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NONPARAMETRIC TESTS FOR SHIFT AT AN UNKNOWN TIME POINT[1]

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1. Introduction and summary. This work is an investigation of a nonparametric approach to the problem of testing for a shift in the level of a process occurring at an unknown time point when a fixed number of observations are drawn consecutively in time. We observe successively the independent random variables $X_1, X_2, \cdots, X_N$ which are distributed according to the continuous cdf $F_i, i = 1, 2, \cdots, N$. An upward shift in the level shall be interpreted to mean that the random variables after the change are stochastically larger than those before. Two versions of the testing problem are studied. The first deals with the case when the initial process level is known and the second when it is unknown. In the first case, we make the simplifying assumption that the distributions F_i are symmetric before the shift and introduce the known initial level by saying that the point of symmetry γ_0 is known. Without loss of generality, we set $\gamma_0 = 0$. Defining a class of cdf's $\mathfrak{F}_0 = \{F : F \text{ continuous}, F \text{ symmetric about origin}\}$, the problem of detecting an upward shift becomes that of testing the null hypothesis

$$H_0 : F_0 = F_1 = \cdots = F_N, \quad \text{some} \quad F_0 \varepsilon \mathfrak{F}_0,$$

against the alternative

$$H_1 : F_0 = F_1 = \cdots = F_m > F_{m+1} = \cdots = F_N, \quad \text{some} \quad F_0 \varepsilon \mathfrak{F}_0$$

where $m(0 \leqq m \leqq N - 1)$ is unknown and the notation $F_m > F_{m+1}$ indicates that X_{m+1} is stochastically larger than X_m.

For the second situation with unknown initial level, the problem becomes that of testing the null hypothesis $H_0^* : F_1 = \cdots = F_N$, against the alternatives $H_1^* : F_1 = \cdots = F_m > F_{m+1} = \cdots = F_N$, where $m(1 \leqq m \leqq N - 1)$ is unknown. Here the distributions are not assumed to be symmetric.

The testing problem in the case of known initial level has been considered by Page [11], Chernoff and Zacks [2] and Kander and Zacks [7]. Assuming that the observations are initially from a symmetric distribution with known mean γ_0, Page proposes a test based on the variables sgn $(X_i - \gamma_0)$. Chernoff and Zacks assume that the F_i are normal cdf's with constant known variance and they derive a test for shift in the mean through a Bayesian argument. Their approach is extended to the one parameter exponential family of distributions by Kander and Zacks. Except for the test based on signs, all the previous work lies within the framework of parametric statistics. The second formulation of the testing prob-

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lem, the case of unknown initial level, has not been treated in detail. The only test proposed thus far is the one derived by Chenoff and Zacks for normal distributions with constant known variance. In both problems, our approach generally is to find optimal invariant tests for certain local shift alternatives and then to examine their properties. Our optimality criterion is the maximization of local average power where the average is over the space of the nuisance parameter m with respect to an arbitrary weighting $\{q_i, i = 1, 2, \cdots, N: q_i \geqq 0, \sum_{i=1}^N q_i = 1\}$. From the Bayesian viewpoint, q_i may be interpreted as the prior probability that X_i is the first shifted variate. Invariant tests with maximum local average power are derived for the case of known initial level in Section 2 and for the case of unknown initial level in Section 3. In both cases, the tests are distribution-free and they are unbiased for general classes of shift alternatives. They all depend upon the weight function $\{q_i\}$. With uniform weights, certain tests in Section 3 reduce to the standard tests for trend while a degenerate weight function leads to the usual two sample tests. In Section 4, we obtain the asymptotic distributions of the test statistics under local translation alternatives and investigate their Pitman efficiencies. Some small sample powers for normal alternatives are given in Section 5.

2. Locally best invariant test (initial process level known). For testing H_0 versus H_1, we use invariance considerations to reduce the data and then develop distribution-free tests which maximize local average power against specific translation alternatives. The problem remains invariant under the group of all transformations $x_i' = h(x_i)$, $i = 1, 2, \cdots N$ where h is continuous, odd and strictly increasing. A maximal invariant under the group is $(\mathbf{R}, \mathbf{A})$ where $\mathbf{R} = (R_1, R_2, \cdots, R_N)$ is the vector of ranks of $|X_1|, \cdots, |X_N|$ and $\mathbf{A} = (A_1, A_2, \cdots, A_N)$ with $A_i = 0$ (1) if $X_i < 0$ (>0). If $\alpha = k/2^N N!$, any invariant test of size α will reject H_0 for exactly k realizations of $(\mathbf{R}, \mathbf{A})$.

Let $F(x)$ denote the common cdf under H_0. For the subfamily of translation alternatives, $F_{m+1}(x) = F(x - \Delta)$, $\Delta > 0$, the power $\beta(\Delta \mid m)$ depends not only on F and the amount of translation Δ, but also on the nuisance parameter m. In order to remove the parameter m, we turn our attention to the average power $\bar{\beta}(\Delta) = \sum_{i=1}^N q_i\beta(\Delta \mid i - 1)$ where the weights satisfy $q_i \geqq 0$ and $\sum_{i=1}^N q_i = 1$. The structure of the invariant test which maximizes $\bar{\beta}(\Delta)$ is exhibited in the following theorem.

