

Large deviations for stable like random walks on \mathbb{Z}^d with applications to random walks on wreath products*

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Abstract

We derive Donsker-Vardhan type results for functionals of the occupation times when the underlying random walk on \mathbb{Z}^d is in the domain of attraction of an operator-stable law on \mathbb{R}^d . Applications to random walks on wreath products (also known as lamplighter groups) are given.

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1 Introduction

This work addresses two closely related questions of independent interests. From the point of view of random walks on the lattices \mathbb{Z}^d , we extend the well-known large deviation theorem of Donsker and Varadhan regarding the Laplace transform of the number D_n of visited points before time n . The theorem of Donsker and Varadhan, [8], treats random walks driven by measure μ in the domain of normal attraction of a symmetric stable law of index $\alpha \in (0, 2)$ (as well as the Gaussian case).

We generalize this result to random walks driven by a measure in the domain of attraction of an *operator-stable law*. For instance, this includes laws that are “stable” with respect to anisotropic dilations of the type

$$\delta_t(x_1, \dots, x_d) = (t^{1/\alpha_1} x_1, \dots, t^{1/\alpha_d} x_d) \tag{1.1}$$

with $\alpha_i \in (0, 2)$, $1 \leq i \leq d$. In this case, the generalization of the theorem of Donsker and Varadhan reads as follows. Let $\hat{\eta}$ denote the Fourier transform of the distribution η on either \mathbb{Z}^d or \mathbb{R}^d .

Theorem 1.1. *Referring to the anisotropic dilations at (1.1), assume that μ is a symmetric measure on \mathbb{Z}^d such that, uniformly on compact sets in \mathbb{R}^d ,*

$$n[1 - \hat{\mu}(\delta_n^{-1}\xi)] \rightarrow \Theta(\xi)$$

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where $\Theta(\xi)$ as the form

$$\Theta(\xi) = \int_{\mathbb{S}^{d-1}} \int_0^\infty (1 - \cos(\xi, \delta_r y)) \frac{M(dy)}{r} \frac{dr}{r} \tag{1.2}$$

for some symmetric finite measure M on \mathbb{S}^{d-1} whose support generates \mathbb{R}^d . Then

$$\lim_{n \rightarrow \infty} \frac{1}{n^{d/(d+\alpha)}} \log \mathbf{E}(e^{-\nu D_n}) = k(\nu, \eta) \in (0, \infty)$$

where η is the probability distribution on \mathbb{R}^d such that $\hat{\eta} = e^{-\Theta}$ and $\frac{d}{\alpha} = \sum_1^d \frac{1}{\alpha_i}$.

As in the classical Donsker-Varadhan theorem, the constant $k(\nu, \eta)$ is described by a variational formula. In its most natural generality (see Theorem 3.9 with $F(s) = \mathbf{1}_{(0, \infty)}(s) = 1 - \delta_0(s)$), this theorem involves more general dilation semigroups of the form $t^E = \sum_0^\infty \frac{(\log t)^n E^n}{n!}$ where E is an invertible matrix with eigenvalues in $[1/2, \infty)$ (E may not be diagonalizable and, even if E is diagonalizable, it may not be diagonalizable in a basis of \mathbb{Z}^d vectors). In this case, the associated limit law η is “operator-stable” with respect to the dilation structure t^E , $t > 0$, and the real $\alpha \in (0, 2)$ is given by $\alpha = \text{tr}(E)/d$.

