The Annals of Applied Probability
1991, Vol. 1, No. 1, 118–125

ON THE PROPERTIES OF A TREE-STRUCTURED SERVER PROCESS

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Let X_0 be a nonnegative integer-valued random variable and let an independent copy of X_0 be assigned to each leaf of a binary tree of depth k. If X_0 and X_0' are adjacent leaves, let $X_1 = (X_0 - 1)^+ + (X_0' - 1)^+$ be assigned to the parent node. In general, if X_j and X_j' are assigned to adjacent nodes at level $j = 0, \dots, k-1$, then X_j and X_j' are, in turn, independent and the value assigned to their parent node is then $X_{j+1} = (X_j - 1)^+ + (X_j' - 1)^+$. We ask what is the behavior of X_k as $k \to \infty$. We give sufficient conditions for $X_k \to \infty$ and for $X_k \to 0$ and ask whether these are the only nontrivial possibilities. The problem is of interest because it asks for the asymptotics of a nonlinear transform which has an expansive term (the + in the sense of addition) and a contractive term (the + in the sense of positive part).

1. Introduction. In this paper we give some partial results about the following process: Let $\{X_0(i),\ i = 0, 1, \dots\}$ be independent, identically distributed, nonnegative integer variables. For $j = 1, 2, \dots$, we define recursively,

$$X_j(i) = \big(X_{j-1}(2i) - 1\big)^+ + \big(X_{j-1}(2i+1) - 1\big)^+, \qquad i = 0, 1, \dots .$$

We may think of the $\{X_0(\cdot)\}$ as being the number of customers associated with the leaves of a complete binary tree. At each epoch, the number of customers at each leaf is diminished by one, if it is positive. The customers remaining at each pair of leaves are handed down and collected at the parent node, which now becomes a leaf. We want to study the behavior of $X_k = X_k(0)$ as $k \to \infty$. The model originally arose as a crude model of the Aloha network as well as of a resource allocation model slightly related to that of [2] and became interesting in its own right.

One obvious question is to determine those probability laws governing X_0 for which X_k tends to zero as $k \to \infty$. In general, we might ask what types of limiting behavior can occur.

In particular, consider the case where X_0 is Poisson with mean λ. While this arrangement bears a superficial resemblance to the tree-structured contention resolution algorithms introduced by Capetenakis [1], there appears to be no real connection. In any event, for Poisson variables we are interested in that value of λ below which $X_k \to 0$. We find, using the results to follow and

Received January 1990; revised March 1990.
AMS 1980 *subject classification*. 60K99.
Key words and phrases. Aloha, Poisson tree, nonlinear recurrence.

extensive numerical calculations, that for $\lambda \leq 0.999$, $X_k \to 0$ and for $\lambda \geq 1.001$, $X_k \to \infty$, but have been unable to prove or disprove the tempting conjecture that the critical value of λ is 1.

In the sequel, we prove three results concerning the limiting behavior of X_k.

THEOREM 1. *If the probability distribution of X_k approaches a limit X_∞, then either*

(i) $\Pr[X_\infty = 0] = 1$, *i.e.*, $X_\infty \equiv 0$,
(ii) $\Pr[X_\infty = 2] = 1$, *i.e.*, $X_\infty \equiv 2$, *or*
(iii) $\Pr[X_\infty < t] = 0$ *for all* t, *i.e.*, $X_\infty \equiv \infty$.

Furthermore, case (ii) only occurs if for all $k = 0, 1, \dots,$ $\Pr[X_k \equiv 2] \equiv 1$. Now for any j, define

$$g_j(\alpha) = E[\alpha^{X_j}],$$

where X_j is a generic jth generation variable. We shall prove:

THEOREM 2. *If for some $\alpha > 2$ and some j, $g_j(\alpha)$ satisfies*

$$g_j(\alpha) < (\alpha - 1)^2,$$

then

$$X_k \to 0 \quad a.s.$$

On the other hand:

THEOREM 3. *If for some j, the mean of X_j satisfies*

$$E[X_j] > 2,$$

then X_k blows up, that is,

$$X_k \to \infty \quad a.s.$$

The corollary to Theorem 3 is a form parallel to that of Theorem 2:

COROLLARY 3A. *If for some $\alpha < 1$ and some j, $g_j(\alpha)$ satisfies*

$$g_j(\alpha) < \alpha^2,$$

then X_k blows up.