THEOREM 2.1. *Let the* cdf $F \varepsilon \mathfrak{F}_0$ *possess a density* $f(x)$ *having the following properties*:

(A) $f(x) > 0$ a.e. (Lebesgue) *and f is absolutely continuous.*

(B) *For a sufficiently small* $\epsilon > 0$, *there exists a function* $H(x)$ *with* $\int_{-\infty}^{\infty} H(x)\,dx < \infty$ *and for almost all* x

$$\sup_{|\delta| \leqq \epsilon} |\{f(x + \delta) - f(x)\}\delta^{-1}| \leqq H(x).$$

Then the invariant test which maximizes the derivative of the average power at $\Delta = 0$ *has a rejection region of the form*

$$T = \sum_{i=1}^N Q_i \operatorname{sgn}(X_i)E[-f'(V^{(R_i)})/f(V^{(R_i)})] > C \tag{2.1}$$

where the $Q_i = \sum_{m=1}^{i} q_m$ *are the cumulative weights and* $V^{(1)} < V^{(2)} \cdots < V^{(N)}$ *is an ordered sample from a population having density* $2f(x)$, $x > 0$. (2.1) *also maximizes the average power itself for all sufficiently small* $\Delta > 0$.

PROOF. Let $n = N - m$ where X_{m+1} is the first shifted variate. With amount of translation Δ, the probability of any specific realization $\mathbf{A} = \mathbf{a}$ is given by

$$(2.2) \qquad P_\Delta(\mathbf{a} \mid m) = 2^{-m}[F(-\Delta)]^{n-a_0}[F(\Delta)]^{a_0}$$

where $a_0 = \sum_{i=m+1}^{N} a_i$. Due to the symmetry of F, if X has cdf $F(x - \Delta)$, the conditional density of $|X|$ given $X > 0$ is $f(x - \Delta)/F(\Delta)$ and given $X < 0$, it is $f(x + \Delta)/F(-\Delta)$. Using this together with condition (A), we follow Lehmann [8], p. 254, and express the conditional probability of any specific realization $\mathbf{r}$ of $\mathbf{R}$ given $\mathbf{A} = \mathbf{a}$ as

$$(2.3) \quad P_\Delta(\mathbf{r} \mid \mathbf{a}, m) = \{N!\,2^{N-m}[F(-\Delta)]^{n-a_0}[F(\Delta)]^{a_0}\}^{-1} \cdot E[\textstyle\prod_{i=m+1}^{N} f(V^{(r_i)} + (1 - 2a_i)\Delta)/f(V^{(r_i)})]\cdot$$

From (2.2) and (2.3), we obtain

$$(2.4) \qquad P_\Delta(\mathbf{r}, \mathbf{a} \mid m) = (N!\,2^N)^{-1}E[\textstyle\prod_{i=1}^{N} f(V^{(r_i)} + b_{mi}\Delta)/f(V^{(r_i)})]$$

where $b_{mi} = 0$ for $i \leqq m$ and $b_{mi} = (1 - 2a_i)$ for $i > m$. If $P_\Delta(\mathbf{r}, \mathbf{a}) = \sum_{m=1}^{N} q_m P_\Delta(\mathbf{r}, \mathbf{a} \mid m - 1)$, the average power of an invariant test for shift Δ is obtained by summing $P_\Delta(\mathbf{r}, \mathbf{a})$ over all $(\mathbf{r}, \mathbf{a})$ belonging to the critical region. Letting S denote the space of the ordered N-tuples $v^{(1)} < v^{(2)} < \cdots < v^{(N)}$, we have $P_\Delta(\mathbf{r}, \mathbf{a} \mid m) = \int_S [\prod_{i=1}^{N} f(v^{(r_i)} + b_{mi}\Delta)\, dv^{(i)}]$. For all $|\Delta| \leqq \epsilon \leqq 1$, $G(x) = H(x) + f(x)$ dominates both $f(x + b_{mi}\Delta)$ and $|\Delta^{-1}[f(x + b_{mi}\Delta) - f(x)]|$ almost everywhere. This yields

$$(2.5) \qquad \Delta^{-1}\,|\textstyle\prod_{i=1}^{N} f(v^{(r_i)} + b_{mi}\Delta) - \prod_{i=1}^{N} f(v^{(r_i)})| \leqq N \prod_{i=1}^{N} G(v^{(r_i)})$$

and the right hand side is integrable. Applying the dominated convergence theorem and Neyman-Pearson's Lemma, it follows essentially from Lehmann [8], p. 237, that the rejection region which maximizes the local average power is given by (2.1). This completes the proof of the theorem.

For a few specific choices of the distribution F, the test statistics T of (2.1) are given in Table 1. Large values of the test statistic are critical in each case. The uniform weighting used in the third column allows for the possibility that a shift might occur before the observations are taken. Chernoff and Zacks [2] and also Kander and Zacks [7] have assumed that the known process level corresponds to the distribution of X_1 and this has led them to the uniform prior $q_i = (N - 1)^{-1}$, $i = 2, 3, \cdots, N$. Apart from this minor difference, our optimal invariant test $T_{(1)}$ with uniform weighting coincides with Kander and Zacks' test which is based on the marginal likelihood ratio for a binomial sample. Some power comparisons between this and Page's test have been made in [2]. Hájek [5] and Adichie [1] have studied the large sample properties of test statistics of the form (2.1) which arise in connection with a linear regression model having the Q_i's as values of the independent variable.

