma^{4k-4i}]^{\frac{1}{4}} \to 0 \end{aligned}$$

for $n \to \infty$, i.e.

$$\int_A [\int_0^1 \varphi_n(t)\, dz(t)]^k\, dP \to \int_A [\int_0^1 \varphi(t)\, dz(t)]^k\, dP. \tag{5.2}$$

The relation

$$\int_{C_k} G(t_1, \cdots, t_k)\, d\varphi_n(t_1) \cdots d\varphi_n(t_k) \to \int_{C_k} G(t_1, \cdots, t_k)\, d\varphi(t_1) \cdots d\varphi(t_k)$$

is well-known (in the one-dimensional case it is Helly's second theorem). The proof is complete.

From Theorem 5.1 there follows

THEOREM 5.1′. *Let $\varphi \,\varepsilon\, L_2^0$ be absolutely continuous. Then*

$$\int_A [\int_0^1 \varphi(t)\, dz(t)]^k\, dP = (-1)^k \int_{C_k} G(t_1, \cdots, t_k)\varphi'(t_1) \cdots \varphi'(t_k)\, dt_1 \cdots dt_k .$$

REMARK 5.1. Heuristically, the idea of proof of Theorem 5.1 is as follows: Integrating by parts, we get

$$[\int_0^1 \varphi(t)\, dz(t)]^k = [-\int_0^1 z(t)\, d\varphi(t)]^k$$
$$= (-1)^k \int_{C_k} z(t_1) \cdots z(t_k)\, d\varphi(t_1) \cdots d\varphi(t_k).$$

Now integration over A with respect to P and the interchange of both integrals on the right side gives (5.1).

REMARK 5.2. In the proof of Theorem 5.1 we have proved this more general assertion: Let $\{\varphi_n(t)\}$, $\varphi_n \,\varepsilon\, L_2^0$, be a sequence of functions such that $\{\varphi_n\}$ converges in quadratic mean to the function $\varphi \,\varepsilon\, L_2^0$. Denote

$$\xi_n = \int_0^1 \varphi_n(t)\, dz(t), \qquad \xi = \int_0^1 \varphi(t)\, dz(t).$$

Then for any $A \,\varepsilon\, \mathfrak{A}$ and any natural number k the relation

$$\int_A \xi_n^k\, dP \to \int_A \xi^k\, dP \tag{5.3}$$

holds. Actually, the proof of (5.2) holds for (5.3), too.

LEMMA 5.4. *Let $\varphi \,\varepsilon\, L_2^0$ be monotone on the intervals $(0, \epsilon]$, $[1 - \epsilon, 1)$ for some $\epsilon \,\varepsilon\, (0, 1)$. Let $[a_n, b_n]$, $n = 1, 2, \cdots$, be intervals in $(0, 1)$ such that $a_n \to 0^+$, $b_n \to 1^-$. Let*

$$\begin{aligned} \varphi_n^*(t) &= \varphi(a_n), && t \,\varepsilon\, [0, a_n), \\ &= \varphi(t), && t \,\varepsilon\, [a_n, b_n], \\ &= \varphi(b_n), && t \,\varepsilon\, (b_n, 1] \end{aligned}$$

and $\varphi_n(t) = \varphi_n^*(t) - \int_0^1 \varphi_n^*(t)\, dt$. Then $\varphi_n \,\varepsilon\, L_2^0$ and $\varphi_n \to \varphi$ in quadratic mean.

PROOF. The reader may prove for himself that $\varphi_n^* \to \varphi$ in quadratic mean. As $\int_0^1 \varphi_n^*(t)\, dt \to \int_0^1 \varphi(t)\, dt = 0$, Lemma 5.4 obviously holds.

