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ASYMPTOTIC NORMALITY OF SAMPLE QUANTILES FOR m-DEPENDENT PROCESSES[1]

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1. Introduction and summary. The usual technique of deriving the asymptotic normality of a quantile of a sample in which the random variables are all independent and identically distributed [cf. Cramér (1946), pp. 367–369] fails to provide the same result for an m-dependent (and possibly non-stationary) process, where the successive observations are not independent and the (marginal) distributions are not necessarily all identical. For this reason, the derivation of the asymptotic normality is approached here indirectly. It is shown that under certain mild restrictions, the asymptotic almost sure equivalence of the standardized forms of a sample quantile and the empirical distribution function at the corresponding population quantile, studied by Bahadur (1966) [see also Kiefer (1967)] for a stationary independent process, extends to an m-dependent process, not necessarily stationary. Conclusions about the asymptotic normality of sample quantiles then follow by utilizing this equivalence in conjunction with the asymptotic normality of the empirical distribution function under suitable restrictions. For this purpose, the results of Hoeffding (1963) and Hoeffding and Robbins (1948) are extensively used. Useful applications of the derived results are also indicated.

2. The main results. Let $\omega = (X_1, X_2, \cdots)$ be a sequence of random variables forming an m-dependent process (not necessarily stationary); that is, the random vectors $(X_1, \cdots, X_i)$ and $(X_j, X_{j+1}, \cdots)$ are stochastically independent if $j - i > m$, where m is a non-negative integer. The marginal cumulative distribution function (cdf) of X_i is denoted by $F_i(x)$, and the joint cdf of (X_i, X_{i+h}) by $F_{i,h}(x, y)$ for $h = 1, \cdots, m$ and $i = 1, 2, \cdots$. For any $p\colon 0 < p < 1$, let $Y_n = Y_n(\omega)$ be the sample p-quantile when the sample is $(X_1, \cdots, X_n)$. Define the empirical cdf $F_n(x, \omega)$ by

$$(2.1)\qquad F_n(x, \omega) = (1/n) \quad (\text{the number of } X_i \leqq x, \text{ in the sample}).$$

Thus, $F_n(x, \omega)$ is an unbiased estimator of the cdf

$$(2.2)\qquad \bar{F}_{(n)}(x) = (1/n) \textstyle\sum_{i=1}^{n} F_i(x).$$

Define ξ_n (the p-quantile of $\bar{F}_{(n)}$) by

$$(2.3)\qquad \bar{F}_{(n)}(\xi_n) = p.$$

It is assumed that

$$(2.4)\qquad \sup_n |\xi_n| < \infty,$$

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and that in the neighbourhood of ξ_n, $F_i(x)$ is absolutely continuous $\forall_i$. Further, one or both of the following assumptions will also be made in the sequel:

(a) $f_i(x) = (d/dx)F_i(x)$ is continuous in some neighbourhood of $\xi_i \forall i$, with

(2.5a) $0 < \inf_{1 \leqq i \leqq n} f_i(\xi_n) \leqq \sup_{1 \leqq i \leqq n} f_i(\xi_n) < \infty$, uniformly in n,

(which implies that

(2.5b) $$0 < \inf_n \bar{f}_{(n)}(\xi_n) \leqq \sup_n \bar{f}_{(n)}(\xi_n) < \infty,$$

where $\bar{f}_{(n)} = (1/n) \sum_{i=1}^{n} f_i = (d/dx)\bar{F}_{(n)})$, and

(b) $\bar{F}''_{(n)}(x)$ is bounded in the same neighbourhood of ξ_n.

It will be seen later on that for an independent process (i.e. $m = 0$), (2.5a) is redundant and we may simply work with (2.5b). Also for a stationary m-dependent process, we have $F_i \equiv F \forall i$, and hence, (2.5a), (2.5b) simplifies to saying that F has a continuous derivative in the neighbourhood of its p-quantile, and at the p-quantile the density is strictly positive. [It may be noted that neither ξ_n nor $\bar{F}_{(n)}(x)$ need converge to some ξ or $\bar{F}(x)$, respectively, as $n \to \infty$.] Define

(2.6) $$p_{n,i} = F_i(\xi_n), \quad i = 1, \cdots, n; \quad \bar{F}_{(n)h}(x, y) = (n - h)^{-1} \sum_{i=1}^{n-h} F_{i,h}(x, y);$$

