The Annals of Mathematical Statistics
1971, Vol. 42, No. 4, 1254–1261

NONCENTRAL DISTRIBUTION OF WILKS' STATISTIC IN MANOVA[1]

BY A. K. GUPTA
University of Arizona

1. Summary. In this paper, the exact distribution of Wilks' likelihood ratio criterion, Λ, for MANOVA in the noncentral linear case i.e. when the alternative hypothesis is of unit rank, has been obtained and explicit expressions for the same for $p = 2, 3, 4$ and 5, where p is the number of variables and for general f_1 and f_2 are given. A general form of the distribution of Λ in this case, for any p, is also given. It has been shown that the total integral of the series obtained by taking a few terms only, rapidly approaches the theoretical value one as more terms are taken into account. Further the accuracy of the approximation, suggested by Posten and Bargmann [11], is examined numerically and it has been shown that the approximation is excellent except when f_1 and f_2 are both small and the noncentrality parameter λ^2 is large.

2. Introduction. Let $\mathbf{X}_1(p \times f_1)(f_1 \geqq p)$ and $\mathbf{X}_2(p \times f_2)$ be distributed in the form

$$(2.1)\quad (2\pi)^{-\frac{1}{2}p(f_1+f_2)}|\mathbf{\Sigma}|^{-\frac{1}{2}(f_1+f_2)}\exp[-\tfrac{1}{2}\operatorname{tr}\Sigma^{-1}\{\mathbf{X}_1\mathbf{X}_1'+(\mathbf{X}_2-\boldsymbol{\mu})(\mathbf{X}_2-\boldsymbol{\mu})'\}],$$

and let the nonzero roots of the determinantal equation $|\mathbf{A}_2-\lambda\mathbf{A}_1| = 0$, be denoted by $0 < \lambda_1 \leqq \cdots \leqq \lambda_s < \infty$, where $s = \min(p, f_2)$, and

$$\mathbf{A}_1(p \times p) = \mathbf{X}_1\mathbf{X}_1'$$

$$\mathbf{A}_2(p \times p) = \mathbf{X}_2\mathbf{X}_2'.$$

The likelihood-ratio statistic for testing $H_0: \boldsymbol{\mu}(p \times f_2) = \mathbf{0}$ against $H_1: \boldsymbol{\mu} \neq \mathbf{0}$ can be expressed in terms of the following criterion suggested by Wilks [16], and Pearson and Wilks [9]:

$$(2.2)\quad \Lambda = |\mathbf{A}_1|/|\mathbf{A}_1+\mathbf{A}_2| = \textstyle\prod_{i=1}^{s}\{1/(1+\lambda_i)\}.$$

It may be noted that in the context of multivariate analysis of variance, $\mathbf{A}_1$ and $\mathbf{A}_2$ are the sums of product matrices for error and hypothesis respectively, and f_1 and f_2 are the corresponding degrees of freedom.

We can write $\mathbf{A}_1 = \mathbf{CLC}'$, where $\mathbf{C}$ is a lower triangular matrix such that $\mathbf{A}_1+\mathbf{A}_2 = \mathbf{CC}'$. When the rank of $\boldsymbol{\mu} = 1$, i.e. in the linear case (Anderson [1], Anderson and Girshick [2]), it has been shown (Kshirsagar [8]) that the density of $\mathbf{L}$ is given by

Received July 22, 1968; revised December 7, 1970.

[1] This research was supported (in part) by the National Science Foundation, Grant No. GP-7663.
Reproduction in whole or in part is permitted for any purpose of the United States Government.