THEOREM 5.2. *Let $\varphi \,\varepsilon\, L_2^0$ be monotone on $(0, \epsilon]$ and $[1 - \epsilon, 1)$, $\epsilon \,\varepsilon\, (0, 1)$. Let $\varphi(t)$ have bounded variation on every interval $[a, b] \subset (0, 1)$. As in Lemma 5.4 we define the sequence $\{\varphi_n(t)\}$ with $b_n = 1 - a_n$. Then for $A \,\varepsilon\, \mathfrak{A}$ and k natural the limit*

$$lim_{n\to\infty} (-1)^k \int_{C_k} G(t_1, \cdots, t_k)\, d\varphi_n(t_1) \cdots d\varphi_n(t_k)$$

exists and is equal to $\int_A [\int_0^1 \varphi(t)\, dz(t)]^k\, dP$. *This limit can be written as*

$$(-1)^k \int_{C_k} G(t_1, \cdots, t_k)\, d\varphi(t_1) \cdots d\varphi(t_k).$$

Proof. The existence of the limit and its equality to $\int_A [\int_0^1 \varphi(t)\, dz(t)]^k\, dP$ follows from Theorem 5.1 and Remark 5.2.

Theorem 5.2′. *Suppose that* $\varphi(t)$ *is absolutely continuous on every interval* $[a, b]$, $0 < a < b < 1$. *Then under the assumptions of Theorem* 5.2

$$\int_A [\int_0^1 \varphi(t)\, dz(t)]^k\, dP = (-1)^k \int_{C_k} G(t_1, \cdots, t_k)\varphi'(t_1) \cdots \varphi'(t_k)\, dt_1 \cdots dt_k \tag{5.4}$$

where the integral on the right side is taken in the sense

$$\lim_{\epsilon\to 0^+} \int_\epsilon^{1-\epsilon} \cdots \int_\epsilon^{1-\epsilon} G(t_1, \cdots, t_k)\varphi'(t_1) \cdots \varphi'(t_k)\, dt_1 \cdots dt_k .$$

Proof. We see that

$$(-1)^k \int_{C_k} G(t_1, \cdots, t_k)\, d\varphi_n(t_1) \cdots d\varphi_n(t_k) = (-1)^k \int_{a_n}^{1-a_n} \cdots \int_{a_n}^{1-a_n} G(t_1, \cdots, t_k)\varphi'(t_1) \cdots \varphi'(t_k)\, dt_1 \cdots dt_k .$$

For $n \to \infty$ the left side converges to $\int_A [\int_0^1 \varphi(t)\, dz(t)]^k\, dP$, the right side to the integral

$$(-1)^k \int_0^1 \cdots \int_0^1 G(t_1, \cdots, t_k)\varphi'(t_1) \cdots \varphi'(t_k)\, dt_1 \cdots dt_k .$$

Remark 5.3. It may be proved that the integral on the right side of (5.4) exists. But the proof does no contain any new ideas.

For example we compute $G(s, t)$ for A_0. The probability $P(y, z)$ that the Brownian bridge does not exceed the barrier $a + b\tau$ on [0, 1] under the conditions $z(s) = y$, $z(t) = z$, $0 < s < t < 1$, is

$$\begin{aligned} P(y, z) = [1 &- \exp\{-(2a/s)(a + bs - y)\}] \\ &\cdot[1 - \exp\{-(2(a + b)/(1 - t))(a + bt - z)\}] \\ &\cdot[1 - \exp\{-(2/(t - s))(a + bs - y)(a + bt - z)\}] \end{aligned}$$

for $y < a + bs$, $z < a + bt$ and $P(y, z) = 0$ otherwise. The density of $y = z(s)$ and $z = z(t)$ is

$$\begin{aligned} f(y, z) = &\{2\pi[s(1 - t)(t - s)]^{\frac{1}{2}}\}^{-1} \\ &\cdot\exp\{-\tfrac{1}{2}[y^2t(1 - t) - 2yzs(1 - t) + z^2s(1 - s)]/s(1 - t)(t - s)\}. \end{aligned}$$

We have, for $s < t$,

$$G(s, t) = \int_{-\infty}^{a+bs} \int_{-\infty}^{a+bt} yzP(y, z)f(y, z)\, dy\, dz.$$

The symmetry gives $G(s, t) = G(t, s)$. For $s = t$ we find as in Section 4

$$G(t, t) = \int_{-\infty}^{a+bt} y^2h(y)\, dy,$$

where $h(y)$ is defined in Section 4.