In fact, we are also interested in a different generalization of the Donsker-Varadhan Theorem. Given a random walk on \mathbb{Z}^d driven by a symmetric measure μ , let $l(n, x)$ denotes the number of visits at x up to time n . We are interested in obtaining a large deviation result for the Laplace transform of more general functionals of the occupation time vector $(l(n, x))_{x \in \mathbb{Z}^d}$ than the number of visited sites, $D_n = \#\{x : l(n, x) \neq 0\}$. For instance, we are interested in the asymptotic behavior of

$$-\log \mathbf{E} \left(e^{-\lambda \sum_{x \in \mathbb{Z}^d} \log l(n, x)} \right)$$

and, more generally,

$$-\log \mathbf{E} \left(e^{-\lambda \sum_{x \in \mathbb{Z}^d} F(l(n, x))} \right)$$

when F belongs to some appropriate class of functions. For simplicity, in the next theorem, we consider the case where the function F is simply $F(s) = s^\gamma$, $\gamma \in (0, 1)$ and the dilation structure is given by (1.1).

Theorem 1.2. Referring to the anisotropic dilations at (1.1), assume that μ is a symmetric measure on \mathbb{Z}^d such that, uniformly on compact sets of \mathbb{R}^d ,

$$n[1 - \hat{\mu}(\delta_n^{-1} \xi)] \rightarrow \Theta(\xi)$$

where Θ has the form (1.2). Then, for $\gamma \in (0, 1)$,

$$\lim_{n \rightarrow \infty} n^{-\frac{\gamma + \tau(1-\gamma)}{1 + \tau(1-\gamma)}} \log \mathbf{E} \left(e^{-\nu \sum_x \ell(n, x)^\gamma} \right) = k(\nu, \eta, \gamma) \in (0, \infty)$$

where η is the probability distribution on \mathbb{R}^d such that $\hat{\eta} = e^{-\Theta}$ and $\tau = \sum_1^d \frac{1}{\alpha_i}$.

In the case of where μ is symmetric finitely supported and the dilation structure is the isotropic $\delta_t(x) = \sqrt{t}x$ (in this case, the limit law η is Gaussian), this result is contained in [3]. See Theorem 1.2 (with $p = 1$ and $H(s) = s^\gamma$) and Section 2.3 in [3]. Indeed, one of the contributions of [3] is to show how to deduce results such as Theorem 1.2 from the Donsker-Varadhan large deviation principle for the scaled version of the occupation measure of the underlying process. In order to treat processes that fall in the operator stable realm, we modify some of the arguments in [3] and rely more on the original techniques of Donsker and Varadhan.

The version of Theorem 1.2 which treats dilations of the form t^E and functionals $\sum_x F(\ell(n, x))$ associated to more general functions F than power functions is given in Theorem 3.9 and in Section 5.2. Except for a few technical adaptations to the operator stable context, the proofs of Theorem 1.1 is a routine generalizations of the proof given by Donsker and Varadhan in the stable context. Similarly, the proofs of Theorems 1.2–Theorem 3.9 involves an adaptation of the techniques of Donsker and Varadhan and [3].

In developing these results in the operator stable context, we are motivated by applications to the study of random walks on a class of groups called wreath products. These groups are also known as lamplighter groups. The wreath product $K \wr H$, i.e., the lamplighter group with base-group H and lamp-group K , will be defined precisely below. If we think of the elements of K as representing different colors (possibly countably many different colors), then an element of $K \wr H$ can be viewed as a pair (h, η) where h is an element of H (the position of the lamplighter on the base H) and $\eta = (k_h)_{h \in H} \in K^H$ is a finite configuration of colors on H in the sense that only finitely many $h \in H$ have $k_h \neq e_K$ where e_K is the identity element in K (only finitely many lamps are turned on). This description does not explain the group law on $K \wr H$ but captures the nature of the elements of the wreath product $K \wr H$. The identity element in $K \wr H$ has the lamplighter sitting at e_H and all the lamps turned off ($k_h = e_K$ for all $h \in H$). In one of the simplest instance of this construction, $H = \mathbb{Z}$ (a doubly infinite street) and $K = \mathbb{Z}/2\mathbb{Z}$ (lamps are either off (0) or on (1)).

