Electronic Journal of Probability

Electron. J. Probab. **18** (2013), no. 57, 1–15.
ISSN: 1083-6489 DOI: 10.1214/EJP.v18-2807

Uniqueness in Law of the stochastic convolution process driven by Lévy noise*

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Abstract

We will give a proof of the following fact. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$, $\tilde{\eta}_1$ and $\tilde{\eta}_2$, ξ_1 and ξ_2 are two examples of filtered probability spaces, time homogeneous compensated Poisson random measures, and progressively measurable Banach space valued processes such that the laws on $L^p([0,T], L^p(Z,\nu;E)) \times \mathcal{M}_I([0,T]\times Z)$ of the pairs (ξ_1, η_1) and (ξ_2, η_2) are equal, and u_1 and u_2 are the corresponding stochastic convolution processes, then the laws on $(\mathbb{D}([0,T];X) \cap L^p([0,T];B)) \times L^p([0,T], L^p(Z,\nu;E)) \times \mathcal{M}_I([0,T]\times Z)$, where $B \subset E \subset X$, of the triples (u_i, ξ_i, η_i), $i = 1, 2$, are equal as well. By $\mathbb{D}([0,T];X)$ we denote the Skorokhod space of X-valued processes.

Keywords: Poisson random measure, stochastic convolution process, uniqueness in law, stochastic partial differential equations..
AMS MSC 2010: Primary 60H15; Secondary 60G57..
Submitted to EJP on June 10, 2010, final version accepted on May 2, 2013.

1 Introduction

A solution to a stochastic partial differential equation (SPDE) driven by a martingale can be represented as a stochastic convolution process with respect to that martingale. If the coefficients of this SPDE are globally Lipschitz continuous, the solution can be found by an application of the Banach Fixed Point Theorem, see for instance Da Prato and Zabczyk [13], Brzeźniak [5], Hausenblas [16] or St. Loubert Bié [2]. However, if the coefficients are only continuous, a typical approach to proof the existence of a solution is via the use the Skorokhod Theorem on representation of a weakly convergent sequences of measures by a.s. convergent random variables. But then the original probability space becomes lost and the solution is defined on a new probability space. For this reason it is important to know whether the joint law of the triplet consisting of the driving martingale, the integrand and the stochastic convolution process, remains

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the same. If the driving martingale is continuous one can use various versions of the martingale representation theorem to justify this equality, see for instance Da Prato and Zabczyk [13, Section 8] and the references therein. If the driving martingale is purely discontinuous we have been unable to find an appropriate embedding theorem to proof the required equality. The aim of the present paper is to fill this gap in the existing literature. To be precise, we will show that changing the underlying probability space without changing the laws of the integrand and the driving martingale does not lead to a change of the law of the corresponding triplet (consisting of the driving martingale, the integrand and the stochastic convolution process).

We encountered this sort of a difficulty in our recent paper [9] (see also an older version [9]) in which we studied the existence of a solution to a stochastic reaction diffusion equation, with only continuous coefficient, driven by a purely discontinuous Lévy process. It turned out that the use of the result presented in this paper was essential. We believe that this result is interesting on its own (as well as it will be applicable in other situations) and, hence, we have decided to publish it separately.

Let us stress here that the difficulty lies in the problem of the existence of a càdlàg modification of the stochastic convolution processes. Let us notice that this is not always true as has been recently shown by Brzeźniak et al [6]. However, in paper [12] the authors give certain positive answer to this problem in the case when a martingale type Banach space satisfies some additional rather restrictive assumption. The crucial point is the proof of appropriate maximal inequalities for stochastic convolution. In the general case, this problem seems to be open.

We finish this introduction with a brief summary of our results. But for that aim we need to introduce the basic notation we will be using throughout the whole paper.

Notation: By $\mathbb{N}$ we denote the set of natural numbers (including 0) and by $\bar{\mathbb{N}}$ we denote the set $\mathbb{N}\cup\{\infty\}$. For a measurable space $(Z,\mathcal{Z})$ by $M_I(Z)$ we denote the family of all $\bar{\mathbb{N}}$-valued measures on $(Z,\mathcal{Z})$ and by $\mathcal{M}_I(Z)$ the σ-field on $M_I(Z)$ generated by functions $i_B : M_I(Z) \ni \mu \mapsto \mu(B) \in \bar{\mathbb{N}}$, $B \in \mathcal{Z}$. By $M_+(Z)$ we denote the set of all non negative and σ-finite measures on $(Z,\mathcal{Z})$.

By $-A$ we denote an infinitesimal generator of a C_0 semigroup $\big(S(t)\big)_{0\le t<\infty}$ on a Banach space $(E, |\cdot|_E)$. We assume that $p \in (1,2]$ is fixed and that E is a separable martingale type p Banach space, see for instance [11, Appendix A].

We assume that $(Z,\mathcal{Z})$ is a measurable space, $\mathfrak{A} = (\Omega,\mathcal{F},\mathbb{F},\mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{0\le t<\infty}$, is a filtered probability space and $\tilde{\eta}$ is a time homogeneous compensated Poisson random measure on $(Z,\mathcal{Z})$ over $\mathfrak{A}$. Let $\xi : \mathbb{R}_+ \times \Omega \times Z \to E$ be a progressively measurable process such that for every $T > 0$

$$\mathbb{E}\int_0^T \int_Z |\xi(r,z)|_E^p \,\nu(dz)\,dr < \infty,$$

where ν is the intensity measure of $\tilde{\eta}$. We consider the following Itô type SPDE in the space E

$$\left\{ \begin{array}{l} du(t) + Au(t)\,dt = \int_Z \xi(t;z)\,\tilde{\eta}(dt;dz), \quad t > 0, \\ u(0) = 0. \end{array} \right. \tag{1.1}$$

A solution to problem (1.1) can be defined to be the following stochastic convolution process with respect to $\tilde{\eta}$:

$$u(t) := \int_0^t \int_Z S(t-r)\,\xi(r,z)\,\tilde{\eta}(dr;dz), \quad t > 0. \tag{1.2}$$

In general, we can expect that u has a càdlàg modification provided that it is considered as a process in an appropriately chosen Banach space X such that $E \subset X$.

Moreover, by smoothing properties of some types of semigroups u, considered as a process in some smaller Banach space $B \subset E$, has p-integrable paths. So, we formulate our results in the general framework. However, we present also applications and give examples of the spaces X and B.

The aim of this paper is to prove the following result. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$, $\tilde{\eta}_1$ and $\tilde{\eta}_2$, ξ_1 and ξ_2 are two examples of filtered probability spaces, time homogeneous compensated Poisson random measures, and progressively measurable processes such that the laws on $L^p([0,T], L^p(Z,\nu;E)) \times M_I([0,T] \times Z)$ of the pairs (ξ_1, η_1) and (ξ_2, η_2) are equal, and u_1 and u_2 are the corresponding stochastic convolution processes, then the laws on $(\mathbb{D}([0,T];X) \cap L^p(0,T;B) \times L^p([0,T], L^p(Z,\nu;E)) \times M_I([0,T] \times Z))$ of the triplets (u_i, ξ_i, η_i), $i = 1, 2$, are equal as well. Here, by $\mathbb{D}([0,T];X)$ we denote the Skorokhod space of X-valued processes. The case of stochastic integrals, i.e. the stochastic convolution processes with $A = 0$, was studied in [10].