6. One special case of $\varphi(t)$. We shall choose a special case of the function $\varphi(t)$ to compare our approximations with the true value of the probability of exceeding a given barrier.

Consider the Brownian bridge with the mean value

$$\Delta\psi(t) = \Delta t \quad \text{for} \quad 0 \leqq t \leqq \tfrac{1}{2}, \qquad = \Delta(1-t) \quad \text{for} \quad \tfrac{1}{2} \leqq t \leqq 1. \tag{6.1}$$

(This corresponds to a double exponential distribution.) We are looking for the probability of the event that this process does not exceed the constant barrier $a > 0$ on [0, 1]. It is equal to the probability that the Brownian bridge with vanishing mean value does not exceed on [0, 1] the barrier $a - \Delta\psi(t)$.

Consider the conditional Brownian bridge with $z(\frac{1}{2}) = y$. This conditional process does not exceed, on $[0, \frac{1}{2}]$, the barrier of $a - \Delta t$ with probability

$$P_1^*(\Delta; y) = 1 - \exp\{-4a(a - \tfrac{1}{2}\Delta - y)\}, \quad \text{if} \quad y \leqq a - \tfrac{1}{2}\Delta,$$
$$= 0, \qquad \text{if} \quad y \geqq a - \tfrac{1}{2}\Delta,$$

and on $[\frac{1}{2}, 1]$ the barrier of $a - \Delta + \Delta t$ with probability

$$P_2^*(\Delta; y) = 1 - \exp\{-4a(a - \tfrac{1}{2}\Delta - y)\}, \quad \text{if} \quad y \leqq a - \tfrac{1}{2}\Delta,$$
$$= 0, \qquad \text{if} \quad y \geqq a - \tfrac{1}{2}\Delta.$$

The density of $y = z(\frac{1}{2})$ is

$$h(y) = 2(2\pi)^{-\frac{1}{2}}e^{-2y^2}.$$

Hence, the process $z(t)$ does not exceed the barrier $a - \Delta\psi(t)$ on [0, 1] with probability

$$R(\Delta) = \int_{-\infty}^{a-\frac{1}{2}\Delta} P_1^*(\Delta; y)P_2^*(\Delta; y)h(y)\,dy.$$

Evaluating this integral we get

$$R(\Delta) = \Phi(2a - \Delta) - 2\exp\{2a(\Delta - a)\}\Phi(-\Delta) + \exp\{4a\Delta\}\Phi(-2a - \Delta). \tag{6.2}$$

Clearly, $z(t)$ exceeds the barrier $a - \Delta\psi(t)$ with probability

$$Q(\Delta) = 1 - R(\Delta). \tag{6.3}$$

Let us return to the general case of $\psi(t)$. Consider the probability $Q(\Delta)$ that $z(t)$ exceeds the barrier $a - \Delta\psi(t)$; put $R(\Delta) = 1 - Q(\Delta)$.

According to Theorem 3.1 we have

$$R(\Delta) = \textstyle\sum_{i=0}^{\infty} c_i\Delta^i, \qquad Q(\Delta) = \sum_{i=0}^{\infty} d_i\Delta^i, \tag{6.4}$$

where

$$d_0 = 1 - c_0, \quad d_i = -c_i \quad \text{for} \quad i = 1, 2, \cdots.$$

The series in (6.4) are absolutely convergent. This implies

$$Q(0) = d_0, \qquad Q^{(k)}(0) = k!\,d_k, \qquad k = 1, 2, \cdots.$$

Let the values d_0 and d_1 be known. Choose coefficients ρ_0 and ρ_1 in such a way that the functions $\Phi(\rho_0 + \rho_1\Delta)$ and $Q(\Delta)$ have the same values at the point $\Delta = 0$ and the same values of their first derivatives at $\Delta = 0$. This leads to the following equations:

$$\Phi(\rho_0) = d_0, \qquad \rho_1(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}\rho_0^2} = d_1.$$

