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# A NECESSARY CONDITION ON THE INFINITE DIVISIBILITY OF PROBABILITY DISTRIBUTIONS

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For a probability distribution function on the real line, a necessary condition on its infinite divisibility is given which deals with its one-sided asymptotic behavior; the proof is based on properties of characteristic functions which are analytic in the upper (lower) half plane.

**1. Introduction.** In a recent note (1970), the author has generalized the familiar fact that a non-degenerate infinitely divisible (i.d.) distribution function (df) cannot be finite, i.e. cannot have its entire mass concentrated on a finite interval. Properties of characteristic functions (ch. f.'s) and of entire functions were used to derive asymptotic conditions on the tail $T(x) = 1-F(x)+F(-x)$ which force the i.d. df $F$ to be either normal or degenerate. (An elegant generalization of these results has been obtained by R. A. Horn (1971), whose demonstration is based on quite elementary methods.)

As the examples of the Poisson and of the $\Gamma$-distribution show, there exist one-sided (i.e. bounded either to the left or to the right) i.d. df's which are non-degenerate. The purpose of the present note is to investigate the one-sided asymptotic behavior of an i.d. df $F$; it is shown that, although results analogous to those found in the two-sided case are no longer true, a certain necessary condition (Theorem 3.5) on the rate of decrease of $F(-x)$ can still be obtained (a similar result holds of course for $1-F(x)$).

In analogy to the methods used in Ruegg (1970), the present proof is based on ch.f.'s which are analytic at least in the upper half plane; the necessary properties of these functions are given in Section 2. Although the theorems on entire functions used in Ruegg (1970) are no longer applicable, the concepts of "order" and of "form" of a ch.f. "with respect to the upper half plane" permit the presentation of both the results and their proofs in a rather simple and transparent way.

**2. Results on** ch.f.'s **which are analytic in the upper half plane.** Throughout this paper, $F$ will always denote a df and $f$ the corresponding ch.f. Let $\mathfrak{A}^+$ denote the class of all ch.f.'s which can be continued analytically in the upper half plane:

$$f(z) = \int_{-\infty}^{+\infty} e^{izu}\, dF(u), \qquad \operatorname{Im} z \geqq 0;$$

it is well known that this is possible if and only if

$$f(iy) = \int_{-\infty}^{+\infty} e^{-yu}\, dF(u) < \infty$$

for every positive $y$ (see Esseen (1965) Theorem 1 or Ramachandran (1967) Theorem 2.2.1). By $\mathscr{B}^+$, we denote the subclass of $\mathfrak{A}^+$ formed by all those ch.f.'s which are

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not bounded (by one) on $\operatorname{Im} z \geqq 0$, i.e. whose df's do not vanish identically on $(-\infty, 0)$.

For $f \in \mathscr{B}^+$, $f(iy)$ is strictly convex on $[0, +\infty)$, and $f(iy) \to +\infty$ as $y \to +\infty$ (Lukacs (1960) page 136). We denote by $M^+(r, f)$ the maximum of $|f(z)|$ for $|z| \leqq r$ and $\operatorname{Im} z \geqq 0$. It then follows easily from the maximum modulus principle and from a well known maximum property of analytic ch.f.'s (Lukacs (1960) page 134) that the following result holds.

THEOREM 2.1. *If $f \in \mathscr{B}^+$, then there exists $R \geqq 0$ such that $M^+(r, f) = f(ir)$ for all $r \geqq R$.*

From this theorem we obtain immediately the next result.

THEOREM 2.2. *If $f \in \mathscr{B}^+$, then $M^+(r, f) \geqq F(-x)\, e^{rx}$ for every $r \geqq R$ and every positive $x$.*

We define the order of a ch.f $f \in \mathscr{B}^+$ with respect to the upper half plane by

$$\rho^+(f) = \limsup_{r\to+\infty} [\log\log M^+(r,f)/\log r];$$

in case $\rho^+(f) = \infty$, the concept of "form with respect to the upper half plane," given by

$$\lambda^+(f) = \limsup_{r\to+\infty} [\log\log\log M^+(r,f)/\log r],$$

is a useful refinement in measuring the rate of growth of $f$ on $\operatorname{Im} z \geqq 0$.