This paper is organised as follows. Section 2 contains some probabilistic preliminaries. The main results are stated in Section 3. In Section 4 we present applications of the general results to stochastic differential equations. Auxiliary results on the Haar projection in the space $L^p([0,T];E)$ and the dyadic projection in the Skorokhod space are contained in Appendices B and C, respectively.

Acknowledgments

The authours would like to thank an anonymous referee for reading the paper twice. His comment, questions and suggestions have lead to a substantial improvement of manner of presentation of the paper.

2 Preliminaries

Let us first recall a definition of time homogeneous Poisson random measures over a filtered probability space.

Definition 2.1. *Let $(Z, \mathcal{Z})$ be a measurable space, $\nu \in M_+(Z)$ and let $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space with the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t < \infty}$. A time homogeneous Poisson random measure η on $(Z, \mathcal{Z})$ with intensity measure ν over $\mathfrak{A}$ is a measurable map*

$$\eta : (\Omega, \mathcal{F}) \to (M_I(\mathbb{R}_+ \times Z), \mathcal{M}_I(\mathbb{R}_+ \times Z))$$

satisfying the following conditions

(i) for all $B \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Z}$, $\eta(B) := i_B \circ \eta : \Omega \to \overline{\mathbb{N}}$ is a Poisson random measure with parameter $\mathbb{E}[\eta(B)]$,

(ii) η is independently scattered, i.e. if the sets $B_j \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{Z}$, $j = 1, ..., n$, are disjoint then the random variables $\eta(B_j)$, $j = 1, ..., n$, are independent,

(iii) for all $U \in \mathcal{Z}$ and $I \in \mathcal{B}(\mathbb{R}_+)$

$$\mathbb{E}[\eta(I \times U)] = \lambda(I)\nu(U),$$

where λ is the Lebesgue measure,

(iv) for all $U \in \mathcal{Z}$ the $\overline{\mathbb{N}}$-valued process $(N(t,U))_{t \ge 0}$ defined by

$$N(t,U) := \eta((0,t] \times U), \qquad t \ge 0,$$

is $\mathbb{F}$-adapted and its increments are independent of the past, i.e. if $t > s \ge 0$, then $N(t,U) - N(s,U) = \eta((s,t] \times U)$ is independent of $\mathcal{F}_s$.

Let η be a time homogeneous Poisson random measure with intensity $\nu \in M_+(Z)$ over $\mathfrak{A}$. We will denote by $\tilde{\eta}$ the *compensated Poisson random measure* defined by $\tilde{\eta} := \eta - \gamma$, where the compensator $\gamma : \mathcal{B}(\mathbb{R}_+) \times \mathcal{Z} \to \mathbb{R}_+$ satisfies in our case the following equality

$$\gamma(I \times A) = \lambda(I)\nu(A), \qquad I \in \mathcal{B}(\mathbb{R}_+), \quad A \in \mathcal{Z}.$$

Let us now state an assumption we will be using throughout the whole paper.

Assumption 2.2. *We suppose that* $\big(E, |\cdot|_E\big)$ *is a separable Banach space of martingale type* p*, where* $1 < p \leq 2$*.*

In [11] there is proven that there exists a unique continuous linear operator I which associates to each progressively measurable process $\xi : \mathbb{R}_+ \times \Omega \times Z \to E$ with

$$\mathbb{E}\Big[\int_0^T \int_Z |\xi(r,x)|_E^p \, \nu(dx)dr\Big] < \infty \tag{2.1}$$

for every $T > 0$, an adapted E-valued càdlàg process

$$I_{\xi,\tilde{\eta}}(t) := \int_0^t \int_Z \xi(r,x)\tilde{\eta}(dr,dx), \quad t \geq 0$$

such that if a process ξ satisfying the above condition (2.1) is a random step process with representation

$$\xi(r,x) = \sum_{j=1}^n 1_{(t_{j-1},t_j]}(r)\, \xi_j(x), \quad x \in Z, \quad r \geq 0,$$

where $\{t_0 = 0 < t_1 < \ldots < t_n < \infty\}$ is a finite partition of $[0,\infty)$ and for all $j \in \{1,\ldots,n\}$, ξ_j is an E-valued $\mathcal{F}_{t_{j-1}}$–measurable p-summable simple random variable, then

$$I_{\xi,\tilde{\eta}}(t) = \sum_{j=1}^n \int_Z \xi_j(x)\, \tilde{\eta}\left((t_{j-1} \wedge t, t_j \wedge t], dx\right), \quad t \geq 0. \tag{2.2}$$

In the recent paper [10] there is shown that this continuous linear operator is unique in a weak sense. In particular, there is proved that for every $T > 0$ the law of the triplet $(I_{\xi,\tilde{\eta}}, \xi, \eta)$ on

$$L^p([0,T],E) \cap \mathbb{D}([0,T],E) \times L^p([0,T], L^p(Z,\nu;E)) \times M_I([0,T] \times Z)$$

depends only on the law of the pair (ξ,η) on the corresponding space. An important point to mention here is that η is a time homogeneous Poisson random measure and that the filtration generated by ξ is nonanticipative with respect to η. In this paper, we will show that a similar results hold also for the stochastic convolution process defined in (1.2).

3 Main results

We assume that $(Z,\mathcal{Z})$ is a measurable space, $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{0 \leq t < \infty}$, is a filtered probability space and $\tilde{\eta}$ is a time homogeneous compensated Poisson random measure on $(Z,\mathcal{Z})$ over $\mathfrak{A}$. Recall that we are working under Assumption 2.2.

3.1 Stochasic convolution with a $\mathcal{C}_0$-semigroup

The following will be a standing assumption in most of this section.

Assumption 3.1. *The operator $-A$ is an infinitesimal generator of a $\mathcal{C}_0$-semigroup $(S(t))_{0\le t<\infty}$ on the space E.*

Let $\xi : \Omega\times\mathbb{R}_+\times Z \to E$ be a progressively measurable process such that for all $T > 0$

$$\mathbb{E}\int_0^T \int_Z |\xi(r,z)|_E^p \,\nu(dz)\,dr < \infty, \tag{3.1}$$

where ν is the intensity measure of $\tilde\eta$. Let us consider the stochastic convolution process $u = u_{\xi,\tilde\eta}$ defined by

$$u(t) = \int_0^t \int_Z S(t-s)\xi(s,z)\,\tilde\eta(ds,dz), \quad t \ge 0. \tag{3.2}$$

Remark 3.2. *Note that $\{u(t), t\ge 0\}$ given in* (3.2) *is well defined process such that for every $T > 0$, $\mathbb{P}$-a.s., $u \in L^p([0,T];E)$.*

Let us recall the following elementary definition.