For the event A_1, d_0 is equal to the probability that the Brownian bridge with vanishing mean value exceeds the barrier $a > 0$. We know that this probability is equal to e^{-2a^2}. For $\psi(t)$ given in (6.1) we get

$$d_1 = -\int_0^1 \varphi(t)g_1(t)\,dt = -2G_1(\tfrac{1}{2}).$$

Formula (4.2) gives

$$d_1 = 2a[d_0 - 2\Phi(-2a)].$$

Table 3 contains for $d_0 = 0.05$, $d_0 = 0.01$ and $d_0 = 0.001$ the values of a, d_1, ρ_0 and ρ_1.

$Q(\Delta)$ and $\Phi(\rho_0 + \rho_1\Delta)$ are compared in Table 4. If we know d_0, d_1, d_2, we may

TABLE 3

d_0	0.05	0.01	0.001
d_1	0.08723	0.02305	0.00297
$2d_2$	0.12741	0.04781	0.00824
ρ_0	−1.645	−2.326	−3.090
ρ_1	0.84594	0.86407	0.88084
ρ_2	0.02924	0.02788	0.02419
a	1.223873	1.517427	1.858461

TABLE 4

Δ	$\alpha = 0.05$			$\alpha = 0.01$			$\alpha = 0.001$		
	$Q(\Delta)$	$\Phi(\rho_0 + \rho_1\Delta)$	$\Phi(\rho_0 + \rho_1\Delta + \rho_2\Delta^2)$	$Q(\Delta)$	$\Phi(\rho_0 + \rho_1\Delta)$	$\Phi(\rho_0 + \rho_1\Delta + \rho_2\Delta^2)$	$Q(\Delta)$	$\Phi(\rho_0 + \rho_1\Delta)$	$\Phi(\rho_0 + \rho_1\Delta + \rho_2\Delta^2)$
0.0	0.050	0.050	0.050	0.010	0.010	0.010	0.001	0.001	0.001
0.5	0.112	0.111	0.112	0.030	0.029	0.030	0.005	0.004	0.004
1.0	0.220	0.212	0.221	0.075	0.072	0.076	0.015	0.014	0.014
1.5	0.373	0.353	0.378	0.163	0.152	0.167	0.046	0.038	0.043
2.0	0.554	0.519	0.565	0.302	0.275	0.313	0.105	0.092	0.109
2.5	0.726	0.681	0.743	0.481	0.434	0.503	0.217	0.187	0.231
3.0	0.858	0.814	0.876	0.664	0.605	0.698	0.380	0.327	0.409
3.5	0.938	0.906	0.953	0.817	0.758	0.851	0.569	0.497	0.614
4.0	0.978	0.959	0.986	0.916	0.871	0.943	0.744	0.668	0.794
4.5	0.995	0.985	0.997	0.969	0.941	0.983	0.872	0.809	0.914
5.0	0.999	0.995	1.000	0.991	0.977	0.996	0.949	0.905	0.973

choose the coefficients ρ_0, ρ_1, ρ_2 such that the functions $Q(\Delta)$ and $\Phi(\rho_0 + \rho_1\Delta + \rho_2\Delta^2)$ have the same values at $\Delta = 0$ and the first two derivatives at $\Delta = 0$ are equal. Thus we obtain the following equations:

$$\Phi(\rho_0) = d_0, \quad \rho_1(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}\rho_0^2} = d_1, \quad 2\rho_2(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}\rho_0^2} - \rho_0\rho_1^2(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}\rho_0^2} = 2d_2.$$