TABLE 1

Tests with locally best average power against translation alternatives in a process with known initial level

F	Test Statistic: General weights	Test Statistic: Uniform weights $q_i = 1/N$	Test Statistic: Degenerate weights $q_{m+1} = 1$, $q_i = 0, i \neq m+1$
Double exponential	$T_{(1)} = \sum_{i=1}^{N} Q_i \operatorname{sgn}(X_i)$	$N^{-1} \sum_{i=1}^{N} i \operatorname{sgn}(X_i)$	$\sum_{i=m+1}^{N} \operatorname{sgn}(X_i)$
Logistic	$T_{(2)} = \sum_{i=1}^{N} Q_i \operatorname{sgn}(X_i) R_i$	$N^{-1} \sum_{i=1}^{N} i \operatorname{sgn}(X_i) R_i$	$\sum_{i=m+1}^{N} \operatorname{sgn}(X_i) R_i$
Normal	$T_{(3)} = \sum_{i=1}^{N} Q_i \operatorname{sgn}(X_i) E(W^{(R_i)})$*	$N^{-1} \sum_{i=1}^{N} i \operatorname{sgn}(X_i) E(W^{(R_i)})$	$\sum_{i=m+1}^{N} \operatorname{sgn}(X_i) E(W^{(R_i)})$

* $W^{(1)} < W^{(2)} \cdots < W^{(N)}$ is an ordered sample from a χ_1 distribution.

When the point of possible shift $m + 1$ is known, the weight function becomes $q_{m+1} = 1$, $q_i = 0$ for $i \neq m + 1$. The three test statistics for this case are given in the fourth column of Table 1. $T_{(1)}$ reduces to the sign test for location based on the observations $X_{m+1}, \cdots, X_N$. The forms of $T_{(2)}$ and $T_{(3)}$ in this case are structurally similar to the Wilcoxon signed rank and the one sample normal score tests based on the above $N - m$ observations. The intrinsic difference is that for $T_{(2)}$ and $T_{(3)}$, the ranking is considered over all N observations. It is interesting to note that in a two sample shift problem where one sample is known to be from a distribution symmetric about 0, the locally optimal invariant tests for logistic and normal distributions are $T_{(2)}$ and $T_{(3)}$ and not the Wilcoxon and the normal score tests. The reason is that a smaller invariance group is appropriate here.

We now investigate the unbiasedness of the class of tests (2.1) and more generally, of any tests of the form

$$T(\mathbf{X}) = \sum_{i=1}^{N} Q_i \operatorname{sgn}(X_i) U(R_i) \tag{2.6}$$

where $0 \leqq Q_1 \leqq Q_2 \cdots \leqq Q_N \leqq 1$ is a given set of constants and $U(\cdot)$ is a function of the ranks of the $|X_i|$, $i = 1, 2, \cdots, N$.

THEOREM 2.2. *If $U(\cdot)$ is a nondecreasing function, any test which rejects H_0 for large values of $T(\mathbf{X})$ is unbiased for testing H_0 against H_1.*

PROOF. Let $m(0 \leqq m \leqq N - 1)$ be arbitrary but fixed. Define a class of mappings $\mathcal{C}$: $(x_1, x_2, \cdots, x_N) \rightarrow (x_1', x_2', \cdots, x_N')$ by $x_i' = x_i$ for $i \leqq m$ and $x_i' = h(x_i)$ for $i > m$ where h is continuous, nondecreasing and $h(x) \geqq x$ for all x. For any cdf $\prod_{i=1}^{N} F_i$ under H_1, there exists an $F_0 \varepsilon \mathfrak{F}_0$ and an h such that if $(X_1, X_2, \cdots, X_N)$ is distributed as $\prod_{i=1}^{N} F_0$, $(X_1', X_2', \cdots, X_N')$ will be distributed as $\prod_{i=1}^{N} F_i$. It is then sufficient to show that for each map of $\mathcal{C}$, $T(\mathbf{x}') \geqq T(\mathbf{x})$ a.e. (Lebesgue), ([8], p. 256).

Consider first a point x where the map is sign preserving in addition to having the above properties. Let $\mathbf{r}$ and $\mathbf{r}'$ denote the vectors of ranks of the absolute values for $\mathbf{x}$ and $\mathbf{x}'$ respectively. Introduce the index sets

$$\begin{aligned} (2.7)\quad I_1 &= \{i: x_i > 0, i \leqq m\}; & I_2 &= \{i: x_i > 0, i > m\}; & I &= I_1 \cup I_2; \\ J_1 &= \{i: x_i < 0, i \leqq m\}; & J_2 &= \{i: x_i < 0, i > m\}; & J &= J_1 \cup J_2. \end{aligned}$$

Consider a new vector of ranks $\mathbf{r}^*$ obtained in the following way: allocate to $\{x_i': i \varepsilon I\}$ the same set of ranks $\{r_i : i \varepsilon I\}$ but permuted according to the ordering of $\{|x_i'|: i \varepsilon I\}$. Follow the same procedure for $\{x_j': j \varepsilon J\}$. Setting $T^* = \sum_{i=1}^{N} Q_i \operatorname{sgn}(x_i') U(r_i^*)$, we have

$$T^* - T(\mathbf{x}) = \sum_{i \varepsilon I} Q_i[U(r_i^*) - U(r_i)] - \sum_{j \varepsilon J} Q_j[U(r_j^*) - U(r_j)]. \tag{2.8}$$

Clearly $r_i^* \leqq (\geqq) r_i$ if $i \varepsilon I_1(I_2)$. Also $\sum_{i \varepsilon I} [U(r_i^*) - U(r_i)] = 0$ because $\{r_i^*: i \varepsilon I\} = \{r_i : i \varepsilon I\}$. This, together with the fact that Q_i is nondecreasing in i, implies that $\sum_{i \varepsilon I} Q_i[U(r_i^*) - U(r_i)] \geqq 0$. The same type of reasoning shows that the second sum in (2.8) is not positive and hence we have $T^* \geqq T(\mathbf{x})$. It is then sufficient to show that $T(\mathbf{x}') \geqq T^*$.