(2.7) $$\alpha_{n,h} = \bar{F}_{(n)h}(\xi_n, \xi_n) - p^2, \qquad \beta_{n,h} = (n - h)^{-1} \sum_{i=1}^{n-h} [p_{n,i}p_{n,i+h} - p^2],$$

for $h = 1, \cdots, m$, and let

(2.8) $$\alpha_{n,0} = p(1 - p), \beta_{n,0} = \sigma_n^2(p) = (1/n) \sum_{i=1}^{n} (p_{n,i} - p)^2.$$

Further, let

(2.9) $$\nu_{n,m}^2 = (\alpha_{n,0} - \beta_{n,0}) + 2 \sum_{h=1}^{m} (n - h)(\alpha_{n,h} - \beta_{n,h})/n.$$

It follows from (2.2), (2.3) and (2.6) through (2.9) that

(2.10) $$n \operatorname{Var} \{F_n(\xi_n, \omega)\} = \nu_{n,m}^2.$$

Finally, let

(2.11) $$I_n = \{x: \xi_n - a_n \leqq x \leqq \xi_n + a_n\} \quad \text{where} \quad a_n \sim n^{-\frac{1}{2}} \log n \quad \text{as} \quad n \to \infty.$$

Then, the main theorem of the paper may be stated as follows.

THEOREM 2.1. *If the condition* (a) *is satisfied, then as* $n \to \infty$

(2.12) $$\sup \{|[\bar{F}_{(n)}(x) - p] + [F_n(\xi_n, \omega) - F_n(x, \omega)]|: x \varepsilon I_n\} = O(n^{-\frac{3}{4}} \log n),$$

with probability one. If in addition, $\inf_n \nu_{n,m}^2 > 0$,

(2.13) $$\mathcal{L}(n^{\frac{1}{2}}\bar{f}_{(n)}(\xi_n)[Y_n(\omega) - \xi_n]/\nu_{n,m}) \to N(0, 1).$$

Finally, if both the conditions (a) *and* (b) *are satisfied, then*

(2.14) $$[Y_n(\omega) - \xi_n]\bar{f}_{(n)}(\xi_n) + [F_n(\xi_n, \omega) - p] = R_n(\omega),$$

where as $n \to \infty$ $R_n(\omega) = O(n^{-\frac{3}{4}} \log n)$, *with probability one.*

The proof of this theorem rests on the following lemmas.

LEMMA 2.1. *Let $\{Z_i\}$ be a sequence of m-dependent binomial variables, where $E(Z_i) = p_i$, $i = 1, 2, \cdots$, and let $\bar{p}_n = (1/n)\sum_{i=1}^n p_i$. Define the partial averages $p_{j,n}^* = (1/n_j^*).\ (p_j + p_{j+(m+1)} + \cdots + p_{j+(n_j^*-1)(m+1)})$, where*

$$n_j^* = [(n + m + 1 - j)/(m + 1)], \qquad j = 1, \cdots, m + 1.$$

Finally, let

$$\gamma_n \geqq \max_j C_1(n^*)^{-\frac{1}{2}}\{(\log n^*)p_{j,n}^*(1 - p_{j,n}^*)\}^{\frac{1}{2}}; \quad C_1^2 > 2, n^* = \min_j n_j^*. \tag{2.15}$$

Then

$$\sum_{i=1}^n P\{|(1/n)\sum_{i=1}^n Z_i - \bar{p}_n| \geqq \gamma_n\} < \infty. \tag{2.16}$$

PROOF. Consider the partial sums

$$S_{j,n} = Z_j + Z_{j+(m+1)} + \cdots + Z_{j+(n_j^*-1)(m+1)}, \qquad j = 1, \cdots, m + 1, \tag{2.17}$$

and define $l_j = n_j^*/n$. Thus, $0 < l_1, \cdots, l_{m+1} < 1$, $\sum_{j=1}^{m+1} l_j = 1$. On using then (5.21) of Hoeffding (1963), it follows easily that

$$\begin{aligned} P\{n^{-1}\textstyle\sum_{i=1}^n Z_i - \bar{p}_n \geqq \gamma_n\} \\ \leqq \textstyle\sum_{j=1}^{m+1} l_j\{\exp[-hn_j^*\gamma_n]\}E\{\exp[h(S_{j,n} - n_j^*p_{j,n})]\}. \end{aligned} \tag{2.18}$$