First of all we shall compute (as in Section 4)

$$a_2 = \int_{A_1} \xi^2\, dP = \int_{A_1} [\int_0^1 \varphi(t)\, dz(t)]^2\, dP = 4\int_{A_1} [z(\tfrac{1}{2})]^2\, dP$$
$$= 1 - e^{-2a^2} - 4a^2e^{-2a^2} + 16a^2\Phi(-2a).$$

We know (see Section 3) that $c_0 = a_0$, $c_1 = a_1$, $c_2 = \frac{1}{2}(a_2 - a_0\sigma^2)$. In our case $\sigma^2 = 1$. Then

$$2d_2 = -2c_2 = a_0\sigma^2 - a_2 = c_0 - a_2 = 1 - d_0 - a_2.$$

Numerical results are in Table 3 and Table 4.

7. Approximations based on the generalized Neyman-Pearson lemma.

LEMMA 7.1 (Generalized Neyman-Pearson lemma.) *Let $f_1, \cdots, f_{m+1}$ be real-valued functions defined on a Euclidean space and integrable with respect to μ, and suppose that for given constants $r_1, \cdots, r_m$ there exists a critical function c satisfying*

$$\int cf_i\, d\mu = r_i, \qquad i = 1, \cdots, m. \tag{7.1}$$

Let $\mathfrak{C}$ be the class of critical functions c for which (7.1) *holds.*

I. *Among all members of $\mathfrak{C}$ there exists one that maximizes*

$$\int cf_{m+1}\, d\mu.$$

II. *A sufficient condition for a member of $\mathfrak{C}$ to maximize*

$$\int cf_{m+1}\, d\mu$$

is the existence of constants $k_1, \cdots, k_m$ such that

$$c(x) = 1 \quad \text{when} \quad f_{m+1}(x) > \sum_{i=1}^m k_if_i(x),$$
$$c(x) = 0 \quad \text{when} \quad f_{m+1}(x) < \sum_{i=1}^m k_if_i(x).$$

PROOF. See [8].

Let ξ be the random variables introduced in Theorem 2.1. Consider $\sup_{A\varepsilon\mathfrak{A}} \int_A dP_\Delta$ under conditions

$$\int_A \xi^i\, dP = r_i, \qquad i = 0, 1, \cdots, k. \tag{7.2}$$

Denote by $\mathfrak{D}$ the class of critical functions such that for any $c\, \varepsilon\, \mathfrak{D}$

$$\int c\xi^i\, dP = r_i, \qquad i = 0, 1, \cdots, k, \tag{7.3}$$

holds.

Obviously under (7.2)

$$\sup_{A\varepsilon\mathfrak{A}} \int_A dP_\Delta \leqq \sup_{c\varepsilon\mathfrak{D}} \int c\, dP_\Delta, \qquad \inf_{A\varepsilon\mathfrak{A}} \int_A dP_\Delta \geqq \inf_{c\varepsilon\mathfrak{D}} \int c\, dP_\Delta.$$

But (7.3) is equivalent to

$$(7.4)\qquad ((2\pi)^{\frac{1}{2}}\sigma)^{-1}\int_{-\infty}^{\infty} c^*(x)x^i \exp\{-x^2/2\sigma^2\}\,dx = r_i,\quad i = 0,\cdots,k,$$

where $c^*(x) = E[c(\omega) \mid \xi = x]$, as ξ is a sufficient statistic (see Theorem 2.1). Let us look for a critical function $c^*(x)$ such that

$$(7.5)\qquad ((2\pi)^{\frac{1}{2}}\sigma)^{-1}\int_{-\infty}^{\infty} c^*(x) \exp\{-\tfrac{1}{2}(x/\sigma - \Delta\sigma)^2\}\,dx$$

is maximal under (7.4).