We are interested in a large collection of random walks on wreath products which can be described collectively as the “switch-walk-switch” walks. See also [16, 22]. Namely, we are given two probability measures, one on H , call it μ , and one on K , call it ν . The measure μ drives a random walk on H which describes the moves of the lamplighter (i.e., the first coordinate, h , in the pair $(h, \eta) \in K \wr H$). The measure ν drives a random walk on K whose basic step is interpreted as “switching” between lamp colors. Based on this input, we construct a probability measure $q = q(\mu, \nu)$ on $K \wr H$ (this measure q is defined precisely later in the paper). The basic step of the walk driven by q can be accurately describes as follows: the lamplighter switches the color of the lamp at its standing position (using ν), takes a step in H (using μ) and switches the color of the lamp at its new position (using ν). These different moves are, in the appropriate sense, made independently of each other hence the name, *switch-walk-switch*. Let us insist on the fact that we will be interested here in cases when the measures μ and ν are not necessarily finitely supported. Now, an elementary argument shows that the probability of return $q^{(n)}(e)$ of the random walk driven by q on $K \wr H$ is given by

$$q^{(n)}(e) = \mathbf{E} \left(\prod_h \nu^{(2l_*(n,h))}(e_K) \mathbf{1}_{\{X_n=e_h\}} \right)$$

where $(X_m)_0^\infty$ is the random walk on H driven by μ and $l_*(n, h)$ is an essentially trivial modification of the number of visits of $(X_m)_0^\infty$ to h up to time n . The expectation is relative to the random walk $(X_m)_0^\infty$ on H , started at e_H . This observation goes back to [22] and is the basis of the analysis developed in [16]. If we set $F(m) = -\log \nu^{(2m)}(e_K)$ then it follows under mild assumptions that

$$\log q^{(n)}(e) \sim \log \mathbf{E} \left(e^{-\sum_h F(l(n,h))} \right). \tag{1.3}$$

In words, the log-asymptotic of the probability of return of a switch-walk-switch random walk on the wreath product $K \wr H$ is given by the appropriate version of the Donsker-Varadhan large deviation theorem for the random walk on the base H driven by μ . The particular functional $\sum_h F(l(n, h))$ that needs to be treated depends on the nature of the

lamp-group K and the measure ν . Formula (1.3) is particularly interesting because, in the general context of random walks on groups, precise log-asymptotic of the probability of return are hard to obtain. The following result serves to illustrate this point.

Theorem 1.3 (Log-asymptotics on $\mathbb{Z}^D \wr \mathbb{Z}^d$). *Fix two integers $D, d \geq 1$. Let ν be any finite symmetric measure on \mathbb{Z}^D with $\nu(0) > 0$ and generating support. Let δ_t be the anisotropic dilation on \mathbb{R}^d defined at (1.1). Let μ be a symmetric measure on \mathbb{Z}^d as in Theorem 1.1 with $\delta_n^{-1}(\mu^{(n)}) \implies \eta$ and $\hat{\eta} = e^{-\Theta}$. On the wreath product $\mathbb{Z}^D \wr \mathbb{Z}^d$ consider the switch-walk-switch random walks $q = \nu * \mu * \nu$. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{(2n)^{\frac{d}{d+\alpha}} (\log(2n))^{\frac{\alpha}{d+\alpha}}} \log q^{(2n)}(e) = -c(\alpha, d, \Theta, D)$$

where

$$c(\alpha, d, \Theta, D) = (D/2)^{\frac{\alpha}{d+\alpha}} \left(1 + \frac{d}{\alpha}\right) \left(\frac{\alpha \lambda_\Theta}{d}\right)^{\frac{d}{d+\alpha}},$$

$$\frac{1}{\alpha} = \frac{1}{d} \sum_1^d \frac{1}{\alpha_i} \text{ and } \lambda_\Theta = \inf_{U:|U|=1} \{\lambda_1(\Theta, U)\}.$$

Here, $\lambda_1(\Theta, U)$ is the principle eigenvalue of the infinitesimal generator L_Θ with Dirichlet boundary condition in U (By definition, $\widehat{L_\Theta f} = \Theta \hat{f}$).