Then, by Hoeffding's (1963) Lemma 1 and by the proof of his Theorem 1, it easily follows that

$$\begin{aligned} P\{n^{-1}\textstyle\sum_{i=1}^n Z_i - \bar{p}_n \geqq \gamma_n\} \\ \leqq \textstyle\sum_{j=1}^{m+1} l_j\{(1 + \gamma_n(p_{j,n}^*)^{-1})^{p_{j,n}^*+\gamma_n}(1 - \gamma_n(1 - p_{j,n}^*)^{-1})^{1-p_{j,n}^*-\gamma_n}\}^{-n_j^*}, \end{aligned} \tag{2.19}$$

(where the jth term on the right hand side of (2.19) is to be replaced by 0 if $p_{j,n}^*$ is equal to 0 or 1). On making use of the simple relation that

$$\log(1 + x) = x - \tfrac{1}{2}x^2 + o(x^2) \quad \text{as} \quad x \to 0, \tag{2.20}$$

we obtain from (2.19) and (2.20) that

$$\begin{aligned} P\{n^{-1}\textstyle\sum_{i=1}^n Z_i - \bar{p}_n \geqq \gamma_n\} \\ \leqq \textstyle\sum_{j=1}^{m+1} l_j \exp\{-n_j^*\gamma_n^2[1 + o(1)]/2p_{j,n}^*(1 - p_{j,n}^*)\}. \end{aligned} \tag{2.21}$$

Now, by virtue of (2.15)

$$n_j^*\gamma_n^2/2p_{j,n}^*(1 - p_{j,n}^*) > (1 + \delta)\log n^*; \qquad \delta > 0, \quad (m + 1)n^* \sim n. \tag{2.22}$$

From (2.21) and (2.22), we have for adequately large values of n

$$P\{n^{-1}\textstyle\sum_{i-1}^n Z_i - \bar{p}_n \geqq \gamma_n\} \leqq Cn^{-(1+\delta)}, \tag{2.23}$$

where C is a finite positive quantity. Similarly, for adequately large n

$$P\{n^{-1}\textstyle\sum_{i=1}^n Z_i - \bar{p}_n \leqq -\gamma_n\} \leqq Cn^{-(1+\delta)}. \tag{2.24}$$

(2.23) and (2.24) along with $\sum_{n\geqq 1} n^{-(1+\delta)} < \infty$ imply (1.16). Q.E.D.

REMARK. For $m = 0$, $p^*_{j,n}$ are redundant and $\gamma_n \geqq C_1\{\bar{p}_n(1 - \bar{p}_n)(\log n)/n^{\frac{1}{2}}\}$, $C_1^2 > 2$.

The following lemma is a slight generalization of Theorem 1 of Hoeffding and Robbins (1948), whose condition (b) is relaxed here.

LEMMA 2.2. *Let* $\{Y_i\}$ *be a sequence of m-dependent random variables, with* $E(Y_i) = 0$ *and* $E|Y_i|^3 < R < \infty$ *for all* $i = 1, 2, \cdots$. *Let* $S_n = \sum_{i=1}^n Y_i$ *and* $\sigma_n^2 = (n^{-1})(\sum_{i=1}^n E(Y_i^2) + 2\sum_{h=1}^m \sum_{i=1}^{n-h} E(Y_i \cdot Y_{i+h}))$. *Then, if* $\inf_n \sigma_n > 0$, $n^{-\frac{1}{2}}S_n/\sigma_n$ *converges in law to a standard normal distribution.*

PROOF. Define

$$U_i = Y_{(i-1)k+1} + \cdots + Y_{ik-m}, \qquad i = 1, \cdots, \nu,$$

$$T = \sum_{i=1}^{\nu-1} (Y_{ik-m+1} + \cdots + Y_{ik}) + (Y_{\nu k-m+1} + \cdots + Y_n),$$

where $k = [n^\alpha]$, $0 < \alpha < \frac{1}{4}$ and $\nu = [n/k]$, so that $n = \nu k + r$, $0 \leqq r < k$. Proceeding as in Hoeffding and Robbins (1948), p. 775, it follows that

$$\text{(2.25)} \qquad \text{(i)} \quad E(T^2) = o(n) \Rightarrow |T| = o_p(n^{\frac{1}{2}});$$

$$\text{(2.26)} \qquad \text{(ii)} \quad \sum_{i=1}^{\nu} E|k^{-\frac{1}{2}}U_i|^3 = o(\nu^{\frac{3}{2}}).$$

