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THE DISCRETE STUDENT'S t DISTRIBUTION

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0. Summary. The discrete Student's t distribution is discussed. In particular, we find that all finite odd order moments are zero, although the distribution need not be symmetric. A numerical comparison with the symmetric binomial shows the higher kurtosis of the Student form.

1. Introduction. In Ord (1967a), (1967b), we discussed the discrete system of distributions defined for the density $f_r = \text{Prob}(R = r)$, R a random variable taking integer values in an interval $[c, d]$ say, by the difference equation,

$$\Delta f_{r-1} = (a - r)f_{r-1}(b_0 + b_1 r + b_2 r(r-1))^{-1}, \tag{1}$$

a and b_i being parameters. These parameters are unrestricted, except that they must be real and satisfy

$$b_2 r^2 + r(b_1 - b_2 + 1) + b_0 - a \geqq 0 \tag{2}$$

for all integer $r \in [c, d]$.

This is the discrete analogue of Pearson's differential equation and has been discussed by Carver (1924) and Katz (1948), (1965) among others.

When the roots of the denominator of (1) are imaginary, we find

$$f_r/f_{r-1} = [\text{ratio of quadratic forms with imaginary roots}]$$

yielding a discrete Type IV distribution (following the usual notation for Pearson curves). A special case of this distribution is the Type VII (discrete Student's t) with

$$f(r; k, a, b) = f_r = \alpha_k / \textstyle\prod_{p=0}^{k} \{(r + p + a)^2 + b^2\} \tag{3}$$

where k is a non-negative integer and $0 \leqq a \leqq 1$, $0 < b^2 < \infty$, which we write as

$$f_r = \alpha_k / Q(r). \tag{4}$$

We now proceed to develop various properties of this distribution.

2. Moments. From (1), we may readily derive iterative relations for the factorial moments; μ_j ($\equiv \mu_j(k)$ for parameter k) are

$$\begin{aligned} \mu_1' &= -(k/2 + a), \\ (2k-1)\mu_2 &= k^2/4 + b^2, \qquad (5)\\ \mu_3 &= 0, \\ (2k-1)(2k-3)\mu_4 &= k^3(k-4)/16 + (3k-2)kb^2/2 + 3b^4, \end{aligned}$$

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and the ratio β_2 has limits $2(2-k)/k(2k-3)$ as $b \to 0$, $6/(2k-3)$ as $b \to \infty$, having a minimum of $2\frac{7}{9}$ for $k = 3$ as $b \to 0$; the latter limit corresponds to the continuous Student form.

3. Evaluation of probabilities. To evaluate the individual terms of (3) we require a closed form for α_k. We may then evaluate f_r directly, or, for large k, use an approximation due to Cohen (1940).

If we define the digamma function

$$\psi(z) = d \ln \Gamma(z)/dz \tag{6}$$

$$= -(1/z + 1/(1+z) + 1/(2+z) + \cdots) \tag{7}$$

which has imaginary part

$$I[\psi(z)] \equiv I(z). \tag{8}$$

We find, on using the relation

$$\alpha_k/\alpha_{k-1} = \mu_2(k) + k^2/4 + b^2, \tag{9}$$

which may be derived from (5), that

$$\alpha_k = b \prod_{j=1}^{k} (j^2 + 4b^2)/\binom{2k}{k} w(a, b), \tag{10}$$

where

$$\begin{aligned} w(a, b) = {} & I(1 + a + bi) + I(2 - a + bi) \\ & + b[(a^2 + b^2)^{-1} + \{(1-a)^2 + b^2\}^{-1}]. \end{aligned} \tag{11}$$

Results (9) and (10) are derived in Ord (1967a), Chapter 3.

In Table 1, we give values of w for various a, b. For $b \geqq 2$, taking $w = \pi = 3.14159 \cdots$, is accurate to within 1 unit in the fourth decimal place.

4. Values of the odd order moments. Let $\mu_{2j+1,k}$ denote the $(2j+1)$th odd order central moment of the distribution with parameter k. For every k, moments of order up to $2k$ exist.