Definition 3.3. *Let $(X,\mathcal{X})$ be a measurable space. When we say that ξ_1 and ξ_2 have the same law on X (and write $\mathcal{L}aw(\xi_1) = \mathcal{L}aw(\xi_2)$ on X), we mean that ξ_i, $i = 1,2$, are X-valued random variables defined over some probability spaces $(\Omega_i,\mathcal{F}_i,\mathbb{P}_i)$, $i = 1,2$, such that*

$$\mathbb{P}_1\bar{\circ}\,\xi_1 = \mathbb{P}_2\bar{\circ}\,\xi_2,$$

where $\mathbb{P}_i\bar{\circ}\,\xi_i(A) = \mathbb{P}_i(\xi_i^{-1}(A))$, $A \in \mathcal{X}$, $i = 1,2$, is a probability measure on $(X,\mathcal{X})$ called the law of ξ_i.

Remark. (see [21, Theorem II. 3.2]) If X is a separable metric space then $\mathbb{P}_i\bar{\circ}\,\xi_i$, $i = 1,2$, are Radon measures.

Now, we are ready to state our main result. We will formulate it in the most general form possible. In the next section we will present a couple of important applications.

Theorem 3.4. *Under Assumptions 2.2 and 3.1, suppose that for $i = 1,2$*

(a) $\mathfrak{A}_i = (\Omega_i,\mathcal{F}_i,\mathbb{F}_i,\mathbb{P}_i)$, where $\mathbb{F}_i = (\mathcal{F}_t^i)_{0\le t<\infty}$, is a complete filtered probability space;

(b) η_i is a time homogeneous Poisson random measure on $(Z,\mathcal{Z})$ with intensity measure ν over $\mathfrak{A}_i$;

(c) ξ_i is a progressively measurable process over $\mathfrak{A}_i$ satisfying condition (3.1).

Let $T > 0$. We put for $i = 1,2$

$$u_i(t) := u_{\xi_i,\tilde\eta_i}(t) = \int_0^t \int_Z S(t-r)\,\xi_i(r;z)\,\tilde\eta_i(dr,dz), \quad t \in [0,T]. \tag{3.3}$$

Assume finally that $\mathcal{L}aw((\xi_1,\eta_1)) = \mathcal{L}aw((\xi_2,\eta_2))$ on $L^p([0,T];L^p(Z,\nu;E))\times M_I([0,T]\times Z)$.

(i) If X is a separable Banach space such that $E \subset X$ continuously and the processes u_i, $i = 1,2$, have X-valued càdlàg modifications (denoted by the same symbols), then $\mathcal{L}aw((u_1,\xi_1,\eta_1)) = \mathcal{L}aw((u_2,\xi_2,\eta_2))$ on

$$\mathbb{D}([0,T];X)\times L^p([0,T];L^p(Z,\nu;E))\times M_I([0,T]\times Z).$$

(ii) If moreover, B is separable Banach spaces such that $B \subset E$ continuously and the processes u_i, $i = 1, 2$, have $\mathbb{P}_i$-almost all paths in $L^p(0,T;B)$, then $\mathcal{L}aw((u_1,\xi_1,\eta_1)) = \mathcal{L}aw((u_2,\xi_2,\eta_2))$ on

$$(\mathbb{D}([0,T],X) \cap L^p([0,T];B)) \times L^p([0,T];L^p(Z,\nu;E)) \times M_I([0,T] \times Z).$$

Remark 3.5. *The claim that $\mathcal{L}aw((u_1,\xi_1,\eta_1)) = \mathcal{L}aw((u_2,\xi_2,\eta_2))$ on*

$$(\mathbb{D}([0,T],X) \cap L^p([0,T];B)) \times L^p([0,T];L^p(Z,\nu;E)) \times M_I([0,T] \times Z)$$

is essentially stronger than a similar claim that $\mathcal{L}aw(u_1) = \mathcal{L}aw(u_2)$ on $\mathbb{D}([0,T];X) \cap L^p([0,T];B)$.

Before we embark on with the proof of Theorem 3.4 let us introduce some useful notation. Given a filtered probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and a Banach space Y, by $\mathcal{N}([0,T] \times \Omega; Y)$ we denote the space of (equivalence classes of) progressively measurable functions $\xi : [0,T] \times \Omega \to Y$.

For $q \in (1,\infty)$ we set

$$\mathcal{M}^q([0,T] \times \Omega; Y) = \Big\{ \xi \in \mathcal{N}([0,T] \times \Omega; Y) : \mathbb{E} \int_0^T |\xi(t)|_Y^q \, dt < \infty \Big\}. \tag{3.4}$$

Let $\mathcal{N}_{\text{step}}([0,T] \times \Omega; Y)$ be the space of all $\xi \in \mathcal{N}([0,T] \times \Omega; Y)$ for which there exists a partition $0 = t_0 < t_1 < \cdots < t_n = T$ such that for $k \in \{1, \cdots, n\}$ and for $t \in (t_{k-1}, t_k]$, $\xi(t) = \xi(t_k)$ is $\mathcal{F}_{t_{k-1}}$-measurable. We put $\mathcal{M}^q_{\text{step}} = \mathcal{M}^q \cap \mathcal{N}_{\text{step}}$. Note that $\mathcal{M}^q([0,T] \times \Omega; Y)$ is a closed subspace of $L^q([0,T] \times \Omega; Y) \cong L^q(\Omega; L^q([0,T];Y))$.

Proof of Theorem 3.4. Note that the second assertion is a consequence the first one. Indeed, by the first claim, the triplets $(u_{\xi_i,\tilde{\eta}_i}, \xi_i, \eta_i)$, $i = 1, 2$, have the same law on $\mathbb{D}([0,T];X) \times L^p([0,T];L^p(Z,\nu;E)) \times M_I([0,T] \times Z)$. Since $\mathbb{D}([0,T];X) \cap L^p(0,T;B) \hookrightarrow \mathbb{D}([0,T];X)$ continuously, in view of Proposition A.1 the same triplets have equal laws on the space

$$\mathbb{D}([0,T];X) \cap L^p(0,T;B) \times L^p([0,T];L^p(Z,\nu;E)) \times M_I([0,T] \times Z).$$

Thus it is enough to prove assertion (i). In order to make the use of the Haar projection more transparent we will assume in this part of the paper that $T = 1$. Let us fix a filtered probability space $\mathfrak{A} = (\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,1]}$, and let us introduce the family of linear operators $(\Phi_t)_{t \in [0,1]}$ from $L^p([0,1];L^p(Z,\nu;E))$ to $L^p([0,1];L^p(Z,\nu;X))$, defined by the following formula

$$(\Phi_t \xi)(s) := 1_{[0,t)}(s) S(t-s) \xi(s), \quad \xi \in L^p([0,1];L^p(Z,\nu;E)), \quad s \in [0,1].$$