(2.3) $f(\mathbf{L}) = K \exp(-\frac{1}{2}\lambda^2) {}_1F_1\{\frac{1}{2}(f_1+f_2), \frac{1}{2}f_2, \frac{1}{2}\lambda^2(1-l_{11})\} |\mathbf{L}|^{\frac{1}{2}(f_1-p-1)} \cdot |\mathbf{I}-\mathbf{L}|^{\frac{1}{2}(f_2-p-1)}$,

where

(2.4) $K = \pi^{-\frac{1}{4}p(p-1)} \prod_{i=1}^{p} \Gamma[\frac{1}{2}(f_1+f_2+1-i)]/\{\Gamma[\frac{1}{2}(f_1+1-i)]\Gamma[\frac{1}{2}(f_1+1-i)]\}$,

and $\lambda^2 = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ is the single noncentrality parameter, l_{11} is the element in the top left corner of the matrix $\mathbf{L}$, and ${}_1F_1$ denotes the confluent hypergeometric function, defined below.

$$(2.5) \qquad {}_1F_1(a, b, Z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \cdot \frac{Z^n}{n!}$$

where $(a)_n = a(a+1) \cdots (a+n-1)$.

Further, we can write $\mathbf{L} = \mathbf{TT}'$ where $\mathbf{T}$ is a lower triangle matrix $[t_{ij}]$ of order p. Then it has been shown by Kshirsagar [8] that the diagonal elements t_{ii} are independently distributed and that t_{ii}^2 $(i = 2, \cdots p)$ follows the distribution

$$(2.6) \qquad \mathscr{L}(t_{ii}^2) = \beta[f_1 - i + 1, f_2; t_{ii}^2],$$

where

$$(2.7) \qquad \beta[a, b; x] = [1/B(\tfrac{1}{2}a, \tfrac{1}{2}b)] x^{\frac{1}{2}a-1}(1-x)^{\frac{1}{2}b-1}, \qquad 0 \leqq x \leqq 1,$$

while t_{11}^2 is distributed as

$$(2.8) \quad \mathscr{L}(t_{11}^2) = \frac{\exp(-\frac{1}{2}\lambda^2)}{B(\frac{1}{2}f_1, \frac{1}{2}f_2)} (t_{11}^2)^{\frac{1}{2}f_1-1}(1-t_{11}^2)^{\frac{1}{2}f_2-1} {}_1F_1(\tfrac{1}{2}(f_1+f_2), \tfrac{1}{2}f_2, \tfrac{1}{2}\lambda^2(1-t_{11}^2)), \quad 0 \leqq t_{11}^2 \leqq 1.$$

Observe that

$$(2.9) \qquad \Lambda = |\mathbf{L}| = \textstyle\prod_{i=1}^{p} t_{ii}^2 = \prod_{i=1}^{p} X_i \quad \text{(say)}$$

where the distribution of X_1 is the same as that of t_{11}^2 and is given by (2.8) and $\mathscr{L}(x_i) = \beta[f_1+1-i, f_2; X_i]$, $i = 2, \cdots, p$.

For purposes of notational ease, the symbol Λ will be replaced by U_{p,f_2,f_1}. Then the above results can be restated in the following form.

THEOREM 2.1. *In the noncentral linear case, U_{p,f_2,f_1} is distributed like $X_1 \cdots X_p$ where X_1 is independently distributed as in (2.8) and X_i $(i = 2, \cdots, p)$ are independently distributed as $\beta[f_1+1-i, f_2; X_i]$.*

The connection between a beta variate and its square gives us the next theorem.

THEOREM 2.2. *In the noncentral linear case, U_{2r,f_2,f_1} is distributed like $X_1 Y_1^2 \cdots Y_{r-1}^2 X_{2r}$ where X_1 is independently distributed as in (2.8), Y_i $(i = 1, 2, \cdots, r-1)$ are independently distributed as $\beta[2(f_1-2i), 2f_2; Y_i]$ and X_{2r} is independently distributed as $\beta[f_1+1-p, f_2; X_{2r}]$. And U_{2s+1,f_2,f_1} is distributed like $X_1 Y_1^2 \cdots Y_s^2$,* *where* X_1 *is independently distributed as in* (2.8) *and* Y_i $(i = 1, \cdots, s)$ *are independently distributed as* $\beta[2(f_1-2i), 2f_2; Y_i]$.

The method employed in the next section relies on Theorems 2.1 and 2.2.