We show also that there is a gap between the hypotheses of Theorems 2 and 3, that is, there are X_0 satisfying neither hypothesis. We nevertheless think it may be true that one of $X_k \to 0$, $X_k \to \infty$ or $X_k \equiv 2$ always holds.

PROOFS OF THE RESULTS. We first prove Theorem 1. Note that if X_k approaches a limit, then for all $0 \leq \alpha \leq 1$, $g_k(\alpha)$ approaches a limit $g_\infty(\alpha)$.

Now, since

$$X_{j+1}(i) = \left(X_j(2i) - 1\right)^+ + \left(X_j(2i+1) - 1\right)^+,$$

then for

$$g_j(\alpha) = \sum_0^\infty \Pr\left[X_j = i\right]\alpha^i,$$

we get the recurrence

$$g_{j+1}(\alpha) = \left[\frac{g_j(\alpha) + (\alpha - 1)p_j}{\alpha}\right]^2, \tag{1}$$

where

$$p_j \triangleq \Pr\left[X_j = 0\right].$$

Now if $g_j(\alpha) \to g_\infty(\alpha)$ for $0 \le \alpha \le 1$, then $p_j \to p_\infty$ and

$$g_\infty(\alpha) = \left[\frac{g_\infty(\alpha) + (\alpha - 1)p_\infty}{\alpha}\right]^2.$$

This quadratic equation can be solved to yield

$$g_\infty(\alpha) = (1 - \alpha)p_\infty + \frac{\alpha^2}{2} \pm \frac{1}{2}\alpha\sqrt{\alpha^2 - 4p_\infty\alpha + 4p_\infty}\,. \tag{2}$$

If $p_\infty = 0$, this has the solutions

$$g_\infty(\alpha) = 0, \qquad \alpha^2,$$

corresponding to cases (ii) and (iii). If $p_\infty = 1$, then (2) has solutions

$$g_\infty(\alpha) = 1, \qquad (1 - \alpha)^2.$$

The former corresponds to case (i) and the latter is spurious (i.e., not increasing and so clearly not a probability generating function). We can further see that case (ii) cannot be approached, but can only arise if $\Pr[X_0(\cdot) = 2] = 1$. To see this, let $p_{ji} = P(X_j = i)$ and thinking of j as fixed, set

$$\alpha_i = \Pr\left[X_j = i\right]$$

and

$$\beta_i = \Pr\left[X_{j+1} = i\right].$$

Since rule (1) gives

$$\beta_2 = 2(\alpha_0 + \alpha_1)\alpha_3 + \alpha_2^2$$

and since $(\alpha_0 + \alpha_1) + \alpha_3 \le 1 - \alpha_2$, we have

$$\beta_2 \le 2\left(\frac{1 - \alpha_2}{2}\right)^2 + \alpha_2^2 < \alpha_2 \quad \text{if } \tfrac{1}{3} < \alpha_2 < 1.$$

Thus if p_{j2} satisfies $\frac{1}{3} < p_{j2} < 1$, then $p_{j+1,2} < p_{j2}$, so that p_{j2} cannot approach 1 from below.

Note also that a solution to equation (2) for which $0 < p_\infty < 1$ cannot be a generating function. The reason for this is that a generating function, viewed as a function on the complex plane, must have its innermost singularity on the real line (see [3]; Section 7.2). However, since the solution of equation (2) has singularities at

$$\alpha = 2p_\infty \pm 2\sqrt{p_\infty(p_\infty - 1)}\,,$$

these singularities are complex unless $p_\infty = 0$ or $p_\infty = 1$.