Take any $a \varepsilon I$ and suppose that x_a' is the tth smallest of $\{x_i' : i \varepsilon I\}$. If x_l is the tth smallest of $\{x_i : i \varepsilon I\}$, then $r_a^* = r_l$ by definition. There are $r_l - (t - 1) - 1$ negative x with $|x| \leqq x_l$ and since the map moves points only to the right, at least $r_l - t$ negative x' would satisfy the inequality $|x'| \leqq x_a'$. It follows that $r_a' \geqq r_a^*$. Similarly for any $j \varepsilon J$, we have $r_j' \leqq r_j^*$ and consequently $T(\mathbf{x}') - T^* \geqq 0$.

Finally, if the map is not sign preserving, some x_i, $i > m$ could be mapped across zero. Introduce the notation $J_0 = \{i : x_i < 0, x_i' > 0\}$. In this case, h may be expressed as $h_3 \circ h_2 \circ h_1$ where $h_i \varepsilon \mathfrak{C}$, $i = 1, 2, 3$. The h_i are partially specified below and their definition can be completed by making h_i linear between consecutive points. h_1 maps x_i, $i \varepsilon J_0$ into the interval $(-a, 0)$ and the other x_i into x_i' where $a = \min\{\min_{1 \leqq i \leqq n} |x_i|, \min_{1 \leqq i \leqq N} |x_i'|\}$. Next, h_2 takes the points $h_1(x_i)$, $i \varepsilon J_0$, into $(0, a)$ and leaves the other points unchanged. Finally, h_3 takes $h_2 \circ h_1(x_i)$, $i \varepsilon J_0$, into $h(x_i')$ leaving other components unchanged. The maps h_1 and h_3 are of the type considered above and h_2 makes the negative terms corresponding to $i \varepsilon J_0$ positive. This completes the proof.

The Q_i represent cumulative weights and hence are nondecreasing. Therefore any test of the form (2.1) is unbiased for every weight function $\{q_i\}$ provided that $E[-f'(V^{(i)})/f(V^{(i)})]$ is nondecreasing in i. In particular, the tests in Table 1 are all unbiased.

Except for $T_{(1)}$, any statistic of the form (2.1) will generally have a sample space consisting of $2^N N!$ points so that the setting of the exact critical region might be very difficult even for moderate sample sizes. To obtain a large sample approximation to the null distribution, consider the sequence of test statistics $T_N = \sum_{i=1}^N Q_{Ni} \operatorname{sgn}(X_{Ni}) E[-g'(V^{(R_{Ni})})/g(V^{(R_{Ni})})]$ where g is a known density having cdf $G \varepsilon \mathfrak{F}_0$ and satisfying the conditions of Theorem 2.1. Let $Z_{N1} < Z_{N2} < \cdots < Z_{NN}$ be an ordered sample from a uniform distribution on $(0, 1)$ and define a function $\psi(u)$ on $0 < u < 1$ by

$$(2.9) \qquad \psi(u) = -g'(G^{-1}(\tfrac{1}{2}(u + 1)))/g(G^{-1}(\tfrac{1}{2}(u + 1))).$$

In terms of ψ the test statistic T_N can be written as

$$T_N = \sum_{i=1}^N Q_{Ni} \operatorname{sgn}(X_{Ni}) E\psi(Z_{Ni}).$$

Under H_0, the distribution of T_N depends only on the choice of g and the weight function $\{q_{Ni}\}$ and not on the particular population cdf. We may therefore assume that the common cdf of X_i under H_0 is G. Define a class of cdf's by

$$(2.10) \qquad \mathfrak{F} = \{F : \textstyle\int_{-\infty}^{\infty} (f'(x)/f(x))^2 f(x)\, dx < \infty, \quad \text{(A) and (B) of Theorem 2.1 hold}\}.$$

The following theorem is a direct consequence of Theorem 7.1 of Hájek [5].

THEOREM 2.3. *If $G \varepsilon \mathfrak{F}$ is symmetric and if the sequence of weights $\{q_{Ni}\}$ satisfies $\lim_{N\to\infty} \sum_{i=1}^N Q_{Ni}^2/N = b^2$, $0 < b^2 < \infty$, then under H_0,*

$$(2.11) \qquad T_N/[N^{\frac{1}{2}} b\, [\textstyle\int_0^1 \psi^2(u)\, du]^{\frac{1}{2}}] \to_{\mathfrak{L}} \mathfrak{N}(0, 1).$$

The limiting distribution can be employed to obtain approximate size α tests for large N. For instance, the statistic $T_{(2)}$ uses $\psi(u) = u$ and with uniform weights $b^2 = \frac{1}{3}$, so that $3N^{-\frac{1}{2}}T_{(2)N}$ is asymptotically $N(0, 1)$ under H_0.

We conclude this section with a remark about the assumption of symmetry. If instead of symmetry, we only assume that some percentile of the initial distribution is known, it is still possible to obtain tests having maximum local average power. Without loss of generality we set the known percentile equal to zero. The new problem is invariant under any transformation $x_i' = h(x_i)$, all i, where h is continuous and strictly increasing with $h(0) = 0$ The main difference in the derivation would occur in the expressions for average power and its derivative. They would not simplify nearly to the extent they do when symmetry is assumed.

3. Locally best invariant test (initial process level unknown). Here we employ procedures similar to those of Section 2 to develop optimal invariant tests for H_0^* vs. H_1^*. When the initial level is unknown, we test whether a jump occurs at some time after the first observation and accordingly any system of weights $\{q_i\}$ on the nuisance parameter m should have $q_1 = 0$. The problem remains invariant under the group of all transformations $x_i' = h(x_i)$, $i = 1, 2, \cdots, N$ where h is continuous and strictly increasing. A maximal invariant is the vector of ranks $\mathbf{S} = (S_1, S_2, \cdots, S_N)$ of $X_1, X_2, \cdots, X_N$.