3. Method of derivation. An immediate consequence of Theorem 2.1 is that, since $-\log U_{p,f_2,f_1} = \sum_{i=1}^{p}(-\log X_i) = \sum_{i=1}^{p} Y_i$ (say), the distribution problem in hand can be reduced to that of a sum of independently distributed random variables. The latter distribution can be handled by taking successive convolutions provided that the procedure yields expressions which can be easily integrated at each stage. Schatzoff [14], [15], has proved that this is in fact the case, and the technique has been used by Pillai and Gupta [10] and by Schatzoff [15] to derive the central distribution of Λ. Consul, [3], applied operational calculus to obtain the exact distribution in the central case for $p = 1, 2, 3$ and 4.

Consider the density of X_1, obtained from (2.8) by substituting X_1 for t_{11}^2. Substitute for ${}_1F_1$ from (2.5) and make the transformation $Y_1 = -\log X_1$. Then the density of Y_1, after binomial expansion, is given by

$$\text{(3.1)} \qquad \frac{e^{-\frac{1}{2}\lambda^2}}{B(\frac{1}{2}f_1, \frac{1}{2}f_2)} \sum_{j=0}^{\infty} \frac{(\frac{1}{2}v)_j}{(\frac{1}{2}f_2)_j} \frac{(\frac{1}{2}\lambda^2)^j}{j!} \sum_{k=0}^{b+j} (-1)^k \binom{b+j}{k} e^{-(\frac{1}{2}f_1+k)Y_1} \qquad Y_1 \geqq 0,$$

where $v = f_1+f_2$ and $b = \frac{1}{2}f_2-1$.

Further consider the beta random variable X_i $(i = 2, \cdots, p)$, when the density is given by

$$\text{(3.2)} \qquad \beta[f_1-i+1, f_2; X_i] = K_i X_i^{\frac{1}{2}(f_1-1-i)}(1-X_i)^{\frac{1}{2}f_2-1} \qquad 0 \leqq X_i \leqq 1, f_1 \geqq i$$

where $K_i = [B(\frac{1}{2}(f_1-i+1), \frac{1}{2}f_2)]^{-1} = \Gamma[\frac{1}{2}(f_1-i+1+f_2)]/\Gamma[\frac{1}{2}(f_1-i+1)]\Gamma(\frac{1}{2}f_2)$. When f_2 is even, $b = \frac{1}{2}(f_2-2)$ is an integer, and using the binomial theorem and transforming $Y_i = -\log X_i$, we get the density of Y_i as

$$\text{(3.3)} \qquad \mathscr{L}(Y_i) = K_i \textstyle\sum_{l=0}^{b}(-1)^l\binom{b}{l}\exp[-\frac{1}{2}Y_i(f_1+1+2l-i)], \qquad Y_i \geqq 0, i = 2, \cdots, p.$$

Similarly in the light of Theorem 2.2 we consider the random variable $Z_i = X_{2i}X_{2i+1}$, then the density function of Z_i is given by

$$\text{(3.4)} \qquad C_i Z_i^{\frac{1}{2}(f_1-2i-2)}[1-(Z_i)^{\frac{1}{2}}]^{f_2-1},$$

where $C_i = [2B(f_1-2i, f_2)]^{-1}$.

Now make the transformation $Y_i' = -\log Z_i$, and as before, from (3.4), we get the density of Y_i' as

$$\text{(3.5)} \qquad \mathscr{L}(Y_i') = C_i \textstyle\sum_{l=0}^{f_2-1}(-1)^l\binom{f_2-1}{l}\exp[-\frac{1}{2}Y_i'(f_1+l-2i)], \qquad Y_i' \geqq 0.$$

Finally, following Schatzoff [15], consider the density of $V = V_1 + V_2$, where the density of V_1 is given by

$$\text{(3.6)} \qquad \{a^{k+1}/\Gamma(k+1)\}v_1^k e^{av_1}, v_1 > 0, \qquad k = \text{nonnegative integer},$$

and that of V_2 by

$$\text{(3.7)} \qquad b\, e^{bv_2}, v_2 > 0.$$

Schatzoff [15] has shown that the density of V takes the form

$$b^{k+2} e^{bv} v^{k+1}/\Gamma(k+2), \qquad a = b \tag{3.8}$$

and

$$\{a^{k+1} b/\Gamma(k+1)\}\left\{\left[e^{av} \sum_{r=1}^{k+1} (-1)^{r+1} \frac{k!}{(k-r+1)!} \frac{v^{k-r+1}}{(a-b)^r}\right] + e^{bv}(b-a)^{-(k+1)} k!\right\}, \qquad a \neq b. \tag{3.9}$$

The above results can be readily applied to obtain the distribution of U_{p,f_2,f_1} in the noncentral linear case in the following section.