(A) $k = 0$. Using Lemma 7.1 we get that

$$c^*(x) = 1 \quad\text{if}\quad x > e_0,$$
$$= 0 \quad\text{if}\quad x < e_0,$$

where e_0 is a constant. Formula (7.4) gives $e_0 = \sigma u_{1-r_0}$. Hence

$$P_\Delta(A) \leqq ((2\pi)^{\frac{1}{2}}\sigma)^{-1}\int_{\sigma u_{1-r_0}}^{\infty} \exp\{-\tfrac{1}{2}(x/\sigma - \Delta\sigma)^2\}\,dx$$
$$= \Phi(\Delta\sigma - u_{1-r_0}) = \bar{Q}(\Delta).$$

If we are looking for a critical function $\bar{c}$ such that $\int \bar{c}f_{m+1}\,d\mu$ is minimal under (7.1), it suffices to substitute $-f_{m+1}$ for f_{m+1} in Lemma 7.1, i.e. to choose a $\bar{c}$ which maximizes $-\int \bar{c}f_{m+1}\,d\mu$. Hence, the lower bound $\underline{Q}(\Delta)$ may be found in the same manner as $\bar{Q}(\Delta)$ and we obtain

$$\underline{Q}(\Delta) = \Phi(u_{r_0} - \Delta\sigma).$$

(B) $k = 1$.

$$c^*(x) = 1,\quad\text{if}\quad x \in B_1,$$
$$B_1 = \{x\colon \exp\{-\tfrac{1}{2}\Delta^2\sigma^2 + \Delta x\} > k_0 + k_1x\},$$
$$k_0, k_1 \quad\text{are unknown constants.}$$

As we suppose $\Delta > 0$, we see that B_1 has the form of

$$B_1 = (-\infty, b_1) \cup (b_2, \infty),\qquad -\infty \leqq b_1 \leqq b_2 \leqq \infty.$$

Our conditions (7.4) give the following equations:

$$(7.6)\qquad \Phi(b^2/\sigma) - \Phi(b_1/\sigma) = 1 - r_0,$$
$$\exp\{-b_2^2/2\sigma^2\} - \exp\{-b_1^2/2\sigma^2\} = (r_1/\sigma)(2\pi)^{\frac{1}{2}}.$$

If $A = \{\sup_{0\leqq t\leqq 1}[z(t) - a] > 0\}$, $a > 0$, i.e. $A = A_1^c$, then

$$r_0 = \int_A dP = 1 - \int_{A_1} dP = d_0,$$
$$r_1 = \int_A \xi\,dP = \int_\Omega \xi\,dP - \int_{A_1}\xi\,dP = -a_1 = d_1.$$

Further

$$r_2 = \int_A \xi^2\,dP = \int_\Omega \xi^2\,dP - \int_{A_1}\xi^2\,dP = \sigma^2 - a_2.$$

In the example given in Section 6 we have $\sigma^2 = 1$ and if $d_0 = 0.05$, then $d_1 =$

0.087226. From (7.6) we get $b_1 = -2.707$, $b_2 = 1.679$. We shall denote the upper bound (7.5) as $\bar{Q}^1(\Delta)$. (See Table 5.)

As for the lower bound $\underline{Q}^1(\Delta)$ we obtain $\bar{c}(x) = 1$ on (β_1, β_2), where β_1 and β_2 are solutions of equations:

$$\Phi(\beta_2/\sigma) - \Phi(\beta_1/\sigma) = r_0, \tag{7.7}$$

$$-\exp\{-\beta_2^2/2\sigma^2)\} + \exp\{-\beta_1^2/2\sigma^2\} = (r_1/\sigma)(2\pi)^{\frac{1}{2}}.$$

Thus we have

$$\underline{Q}_1(\Delta) = ((2\pi)^{\frac{1}{2}}\sigma)^{-1} \int_{\beta_1}^{\beta_2} \exp\{-\tfrac{1}{2}(x/\sigma - \Delta\sigma)^2\}\, dx.$$

In our example $\beta_1 = 1.494$, $\beta_2 = 2.107$.

(C) $k = 2$. As in the previous cases $c^*(x) = 1$, if $x \varepsilon B_2$,

$$B_2 = \{x: \exp\{-\tfrac{1}{2}\Delta^2\sigma^2 + \Delta x\} > k_0 + k_1 x + k_2 x^2\},$$

i.e. $B_2 = (b_1, b_2) \cup (b_3, \infty)$, $b_1 \leqq b_2 \leqq b_3$.