The following theorem gives the structure of the test having maximum local average power against the translation alternatives $F_i(x) = F(x)$ for $i = 1, 2, \cdots, m$, $F_i(x) = F(x - \Delta)$ for $i = m + 1, \cdots, N$ where $\Delta > 0$ and m is unknown.

THEOREM 3.1. *Let X_1 have density f which satisfies conditions* (A) *and* (B) *of Theorem* 2.1. *Then the invariant test which maximizes the average power for all sufficiently small $\Delta > 0$ has a rejection region of the form*

$$T = \sum_{i=1}^{N} Q_i E[-f'(V^{(S_i)})/f(V^{(S_i)})] > C \tag{3.1}$$

where $Q_i = \sum_{m=1}^{i} q_m$ and $V^{(1)} < V^{(2)} < \cdots < V^{(N)}$ is an ordered sample from F. The proof is similar to that of Theorem 2.1 and hence is omitted.

The simplified forms of the test statistic (3.1) for logistic, normal and double exponential distributions are $T^{(1)} = \sum_{i=1}^{N} Q_i S_i$, $T^{(2)} = \sum_{i=1}^{N} Q_i E_\Phi(V^{(S_i)})$ and $T^{(3)} = \sum_{i=1}^{N} Q_i E[\operatorname{sgn} W^{(S_i)}]$ respectively, where $E_\Phi(V^{(S_i)})$ are the normal scores and $W^{(1)} < W^{(2)} < \cdots < W^{(N)}$ is an ordered sample from the double exponential distribution.

Chernoff and Zacks [2] obtained the test $\sum_{i=1}^{N} (i - 1)(X_i - \bar{X}) > C$ from the marginal likelihood ratio for normal observations with known variance. For the special case of uniform weights the test statistics (3.1) have the same structure except that functions of ranks are involved instead of the actual observations. Note also that $T^{(1)}$ becomes $\sum_{i=1}^{N} (i - 1)S_i$ and the test is equivalent to Spearman's rank correlation test for trend. Because of this correspondence it is expected that our tests would perform well even when more than one jump occurs in the same direction. With the weight function $q_{m+1} = 1$, $q_i = 0$, $i \neq m + 1$, $T^{(1)}$ and

$T^{(2)}$ reduce to the standard two sample Wilcoxon and normal score tests respectively.

THEOREM 3.2. *If* $U(\cdot)$ *is a nondecreasing function, any test which rejects* H_0^* *for large values of* $M(\mathbf{X}) = \sum_{i=1}^{N} Q_i U(S_i)$ *is unbiased for testing* H_0^* vs. H_1^*.

PROOF. Let m $(1 \leqq m < N)$ be arbitrary but fixed. Consider the same class of transformations $\mathfrak{C}$ introduced in the proof of Theorem 2.2. It is sufficient again to show that $M(\mathbf{x}') \geqq M(\mathbf{x})$ a.e. Let $\mathbf{s}' = (s_1', s_2', \cdots, s_N')$ be the vector of ranks of the x_i'. Clearly $i > m$ $(\leqq m) \Rightarrow s_i' \geqq (\leqq) s_i \Rightarrow U(s_i') \geqq (\leqq) U(s_i)$. Hence

$$\begin{aligned} &T(\mathbf{x}') - T(\mathbf{x}) \\ (3.2) \quad &= \textstyle\sum_{i=1}^{m} Q_i[U(s_i') - U(s_i)] + \sum_{i=m+1}^{N} Q_i[U(s_i') - U(s_i)] \\ &\geqq Q_m \textstyle\sum_{i=1}^{m} [U(s_i') - U(s_i)] + Q_{m+1} \sum_{i=m+1}^{N} [U(s_i') - U(s_i)] \\ &= (Q_{m+1} - Q_m) \textstyle\sum_{i=m+1}^{N} [U(s_i') - U(s_i)] \geqq 0. \qquad \text{QED.} \end{aligned}$$

We now consider the asymptotic distribution of the statistics

$$(3.3) \qquad T_N = \textstyle\sum_{i=1}^{N} Q_{Ni} E[-g'(V^{(S_{Ni})})/g(V^{(S_{Ni})})].$$

Defining a function ψ by

$$(3.4) \qquad \psi(u) = -g'(G^{-1}(u))/g(G^{-1}(u)), \qquad 0 < u < 1,$$

and letting $Z_{N1} < \cdots Z_{NN}$ be an ordered sample from the uniform distribution on $(0, 1)$, the test statistic T_N can be expressed as $T_N = \sum_{i=1}^{N} Q_{Ni} E[\psi(Z_{Ni})]$. The next theorem follows directly from Hájek [4], Section 6.

THEOREM 3.3. *Let* $\bar{Q}_N = \sum_{i=1}^{N} Q_{Ni}/N$. *If* $G \varepsilon \mathfrak{F}$ *where* $\mathfrak{F}$ *is defined by* (2.10) *and if the sequence of weights* $\{q_{Ni}\}$ *satisfies*

$$(\mathrm{A}_1) \qquad \lim_{N\to\infty} N^{-1} \textstyle\sum_{i=1}^{N} (Q_{Ni} - \bar{Q}_N)^2 = c^2, \qquad 0 < c^2 < \infty,$$

then under H_0^*,

$$[T_N - E(T_N)]/[N^{\frac{1}{2}} c [\textstyle\int_0^1 \psi^2(u)\, du]^{\frac{1}{2}}] \to_{\mathfrak{L}} \mathfrak{N}(0, 1).$$

4. Asymptotic distribution under local alternatives and Pitman ARE. Although desirable, an exact power comparison of our tests with those of [11] and [2] for various parent distributions would involve tremendous computational difficulties even for moderate sample sizes. Consequently, we devote this section to the derivation of the Pitman asymptotic relative efficiency (ARE). The usefulness of this measure in our time series situation is somewhat questionable because the assumption of a single shift in the process level may make little sense when the sample size can be increased only by taking observations over an extended period of time. However, this objection could be ruled out in many cases where it is possible to increase the size by sampling more frequently in a fixed time period. We will treat in detail the class of tests (3.3). The development of the corresponding results for the tests of Section 2 is similar.