4. Exact distribution of U_{p,f_2,f_1} in the noncentral linear case. In this section, we derive the density and cdf of U_{p,f_2,f_1} for $p = 2, 3, 4$ and 5. We also give general form of the distribution for any p. We start with $p = 2$.

Case i, $p = 2$. We write Wilks' statistic as a product of independent variables, $U_{2,f_2,f_1} = X_1 X_2$, and hence

$$-\log U_{2,f_2,f_1} = Y_1 + Y_2 \tag{4.1}$$

where Y_1 is distributed as in (3.1) and Y_2 as in (3.3) for $i = 2$. Then making use of (3.8) and (3.9), we get the density of U_{2,f_2,f_1} in the following form:

$$K e^{-\frac{1}{2}\lambda^2} U^{\frac{1}{2}f_1 - 1} \sum_{j=0}^{\infty} a_j \left\{ \sum_{k=0}^{b+j} \sum_{l=0}^{b} \frac{(-1)^{l+k}}{(2l-2k-1)} \binom{b+j}{k} \binom{b}{l} (U^k - U^{l-\frac{1}{2}}) \right\}, \quad 0 \leqq U \leqq 1, \tag{4.2}$$

where

$$K = 2 \textstyle\prod_{i=1}^{2} [1/B\{\frac{1}{2}(f_1 - i + 1), \frac{1}{2}f_2\}] \tag{4.3}$$

and

$$a_j = [(\tfrac{1}{2}v)_j (\tfrac{1}{2}\lambda^2)^j / (\tfrac{1}{2}f_2)_j j!]. \tag{4.4}$$

The cdf of U_{2,f_2,f_1} can be easily obtained by integrating (4.2) between $(0, u)$, $0 < u \leqq 1$ (see Gupta [6]).

Case ii, $p = 3$. In this case, $-\log U_{3,f_2,f_1} = Y_1 + Y_1'$, where Y_1 is distributed as in (3.1) and Y_1' as in (3.5) for $i = 1$. As in Case i, we obtain the density of U_{3,f_2,f_1} in the following form:

$$\begin{aligned} K e^{-\frac{1}{2}\lambda^2} U^{\frac{1}{2}f_1 - 1} \sum_{j=0}^{\infty} a_j \Bigg\{ & \sum_{k=0}^{b+j} (-U)^k \binom{f_2 - 1}{2k+2} \binom{b+j}{k} \log U \\ & + 2 \sum_{k=0, l \neq 2k+2}^{b+j} \sum_{l=0}^{f_2 - 1} \frac{(-1)^{l+k}}{(l-2k-2)} \binom{f_2 - 1}{l} \binom{b+j}{k} (U^k - U^{\frac{1}{2}l - 1}) \Bigg\}, \end{aligned} \tag{4.5}$$

where

$$K = [2B(\tfrac{1}{2}f_1, \tfrac{1}{2}f_2) B(f_1 - 2, f_2)]^{-1}. \tag{4.6}$$

The cdf of U_{3,f_2,f_1} can be obtained by straightforward integration (see Gupta [6]).