Note that for any $t \in [0,1]$, the operator Φ_t is well defined bounded and linear. Indeed, there exists a constant $M > 0$ such that for all $t \in [0,1]$, $|S(t)|_{\mathcal{L}(E)} \le M$. Hence by the continuity of the embedding $\iota : E \hookrightarrow X$ we obtain for any $t \in [0,1]$

$$\begin{aligned} &\int_0^1 \|(\Phi_t \xi)(s)\|^p_{L^p(Z,\nu;X)} = \int_0^1 \|1_{[0,t)}(s) \iota \circ S(t-s) \xi(s)\|^p_{L^p(Z,\nu;X)} \, ds \\ &= \int_0^1 \int_Z \|1_{[0,t)}(s) \iota \circ S(t-s) \xi(s)(z)\|^p_X \, d\nu(z) ds \\ &\le C M^p \int_0^1 |\xi(s)(z)|^p_E \, d\nu(z) ds = C M^p \|\xi\|^p_{L^p(0,1;L^p(Z,\nu;E))}. \end{aligned} \tag{3.5}$$

Thus the operator Φ_t is well defined and bounded.

Notice that the stochastic convolution process $u_{\xi,\tilde{\eta}}$ defined in (3.2) can be expressed in terms of the map Φ_t. Indeed, identifying a process ξ with a map

$$\xi : \Omega \to L^p([0,1]; L^p(Z,\nu; E))$$

we have the following equality

$$u_{\xi,\tilde{\eta}}(t) = I_{\Phi_t\circ\xi,\tilde{\eta}}(t) = \int_0^t \int_Z (\Phi_t \circ \xi)(s,z)\, \tilde{\eta}(ds,dz), \quad t \in [0,1].$$

Going back to our original problem we assume that $\mathfrak{A}_i = (\Omega_i, \mathcal{F}_i, \mathbb{F}_i, \mathbb{P}_i)$, $i = 1,2$, where $\mathbb{F}_i = (\mathcal{F}^i_t)_{0\le t<\infty}$, are two fixed complete, filtered probability spaces and $\xi_i \in \mathcal{M}^p([0,1] \times \Omega_i; L^p(Z,\nu,E))$, $i = 1,2$. For fixed $t \in [0,1]$ we approximate $\Phi_t \circ \xi_i$, $i = 1,2$, by a sequence

$$(\mathfrak{h}^s_n \circ \Phi_t \circ \xi_i)_{n\in\mathbb{N}} \subset \mathcal{M}^p_{\text{step}}([0,1] \times \Omega_i; L^p(Z,\nu,E)), \quad i = 1,2,$$

where $\mathfrak{h}^s_n$ is the shifted Haar projection operator in the space $L^p([0,1]; L^p(Z,\nu,X))$, defined in (B.1). Let us note that by Proposition B.2-(i) the shifted Haar projection $\mathfrak{h}^s_n$ is also a continuous operator from $L^p([0,1]; L^p(Z,\nu,X))$ into itself. This continuity implies that for any $t \in [0,1]$ the random variables $\mathfrak{h}^s_n \circ \Phi_t \circ \xi_1$ and $\mathfrak{h}^s_n \circ \Phi_t \circ \xi_2$ have the same laws on $L^p([0,1]; L^p(Z,\nu; X))$. Moreover, by (B.2)-(ii) $\mathfrak{h}^s_n \circ \Phi_t \circ \xi_i \to \Phi_t \circ \xi_i$ in $\mathcal{M}^p([0,1] \times \Omega_i; L^p(Z,\nu,X))$. Taking into account the assumption that $\mathcal{L}aw(\xi_1,\eta_1) = \mathcal{L}aw(\xi_2,\eta_2)$ on $L^p([0,1]; L^p(Z,\nu;E)) \times M_I([0,1] \times Z)$, we conclude by Corollary A.8 of [10] that for any $(t_1,\dots,t_m) \in [0,1]^m$

$$\begin{aligned}&\mathcal{L}aw\big(I_{\Phi_{t_1}\circ\xi_1,\tilde{\eta}_1}(t_1), I_{\Phi_{t_2}\circ\xi_1,\tilde{\eta}_1}(t_2), \dots, I_{\Phi_{t_m}\circ\xi_1,\tilde{\eta}_1}(t_m), \xi_1, \tilde{\eta}_1\big)\\ &= \mathcal{L}aw\big(I_{\Phi_{t_1}\circ\xi_2,\tilde{\eta}_2}(t_1), I_{\Phi_{t_2}\circ\xi_2,\tilde{\eta}_2}(t_2), \dots, I_{\Phi_{t_m}\circ\xi_2,\tilde{\eta}_2}(t_m), \xi_2, \tilde{\eta}_2\big)\end{aligned} \tag{3.6}$$

on $X^m \times L^p([0,1]; L^p(Z,\nu;E)) \times M_I([0,1] \times Z)$.

We have to show that the triplets (u_1,ξ_1,η_1) and (u_2,ξ_2,η_2) have the same laws on $\mathbb{D}([0,1];X) \times L^p([0,1]; L^p([0,1],\nu;E)) \times M_I([0,1] \times Z)$. To proceed further, let us recall the definition of the dyadic projection $\pi_n : \mathbb{D}([0,1];X) \to \mathbb{D}([0,1];X)$ of order $n \in \mathbb{N}$ introduced in (C.2), i.e.

$$(\pi_n x)(t) := \sum_{j=0}^{2^n-1} 1_{(j2^{-n},(j+1)2^{-n}]}(t)\, x(j2^{-n}),\ t \in [0,1]. \tag{3.7}$$

Note, that for $i = 1,2$ and $u_{\xi_i,\tilde{\eta}_i}$ defined in (3.3) the following identity holds

$$\left[\pi_n \circ u_{\xi_i,\tilde{\eta}_i}\right](t) = \sum_{j=0}^{2^n-1} 1_{(j2^{-n},(j+1)2^{-n}]}(t)\, I_{\Phi_{j2^{-n}}\circ\xi_i,\tilde{\eta}_i}, \quad t \in [0,1].$$

By (3.6), it follows that $\pi_n \circ u_{\xi_1,\tilde{\eta}_1}$ and $\pi_n \circ u_{\xi_2,\tilde{\eta}_2}$ have the same laws on $\mathbb{D}([0,1];X)$. Finally, since by Proposition C.2 the family of dyadic projections $\{\pi_n, n \in \mathbb{N}\}$ converges pointwise to the identity, i.e. $d_0(\pi_n x, x) \to 0$ [1], for all $x \in \mathbb{D}([0,1];X)$, we infer by Proposition 2.6 of [10] that the laws of the triplets (u_i,ξ_i,η_i), $i = 1,2$, are equal on

$$\mathbb{D}([0,1];X) \times L^p([0,1]; L^p(Z,\nu;E)) \times M_I([0,1] \times Z).$$

Hence the proof of Theorem 3.4 is complete. □

[1] The Prohorov metric d_0 is defined in Appendix C in formula (C.1).