Define a sequence of local translation alternatives $\{K_N\}$ by

$$(4.1)\qquad \begin{aligned} K_N : F_i(x) &= F(x), \qquad i = 1, 2, \cdots, m, \\ &= F(x - \theta N^{-\frac{1}{2}}), \qquad i = m + 1, \cdots, N, \qquad F \varepsilon \mathfrak{F}, \\ \lim_{N\to\infty} (m/N) &= \lambda, \qquad 0 < \lambda < 1, \end{aligned}$$

where $\mathfrak{F}$ is defined by (2.10). Let $\psi_N(i/(N + 1)) = E[-g'(V^{(Ni)})/g(V^{(Ni)})]$, where $V^{(N1)} < V^{(N2)} < \cdots < V^{(NN)}$ is an ordered sample from $G \varepsilon \mathfrak{F}$. Noting that $\sum_{i=1}^{N} \psi_N(S_{Nj}/(N + 1))$ is constant for every N, we express the test statistic (3.3) as

$$(4.2)\qquad S_N^0 = N^{-\frac{1}{2}} \textstyle\sum_{i=1}^{N} (Q_{Ni} - \bar{Q}_N)\psi_N(S_{Ni}/(N + 1)).$$

Set $d_\psi^2 = \int_0^1 \psi^2(u)\, du$ and $d_\phi^2 = \int_0^1 \phi^2(u)\, du$, where $\psi(u)$ is defined by (3.4), and $\phi(u)$ is the same function with g replaced by f and G by F.

THEOREM 4.1. *Let $G \varepsilon \mathfrak{F}$ where $\mathfrak{F}$ is defined by* (2.10). *If the sequence of weights $\{q_{Ni}\}$ satisfies*

$$(\mathrm{A}_2)\qquad \lim_{N\to\infty} \textstyle\sum_{i=m+1}^{N} (Q_{Ni} - \bar{Q}_N)/N = a < \infty$$

in addition to (A_1) *of Theorem* 3.3, *then* $\lim \mathfrak{L}(S_N^0 \mid K_N) = \mathfrak{N}(\mu, c^2 d_\psi^2)$, *where*

$$(4.3)\qquad \mu = \theta a \textstyle\int_0^1 \phi(u)\psi(u)\, du.$$

PROOF. The proof uses the principle of contiguity and is methodically based on Hájek [5]. The important difference is that the coefficients $(Q_{Ni} - \bar{Q}_N)$ occurring in the test statistic (4.2) do not appear in the alternatives $\{K_N\}$ while in [5] they do. We will sketch the main steps leaving out the details. Introduce

$$\begin{aligned} s(x) &= f^{\frac{1}{2}}(x), \\ U_N &= -N^{-\frac{1}{2}} \textstyle\sum_{i=m+1}^{N} [f'(X_i)/f(X_i)], \\ W_N &= 2 \textstyle\sum_{i=1}^{N} [r_{Ni}^{\frac{1}{2}}(X) - 1] \quad \text{and} \\ L_N &= \textstyle\sum_{i=1}^{N} \log r_{Ni}, \end{aligned}$$

where $r_{Ni}(x) = f(x - \theta N^{-\frac{1}{2}})/f(x)$ for $i = m + 1, \cdots, N$ and $r_{Ni}(x) = 1$ for $i = 1, \cdots, m$. We have $E(U_N \mid H_0^*) = 0$ and $\lim_{N\to\infty} \mathrm{Var}\,(U_N \mid H_0^*)/\{(1 - \lambda)\, d_\phi^2\} = 1$. Approximating W_N in mean square as in Section 5 of Hájek [5], we obtain

$$(4.4)\qquad W_N + \tfrac{1}{4}(1 - \lambda)\theta^2 d_\phi^2 + \theta U_N \to_P 0 \quad \text{under} \quad H_0^*.$$

The central limit Theorem applied to $\{U_N\}$ gives $\lim_{N\to\infty} \mathfrak{L}(W_N \mid H_0^*) = \mathfrak{N}(-(1 - \lambda)\theta^2 d_\phi^2/4, (1 - \lambda)\theta^2 d_\phi^2)$. The conditions of Lemma 4.2 of [5] are then satisfied and consequently

$$(4.5)\qquad \begin{aligned} W_N - L_N &\to_P \tfrac{1}{4}(1 - \lambda)\theta^2 d_\phi^2 \quad \text{under} \quad H_0^*, \\ \lim_{N\to\infty} \mathfrak{L}(L_N \mid H_0^*) &= \mathfrak{N}(-(1 - \lambda)\theta^2 d_\phi^2/2, (1 - \lambda)\theta^2 d_\phi^2), \end{aligned}$$

and the probability measures are contiguous.