Case iii, $p = 4$. For $p = 4$, $-\log U_{4,f_2,f_1} = Y_1 + Y_1' + Y_4$, where Y_1 is distributed as in (3.1), Y_1' as in (3.5) for $i = 1$ and Y_4 as in (3.3) for $i = 4$. As before we obtain the density of U_{4,f_2,f_1} as

$$K e^{-\frac{1}{2}\lambda^2} U^{\frac{1}{2}f_1 - 1} \sum_{j=0}^{\infty} a_j (T_{1j} + 2T_{2j}) \tag{4.7}$$

where

$$T_{1j} = \sum_{k,n} \frac{(-1)^{k+n} U^k}{(2n-2k-3)} f(0,n,k) \left\{ \binom{f_2-1}{2k+2} \left(\frac{2U^{n-k-\frac{3}{2}} - 2}{2n-2k-3} - \log U \right) - \binom{f_2-1}{2n-1} U^{n-k-\frac{3}{2}} \log U \right\},$$

$$T_{2j} = \sum_{k,l,l \neq 2k+2} \frac{(-1)^{l+k}}{(l-2k-2)} f(l,0,k) \left\{ \sum_n \frac{(-1)^n}{2n-2k-3} \binom{b}{n} (U^k - U^{n-\frac{3}{2}}) \right.$$
$$\left. - \sum_{n,n \neq \frac{1}{2}(l+1)} \frac{(-1)^n}{2n-l-1} \binom{b}{n} (U^{\frac{1}{2}l-1} - U^{n-\frac{3}{2}}) \right\},$$

$K = [B(f_1-2, f_2)]^{-1} \prod_{i=1}^{2} [B(\frac{1}{2}(f_1 - 3i + 3), \frac{1}{2}f_2)]^{-1}$ and $f(l,n,k) = \binom{f_2-1}{l}\binom{b}{n}\binom{b+j}{k}$.

The cdf can be obtained from (4.7) by straightforward integration between $(0, u)$, $0 < u \leqq 1$ and is available in an unpublished report by Gupta [6].

Case iv, $p = 5$. In this case, $-\log U_{5,f_2,f_1} = Y_1 + Y_1' + Y_2'$, and the density of U_{5,f_2,f_1} is given by

$$K e^{-\frac{1}{2}\lambda^2} U^{\frac{1}{2}f_1 - 1} \sum_{j=0}^{\infty} a_j (T_{1j} + 4T_{2j} + 8T_{3j}), \tag{4.8}$$

where

$$T_{1j} = \sum_k (-U)^k f(2k+2, 0, k) \left\{ \binom{f_2-1}{2k+4} (\log U)^2 \right.$$
$$\left. + 4 \sum_{n,n \neq 2k+4} \frac{(-1)^n}{n-2k-4} \binom{f_2-1}{n} \left(\frac{2U^{\frac{1}{2}n-k-2} - 2}{n-2k-4} - \log U \right) \right\},$$

$$T_{2j} = \sum_{l,k,l \neq 2k+2} \frac{(-U)^k}{l-2k-2} f(l,0,k) \log U \left\{ \binom{f_2-1}{l+2} U^{\frac{1}{2}l-k-1} - (-1)^l \binom{f_2-1}{2k+4} \right\},$$

$$T_{3j} = \sum_{l,k,l \neq 2k+2} \frac{(-1)^{l+k}}{l-2k-2} f(l,0,k) \left\{ \sum_{n,n \neq 2k+4} (-1)^n \binom{f_2-1}{n} \left(\frac{U^k - U^{\frac{1}{2}n-2}}{n-2k-4} \right) \right.$$
$$\left. - \sum_{n,n \neq l+2} (-1)^n \binom{f_2-1}{n} \left(\frac{U^{\frac{1}{2}l-1} - U^{\frac{1}{2}n-2}}{n-l-2} \right) \right\},$$

$K = [8B(\frac{1}{2}f_1, \frac{1}{2}f_2)]^{-1} \prod_{i=1}^{2} [B(f_1 - 2i, f_2)]^{-1}$ and $f(l,n,k) = \binom{f_2-1}{l}\binom{f_2-1}{n}\binom{b+j}{k}$.

The cdf can be obtained from (4.8) by integrating between $(0, u)$, $0 < u \leqq 1$, and is available in an unpublished report by Gupta [6]. For $p > 5$, the density function and the corresponding cdf become too involved for presentation.

