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A NOTE ON FIRST PASSAGES FOR $S_n/n^{\frac{1}{2}}$ [1]

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1. Introduction and summary. Let $X_1, X_2, \cdots$ be independent random variables defined on a probability space (Ω, F, P) and with mean 0 and variance 1.

Let $F_0 = \{\Omega, \phi\}$, and for $n \geqq 1$ let $S_n = X_1 + X_2 + \cdots + X_n$ and $F_n = \sigma(X_1, X_2, \cdots, X_n)$. Denote by T the class of all stopping rules with respect to $\{F_n\}$, i.e., the class of all $t: \Omega \to \{1, 2, \cdots, \infty\}$ such that $\{t = k\} \in F_k$ for $k = 1, 2, \cdots$, and let $T_n = \{t \in T: t \geqq n \text{ a.s.}\}$.

If $t \in T$ we adopt the convention of the author of [3] that $|S_t|/t = \limsup_n |S_n|/n$ if $t = \infty$. We remind the reader that for this sequence of random variables $X_1, X_2, \cdots, \limsup_n |S_n|/n = 0$ a.s. Since $E(\sup_n |S_n|/n) < \infty$ (Lemma 9 [1]), it follows at once from the results of [3] that

LEMMA 1. *If* $f_n = \operatorname{ess\,sup}_{t \in T_n} E(|S_t|/t \mid F_n)$, *then* $f_n = \max(|S_n|/n, E(f_{n+1} \mid F_n))$ a.s. *and* $\limsup_n f_n = \limsup_n |S_n|/n = 0$ a.s.

For each $c > 0$ define $t(c) =$ first $n \geqq 1$ for which $|S_n| > cn^{\frac{1}{2}}$, $= \infty$ if no such n exists. In this note we prove

THEOREM 1. *If for each* $c > 0$, $P(t(c) < \infty) = 1$, *then* $P(|S_n| = nf_n \text{ i.o.}) = 1$.

Although this theorem will not be a surprise to readers of the recent literature on optimal stopping problems related to S_n/n, this proof may be of some interest.

2. Two lemmas. Let $s(0) \equiv 0$, and for $j = 1, 2, \cdots$, let $s(j) = j$th time for which $|S_n| = nf_n$, $= \infty$ if no such time exists. The following notation is used below:

$$C_{s(j)} = \{t \in T : t > s(j) \text{ a.s.}\}$$
$$F_{s(j)} = \{A \in F : A \cap \{s(j) = k\} \in F_k,\ k = 1, 2, \cdots\}$$
$$F_{s(j)+1} = \{A \in F : A \cap \{s(j) = k\} \in F_{k+1},\ k = 1, 2, \cdots\}$$
$$V_{s(j)} = \operatorname{ess\,sup}_{t \in C_{s(j)}} E(|S_t|/t \mid F_{s(j)}).$$

LEMMA 2. *For* $j = 0, 1, 2, \cdots$,

$$V_{s(j)} = E(|S_{s(j+1)}|/s(j+1) \mid F_{s(j)}) \quad \text{a.s.}$$

PROOF. Since $s(j+1) \in C_{s(j)}$, and $f_{s(j+1)} = |S_{s(j+1)}|/s(j+1)$ it suffices to prove

$$V_{s(j)} \leqq E(f_{s(j+1)} \mid F_{s(j)}) \quad \text{a.s.}$$

In view of Lemma 1, for any $t \in C_{s(j)}$ and $A \in F_{s(j)}$ we have:

$$\int_{A, s(j)=n} |S_t|/t \, dP = \int_{A, s(j)=n} E(|S_t|/t \mid F_{n+1}) \, dP$$
$$\leqq \int_{A, s(j)=n} f_{n+1} \, dP = \int_{A, s(j)=n} f_{s(j)+1} \, dP;$$

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hence,

$$\int_A |S_t|/t\,dP = \sum_{n=j}^{\infty} \int_{A,s(j)=n} |S_t|/t\,dP \leqq \sum_{n=j}^{\infty} \int_{A,s(j)=n} f_{s(j)+1}\,dP$$
$$= \int_A f_{s(j)+1}\,dP.$$

It then follows that $V_{s(j)} \leqq E(f_{s(j)+1} \mid F_{s(j)})$ a.s., and hence to complete the proof it suffices to show that

$$f_{s(j)+1} \leqq E(f_{s(j+1)} \mid F_{s(j)+1}) \quad \text{a.s.}$$

or equivalently

$$\int_{A\{s(j)+1=k\}} f_k \leqq \int_{A\{s(j)+1=k\}} f_{s(j+1)}, \tag{1}$$

for $A \in F_{s(j)+1}$, $k = 1, 2, \cdots$.
In [3] it is shown that if

$$\sigma(=\sigma_k) = \text{first } n \geqq k \text{ such that } |S_n|/n = f_n$$
$$= \infty \text{ if no such } n \text{ exists,}$$

then $E(f_\sigma \mid F_k) = f_k$, and since $A\{s(j)+1=k\} \in F_{k-1} \subset F_k$, we have

$$\int_{A\{s(j)+1=k\}} f_k \leqq \int_{A\{s(j)+1=k\}} f_\sigma\,. \tag{2}$$

But on $\{s(j)+1=k\}$, $\sigma = s(j+1)$ and hence (1) follows from (2).