Note that existence of càdlàg modification of the stochastic convolution process is fundamental both in the formulation and in the proof of Theorem 3.4. We will now give some examples of the spaces X and B whose existence is assumed in Theorem 3.4.

For simplicity we may assume that A^{-1} exists and is a bounded operator on E. For every $\beta > 0$, we consider the extrapolation space $E_{-\beta}$ defined as the completion of the space E with respect to the norm $|A^{-\beta} \cdot |_E$. Note that A^β extends to an isometry, still denoted by A^β, between E and $E_{-\beta}$.

Lemma 3.6. *Let $(S(t)_{t\geq 0}$ be a $\mathcal{C}_0$-semigroup generated by $-A$. Then the process u has an E_{-1}-valued càdlàg modification.*

Proof. The assertion follows from [12, Lemma 3.3] according to which the process u satisfies the following equality.

$$A^{-1}u(t) = \int_0^t u(s)\, ds + \int_0^t \int_Z A^{-1}\xi(s,z)\, \tilde{\eta}(ds,dz), \quad t \in [0,T]. \tag{3.8}$$

Hence, since by [11] the second term above has an E-valued càdlàg modification, we infer that the process u has an E_{-1}-valued càdlàg modification. □

Using Theorem 3.4, Lemma 3.6 and Remark 3.2, we obtain the following Corollary.

Corollary 3.7. *Suppose that assumptions of Theorem 3.4 are satisfied. Then we have $\mathcal{L}aw((u_1,\xi_1,\eta_1)) = \mathcal{L}aw((u_2,\xi_2,\eta_2))$ on*

$$(\mathbb{D}([0,T],E_{-1}) \cap L^p([0,T];E)) \times L^p([0,T];L^p(Z,\nu;E)) \times M_I([0,T]\times Z).$$

Proof. By Lemma 3.6 the processes u_i, $i=1,2$, have E_{-1}-valued modifications. By Remark 3.2 $\mathbb{P}_i$-almost all paths of u_i, $i=1,2$, are in $L^p([0,T];E)$. Thus the assertion follows directly from Theorem 3.4. □

3.2 Stochasic convolution with a contraction type semigroup

In paper [12] it is proven that in the case when the space E satisfies some stronger assumption and the semigroup is of contraction type then the stochastic convolution process has an E-valued càdlàg modification. The problem of the existence of a càdlàg modification is closely related to appropriate maximal inequalities. We recall now some of these results.

In addition we assume that the space E satisfies the following condition

Assumption 3.8. *(see [12]) There exists an equivalent norm $\|\cdot\|_E$ on E and $q \in [p,\infty)$ such that the function $\phi : E \ni x \mapsto \|x\|_E^q \in \mathbb{R}$ is of class $\mathcal{C}^2$ and there exist constants k_1, k_2 such that for every $x \in E$, $|\phi'(x)| \leq k_1\|x\|_E^{q-1}$ and $|\phi''(x)| \leq k_2\|x\|_E^{q-2}$.*

Theorem 3.9. *([12, Corollary 4.3]) If E satisfies Assumptions 2.2 and 3.8 and a $\mathcal{C}_0$-semigroup $(S(t))_{t\geq 0}$ is of contraction type, then there exists an E-valued càdlàg modification $\tilde{u}$ of u such that for some constant $C>0$ independent of u and all $t\in[0,T]$ and $0<r\leq p$,*

$$\mathbb{E} \sup_{0\leq s\leq t} \|\tilde{u}\|_E^r \leq C\mathbb{E}\Big(\int_0^t \int_Z \|\xi(s,z)\|_E^p \nu(dz)ds\Big)^{\frac{r}{p}}.$$

In the sequel, in the case the assertion of Theorem 3.9 holds true, by the stochastic convolution process we mean its E-valued càdlàg modification.

Now, we are ready to state the next corollary to our main result Theorem 3.4.

Corollary 3.10. *In addition to the assumptions of Theorem 3.4 we assume that the Banach space $(E, |\cdot|_E)$ satisfies Assumption 3.8 and the semigroup $\big(S(t)\big)_{t\geq 0}$ is of contraction type. Then $\mathcal{L}aw((u_1,\xi_1,\eta_1)) = \mathcal{L}aw((u_2,\xi_2,\eta_2))$ on*

$$(\mathbb{D}([0,T],E)\cap L^p([0,T];E)) \times L^p([0,T];L^p(Z,\nu;E)) \times M_I([0,T]\times Z).$$

Remark 3.11. *The only reason we have assumed in Corollary 3.10 that $\big(S(t)\big)_{t\geq 0}$ is a contraction type semigroup is that we need Theorem 3.9 about the existence of an E-valued càdlàg modification of a stochastic convolution processes. Corollary 3.10 remains valid for such class of semigroups $\big(S(t)\big)_{t\geq 0}$ for which the conclusion of Theorem 3.9 holds true.*

4 Applications

4.1 Equations with a drift

Similarly to the paper [10] let us consider the following problem

$$\begin{cases} du(t) + Au(t)\,dt = b(t)\,dt + \int_Z \xi(t;z)\,\tilde\eta(dt;dz), \quad t\in[0,T], \\ u(0)=0. \end{cases} \tag{4.1}$$

Let us suppose that A is a linear operator satisfying Assumption 3.1. Moreover, assume that

(A.1) $\xi : [0,T]\times\Omega \to L^p(Z,\nu;E)$ is a progressively measurable process such that

$$\mathbb{E}\Big[\int_0^T\int_Z |\xi(s,z)|_E^p\,\nu(dz)ds\Big] < \infty,$$

(A.2) $b : [0,T]\times\Omega\to E$ is a progressively measurable process such that $\mathbb{P}$-a.s. $b\in L^p([0,T];E)$.

Definition 4.1. *Let ξ and b be two processes satisfying assumptions (A.1) and (A.2), respectively. A solution to problem* (4.1) *is an E-valued predictable process u such that $\mathbb{P}$-a.s., $u(0)=0$ and for every $t\in[0,T]$ the following equality*

$$u(t) = \int_0^t S(t-s)b(s)\,ds + \int_0^t\int_Z S(t-s)\xi(s,z)\,\tilde\eta(ds,dz)$$

holds $\mathbb{P}$-a.s.

Lemma 4.2. *Under Assumptions 2.2 and 3.1, let ξ and b be two processes satisfying assumptions (A.1) and (A.2), respectively. Then there exists a solution to problem* (4.1) *with $\mathbb{P}$-almost all paths in $L^p([0,T];E)$. Moreover, the solution has an E_{-1}-valued càdlàg modification.*

If in addition the space E satisfies Assumption 3.8 and the semigroup $(S(t))_{t\geq 0}$ is of contraction type then this solution has an E-valued càdlàg modification.