Case v, *general p*. The techniques of this section enable us to give the form of the noncentral distribution of U_{p,f_2,f_1} in the linear case for any p and any f_2.

THEOREM 4.1. *The probability density function of U_{p,f_2,f_1} in the noncentral linear case is of the form*

$$(4.9)\qquad \left(\prod_{i=1}^{p} K_i\right) e^{-\frac{1}{2}\lambda^2} \sum_{j=0}^{\infty} \frac{(\frac{1}{2}\nu)_j}{(\frac{1}{2}f_2)_j} \frac{(\frac{1}{2}\lambda^2)^j}{j!} \sum_{k=0}^{m_j} C_{jk} U^{\frac{1}{2}(f_1-b_k)}(-\log U)^{d_k},$$

where K_i is defined in (3.2) *and the constants C_{jk} and integers m_j, b_k and d_k are determined from p, f_2 and f_1.*

The theorem can be proved easily by induction. For example, when $p = 1$, this is readily seen to be of the required form where $m_j = \frac{1}{2}f_2 - 1 + j$, $b_k = 2 - 2k$, $d_k = 0$ and $C_{jk} = (-1)^k \binom{\frac{1}{2}f_2 - 1 + j}{k}$.

The Theorem 4.1 does not indicate explicitly how to find the values of the constants m_j, C_{jk}, b_k and d_k, a task which is by no means easy for large values of p or f_2. However, the theorem provides a basis for a recursive algorithm for deriving the density and the distribution function at successive stages of the convolution process. Schatzoff, [14], spelled out an algorithm for computation of the coefficients in the central case. An analogous algorithm can be derived in the present case. Indeed, the present noncentral development essentially parallels the central derivation.

The mathematical simplicity of the density function (4.9) makes possible the derivation of the corresponding cdf by straightforward integration which is simplified to a double summation, given by

$$(4.10)\quad F(u) = \left(\prod_{i=1}^{p} K_i\right) e^{-\frac{1}{2}\lambda^2} \sum_{j=0}^{\infty} \frac{(\frac{1}{2}\nu)_j}{(\frac{1}{2}f_2)_j} \frac{(\frac{1}{2}\lambda^2)^j}{j!} \sum_{h=1}^{M_j} \left\{ a_{hj} \left(\frac{2}{f_1 - b_h}\right)^{r_h} u^{\frac{1}{2}(f_1-b_h)} \cdot (-\log u)^{s_h} \right\},$$

where the constants M_j, a_{hj}, b_h, r_h and s_h can be determined from p, f_2 and f_1.

5. Numerical feasibility of the cdf's obtained and the validity of Posten–Bargmann approximation. It is well known (see Ghosh [5], Kiefer and Schwartz [7]) that Wilks' Λ is unbiased, consistent and admissible and its power function is a monotonic-increasing function of each of the noncentrality parameters (see Dasgupta, Anderson and Mudholkar [4]). Very little is known, however, about the actual magnitude of the power and this is due to the fact that the noncentral distribution of the test criterion has not been expressed in a numerically feasible form. A number of asymptotic approximations has been given. Roy, [12], obtained gamma-series expansion for the power function of Wilks' test, which is convenient to use when the error df is larger and the noncentrality parameter is small. Roy, [13] also obtained exact expression for $p = 2$ and suggested two more approximations for $p > 2$. Posten and Bargmann [11] obtained an approximation to the power of the likelihood ratio test of the multivariate general linear hypothesis.

From a practical standpoint, the results of Section 4 make possible for the first time the calculation of exact percentage points for $p = 2, 3, 4$ and 5 and any f_1 and f_2. It has been verified for $p = 2, 3, 4$ and 5 that the total integral of the series obtained by taking a few terms only, rapidly approaches the theoretical value one as more terms are taken into account. The results also provide the check for the validity of Posten–Bargmann approximation. In particular, the comparisons are made for $p = 2$, 3 and for selected values of λ^2, f_1 and f_2, by computing the percentage points. The results are presented in Table 1.