Further, straightforward computations yield that

$$\text{(2.27)} \qquad |(1/\nu)\sum_{i=1}^{\nu} E(k^{-\frac{1}{2}}U_i)^2 - \sigma_n^2| = o(1),$$

and hence, by the hypothesis $\inf_n \sigma_n > 0$, we obtain that $\sum_{i=1}^{\nu} E(k^{-\frac{1}{2}}U_i)^2 = O(\nu)$. Thus, for the independent sequence of random variables $\{k^{-\frac{1}{2}}U_i, i = 1, 3, \cdots\}$, the Liapounoff's condition on the central limit theorem holds i.e.,

$$\text{(2.28)} \qquad \lim_{n\to\infty} \{\sum_{i=1}^{\nu} E|k^{-\frac{1}{2}}U_i|^3/[\sum_{i=1}^{\nu} E(k^{-\frac{1}{2}}U_i)^2]^{\frac{3}{2}}\} = 0.$$

Noting that $S_n = T + U_1 + \cdots + U_\nu$, the proof of the lemma follows from (2.25), (2.28) and some simple reasonings. Q.E.D.

LEMMA 2.3. *If* $\inf_n \nu_{n,m}^2 > 0$, $\mathcal{L}(n^{\frac{1}{2}}[F_n(\xi_n, \omega) - p]/\nu_{n,m}) \to N(0, 1)$.

PROOF. Let $c(u)$ be 1 or 0 according as u is $\geqq 0$ or not. Then writing $F_n(\xi_n, \omega) = n^{-1}\sum_{i=1}^n c(\xi_n - X_i)$, the proof directly follows from (2.10) and Lemma 2.2 (as further simplified for zero-one random variables). Q.E.D.

LEMMA 2.4. *Let* $k_n = np + o(n^{\frac{1}{2}}\log n)$, *and let* $V_n(\omega)$ *be the* k_nth *smallest observation among* $(X_1, \cdots, X_n)$. *If* $\inf_n \bar{f}_{(n)}(\xi_n) > 0$, *then* $V_n(\omega) \in I_n$, *with probability one, as* $n \to \infty$.

PROOF. By virtue of (2.5b) $\bar{F}_{(n)}(\xi_n - a_n) = p - O(n^{-\frac{1}{2}}\log n)$. Also, $\max_j p^*_{j,n}(1 - p^*_{j,n}) \leqq \frac{1}{4}$, where $p^*_{j,n}$ is defined as in Lemma 2.1, with $p_i = F_i(\xi_n - a_n)$, $i = 1, 2, \cdots$. Hence, it follows from (2.15) and (2.16) that if $Z_1, Z_2, \cdots$, are m-dependent random variables with $E(Z_i) = F_i(\xi_n - a_n)$, $i = 1, 2, \cdots$, then for $\gamma_n = C(\log n)/n^{\frac{1}{2}}$, $C > 0$ (chosen adequately large),

$$\text{(2.29)} \qquad \sum_{n=1}^{\infty} P\{(1/n)\sum_{i=1}^n Z_i - \bar{F}_{(n)}(\xi_n - a_n) \geqq \gamma_n\} < \infty.$$

The rest of the proof of the lemma follows precisely on the same line as in Lemma 2 of Bahadur (1966), with his (11) and (12) replaced by (2.29). Q.E.D.