THEOREM 2.3. *If $f$, $g \in \mathscr{B}^+$, then $\rho^+(f\,g) = \max\,[\rho^+(f), \rho^+(g)]$.*

PROOF. Denoting the last expression by $\rho$, we have

$$\begin{aligned} M^+(r,fg) &\leqq M^+(r,f)M^+(r,g) \\ &\leqq \exp\left[r^{\rho^+(f)+\varepsilon/2} + r^{\rho^+(g)+\varepsilon/2}\right] \\ &\leqq \exp\left[2r^{\rho+\varepsilon/2}\right] \leqq \exp\left[r^{\rho+\varepsilon}\right] \end{aligned}$$

for sufficiently large $r$, i.e. $\rho^+(f\,g) \leqq \rho$, and the result follows at once.

THEOREM 2.4. *If $f \in \mathscr{B}^+$, then $\rho^+(f) \geqq 1$.*

This is an immediate consequence of Theorem 2.2.

The following two theorems are one-sided versions of results obtained by B. Ramachandran (1962), who investigated the relations existing between the asymptotic behavior of the tail $T(x) = 1 - F(x) + F(-x)$ of a df $F$ and the order and form of the corresponding (entire) ch.f.

THEOREM 2.5. *Let $g(x) = \{\log\log[1/F(-x)]\}/\log x$ and $\alpha > 0$. Then* $\liminf_{x\to+\infty} g(x)$ *is* (i) $>$, (ii) $=$ *or* (iii) $< 1+\alpha$ *if and only if $f \in \mathscr{B}^+$ and $\rho^+(f)$ is* (i) $<$, (ii) $=$ *or* (iii) $> 1+1/\alpha$.

REMARK. The conditions (i) and (ii) used in Theorem 7.2.4. (Lukacs (1960), page 142) are sufficient but not necessary for $\liminf_{x\to+\infty} g(x) \geqq 1+\alpha$ and $\liminf_{x\to+\infty} g(x) \leqq 1+\alpha$ to hold.

THEOREM 2.6. *Let* $h(x) = \{\log\log[1/F(-x)] - \log x\}/\log\log x$ *and* $\alpha > 0$. *Then* $\liminf_{x\to+\infty} h(x)$ *is* (i) $>$, (ii) $=$ *or* (iii) $< 1/\alpha$ *if and only if* $f \in \mathscr{B}^+$ *and* (i) $\rho^+(f) < \infty$ *or* $\rho^+(f) = \infty$ *and* $\lambda^+(f) < \alpha$, (ii) $\rho^+(f) = \infty$ *and* $\lambda^+(f) = \alpha$ *or* (iii) $\rho^+(f) = \infty$ *and* $\lambda^+(f) > \alpha$.

The proofs given by Ramachandran for his Theorems 6.1 and 9.3 can be modified in a straightforward way to cover the one-sided case. (Incidentally, the same is true for his other results, but we will not need these in the present context.)

**3. One-sided asymptotic behavior of** i.d. df's. It is well known that the family of i.d. probability laws coincides with those defined by the Lévy canonical representation:

$$\log f(x) = iax - \sigma^2 x^2/2 + \int_{u\neq 0} [e^{iux} - 1 - iux/(1+u^2)]\, dL(u)$$

where $a$, $\sigma$ are real constants and $L$ is defined on the real line, except at the origin, is nondecreasing on $(-\infty, 0)$ and on $(0, +\infty)$ and satisfies $L(-\infty) = L(+\infty) = 0$ and $\int_{0<|u|<\varepsilon} u^2\, dL(u) < \infty$ for some positive $\varepsilon$. For our purpose, the following modification of Lévy's formula, based on the identity $u/(1+u^2) = u - u^3/(1+u^2)$, will be useful:

$$\log f(x) = P_\varepsilon(x) + A_\varepsilon(x) + B_\varepsilon(x) + C_\varepsilon(x)$$

where $\varepsilon$ is an arbitrary positive constant and

$$\begin{aligned} P_\varepsilon(x) &= -\int_{|u|\geqq\varepsilon} dL(u) + ix[a + \int_{0<|u|<\varepsilon} u^3/(1+u^2)\, dL(u) \\ &\quad - \int_{|u|\geqq\varepsilon} u/(1+u^2)\, dL(u)] - \sigma^2 x^2/2, \\ A_\varepsilon(x) &= \int_{0<|u|<\varepsilon} (e^{iux} - 1 - iux)\, dL(u), \\ B_\varepsilon(x) &= \int_{u\leqq-\varepsilon} {}^{iux}\, dL(u), \\ C_\varepsilon(x) &= \int_{u\geqq\varepsilon} e^{iux}\, dL(u). \end{aligned}$$

For the proof of the following theorem, we refer to Esseen (1965) (Theorem 2) or to Ramachandran (1967) (Theorem 2.4.2).