Under H_0^*, S_N^0 can be approximated in mean square (c.f. [4]) by

$$S_N^* = N^{-\frac{1}{2}} \sum_{i=1}^{N} (Q_{Ni} - \bar{Q}_N)\psi(F(X_i)).$$

From (4.5), we have

$$\lim_{N\to\infty} \mathcal{L}(S_N, L_N \mid H_0^*) = \lim_{N\to\infty} \mathcal{L}(S_N^*, -\theta U_N - \tfrac{1}{2}(1-\lambda)\theta^2 d_\phi^2 \mid H_0^*). \tag{4.6}$$

An application of bivariate central limit theorem (Cramér [3], p. 114) to (S_N^*, U_N) shows that under H_0^*, (S_N^0, L_N) is asymptotically bivariate normal with correlation

$$\rho = a(c(1-\lambda)^{\frac{1}{2}})^{-1} \cdot \int_0^1 \phi(u)\psi(u)\, du/(d_\phi\, d_\psi). \tag{4.7}$$

This completes the proof.

In the special case of uniform weights, we have $\bar{Q}_N = \frac{1}{2}$ and it is easy to see that the conditions of Theorem 4.1 are satisfied with $a = \lambda(1-\lambda)/2$ and $c^2 = \frac{1}{12}$. Under $\{K_N\}$ the limiting distribution of S_N^0 is therefore $\mathfrak{N}(\frac{1}{2}\lambda(1-\lambda)\int_0^1 \phi(u)\cdot \psi(u)\, du, d_\psi^2/12)$. In order to arrive at the usual expressions for the ARE we shall assume that the conditions of Lemma 3 of [6] are also satisfied. Under these additional conditions, the application of Theorem 4.1 to

$$T_N^{(1)} = (N-1)^{-1} \sum_{i=1}^{N} (i-1)S_i \quad \text{and}$$
$$T_N^{(2)} = (N-1)^{-1} \sum_{i=1}^{N} (i-1)E_\Phi(V^{(S_i)})$$

yields

$$\begin{aligned} &\lim_{N\to\infty} \mathcal{L}(2N^{-\frac{1}{2}}[T_N^{(1)}(N+1)^{-1} - (N/4)] \mid K_N) \\ &\qquad = \mathfrak{N}(\theta\lambda(1-\lambda)\int_{-\infty}^{\infty} f^2(x)\, dx, \tfrac{1}{36}) \\ &\lim_{N\to\infty} \mathcal{L}(2T_N^{(2)}N^{-\frac{3}{2}} \mid K_N) \\ &\qquad = \mathfrak{N}(\theta\lambda(1-\lambda)\int_{-\infty}^{\infty} f^2(x)\, dx/\phi[\Phi^{-1}(F(x))], \tfrac{1}{3}). \end{aligned} \tag{4.8}$$

When the initial process level is unknown Chernoff-Zacks' test statistic has the form $Z_N = \sum_{i=1}^{N} (i-1)(X_i - \bar{X})$. Application of this test to normal populations requires the knowledge of the standard deviation σ. With σ unknown, a Studentized form $Z_N^* = (N-2)^{\frac{1}{2}} Z_N/(D_N S_e)$, with $D_N^2 = N(N^2-1)/12$ and $S_e^2 = \sum_{i=1}^{N} (X_i - \bar{X})^2 - Z_N^2/D_N^2$, may be used. Under normality, the null distribution of Z_N^* is student's t with $(N-2)$ d.f. The asymptotic distribution of Z_N and Z_N^* under the sequence $\{K_N\}$ is given in the following theorem.

THEOREM 4.2. *If for some $\delta > 0$, F has $(2+\delta)$th absolute moment then*

$$\begin{aligned} &\lim_{N\to\infty} \mathcal{L}(Z_N(D_N\sigma)^{-1} \mid K_N) \\ &\qquad = \lim_{N\to\infty} \mathcal{L}(Z_N^* \mid K_N) = \mathfrak{N}(\theta 3^{\frac{1}{2}}\lambda(1-\lambda)\sigma^{-1}, 1). \end{aligned} \tag{4.9}$$

PROOF. Apply Liapounov's central limit theorem to the sequence of random variables $Y_{Ni} = (i - \frac{1}{2}(N+1))(X_i - \nu)$, $i = 1, 2, \cdots, m$; $Y_{Ni} = (i - \frac{1}{2}(N+1))(X_i - \nu - \theta N^{-\frac{3}{2}})$, $i = m+1, \cdots, N$ where $\nu = \int_{-\infty}^{\infty} x\, dF(x)$, and note that $S_e^2/(N-2) \to_P \sigma^2$ under K_N.

It follows that for uniform weights the ARE of the test T_N of (3.3) relative to Chernoff-Zacks' test is given by

$$e_{T:Z} = \sigma^2 d_\psi^{-2}(\int_0^1 \phi(u)\psi(u)\,du)^2. \tag{4.10}$$

For the particular tests $T_N^{(1)}$ and $T_N^{(2)}$ this reduces to

$$e_{T^{(1)}:Z} = 12\sigma^2(\int_{-\infty}^{\infty} f^2(x)\,dx)^2, \qquad e_{T^{(2)}:Z} = \sigma^2\{\int_{-\infty}^{\infty} f^2(x)\,dx/\phi[\Phi^{-1}(F(x))]\},$$

and these are precisely the ARE of the two sample Wilcoxon and the normal score tests relative to the t-test.

The selection of a test T_N of the form (3.3) or equivalently of an S_N^0 involves the choice of a function ψ defined through a density g as well as a weight function $\{q_i\}$. If two such tests, T_N and T_N^* defined through ψ and ψ^*, are based on identical or asymptotically equivalent weight functions (i.e. a and c of (A_1) and (A_2) are equal), Theorem 4.1 shows that their ARE is given by

$$e_{T:T^*} = [d_\psi^* d_\psi^{-1} \int_0^1 \phi(u)\psi(u)\,du[\int_0^1 \phi(u)\psi^*(u)\,du]^{-1}]^2 \tag{4.11}$$

which is independent of the particular weights used. Therefore, the ARE equals that of the standard two sample rank order tests for shift.