Proof of Lemma 4.2. Since $S(t)$ is $\mathcal{C}_0$-semigroup and b satisfies assumption (A.2), the process

$$v(t) = \int_0^t S(t-s)b(s)\,ds, \quad t\in[0,T]$$

is well defined and $\mathbb{P}$-a.s. $v\in\mathcal{C}([0,T];E)$. Thus the existence of a solution follows from Remark 3.2 and [12, Lemma 3.1]. By Lemma 3.6 the solution has an E_{-1}-valued càdlàg modification.

The existence of an E-valued càdlàg modification in the case when Assumption 3.8 is satisfied and the semigroup $(S(t))_{t\geq 0}$ is of contraction type follows directly from Theorem 3.9. The proof of the Lemma is thus complete. □

By methods similar to those used in the proofs of Theorem 3.4 and Lemma 4.2 the following Corollary can be established.

Corollary 4.3. *Suppose that the assumptions of Theorem 3.4 are satisfied. Let $T>0$ and let u_i, $i=1,2$, be the solutions of the following problems*

$$\begin{cases} du_i(t)+Au_i(t)\,dt=b_i(t)\,dt+\int_Z \xi_i(t,x)\,\tilde{\eta}_i(dt;dz), & t\in[0,T], \\ u_i(0)=0, \end{cases} \tag{4.2}$$

where $b_i:[0,T]\times\Omega_i\to E$, $i=1,2$, is a progressively measurable process such that $\mathbb{P}$-a.s. $b_i\in L^p([0,T];E)$. Assume that $\mathcal{L}aw((b_1,\xi_1,\eta_1))=\mathcal{L}aw((b_2,\xi_2,\eta_2))$ on

$$L^p([0,T],E)\times L^p([0,T];L^p(Z,\nu;E))\times M_I([0,T]\times Z).$$

Then $\mathcal{L}aw((u_1,b_1,\xi_1,\eta_1))=\mathcal{L}aw((u_2,b_2,\xi_2,\eta_2))$ on

$$\begin{aligned}&(\mathbb{D}([0,T],E_{-1})\cap L^p([0,T];E))\\ &\times L^p([0,T];E)\times L^p([0,T];L^p(Z,\nu;E))\times M_I([0,T]\times Z).\end{aligned}$$

Proof of Corollary 4.3. The assertion follows directly from Lemma 4.2 and Theorem 3.4. □

4.2 Equations with a contraction type semigroup

Corollary 4.4. *If in addition to hypotheses of Corollary 4.3 the space E satisfies Assumption 3.8 and the semigroup $(S(t))_{t\geq 0}$ is of contraction type then*

(i) the solution u_1 and u_2 have E-valued càdlàg modifications,

(ii) $\mathcal{L}aw((u_1,b_1,\xi_1,\eta_1))=\mathcal{L}aw((u_2,b_2,\xi_2,\eta_2))$ on

$$\mathbb{D}([0,T],E)\times L^p([0,T],E)\times L^p([0,T];L^p(Z,\nu;E))\times M_I([0,T]\times Z).$$

Proof of Corollary 4.4. The assertion is a direct consequence of Lemma 4.2 and Theorem 3.4. □

4.3 Equations with an analytic semigroup

In the last subsection we will make a stronger assumption than before.

Assumption 4.5. *The operator $-A$ is an infinitesimal generator of an analytic $\mathcal{C}_0$-semigroup $(S(t))_{0\leq t<\infty}$ on the space E.*

We begin with the following useful result. An alternative proof of it could be obtained by applying [11] but for the convenience of the reader we have decided to include a self-contained proof.

Lemma 4.6. *Under Assumptions 2.2 and 4.5, for every $\alpha\in(0,\frac{1}{p})$ there exists a constant $C_{\alpha,p}>0$ such that the stochastic convolution process u defined by*

$$u(t):=\int_0^t\int_Z S(t-s)\xi(s,z)\tilde{\eta}(ds,dz),\qquad t\in[0,T].$$

satisfies the following inequality

$$\mathbb{E}\big[\|A^\alpha u\|^p_{L^p(0,T;E)}\big]\leq C_{\alpha,p}\mathbb{E}\Big[\int_0^T\int_Z|\xi(s,z)|_E^p\,d\nu(z)ds\Big].$$

Proof. Let us fix $\alpha \in (0, \frac{1}{p})$. Then, see [22], there exists a constant $C = C(\alpha, T) > 0$ such that for every $t > 0$ linear operator $A^\alpha S(t)$ is well defined and bounded, and

$$|A^\alpha S(t)|_{\mathcal{L}(E)} \le \frac{C}{t^\alpha}, \qquad t \in (0, T].$$

We will show that for almost all $t \in [0, T]$

$$\mathbb{E}\Big[\int_0^t \int_Z \frac{1}{(t-s)^{\alpha p}} |\xi(s,z)|_E^p \, \nu(dz)\, ds\Big] < \infty. \tag{4.3}$$

To this end it is sufficient to prove that

$$\int_0^T \Big\{\mathbb{E}\Big[\int_0^t \int_Z \frac{1}{(t-s)^{\alpha p}} |\xi(s,z)|_E^p \, \nu(dz)\, ds\Big]\Big\}\, dt < \infty.$$

Using the Fubini Theorem (for non-negative functions) we obtain

$$\begin{aligned}
&\int_0^T \Big\{\mathbb{E}\Big[\int_0^t \int_Z \frac{1}{(t-s)^{\alpha p}} |\xi(s,z)|_E^p \, \nu(dz)\, ds\Big]\Big\}\, dt \\
&= \mathbb{E}\Big[\int_0^T \Big\{\int_0^t \frac{1}{(t-s)^{\alpha p}} \int_Z |\xi(s,z)|_E^p \, \nu(dz)\, ds\Big\}\, dt\Big] \\
&\le \mathbb{E}\Big[\int_0^T \frac{1}{s^{\alpha p}}\, ds \cdot \int_0^T \int_Z |\xi(s,z)|_E^p \, \nu(dz)\, ds\Big] \\
&= \frac{T^{-\alpha p+1}}{-\alpha p+1} \cdot \mathbb{E}\Big[\int_0^T \int_Z |\xi(s,z)|_E^p \, \nu(dz)\, ds\Big] < \infty.
\end{aligned}$$

Thus (4.3) holds. By (4.3) we have for almost all $t \in [0, T]$

$$\begin{aligned}
\mathbb{E}\big[|A^\alpha u(t)|_E^p\big] &= \mathbb{E}\Big[\Big|\int_0^t \int_Z A^\alpha S(t-s)\xi(s,z)\tilde{\eta}(ds,dz)\Big|_E^p\Big] \\
&\le \mathbb{E}\Big[\int_0^t \int_Z |A^\alpha S(t-s)\xi(s,z)|_E^p \, \nu(dz)\, ds\Big] \\
&\le C^p \mathbb{E}\Big[\int_0^t \int_Z \frac{1}{(t-s)^{\alpha p}} |\xi(s,z)|_E^p \, \nu(dz)\, ds\Big].
\end{aligned}$$