Theorem 1 is proven.

Now, to prove Theorem 2, assume that for some j and $\alpha > 2$,

$$g_j(\alpha) = \theta(\alpha - 1)^2,$$

where, by the hypothesis of Theorem 2 and the fact that $g_j(\alpha) > 1$ for $\alpha > 1$,

$$\frac{1}{(\alpha - 1)^2} < \theta < 1. \tag{3}$$

Using (1), we get

$$\begin{aligned} g_{j+1}(\alpha) &= \left[\frac{g_j(\alpha) + p_j(\alpha - 1)}{\alpha}\right]^2 \\ &= \left[\frac{\theta(\alpha - 1)^2 + p_j(\alpha - 1)}{\alpha}\right]^2 \qquad (4) \\ &= (\alpha - 1)^2\left[\frac{\theta(\alpha - 1) + p_j}{\alpha}\right]^2, \end{aligned}$$

or since $p_j < 1$,

$$\frac{g_{j+1}(\alpha)}{g_j(\alpha)} < \frac{1}{\theta}\left[\frac{\theta(\alpha - 1) + 1}{\alpha}\right]^2.$$

The right-hand side is strictly less than 1 for all θ satisfying (3), so that $g_j(\alpha) \downarrow 1$. Now if $g_j(\alpha) = 1 + \delta$, then from (4) since $p_j \le 1$,

$$\begin{aligned} g_{j+1}(\alpha) &\le \left[\frac{1 + \delta + \alpha - 1}{\alpha}\right]^2 \\ &= \left(1 + \frac{\delta}{\alpha}\right)^2 \\ &= 1 + \frac{2}{\alpha}\delta + \frac{\delta^2}{\alpha^2} \\ &= 1 + \delta\left[\frac{2}{\alpha} + \frac{\delta}{\alpha^2}\right]. \end{aligned}$$

Now since $\alpha > 2$, if δ is sufficiently small,

$$g_{j+1}(\alpha) < 1 + \rho\delta$$

for some

$$\rho < 1,$$

hence for all $j > j'$,

$$g_j(\alpha) < 1 + \rho^{j-j'}\delta.$$

Now

$$g_j(\alpha) \ge \Pr[X_j = 0] + \alpha \Pr[X_j > 0] = p_j + \alpha(1 - p_j),$$
$$1 + \rho^{j-j'}\delta \ge p_j + \alpha(1 - p_j) \quad \text{for all } j > j'$$

or

$$(\alpha - 1)(1 - p_j) \le \rho^{j-j'}\delta,$$
$$(\alpha - 1)\sum_{j'}^{\infty} (1 - p_j) \le \delta \sum_{j'}^{\infty} \rho^{j-j'} = \frac{\delta}{1 - \rho} < \infty,$$

so by the Borel–Cantelli lemma, $\Pr[X_j > 0] \to 0$ a.s.

To prove Theorem 3, suppose that

$$E[X_j] = 2 + \delta.$$

Since

$$X_{j+1}(i) = (X_j(2i) - 1)^+ + (X_j(2i + 1) - 1)^+ \ge X_j(2i) + X_j(2i + 1) - 2,$$
$$E[X_{j+1}] \ge 2E[X_j] - 2 = 2 + 2\delta.$$

Thus

$$E[X_{j'}] \ge 2 + 2^{(j'-j)}\delta \quad \text{for all } j' \ge j.$$

Now

$$\mathrm{Var}(X_{j+1}) = 2\,\mathrm{Var}\big[(X_j - 1)^+\big].$$

But

$$\mathrm{Var}\big[(X_j - 1)^+\big] = \mathrm{Var}(X_j) + p_j(1 - p_j - 2EX_j).$$

Since

$$EX_j > 2,$$

the last term is less than 0, so

$$\mathrm{Var}(X_{j+1}) < 2\,\mathrm{Var}(X_j).$$

Hence

$$\mathrm{Var}(X_{j'}) < 2^{j'-j}\,\mathrm{Var}(X_j).$$

Since the mean of X_j grows, we can apply the Chebyshev inequality to show that for any finite t for sufficiently large j',

$$\Pr[X_{j'} < t] \le k\left(\tfrac{1}{2}\right)^{j'},$$

hence goes to zero. A Borel–Cantelli argument shows $X_j \to \infty$ strongly.