It is also of interest to study the sensitivity of the ARE in relation to the choice of the weight function. Suppose T and T' are two tests defined through the same ψ-function but involve two different weight functions $\{q_i\}$ and $\{q_i'\}$ which satisfy the conditions (A_1) and (A_2) with the limits (a, c^2) and (a', c'^2) respectively. From Theorem 4.1, we obtain $e_{T:T'} = (ac'/a'c)^2$ which is independent of ψ. Suppose that T' has the degenerate weight $q_{m+1}' = 1, q_i' = 0, i \neq m + 1$ and that T has uniform weights. If $m/N \to \lambda$ as $N \to \infty$, the ARE is $e_{T:T'} = 3\lambda(1 - \lambda) \leqq \frac{3}{4}$. This indicates that the loss of efficiency incurred in using a uniform weight instead of the correct degenerate weight is at least 25% and could be much higher if the point of shift is near the beginning or the end of the observation period. Some small sample power comparisons for different choices of weight function are given in the next section.

For the sake of completeness, we state the asymptotic distribution of the test statistic $T_N = \sum_{i=1}^N Q_{Ni}\,\mathrm{sgn}\,(X_{Ni})E\psi(Z_{Ni})$ of Section 2 under the sequence of alternatives $\{K_N\}$ with the additional assumption that F is symmetric. In this case ψ is defined by (2.9).

THEOREM 4.3. *Let F and G be symmetric and members of $\mathfrak{F}$ where $\mathfrak{F}$ is defined by* (2.10). *If the sequence of weights $\{q_{Ni}\}$ satisfies*

$$\lim_{N\to\infty} \sum_{i=1}^N Q_{Ni}^2/N = b^2 < \infty \quad \text{and} \quad \lim_{N\to\infty} \sum_{i=m+1}^N Q_{Ni}/N = \xi < \infty$$

then

$$\lim_{N\to\infty} \mathcal{L}(T_N \mid K_N) = \mathfrak{N}(\theta\xi \int_0^1 \phi(u)\psi(u)\,du,\, b^2 d_\psi^2). \tag{4.12}$$

The proof is similar to that of Theorem 4.1.

5. Small sample power. The power of the test $T^{(1)} = \sum_{i=1}^N Q_i S_i$ for testing H_0^* vs. H_1^* is calculated in the special case of translations in the distribution

TABLE 2

Power of the test $\sum_{i=1}^{n} iS_i$ for normal translation alternatives
$\alpha = .05$

N	m	Δ			
		0.2	0.8	1.5	3.0
4	1	.060	.095	.135	.181
4	2	.064	.116	.182	.268
5	1	.060	.094	.132	.174
5	2	.067	.136	.232	.365
6	1	.059	.092	.127	.166
6	2	.068	.141	.244	.384
6	3	.072	.170	.327	.572

TABLE 3

Effect of the weight function on power of $\sum_{i=1}^{n} Q_iS_i$
$N = 5, m = 2, \alpha = .10$

Weight function	Δ			
	0.2	0.8	1.5	3.0
(0, 0, 1, 0, 0)	.137	.296	.540	.921
$(0, \frac{2}{11}, \frac{6}{11}, \frac{2}{11}, \frac{1}{11})$	.135	.283	.498	.813
$(0, \frac{2}{9}, \frac{4}{9}, \frac{2}{9}, \frac{1}{9})$	.135	.278	.484	.777
$(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	.131	.251	.407	.602

$F_1(x) = \Phi(x)$ where Φ is the standard normal cdf. Consider the statistic $\sum_{i=1}^{N} iS_i$ resulting from a choice of uniform weights. Fixing the point of shift $m + 1$, we proceed by coding the critical rank vectors $\mathbf{Z} = (Z_1, Z_2, \cdots, Z_N)$ according to the rule $Z_i = 0$ if $i \varepsilon \{S_1, S_2, \cdots, S_m\}$ and $Z_i = 1$ otherwise. The probabilities of the $\mathbf{Z}$ vectors under various normal translation alternatives are tabulated by Milton [9] to nine decimal places. Power is computed by using Table A of [9] and the fact that $m!(N - m)!$ different rank orders yield the same $\mathbf{Z}$. Table 2 gives the power for $m \leqq N/2$. The powers for $m > N/2$ follow by symmetry.

Since $T^{(1)}$ is designed for the situation where the process level is unknown, a comparison of the power with Page's test [11] would not be relevant. The performances of $\sum_{i=1}^{N} iS_i$ and the Studentized form $Z_N{}^*$ of Chernoff and Zacks' test are being studied.

For the situation where the initial process level is known, some power comparisons between the test $T_{(1)} = \sum_{i=1}^{N} i \operatorname{sgn}(X_i)$ and Page's test were made by Chernoff-Zacks [2] for normal alternatives. $T_{(1)}$ was found to have slightly more power unless the point of shift is near either end in which case Page's test performs better. A table of rank order probabilities for the absolute values of observations from a normal population is required before these tests can be compared with $T_{(2)}$ and $T_{(3)}$ of Table 1.

To illustrate the effect of the selection of weights $\{q_i\}$ on the power, we consider the test $T^{(1)}$ with sample size $N = 5$ and four systems of weights. Powers of each test for normal translation alternatives are calculated as above and are presented in Table 3. The power is maximum for the choice of the correct degenerate weighting and it falls off with the approach towards the uniform weighting. Similarly the entries in Table 2 may be compared to the powers of the corresponding Wilcoxon tests which are available in Milton [10].

The study of the small sample power of the tests derived in this paper is being continued and the results will be communicated later.

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