Our conditions (7.4) lead to

$$1 - \Phi(b_3/\sigma) + \Phi(b_2/\sigma) - \Phi(b_1/\sigma) = r_0,$$

$$\exp\{-b_1^2/2\sigma^2\} - \exp\{-b_2^2/2\sigma^2\} + \exp\{-b_3^2/2\sigma^2\} = (r_1/\sigma)(2\pi)^{\frac{1}{2}},$$

$$1 - \Phi(b_3/\sigma) + \Phi(b_2/\sigma) - \Phi(b_1/\sigma) + ((2\pi)^{\frac{1}{2}}\sigma)^{-1}$$
$$\cdot [b_1 \exp\{-b_1^2/2\sigma^2\} - b_2 \exp\{-b_2^2/2\sigma^2\} + b_3 \exp\{-b_3^2/2\sigma^2\}] = r_2/\sigma^2.$$

In our example $d_0 = 0.05$, $r_2 = 0.17741$, $b_1 = 0.868$, $b_2 = 0.939$, $b_3 = 1.864$. Denote this upper bound as $\bar{Q}^2(\Delta)$. As for the lower bound $\underline{Q}^2(\Delta)$ we have $\bar{c}(x) = 1$ on $(-\infty, \beta_1) \cup (\beta_2, \beta_3)$, $\beta_1 \leqq \beta_2 \leqq \beta_3$. The system of equations for $\beta_1, \beta_2, \beta_3$ is as follows:

$$\Phi(\beta_1/\sigma) - \Phi(\beta_2/\sigma) + \Phi(\beta_3/\sigma) = r_0,$$

$$-\exp\{-\beta_1^2/2\sigma^2\} + \exp\{-\beta_2^2/2\sigma^2\} - \exp\{-\beta_3^2/2\sigma^2\} = (r_1/\sigma)(2\pi)^{\frac{1}{2}},$$

$$\Phi(\beta_1/\sigma) - \Phi(\beta_2/\sigma) + \Phi(\beta_3/\sigma) - ((2\pi)^{\frac{1}{2}}\sigma)^{-1}$$
$$\cdot [\beta_1 \exp\{-\beta_1^2/2\sigma^2\} - \beta_2 \exp\{-\beta_2^2/2\sigma^2\} + \beta_3 \exp\{-\beta_3^2/2\sigma^2\}] = r_2/\sigma^2.$$

TABLE 5

Δ	$Q(\Delta)$	$Q(\Delta)$	$\bar{Q}^1(\Delta)$	$\bar{Q}^2(\Delta)$	$\underline{Q}^1(\Delta)$	$\underline{Q}^2(\Delta)$
0.0	0.050	0.050	0.050	0.050	0.050	0.050
0.5	0.112	0.126	0.120	0.112	0.106	0.109
1.0	0.220	0.259	0.249	0.222	0.177	0.196
1.5	0.373	0.442	0.429	0.382	0.230	0.262
2.0	0.554	0.639	0.626	0.570	0.236	0.284
2.5	0.726	0.804	0.794	0.746	decreasing	decreasing
3.0	0.858	0.912	0.907	0.875		
3.5	0.938	0.968	0.966	0.950		
4.0	0.978	0.991	0.990	0.984		
4.5	0.995	0.998	0.998	0.996		
5.0	0.999	1.000	1.000	0.999		

In our case $\beta_1 = -3.135$, $\beta_2 = 1.551$, $\beta_3 = 2.280$. In Table 5 we see that the $\bar{Q}^2(\Delta)$ may serve as an approximation to $Q(\Delta)$ as well as $\Phi(\rho_0 + \rho_1\Delta + \rho_2\Delta^2)$. In addition we know that $\bar{Q}^2(\Delta)$ is an upper bound for $Q(\Delta)$.

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