Remark 1.4. Assume that the dilations δ_t are isotropic with $\alpha_i = \alpha, i = 1, \dots, d$ and that there is an Euclidean norm $\langle Qx, x \rangle$ such that $\Theta(\xi) = \langle Q\xi, \xi \rangle^\alpha$. Then λ_Θ is achieved on an Euclidean ball for the Euclidean structure provided by Q^{-1} , namely, the Euclidean ball whose volume is one. Note that the volume is computed here with respect to the Lebesgue measure corresponding to the fixed square lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. This fact is well-known when $\alpha = 2$ and follows from [1] when $\alpha \in (0, 2)$.

For any finitely generated group G and any $\alpha \in (0, 2)$, [2] introduces a non-increasing function

$$\tilde{\Phi}_{G, \rho_\alpha} : \mathbb{N} \ni n \rightarrow \tilde{\Phi}_{G, \rho_\alpha}(n) \in (0, \infty)$$

which, by definition, provides the best possible lower bound

$$\exists c > 0, N \in \mathbb{N}, \forall n, \mu^{(2n)}(e) \geq c \tilde{\Phi}_{G, \rho_\alpha}(Nn),$$

valid for every measure μ on G satisfying the weak- α -moment condition

$$W(\rho_\alpha, \mu) = \sup_{s>0} \{s \mu(\{g : \rho_\alpha(g) > s\})\} < \infty.$$

Here $|g|$ is the word-length of G with respect to some fixed finite symmetric generating set and $\rho_\alpha(g) = (1 + |g|)^\alpha$. For instance, it is well know and easy to see that

$$\tilde{\Phi}_{\mathbb{Z}^d, \rho_\alpha}(n) \simeq n^{-d/\alpha}.$$

Here and throughout this paper, we write $f \sim g$ if $\lim f/g = 1$ and $f \simeq g$ if there are constants $c_i, 1 \leq i \leq 4$, such that $c_1 f(c_2 t) \leq g(t) \leq c_3 f(c_4 t)$ on the relevant real interval or on \mathbb{N} . We use \simeq only when at least one of the functions f, g is monotone (or roughly monotone).

The main results of the present work allow us to complement some of the lower bounds proved in [2] for $\tilde{\Phi}_{G, \rho_\alpha}$ with matching upper bounds (note that upper bounds on $\tilde{\Phi}_{G, \rho_\alpha}$ are proved by exhibiting a measure with finite weak- α -moment and the appropriate return probability behavior).

Theorem 1.5. Fix $\alpha \in (0, 2)$. Let G be the group $K \wr \mathbb{Z}^d$.

1. Assume that K is finite. Then

$$\log \tilde{\Phi}_{G, \rho_\alpha}(n) \simeq -n^{d/(d+\alpha)}.$$

2. Assume that K has polynomial volume growth. Then

$$\log \tilde{\Phi}_{G, \rho_\alpha}(n) \simeq -n^{d/(d+\alpha)} (\log n)^{\alpha/(d+\alpha)}.$$

3. Assume that K is polycyclic with exponential volume growth. Then

$$\log \tilde{\Phi}_{G, \rho_\alpha}(n) \simeq -n^{(d+1)/(d+1+\alpha)}.$$

Remark 1.6. The lower bounds are from [2]. The upper bound in the first statement is already in [2] since it is based on the classical large deviation result in [8]. The upper bounds in Statements 2 and 3 make use of the extensions of [8] in the spirit of [3] developed here.

Iterated applications of this technique gives the following Theorem.