Assume

$$\mu_{2j+1,k} = 0 \quad \text{for} \quad k = 1, 2, \cdots, m+2; \quad j = 0, 1, \cdots, s; \quad s < m+2. \tag{12}$$

TABLE 1

a (or $1-a$)	$b = 1$	$b = 2$
0.0	3.15334	3.14162
0.1	3.1511	3.1416
0.2	3.1452	3.1415
0.3	3.1379	3.1415
0.4	3.1321	3.1416
0.5	3.12988	3.14158

Then the expression

$$(13)\quad \mu_{2s+3,m+2}/\alpha_{m+2} - \mu_{2s-1,m}/\alpha_m - \{\tfrac{1}{2}(m+2)^2 - 2b^2\}\mu_{2s+1,m+2}/\alpha_{m+2}$$

can be rearranged as

$$(14)\quad \begin{aligned}&\textstyle\sum_r f(r, m+2)(r+a+m/2)^{2s-1}\\ &\cdot[\{(r+a+m+1)^2+b^2\}\{(r+a-1)^2+b^2\} - (r+a+m/2)^4]/\alpha_{m+2}\end{aligned}$$

which is simply constant $\times\ \mu_{2s-1,m+2}$.

Assumption (12) thus implies that $\mu_{2s+3,m+2}$ is zero.

Since μ_1, μ_3 are zero for all k from (5), we have shown, for all k, that

$$\mu_{2s+1} = 0 \quad \text{for} \quad s = 0, 1, 2, \cdots \quad \text{provided it exists.}$$

Further it is readily shown from (3) that the distribution is only symmetric for $a = 0, \frac{1}{2}$ or 1, so we have shown the existence of a family of distributions for which measures of skewness based on the odd order moments are of no value.

For any finite number N, we may choose $k > N/2$ so that the first N moments exist. Since a symmetric characteristic function implies that the distribution is symmetric and the moment generating function, if it exists, determines the characteristic function, our result is nearly optimal. That is for any finite M we may construct a distribution with its first $2M$ moments finite and all M odd order amounts about the mean zero, although the distribution is itself asymmetric.

We observe that this asymmetry is of a very special form. If f_0 is the modal frequency, f_1 second largest, then

$$(15)\quad f_0 > f_1 > f_{-1} > f_2 > f_{-2} > \cdots.$$

5. Estimation. Sichel (1949) showed that the second frequency moment

$$(16)\quad w_2 = \int f^2\,dx \qquad (w_1 \equiv 1)$$

TABLE 2

Value of r. v.	Binomial ($n=20$)	S (2, 0, 14)	S (10, 0, 70)
0	.176,196	.215,064	.182,503
±1	.160,179	.179,596	.162,363
±2	.120,134	.109,319	.119,044
±3	.073,929	.054,660	.070,257
±4	.036,964	.024,000	.033,431
±5	.014,786	.011,040	.014,380
±6	.004,621	.005,257	.005,561
±7	.001,087	.002,763	.001,892
±8	.000,183	.001,454	.000,586
±9	.000,019	.000,804	.000,100
±10	.000,001	.000,479	.000,051
>10 or <−10	—	.003,091	.000,101

was asymptotically efficient for estimating the scale parameter of the Cauchy curve. For the discrete Cauchy ($k = 0$ in (3)), we find

$$w_2 = \tfrac{1}{2} + (\coth \pi b - \pi b)/2\pi b \coth^2 \pi b \tag{17}$$

while the estimate b^* from (17) has variance

$$n^{-1}\{\tfrac{1}{2}b^2 - (\pi \operatorname{cosech} 2\pi b)/b + \pi^2 \operatorname{sech}^2 \pi b\} \tag{18}$$

asymptotically equal to the variance of the ML estimator.

Generally, it may be shown that if the density function obeys the differential equation

$$df/db = fh(g - f) \tag{19}$$

where h, g are functions of b, then the second frequency moment is an asymptotically efficient estimator of b.

Equation (19) is a Bernoulli equation and thus has the general solution

$$f = \mu/(\textstyle\int \mu h \, db + C(x)) \tag{20}$$

where $\mu = \exp(\int hg \, db)$, C is some function of x only. A particular solution is

$$f(x, b) \propto \{1 + (x/b)^j\}^{-1}, \qquad j > 1. \tag{21}$$

6. Numerical comparison. Denote the distribution of form (3) with parameters, k, a, b^2 by $S(k, a, b^2)$. We compare $S(2, 0, 14)$, $S(10, 0, 70)$ with the symmetric binomial having $n = 20$, taking origins at the mean, and all the distributions have variance 5. The β_2 values are 8.6, 3.2 and 2.9 respectively.

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