THEOREM 3.1. *If* $f$ *is* i.d. *and an element of* $\mathfrak{A}^+$, *then both the original and the modified Lévy representation remain valid if we replace* $x$ *by* $z = x + iy$ $(y \geqq 0)$.

REMARK 1. Provided that $L$ does not vanish identically on the respective intervals, $A_\varepsilon(z)$ is an entire function of exponential type (its second derivative is up to a constant factor the ch.f. of a finite df), whereas $B_\varepsilon(z)$ (if $f \in \mathscr{B}^+$) and $C_\varepsilon(z)$ are, up to positive constant factors, ch.f.'s which are elements respectively of $\mathscr{B}^+$ and of $\mathfrak{A}^+$.

REMARK 2. It is easy to verify that $e^{A_\varepsilon}$ and (if $f \in \mathscr{B}^+$) $e^{B_\varepsilon}$ are (again up to constant factors) i.d. ch.f.'s which are in $\mathscr{B}^+$ (at least if the exponents do not vanish identically), whereas $e^{C_\varepsilon}$ is in $\mathfrak{A}^+$ but not in $\mathscr{B}^+$, and $e^{P_\varepsilon}$ is in $\mathfrak{A}^+$ and possibly in $\mathscr{B}^+$. For $e^{A_\varepsilon}$ for instance, this results from the fact that

$$A_\varepsilon(z) = \int_{0<|u|<\varepsilon} [e^{iuz} - 1 - iuz/(1+u^2)]\, dL(u) - iz \int_{0<|u|<\varepsilon} u^3/(1+u^2)\, dL(u),$$

both terms being logarithms of i.d. ch.f.'s. It follows in particular that Theorem 2.3 applies if we decompose $f \in \mathscr{B}^+$ into factors of the above type, provided we neglect all those factors which are in $\mathfrak{A}^+$ but not in $\mathscr{B}^+$ (for these functions, $M^+(r)$ converges to a finite limit as $r \to +\infty$).

THEOREM 3.2. *Let $f$ be* i.d. *Then $f \in \mathfrak{A}^+$ if and only if there exists a positive $\varepsilon$ such that $B_\varepsilon(iy) < \infty$ for all positive $y$.*

PROOF. Since $A_\varepsilon$ is an entire function and $C_\varepsilon$ is analytic in $\text{Im } z > 0$, it follows that the two conditions $f \in \mathfrak{A}^+$ and $B_\varepsilon \in \mathscr{B}^+$ (up to a constant factor) imply each other, from which the theorem results.

THEOREM 3.3. *Let $f$ be an* i.d. *element of $\mathscr{B}^+$. Then $\rho^+(f) < \infty$ if and only if $L = 0$ on $(-\infty, 0)$. Moreover, we have then $\rho^+(f) \leqq 2$.*

PROOF. If $L = 0$ on $(-\infty, 0)$, then all factors of $f$ as given by the modified Lévy formula are of order less than or equal to two with respect to the upper half plane; for $e^{A_\varepsilon}$, this follows from the inequality $e^{-x}-1+x \leqq x^2/2$ $(x \geqq 0)$. It then results from Theorem 2.3 and Remark 2 above that $\rho^+(f) \leqq 2$.

Conversely, if $L \neq 0$ on $(-\infty, 0)$, then there exists some positive $\varepsilon$ such that $B_\varepsilon \neq 0$, i.e. $B_\varepsilon \in \mathscr{B}^+$ (up to a constant factor), and it follows that $\rho^+(B_\varepsilon) \geqq 1$ (Theorem 2.4), i.e. $\rho^+(e^{B_\varepsilon}) = \infty$ and therefore $\rho^+(f) = \infty$ (Theorem 2.3).