Theorem 1.7. Fix $\alpha \in (0, 2)$ and integers d_1, \dots, d_r . Given a group K , let

$$G = (\dots (K \wr \mathbb{Z}^{d_1}) \wr \dots) \wr \mathbb{Z}^{d_r} \text{ and } d = \sum_1^r d_i.$$

1. Assume that K is finite. Then

$$-\log \tilde{\Phi}_{G, \rho_\alpha}(n) \simeq n^{d/(d+\alpha)}.$$

2. Assume that K has polynomial volume growth. Then

$$-\log \tilde{\Phi}_{G, \rho_\alpha}(n) \simeq n^{d/(d+\alpha)} (\log n)^{\alpha/(d+\alpha)}.$$

2 Operator-stable laws

For $\alpha \in (0, 2)$, the rotationally symmetric α -stable law with density f_α on \mathbb{R}^d is the probability distribution whose Fourier transform is $e^{-|\xi|^\alpha}$. It is embedded in a convolution semigroup with density f_t^α which satisfies $f_t^\alpha(x) = t^{-d/\alpha} f_\alpha \circ \delta_{1/t}^\alpha$ where δ_t^α is the isotropic dilation $\delta_t^\alpha(x) = t^{1/\alpha} x$, $x \in \mathbb{R}^d$, $t > 0$.

More generally, a probability measure μ on \mathbb{R}^d is called a (non-degenerate) symmetric α -stable law if its support is \mathbb{R}^d and it is embedded in a probability semigroup μ^t such that $\delta_t^\alpha(\mu) = \mu^t$. A necessary and sufficient condition for that property is that $\hat{\mu} = e^{-\Theta}$ with

$$\Theta(\xi) = \int_{\mathbb{S}^{d-1}} \int_0^\infty (1 - \cos(\xi, \delta_r^\alpha y)) \frac{M(dy)}{r} \frac{dr}{r}$$

where M is a finite Borel measure on \mathbb{S}^{d-1} whose support generates \mathbb{R}^d (that is, the Lévy measure W of μ satisfies $\delta_t^\alpha(W) = tW$ and its support generates \mathbb{R}^d).

In the next section, we briefly review the definition of operator-stable laws. In this definition, the role of the isotropic dilations is played by more general one-parameter groups of transformations $t^E = \sum_0^\infty \frac{(\log t)^n E^n}{n!}$ where E is an endomorphism of the underlying vector space. For a detail account of the theory of operator-stable laws, see [13, 15]. Given a Borel measure μ , we let $t^E(\mu)$ be the Borel measure defined by $t^E(\mu)(A) = \mu(t^{-E}(A))$.

2.1 Operator-stable laws

Let \mathbb{V} be a finite dimensional vector space equipped with the Euclidean scalar product $\langle \cdot, \cdot \rangle$. Let $\mathcal{M}^1(\mathbb{V})$ denote the set of probability measures on \mathbb{V} . Given $\mu \in \mathcal{M}^1$, let $\hat{\mu} = e^{-\psi}$ denotes its Fourier transform. Let $\mathcal{ID}(\mathbb{V})$ denotes the set of infinitely divisible laws on \mathbb{V} . Throughout this section, we use notation compatible with [13]. Recall that if $\mu \in \mathcal{ID}(\mathbb{V})$ with Fourier transform $e^{-\psi}$ then $e^{-t\psi}$ is the Fourier transform of a probability measure μ^t and $(\mu^t)_{t \geq 0}$ is a continuous convolution semigroup of measure (uniquely determined by μ). Of course, for $\mu \in \mathcal{ID}(\mathbb{V})$, the function ψ admits a Levy-Khinchine representation so that $x \mapsto \psi(x)$ is the sum of three terms, namely, the drift term $-i\langle c, x \rangle$ with $c \in \mathbb{V}$, the Gaussian term $\frac{1}{2}\langle Qx, x \rangle$, where $Q \in \text{End}^+(\mathbb{V})$, and the generalized Poisson term

$$- \int_{\mathbb{V}^*} \left(e^{i\langle x, \xi \rangle} - 1 - \frac{i\langle x, \xi \rangle}{1 + \|\xi\|^2} \right) W(d\xi)$$

where W is a Levy measure. Following [13], we call the triple (c, Q, W) the L-K triple of μ (this triple is uniquely determined by μ). We will be interested in the symmetric case where $c = 0$ and $W(x) = W(-x)$. In this case, the Poisson term of the Levy-Khinchine formula equals

$$\int_{\mathbb{V}^*} (1 - \cos\langle x, \xi \rangle) W(d\xi).$$

In general, we let η_Q be the (Gaussian) law associated with the triple $(0, Q, 0)$ and $e(W)$ the (generalized-Poisson) law associated with $(0, 0, W)$.