TABLE 1

Exact, E, *and approximate*, *A*, *percentage points when* $\alpha = 0.95$

			$p = 2$		$p = 3$	
f_2	f_1	λ^2	E	A	E	A
2	2	0.5	.571752	.572598		
		1.0	.541292	.543222		
		4.0	.385821	.398234		
	8	0.5	.898552	.898554	.775704	.775729
		1.0	.886876	.886881	.760957	.761017
		4.0	.802938	.803031	.670107	.676552
	20	0.5	.959648	.959648	.910887	.910897
		1.0	.954656	.954656	.904028	.904040
		4.0	.915845	.915839	.857452	.857602

The approximate percentage points were computed using (2.4) of Posten and Bargmann [11], neglecting terms of order $O(m^{-3})$ where $m = f_1 + \frac{1}{2}(f_2 - p - 1)$. As is shown by the table, even when we neglect terms of order $O(m^{-3})$ the approximation suggested by Posten and Bargmann is not so good when f_1 and f_2 are small i.e. m is small, and λ^2 is large. However, for large values of m, the agreement between the exact results as derived by the methods of the present paper and by the approximate method of Posten and Bargmann is excellent. The computations in Table 1 were carried on CDC 6400 of the University of Arizona's Computer Center.

Acknowledgment. The author wishes to express his thanks to Professor K. C. S. Pillai of Purdue University for his valuable suggestions. Thanks are also due to the referee and the associate editor for some useful comments on this paper.

REFERENCES

[1] ANDERSON, T. W. (1946). The noncentral Wishart distribution and certain problems of multivariate statistics. *Ann. Math. Statist.* **17** 409–431.

[2] ANDERSON, T. W. and GIRSHICK, M. A. (1944). Some extentions of the Wishart distribution. *Ann. Math. Statist.* **15** 345–357.

[3] CONSUL, P. C. (1966). On the exact distribution of the likelihood ratio criterion for testing linear hypotheses about regression coefficients. *Ann. Math. Statist.* **37** 1319–1330.

[4] DASGUPTA, S., ANDERSON, T. W. and MUDHOLKAR, G. S. (1964). Monotonocity of power functions of some tests of multivariate linear hypothesis. *Ann. Math. Statist.* **35** 200–205.

[5] GHOSH, M. N. (1964). On the admissibility of some tests of MANOVA. *Ann. Math. Statist.* **35** 789–794.

[6] GUPTA, A. K. (1970). Distribution of Wilks' Λ in the noncentral linear case. Report No. 128, Department of Mathematics, Univ. of Arizona.

[7] KIEFER, J. and SCHWARTZ, R. (1965). Admissible Bayes character of T^2-, R^2- and other fully invariant tests for classical multivariate normal problems. *Ann. Math. Statist.* **36** 747–770.

[8] KSHIRSAGAR, A. M. (1961). Noncentral multivariate Beta distribution. *Ann. Math. Statist.* **32** 104–110.

[9] PEARSON, E. S. and WILKS, S. S. (1933). Methods of analysis appropriate for k samples of two variables. *Biometrika* **25** 353–378.

[10] PILLAI, K. C. S. and GUPTA, A. K. (1969). On the exact distribution of Wilks' criterion. *Biometrika* **56** 109–118.

[11] POSTEN, H. O. and BARGMANN, R. E. (1964). Power of the likelihood-ratio test of the general linear hypothesis in multivariate analysis. *Biometrika* **51** 467–476.

[12] ROY, J. (1960). Nonnull distribution of the likelihood-ratio in analysis of dispersion. *Sankhya* **22** 289–294.

[13] ROY, J. (1966). Power of the likelihood-ratio test used in analysis of dispersion, *Multivariate Analysis*. Academic Press, New York, 105–127.

[14] SCHATZOFF, M. (1964). Exact distribution of Wilks' likelihood ratio criterion and sensitivity comparisons with competitive tests. Ph.D. Dissertation, Harvard University.

[15] SCHATZOFF, M. (1966). Exact distribution of Wilks' likelihood-ratio criterion. *Biometrika* **53** 347–358.

[16] WILKS, S. S. (1932). On the independence of k set of normally distributed variables. *Econometrica* **3** 309–326.