PROOF OF THEOREM 2.1. The proof of (2.12) follows along with the same line as in Lemma 1 of Bahadur (1966), provided it can be shown that his (13) also holds for the m-dependent process. Now, by his definition, $G_n(\eta_r, \omega)$ (see (5) and (13) of Bahadur) is equal to $[F_n(\xi_n + rn^{-\frac{3}{4}} \log n) - F_n(\xi_n)] - [\bar{F}_{(n)}(\xi_n + rn^{-\frac{3}{4}} \log n) - \bar{F}_{(n)}(\xi_n)]$, where $r(\leqq [n^{\frac{1}{4}}])$ is a positive integer. The above can also be expressed as $(1/n) \sum_{i=1}^{n} (Z_i - p_i)$, where the Z_i's are m-dependent binomial variables and $p_i = F_i(\xi_n + rn^{-\frac{3}{4}} \log n) - F_i(\xi_n)$, $i = 1, 2, \cdots ; r \leqq [n^{\frac{1}{4}}]$. It therefore follows from (2.5b) and its implications that for n sufficiently large, $0 < \bar{p}_n \leqq C_1 n^{-\frac{1}{2}} \log n$ for all $r \leqq [n^{\frac{1}{4}}]$. Since, $\bar{p}_n = \sum_{j=1}^{m+1} l_j p_{j,n}^*$, where $l_j \sim 1/(m+1)\ \forall j$, this along with (2.5a) implies that $\max_j p_{j,n}^* \leqq O(n^{-\frac{1}{2}} \log n)$ for all $r \leqq [n^{\frac{1}{4}}]$. Hence, it follows from (2.15) that we can select $\gamma_n = C_2 n^{-\frac{3}{4}} \log n$, with $C_2^2/2C_1$ sufficiently large, and this along with (2.16), will extend Bahadur's (1966) expression (13) to the m-dependent case. This completes the proof of (2.12). Again, by virtue of (2.12), Lemma 2.3 and Lemma 2.4, it follows that

$$\mathcal{L}(n^{\frac{1}{2}}[\bar{F}_{(n)}(Y_n(\omega)) - p]/\nu_{n,m}) \sim \mathcal{L}(n^{\frac{1}{2}}[F_n(\xi_n, \omega) - p]/\nu_{n,m}) \rightarrow N(0, 1). \tag{2.30}$$

Since (2.5b) holds and $\inf_n \nu_{n,m} > 0$, it follows by standard techniques that as $n \rightarrow \infty$

$$n^{\frac{1}{2}}[\bar{F}_{(n)}(Y_n(\omega)) - p]/\nu_{n,m} \sim_P n^{\frac{1}{2}}[Y_n(\omega) - \xi_n]\bar{f}_{(n)}(\xi_n)/\nu_{n,m}. \tag{2.31}$$

Therefore, (2.13) follows from (2.30) and (2.31). Finally, (2.14) follows directly from (2.12) and the condition (b), as Lemma 3 of Bahadur (1966) extends directly to an m-dependent process under our condition (b). This completes the proof when $Y_n(\omega)$ is defined by a single order statistic. If $Y_n(\omega)$ is defined as an average of two successive order statistics, say, the k_nth and the $(k_n + 1)$th ones, (where $k_n \leqq np < (k_n + 1)$), upon noting that the difference of the empirical cdf's at these two points is equal to $1/n$, it follows from (2.12) and Lemma 2.4 that the difference between the values of $\bar{F}_{(n)}(x)$ at these two points is also $O(n^{-\frac{3}{4}} \log n)$, with probability one, as $n \rightarrow \infty$. Consequently, both (2.13) and (2.14) hold for the general case when $Y_n(\omega)$ is any inner point of two successive order statistics. Hence the theorem.

REMARK. For an independent process (i.e., when $m = 0$), in Lemma 2.1, $P_{j,n}^*$, $j = 1, \cdots, m + 1$ are all redundant. As such, in the proof of (2.12), we only require (2.5b), which is a single condition on the cdf $\bar{F}_{(n)}$. Also, for a stationary process (even if $m \geqq 1$), as remarked earlier, $\xi_n = \xi$, $\bar{F}_{(n)} \equiv F$, and hence, the usual condition viz. ($f(x)$ is continuous in some neighbourhood of ξ with $f(\xi) > 0$, is sufficient.