Definition 2.1 (Definition 1.3.11 [13]). A law $\eta \in \mathcal{ID}(\mathbb{V})$ is said to be operator-stable if there exist $E \in \text{End}(\mathbb{V})$ and a mapping $a : \mathbb{R}_+^\times \rightarrow \mathbb{V}$ such that

$$t^E(\eta) * \delta_{a(t)} = \eta^t,$$

for all $t \in \mathbb{R}_+^\times$. In this case, E is called an exponent of η . Let $\text{EXP}(\eta)$ denote the set of exponents of η . If $a \equiv 0$, η is said to be strictly operator-stable.

One can always split an operator-stable law into a Gaussian part and a generalized Poisson part that are supported on supplementary linear subspaces of \mathbb{V} .

The subspace supporting the Gaussian part is either trivial or associated with the eigenvalues z of E with $\text{Re}(z) = 1/2$ of E . The subspace spanned by the support of W is associated with the eigenvalues z of E with real part strictly larger than $1/2$. Both the Gaussian part $e(Q)$ and the Poisson part $e(W)$ are operator stable with exponent E . Further, $T^E(W) = tW$. See the splitting theorem, [13, Lemma 1.3.12 and Theorem 1.3.14].

Now we restrict our attention to symmetric operator stable laws (so that $c = 0, W(dx) = W(-dx)$). Since $t^E(W) = tW$, the Fourier transform of $e(W)$ can be written (with $\mathbb{S} \subset \mathbb{V}$, the unit sphere)

$$\int_{\mathbb{V}^*} (1 - \cos\langle x, \xi \rangle) W(d\xi) = \int_0^\infty \int_{\mathbb{S}} (1 - \cos\langle \xi, r^E y \rangle) \frac{M(dy)}{r} \frac{dr}{r}$$

where M is a finite measure on \mathbb{S} . Compare with the hypothesis in Theorem 1.1 and Theorem 1.2.

Choose an orthonormal basis $\{e_i\}$ on \mathbb{V} with respect to inner product $\langle \cdot, \cdot \rangle$. The generating functional A of $(\eta^t)_{t \geq 0}$ (see [13, 1.3.16]) is given for $f \in C^2(\mathbb{V})$ by

$$\begin{aligned} \langle A, f \rangle &= \frac{1}{2} \sum q_{ij} \cdot \frac{\partial^2}{\partial x_i \partial x_j} f(0) \\ &+ \int_{\mathbb{V}^*} \left[f(x) - f(0) - \sum \frac{\partial}{\partial x_i} f(0) \cdot \frac{x_i}{1 + \|x\|^2} \right] W(dx). \end{aligned}$$

One can also write down the Dirichlet form of the continuous convolution semigroup $(\eta^t)_{t \geq 0}$ as

$$\begin{aligned} \mathcal{E}_\eta(f, g) &= \frac{1}{2} \int_{\mathbb{R}^d} \sum q_{ij} \cdot \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_j} dx \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (f(x+y) - f(x))(g(x+y) - g(x)) W(dy) dx, \\ \mathcal{D}(\mathcal{E}_\eta) &= \{f \in L^2(\mathbb{V}) : \mathcal{E}_\eta(f, f) < \infty\}. \end{aligned}$$

From the splitting theorem [13, Theorem 1.3.14] it follows that (q_{ij}) is semi-positive definite and that the subspace where it is positive definite is the support of the Gaussian part $e(Q)$.