To show the equivalence of Theorem 3 and Corollary 3a, note that $g_j(\alpha)$ is continuous, at least on the unit disc. Then since

$$EX_j = \frac{d}{d\alpha} g_j(\alpha)|_{\alpha=1},$$

the function $g_j(\alpha) - \alpha^2$ is continuous on [0, 1], is zero at $\alpha = 1$ and increasing at that point. Hence it must be negative for some interval including the point $\alpha = 1$. Therefore

$$E[X_j] > 2 \Rightarrow g_j(\alpha) < \alpha^2$$

for some α sufficiently near 1.

As for the reverse implication, consider the function

$$h_j(\alpha) = \frac{g_j(\alpha)}{\alpha}.$$

Clearly $h_j(1) = 1$. Now

$$h'_j(\alpha) = \frac{g'_j(\alpha)}{\alpha} - \frac{g_j(\alpha)}{\alpha^2}.$$

Also

$$h''_j(\alpha) = \frac{d^2}{d\alpha^2} \sum_0^\infty p_j \alpha^{j-1}$$

$$(5) \qquad = \frac{d}{d\alpha} \sum_0^\infty (j-1) p_j \alpha^{j-2}$$

$$= \sum_0^\infty (j-1)(j-2) p_j \alpha^{j-3}.$$

Since all the terms in (5) are nonnegative, $h_j(\alpha)$ is convex. If

$$EX_j < 2,$$

then

$$h'_j(1) = \frac{g'_j(1)}{1} - \frac{g_j(1)}{1} < 1.$$

Thus $h_j(\alpha)$, since it is convex must lie above the line α and we have shown

$$E[X_j] \leq 2 \Rightarrow g_j(\alpha) \geq \alpha^2$$

for all $\alpha < 1$. Therefore Theorem 3 is equivalent to Corollary 3a. □

2. Conclusion. We have proved the theorems described in the introduction. In particular, Theorems 2 and 3 applied to a Poisson starting distribution yield that

$$\lambda < 0.999 \Rightarrow X_k \to 0$$

and

$$\lambda > 1.001 \Rightarrow X_k \to \infty.$$

The lower number follows from the fact that $g_{55}(2.03) = 1.059 < (2.03 - 1)^2 = 1.0609$. The upper value is implied by the fact that $E[X_{170}] = 2.311$. The region of uncertainty can be narrowed with more numerical work, but the computations become onerous because of difficulty of dealing with roundoff errors. Our computations were carried out using the multiprecision facilities of the Maple symbolic computation language.

It remains to be seen whether any behavior other than that described in Theorem 1 is possible.

We finally prove there is a gap between the hypotheses of Theorems 2 and 3, that is, there is an X_0 which satisfies neither hypothesis. Nevertheless, it seems likely that the conclusion of Theorem 1 always holds, that is, either $X_k \to 0$, $X_k \to \infty$ or the trivial case $X_k \equiv 2$ holds. We have been unable to prove this.

To show an example it seems worthwhile to slightly change notation by replacing for each $k \geq 0$,

$$Y_k = (X_k - 1)^+,$$

so that the recurrence becomes

$$Y_{k+1} = (Y_k + Y_k' - 1)^+, \qquad k \geq 0.$$

Although X_k represents the original notation, calculations become simpler in the Y_k notation. Theorem 2 and a slightly stronger version of Theorem 3 become in the Y notation:

THEOREM 2′. *If* $E\alpha^{Y_k} \leq \alpha - 1$ *for some* $\alpha > 2$ *and* $k \geq 0$, *then*

$$Y_k \to 0 \quad a.s.$$

THEOREM 3′. *If* $EY_k \geq 1$ *for some* k *and if* $Y_0 \not\equiv 1$, *then*

$$Y_k \to \infty \quad a.s.$$

We will give an example of Y_0, where $E\alpha^{Y_k} > \alpha - 1$ for all k and $\alpha > 0$ but $EY_k < 1$ for all k. To do this, fix $0 < \eta < 1$ and let $Y_0(\eta)$ take values 0 and 2 only with probabilities

$$P(Y_0(\eta) = 2) = \eta,$$
$$P(Y_0(\eta) = 0) = 1 - \eta.$$

We will show that for some η, $Y_0(\eta)$ lies in the gap.

Let η_n be that value of η for which $E\alpha^{Y_n(\eta)}$ is tangent to the line $\alpha - 1$, that is, there is a value $\alpha = \alpha_n$ for which

$$E\alpha_n^{Y_n(\eta_n)} = \alpha_n - 1,$$
$$\frac{d}{d\alpha} E\alpha^{Y_n(\eta_n)} = 1 \quad \text{at } \alpha = \alpha_n.$$

It is easy to verify since $E\alpha^{Y_n(\eta)}$ increases in η for $\alpha > 1$, that $\alpha_n \downarrow$ in n and $\eta_n \uparrow$ in n. Denote $\eta_\infty = \lim \eta_n$.

For each fixed n, $Y_k(\eta_n) \to 0$ as $k \to \infty$ because of Theorem 2′ so that by Theorem 3′,

$$EY_k(\eta_n) < 1 \quad \text{for all } k \geq 0, n \geq 0. \tag{6}$$

Since $Y_k(\eta_n) \to Y_k(\eta_\infty)$, we must have from (6),

$$EY_k(\eta_\infty) \leq 1 \quad \text{for all } k \geq 0.$$

It follows that $Y_0(\eta_\infty)$ satisfies neither the hypothesis of Theorem 2′ nor 3′ and $X_0 = Y_0(\eta_\infty) + 1$ satisfies neither the hypothesis of Theorem 2 nor of Theorem 3.

Nevertheless it seems possible that there is no Y_0 for which Y_k oscillates, that is, neither $Y_k \to 0$ nor $Y_k \to \infty$ (except for $Y_0 \equiv 1$). On the other hand, if there is such an example it probably has the following property:

$$EY_k < 1 \quad \text{for all } k, \tag{7}$$

but if $Y_0' > Y_0$ in the stochastic sense, then

$$Y_k' \to \infty.$$

To see that there are Y_0's with this property, suppose $Y_0^{(0)}$ has $P(Y_0^{(0)} > 1) > 0$ and (7) holds for $Y_0 = Y_0^{(0)}$. Let $P(Y_0^{(1)} = j) = P(Y_0^{(0)} = j)$ for $j > 1$ and let $P(Y_0^{(1)} = 1)$ be as large as possible subject to (7) holding for $Y_0 = Y_0^{(1)}$. If $Y_0^{(n)}$ has been defined, let $P(Y_0^{(n+1)} = j) = P(Y_0^{(n)} = j)$ for $j > n + 1$ and for $j < n$ and let $P(Y_0^{(n+1)} = n + 1)$ be as large as possible subject to (7) holding for $Y_0 = Y_0^{(n+1)}$. Then we have $Y_0^{(n)}$ stochastically increases in n say to $Y_0^{(\infty)}$. Since

$$EY_k^{(n)} < 1$$

for all k and n, we must have

$$EY_k^{(\infty)} \leq 1,$$

but since $P(Y_0^{(0)} > 1) > 0$, we must have

$$EY_k^{(\infty)} < 1 \quad \text{for all } k.$$

It follows that if $Y_0' > Y_0^{(\infty)}$, then $Y_0' > Y_0^{(k)}$ for some k and $Y_k' \to \infty$.

Acknowledgment. The authors thank Mireille Regnier for very fruitful discussions.

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