Hence by the Fubini Theorem we obtain

$$\begin{aligned}
\mathbb{E}\big[|A^\alpha u|_{L^p(0,T;E)}^p\big] &= \mathbb{E}\Big[\int_0^T |A^\alpha u(t)|_E^p \, dt\Big] = \int_0^T \mathbb{E}\big[|A^\alpha u(t)|^p\big]\, dt \\
&\le C^p \int_0^T \mathbb{E}\Big[\int_0^t \int_Z \frac{1}{(t-s)^{\alpha p}} |\xi(s,z)|_E^p \, \nu(dz)\, ds\Big]\, dt \\
&\le C_{\alpha,p} \mathbb{E}\Big[\int_0^T \int_Z |\xi(s,z)|_E^p \, d\nu(z) ds\Big],
\end{aligned}$$

where $C_{\alpha,p} > 0$ is some constant.

□

We continue with the following strengthenings of Lemmas 3.6 and 4.2.

Lemma 4.7. *Under the assumptions of Lemma 4.6 the process u has an $E_{\alpha-1}$-valued modification.*

Proof. From identity 3.8 we get

$$A^{\alpha-1}u(t)=\int_0^t A^{\alpha}u(s)\,ds+\int_0^t\int_Z A^{\alpha-1}\xi(s,z)\,\tilde{\eta}(ds,dz),\quad t\in[0,T]. \tag{4.4}$$

Hence, since by [11] the second term above has an E-valued càdlàg modification and by the previous Lemma 4.6 the first term above has an E-valued continuous modification, we infer that the process u has an $E_{\alpha-1}$-valued càdlàg modification. □

Lemma 4.8. *Suppose that Assumptions 3.1 and 4.5 hold and that $\alpha<\frac{1}{p}$. Let ξ and b be two processes satisfying assumptions (A.1) and (A.2), respectively. Then there exist a solution u to problem* (4.1) *with $\mathbb{P}$-almost all paths in $L^p([0,T];D(A^{\alpha}))$. Moreover, every this solution has an $E_{\alpha-1}$-valued càdlàg modification.*

Proof. The proof is obvious. □

We finish this subsection with its main result.

Corollary 4.9. *Suppose that the assumptions of Theorem 3.4 are satisfied and that Assumption 4.5 holds. Assume also that $\alpha\in(0,\frac{1}{p})$. Let $T>0$ and let u_i, $i=1,2$, be the solutions of the following problems*

$$\begin{cases} du_i(t)+Au_i(t)\,dt=b_i(t)\,dt+\int_Z \xi_i(t,x)\,\tilde{\eta}_i(dt;dz), & t\in[0,T],\\ u_i(0)=0, \end{cases} \tag{4.5}$$

where $b_i:[0,T]\times\Omega_i\to E$, $i=1,2$, is a progressively measurable process such that $\mathbb{P}$-a.s. $b_i\in L^p([0,T];E)$. Assume that $\mathcal{L}aw((b_1,\xi_1,\eta_1))=\mathcal{L}aw((b_2,\xi_2,\eta_2))$ on

$$L^p([0,T],E)\times L^p([0,T];L^p(Z,\nu;E))\times M_I([0,T]\times Z).$$

Then $\mathcal{L}aw((u_1,b_1,\xi_1,\eta_1))=\mathcal{L}aw((u_2,b_2,\xi_2,\eta_2))$ on

$$\begin{aligned}&(\mathbb{D}([0,T],E_{\alpha-1})\cap L^p([0,T];D(A^{\alpha})))\\ &\times L^p([0,T];E)\times L^p([0,T];L^p(Z,\nu;E))\times M_I([0,T]\times Z).\end{aligned}$$

A Equality of Laws

The purpose of this Appendix is to prove the following essential result. This result should be known but we were unable to trace it in the literature.

Proposition A.1. *Assume that X and E are Polish spaces such that $E\subset X$ and the natural embedding $\iota:E\hookrightarrow X$ is continuous. For fixed $k=1,2$, let $\mathfrak{A}_k=(\Omega_k,\mathcal{F}_k,\mathbb{P}_k)$ be a probability space, and ξ_k be an E-valued random variable defined on $\mathfrak{A}_k$. If $\mathcal{L}aw(\iota\circ\xi_1)=\mathcal{L}aw(\iota\circ\xi_2)$ on X, then $\mathcal{L}aw(\xi_1)=\mathcal{L}aw(\xi_2)$ on E.*

Proof. It is enough to prove that $\mathcal{B}(E)=\iota^{-1}(\mathcal{B}(X))$. Since ι is a continuous mapping, the inclusion $\supset$ follows. To prove the inclusion $\subset$ let us take $A\in\mathcal{B}(E)$. Therefore, by the Kuratowski Theorem [19, p. 499], see also [24, Theorem 1.1, p. 5] and [21, §1.3], since ι is an injective continuous (and hence Borel) mapping, the set $\iota(A)$ belongs to $\mathcal{B}(X)$. By injectivity of ι we infer that $A=\iota^{-1}\big(\iota(A)\big)$ which completes the proof. □

Corollary A.2. *Assume that X, E and F are Polish spaces such that $E\subset X$ and $F\subset X$, and the natural embeddings $\iota_E:E\hookrightarrow X$ and $\iota_F:F\hookrightarrow X$ are continuous. If the random variables ξ_i, $i=1,2$ as in Proposition A.1 are $E\cap F$ valued and $\mathcal{L}aw(\xi_1)=\mathcal{L}aw(\xi_2)$ on E then $\mathcal{L}aw(\xi_1)=\mathcal{L}aw(\xi_2)$ on F.*

Proof of Corollary A.2. By assumptions, $\mathcal{L}aw(\iota_E\circ\xi_1)=\mathcal{L}aw(\iota_E\circ\xi_2)$ on X. But $\iota_E\circ\xi_i=\iota_F\circ\xi_i$, $i=1,2$ and hence $\mathcal{L}aw(\iota_F\circ\xi_1)=\mathcal{L}aw(\iota_F\circ\xi_2)$ on X. Applying next Proposition A.1 concludes the proof. □