3. Applications.

(I) *Multisample situation.* In the context of nonparametric tests(univariate as well as multivariate cases), the asymptotic distribution of pooled sample quantiles have been studied by Mood (1954) and Chatterjee and Sen (1964), (1966). These relate to the situation where all the different samples are of large sizes and the corresponding cdf's differ only in sequences of locations all converging to zero as the combined sample size tends to ∞. Chen Pei-de (1966) has considered the case where in $(X_1, \cdots, X_n)$, each X_i can have one of $l(\geq 1)$ different cdf's $F_1, \cdots, F_l$ (i.e., finite mixture). The derivation are quite cumbrous and lengthy too. On the other hand, Theorem 2.1 provides the desired normality for a much more general case, where both independence and identity of marginal cdf's are relaxed to some extent. We remark that for independent process (i.e., $m = 0$), the conditions of Lemma 2.1 simplify, and hence, we require only the continuity of $\bar{f}_{(n)}(x)$ along with $\inf_n \bar{f}_{(n)}(\xi_n) > 0$. As an illustration, consider the sequence of symmetric distributions $F_i(x) = F([x - \mu]/\sigma_i), 0 < \sigma_0 < \sigma_i < \sigma^0 < \infty, \forall\, i$. For this sequence, $\bar{F}_{(n)}$ has the unique median μ and condition (a) is satisfied if $f(0) = F'(0) > 0$. Thus, the asymptotic normality follows directly from Theorem 2.1, where as the other techniques may involve some difficulties.

(II) *Robust-efficiency of sample median.* Suppose $X_1, X_2, \cdots$ are independent random variables with (common) median ($=$ mean) μ and variances $\sigma_1^2, \sigma_2^2, \cdots$. Let $Y_n(\omega)$ and $\bar{X}_n$ be respectively the median and mean of the sample $(X_1, \cdots, X_n)$. Then, we obtain that

$$(3.1)\quad \mathcal{L}(n^{\frac{1}{2}}[\bar{X}_n - \mu]/\bar{\sigma}_n) \to N(0, 1) \quad \text{and} \quad \mathcal{L}(n^{\frac{1}{2}}[Y_n(\omega) - \mu]/\delta_n) \to N(0, 1),$$

where $\bar{\sigma}_n^2 = (1/n) \sum_{i=1}^n \sigma_i^2$ and $\delta_n^2 = \frac{1}{4}[n^{-1} \sum_{i=1}^n f_i(\mu)]^2$. Thus, the asymptotic relative efficiency (ARE) of $Y_n(\omega)$ with respect to $\bar{X}_n$ is the limit (as $n \to \infty$) of

$$(3.2)\quad e_{Y,X}^{(n)} = (\bar{\sigma}_n^2/\delta_n^2) = 4\sigma_n^2(n^{-1} \sum_{i=1}^n f_i(\mu))^2,$$

where the limit is assumed to exist.
The ARE for the parent cdf F_i is equal to $e_i = 4\sigma_i^2 f_i^2(\mu)$, and hence, (3.2) may be written as

$$(3.3)\quad e_{Y,X}^{(n)} = \bar{\sigma}_n^2(n^{-1} \sum_{i=1}^n e_i^{\frac{1}{2}}/\sigma_i)^2.$$

If $F_1, F_2, \cdots$ differ only by scales, viz., $F_i(x) = F([x - \mu]/\sigma_i)$, $i = 1, 2, \cdots$, it easily follows that $e_i = e_0 = 4f^2(0), \forall\, i$, and hence, by elementary inequalities we obtain that

$$(3.4)\quad e_{Y,\bar{X}}^{(n)} = e_0\bar{\sigma}_n^2(n^{-1} \sum_{i=1}^n 1/\sigma_i)^2 \geqq e_0,$$

where the equality sign holds iff $\sigma_1 = \sigma_2 = \cdots$. This clearly indicates the robust-efficiency of $Y_n(\omega)$ for heteroscedastic cdf's.

(III) *Estimation of a density function.* As an estimate of the density function at the population p-quantile, often we consider

$$(3.5)\quad \hat{f} = (r_n - s_n - 1)/n(Y_{r_n} - Y_{s_n}),$$

where Y_{r_n} and Y_{s_n} are the r_nth and s_nth smallest observations in a sample $(X_1, \cdots, X_n)$ [cf. Siddiqui (1960), Sen (1966)], and where $r_n = np + o(n^{\frac{1}{2}})$, $s_n = np - o(n^{\frac{1}{2}})$. It follows from Theorem 2.1 that such an estimate will consistently estimate $\bar{f}_{(n)}(\xi_n)$, even for m-dependent processes.

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