REMARK 1. The following example shows that, in contradiction to the two-sided case, there exist i.d. ch.f.'s $f$ with $\rho^+(f) = \alpha$ for every $\alpha$ between one and two. If $\sigma = 0$, and if we choose $L$ such that $dL(u) = u^{-1-\alpha}du$ on $(0, \varepsilon)$ where $1 < \alpha < 2$, then a simple change of variable shows that

$$A_\varepsilon(ir) = r^\alpha \int_0^{\varepsilon r} (e^{-v}-1+v)v^{-1-\alpha}\, dv,$$

and since the integral converges for $r \to +\infty$, we have

$$\rho^+(f) = \rho^+(e^{A_\varepsilon}) = \limsup_{r\to+\infty} [\log A_\varepsilon(ir)/\log r]$$
$$= \alpha.$$

REMARK 2. Under the assumption that $\sigma = 0$ and $L = 0$ on $(-\infty, 0)$, it can be shown (see Baxter and Shapiro (1960) and Esseen (1965)), that the additional condition requiring that $\int_{0<u<\varepsilon} u\,dL(u) < \infty$ for some positive $\varepsilon$ is necessary and sufficient for $f$ to be "of exponential type with respect to the upper half plane" or, equivalently, for $F$ to be bounded to the left.

THEOREM 3.4. *Let $f$ be an* i.d. *element of $\mathscr{B}^+$. Then $\rho^+(f) = \infty$ implies $\lambda^+(f) \geqq 1$.*

PROOF. By Theorem 3.3, $L \neq 0$ on $(-\infty, 0)$, i.e. there exists a positive $\varepsilon$ such that $B_\varepsilon \in \mathscr{B}^+$ (up to a constant factor), and therefore (Theorem 2.4) $\rho^+(B_\varepsilon) \geqq 1$. But then it follows easily from Theorem 2.1 that $\lambda^+(e^{B_\varepsilon}) \geqq 1$ and therefore $\lambda^+(f) \geqq 1$.

REMARK. The existence of i.d. ch.f.'s $f$ with $\rho^+(f) = \infty$ and $\lambda^+(f) = \alpha$ for every $\alpha \geqq 1$ is an immediate consequence of the relation $\lambda^+(e^{B_\varepsilon}) = \rho^+(B_\varepsilon)$.

THEOREM 3.5. *Let the* df *F be non-bounded to the left and assume that*

(i) $\liminf_{x\to+\infty} \{\log\log[1/F(-x)]\}/\log x < 2$ *and*

(ii) $\liminf_{x\to+\infty} \{\log\log[1/F(-x)]-\log x\}/\log\log x > 1$.

*Then F cannot be* i.d.

PROOF. By Theorem 2.5, (i) implies that $F$ has a ch.f. $f \in \mathscr{B}^+$ such that $\rho^+(f) > 2$, whereas by (ii) and Theorem 2.6, we have either $\rho^+(f) < \infty$ or $\rho^+(f) = \infty$ and $\lambda^+(f) < 1$. But in both these cases, it follows respectively from Theorem 3.3 and Theorem 3.4 that $F$ cannot be i.d.

REMARK 1. A consequence of this result is that $F$ cannot be i.d. if $F(-x) \sim a \exp[-bx^\alpha]$ as $x \to +\infty$ with $a > 0$, $b > 0$ and $1 < \alpha < 2$.

REMARK 2. The remarks following Theorems 3.3 and 3.4 show that the assumptions made in Theorem 3.5 on the two lower limits cannot be weakened.

REMARK 3. If we define $k(x) = \{\log\log[1/L(-x)]\}/\log x$ and $h(x)$ as in Theorem 2.6, we can use Theorems 2.5 and 2.6 to obtain the following relations between the one-sided asymptotic behaviors of $L$ and $F$:

(i) If $L$ is bounded to the left, without vanishing identically on $(-\infty, 0)$, then $\liminf_{x\to+\infty} h(x) = 1$.

(ii) If $L$ is not bounded to the left, then we have (for every positive $\alpha$) $\liminf_{x\to+\infty} k(x) = 1+\alpha$ if and only if $\liminf_{x\to+\infty} h(x) = \alpha/(1+\alpha)$.

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