B L^p-spaces and the Haar system

In this Appendix we assume that $(Y, |\cdot|)$ is a separable Banach space. As in the Proof of Theorem 3.4 we assume for simplicity that $T = 1$. In this section we recall some facts about the approximation properties of the Haar system. For $n \in \mathbb{N}$, let $\Pi^n = \{s_0^n = 0 < s_1^n < \cdots < s_{2^n}^n\}$ be a partition of the interval $[0,1]$ defined by $s_j^n = j\,2^{-n}$, $j = 1, \cdots, 2^n$. Each interval of the form $(s_{j-1}^n, s_j^n]$, where $n \in \mathbb{N}$ and $j = 1, \ldots, 2^n$ is called a *dyadic* interval. For $n \in \mathbb{N}$, the j^{th} *element, for* $j = 1, \ldots, 2^n$*, of the Haar system of order* n is the indicator function of the interval $(s_{j-1}^n, s_j^n]$, i.e. $1_{(s_{j-1}n, s_j^n]}$. For $n \in \mathbb{N}$ let $\mathfrak{h}_n : L^p([0,1], Y) \to L^p([0,1], Y)$ be the *Haar projection* of order n, i.e.

$$\mathfrak{h}_n(x) = \sum_{j=1}^{2^n} 1_{(s_{j-1}^n, s_j^n]} \otimes \iota_{j,n}(x), \quad x \in L^p([0,1], Y), \tag{B.1}$$

where $\iota_{j,n} : L^p([0,1], Y) \to Y$, is the averaging operator over the interval $(s_{j-1}^n, s_j^n]$, i.e.

$$\iota_{j,n}(x) := \frac{1}{s_j^n - s_{j-1}^n} \int_{s_{j-1}^n}^{s_j^n} x(s)\, ds, \; x \in L^p([0,1], Y). \tag{B.2}$$

In the above, for $f \in L^p([0,1], \mathbb{R})$ and $y \in Y$, by $f \otimes y$ we mean an element of $L^p([0,1], Y)$ defined by $[0,1] \ni t \mapsto f(t)y \in Y$.

Remark B.1. *Note that by the Jensen inequality, every map* $\iota_{j,n} : L^p([0,1], Y) \to Y$ *is a linear contraction. Therefore, since* $|1_{(s_{j-1}^n, s_j^n]}|_{L^p([0,1],\mathbb{R})} = 2^{-n}$*, we infer that that for any* $n \in \mathbb{N}$

$$\|\mathfrak{h}_n(x)\|_{L^p([0,1],Y)} \le \|x\|_{L^p([0,1],Y)}, \quad x \in L^p([0,1], Y). \tag{B.3}$$

Therefore, since $\mathfrak{h}_n x \to x$ *in* $L^p([0,1], Y)$ *for any* $x \in C^1([0,1], Y)$ *and* $C^1([0,1], Y)$ *is a dense subspace of* $L^p([0,1]; Y)$*, we infer that*

$$\mathfrak{h}_n(x) \to x \;\; in \;\; L^p([0,1]; Y), \quad \forall x \in L^p([0,1]; Y). \tag{B.4}$$

Let ξ be a Y-valued progressively measurable p integrable stochastic process. One way to get a sequence of step functions is to approximate ξ by the sequence $(\mathfrak{h}_n \circ \xi)_{n\in\mathbb{N}}$, where $\mathfrak{h}_n$ is the Haar projection of order n. The only problem which arises is, that $\mathfrak{h}_n \circ \xi$ is not necessarily progressively measurable. To get this property, we have to shift the projection by one time interval. With $\iota_{j,n}$ being defined in (B.2), the n-th order shifted Haar projection is a linear bounded map in $L^p([0,1]; Y)$ is defined by

$$\mathfrak{h}_n^s x = \sum_{j=1}^{2^n-1} 1_{(s_j^n, s_{j+1}^n]} \otimes \iota_{j-1,n}(x), \quad x \in L^p([0,1]; Y), \tag{B.5}$$

where we put $\iota_{0,n} = 0$ for every $n \in \mathbb{N}$.

Proposition B.2. *The following holds:*

(i) For any $n \in \mathbb{N}$*, the shifted Haar projection* $\mathfrak{h}_n^s : L^p([0,1]; Y) \to L^p([0,1]; Y)$ *is a continuous operator.*

(ii) For all $x \in L^p([0,1]; Y)$*,* $\mathfrak{h}_n^s x \to x$ *in* $L^p([0,1]; Y)$*.*

Proof of Proposition B.2. For each $n \in \mathbb{N}$ let us define

$$(\mathfrak{s}_n x)(s) := \begin{cases} x(0), & \text{if } s \le 2^{-n}, \\ x(s - 2^{-n}), & \text{if } s \in (2^{-n}, 1], \end{cases}$$

Obviously, $\mathfrak{s}_n$ is a linear contraction in $L^p([0,1];Y)$. Moreover,

$$\mathfrak{h}_n^s = \mathfrak{h} \circ \mathfrak{s}_n.$$

Therefore part (i) follows from Remark B.1. Since $\mathfrak{s}_n x \to x$ for all $x \in C^1([0,1],Y)$, assertion (ii) follows by similar arguments as in (B.4). This completes the proof. □

C The Skorokhod space

For an introduction to the Skorokhod space we refer to Billingsley [3], Ethier and Kurtz [15] and Jacod and Shiryaev [17]. In this Section we state only these results which are necessary for our work.

Let $(Y, |\cdot|_Y)$ be a separable Banach space. The space $\mathbb{D}([0,1];Y)$ denotes the space of all right continuous functions $x : [0,1] \to Y$ with left-hand limits. Let Λ denote the class of all strictly increasing continuous functions $\lambda : [0,1] \to [0,1]$ such that $\lambda(0) = 0$ and $\lambda(1) = 1$. Obviously any element $\lambda \in \Lambda$ is a homeomorphism of $[0,1]$ onto itself. Let us define the Prohorov metric d_0 by

$$\begin{aligned} \|\lambda\|_{\log} &:= \sup_{t\neq s\in[0,1]} \left| \log \tfrac{\lambda(t)-\lambda(s)}{t-s} \right|, \\ \Lambda_{\log} &:= \left\{ \lambda \in \Lambda : \|\lambda\|_{\log} < \infty \right\} \\ d_0(x,y) &:= \inf \left\{ \|\lambda\|_{\log} \vee \sup_{t\in[0,1]} |x(t) - y(\lambda(t))|_Y : \lambda \in \Lambda_{\log} \right\}. \end{aligned} \tag{C.1}$$

The space $\mathbb{D}([0,1];Y)$ equipped with the metric d_0 is a separable complete metric space. We recall the notion of the dyadic projection.

Definition C.1. *Assume that $n \in \mathbb{N}$. The n-th order dyadic projection is a linear map $\pi_n : \mathbb{D}([0,1];Y) \to \mathbb{D}([0,1];Y)$ defined by*

$$\pi_n x := \sum_{i=0}^{2^n-1} 1_{(2^{-n}i, 2^{-n}(i+1)]}\, x(2^{-n}i), \quad x \in \mathbb{D}([0,1];Y). \tag{C.2}$$

An important property of the dyadic projection is given in the following result.

Proposition C.2. *(see [10, Proposition B.5]) If $x \in \mathbb{D}([0,1];Y)$ then*

$$\lim_{n\to\infty} d_0(x, \pi_n x) = 0.$$

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