LEMMA 3. *For each $c > 0$ and $j = 0, 1, 2, \cdots$,*

$$P\{2t(c) \leqq s(j+1) \text{ and } s(j) < t(c) < \infty \mid F_{s(j)}\} \leqq 2/c \quad \text{a.s.}$$

PROOF. If for some j and $c > 0$ the assertion of this lemma is not true, then we have:

$$E\{I_{s(j)<t(c)<\infty}\, I_{2t(c)\leqq s(j+1)} \mid F_{s(j)}\} - 2/c > 0$$

with positive probability, which implies that:

$$E\{I_{s(j)<t(c)<s(j+1)}(I_{2t(c)\leqq s(j+1)} - 2/c) \mid F_{s(j)}\} > 0$$

with positive probability. Rewriting the left side of this inequality one obtains:

$$\sum_{n=1}^{\infty} E\{I_{s(j)<n=t(c)<s(j+1)}[E(I_{2n\leqq s(j+1)} \mid F_n) - 2/c] \mid F_{s(j)}\} > 0$$

with positive probability. Hence for some n we must have:

$$E\{I_{s(j)<n=t(c)<s(j+1)}[P(2n \leqq s(j+1) \mid F_n) - 2/c] \mid F_{s(j)}\} > 0 \tag{3}$$

with positive probability. For this n let,

$$B_n = \{s(j) < n = t(c) < s(j+1) \text{ and } P(2n \leqq s(j+1) \mid F_n) > 2/c\};$$

then,

$$P(P\{B_n \mid F_{s(j)}\} > 0) > 0 \tag{4}$$

otherwise (4) is a false statement. On B_n the following things happen:

$$|S_n| > cn^{\frac{1}{2}} \quad \text{and} \quad P\{s(j+1)-n \geqq n \mid F_n\} > 2/c.$$

We now use Lemma 1 and Lemma 2 of [2], together with these facts to prove the second of the following inequalities:

$$\begin{aligned} \text{On} \quad B_n : & E\{|S_{s(j+1)}|/s(j+1) \mid F_n\} \\ (5) \qquad & \leqq E\{(|S_n| + |X_{n+1} + \cdots + X_{n+s(j+1)-n}|)/(n+s(j+1)-n) \mid F_n\} \\ & < |S_n|/n \quad \text{a.s.} \end{aligned}$$

Now define the stopping rule $s^1(j+1)$ by

$$\begin{aligned} s^1(j+1) &= s(j+1) \quad \text{on} \quad B_n^{\ c} \\ &= n \quad \text{on} \quad B_n . \end{aligned}$$

It is easy to check that $s^1(j+1) \in C_{s(j)}$. And then we have that:

$$\begin{aligned} & E\{|S_{s^1(j+1)}|/s^1(j+1) \mid F_{s(j)}\} - V_{s(j)} \\ & \quad = E\{|S_{s^1(j+1)}|/s^1(j+1) - |S_{s(j+1)}|/s(j+1) \mid F_{s(j)}\} \\ & \quad = E\{I_{B_n}[|S_n|/n - E(|S_{s(j+1)}|/s(j+1) \mid F_n)] \mid F_{s(j)}\} > 0 \end{aligned}$$

with positive probability by (4) and (5), which contradicts the definition of $V_{s(j)}$.

COROLLARY. *For each* $c > 0$ *and* $j = 0, 1, \cdots$,

$$P(2t(c) \leqq s(j+1) \ \textit{and} \ s(j) < t(c) < \infty) \leqq 2/c.$$

3. Proof of Theorem 1. By hypothesis, for arbitrary $c > 0$, $t(c) < \infty$ a.s., hence,

$$P(s(1) = \infty) \leqq P(2t(c) \leqq s(1) \ \text{and} \ s(0) \equiv 0 < t(c) < \infty) \leqq 2/c$$

and then it follows that $P(s(1) = \infty) = 0$.

Suppose that $P(s(j) < \infty) = 1$ for $j = 1, 2, \cdots, j_0$. Then by virtue of the corollary we have:

$$P(s(j_0+1) = \infty) = \lim_{c\to\infty} P(s(j_0+1) = \infty \ \text{and} \ s(j_0) < t(c)) \leqq \limsup_{c\to\infty} 2/c = 0.$$

And the proof is completed by induction.

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REFERENCES

[1] DVORETZKY, A. (1965). Existence and properties of certain optimal stopping rules. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* **1** 441–452. Univ. of California Press.

[2] RUIZ-MONCAYO, A. (1968). Optimal stopping for functions of Markov chains. *Ann. Math. Statist.* **39** 1905–1912.

[3] SEIGMUND, D. O. (1967). Some problems in the theory of optimal stopping rules. *Ann. Math. Statist.* **38** 1627–1640.