Example 2.1 (Anisotropic radial operator-stable laws). One can construct operator-stable laws with respect to non-isotropic homogeneous norms. On $\mathbb{V} = \mathbb{R}^d$, let E be a $d \times d$ diagonal matrix with diagonal entries $a_i \in (\frac{1}{2}, \infty)$. We may assume that $a_1 = \min_{1 \leq i \leq d} a_i$. Since

$$t^E = \begin{pmatrix} t^{a_1} & 0 & \dots & 0 \\ 0 & t^{a_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{a_d} \end{pmatrix},$$

we can think of t^E as dilations scaling differently in different coordinates. The following norm was considered in [14]. Let $\mathbb{B} = \{x : \|x\| < 1\}$ be the open Euclidean unit ball, define

$$\|x\|_{*,E} := \inf\{t : t^{-a_1^{-1}E}x \in \mathbb{B}\}.$$

From Theorem 1 in [14], $\|\cdot\|_{*,E}$ is a sub-additive homogeneous norm. Set

$$W(dx) = \frac{c}{\|x\|_{*,E}^{a_1^{-1} + \text{tr}(a_1^{-1}E)}}.$$

Clearly, $t^E(W) = tW$ for all $t \in \mathbb{R}_+^\times$. Let η be the generalized Poisson law with L-K triple $(0, 0, W)$. Then η is operator-stable with exponent E . Note that the assumption $a_1 > \frac{1}{2}$ is needed so that W is a Lévy measure.

Example 2.2 (Anisotropic axial operator-stable laws). Let E be as in the previous example. For $\alpha \in (0, 2)$ let ν_α be the one-dimensional symmetric α -stable law (so that $\hat{\nu}^\alpha(y) = e^{-|y|^\alpha}$). Let η be the product measure on $\mathbb{V} = \mathbb{R}^d$ given by $\eta = \otimes_1^d \nu_{1/a_i}$ so that $\hat{\eta}(\xi) = e^{-\sum_1^d |\xi_i|^{1/a_i}}$. Clearly, η is operator-stable with exponent E . Note that in this case, the Levy measure is supported on the union of the axes.

2.2 Domain of operator-attraction

For full probability laws, the class of operator-stable laws coincides with limit distributions of normalized sums of i.i.d. random variables and convergence in law of normalized sums can be characterized in terms of convergence of Fourier transforms or convergence of generators as in Trotter’s theorem. More precisely, we have the following equivalent characterizations of convergence.

Theorem 2.2 ([13, Theorem 1.6.12 and Corollary 1.6.18]). *Let $\mu, \eta \in \mathcal{M}^1(\mathbb{V})$ with $\eta \in \mathcal{ID}(\mathbb{V})$ and $\hat{\eta} = e^{-\psi}$. Let $T_n \in GL(\mathbb{V})$ and set $\mu_n = T_n\mu$. The following properties are equivalent:*

1. $\mu_n^{(n)} \implies \eta$.

2. $\mu_n^{(\lfloor nt \rfloor)} \implies \eta^t$, uniformly in t over compact subsets of $[0, \infty)$.
3. $n(1 - \hat{\mu}_n) \rightarrow \psi$ uniformly on compact subsets.
4. For any $f \in C^2(\mathbb{V})$, $n(\mu_n - \delta_0) * f(0) \rightarrow \langle A, f \rangle$ where A is the generating functional of η .

Next, we introduce the definition of strict domain of operator-attraction.

Definition 2.3 (Definition 1.6.3 [13]). Let $\eta \in \mathcal{M}^1(\mathbb{V})$. Then the strict domain of operator-attraction $\text{DOA}_s(\eta)$ of η consists of all $\mu \in \mathcal{M}^1(\mathbb{V})$ for which there exists a sequence T_n in $GL(\mathbb{V})$ such that

$$\eta = \lim_{n \rightarrow \infty} T_n(\mu^{(n)}).$$

Remark 2.4. With this definition, $\text{DOA}_s(\eta) \neq \emptyset$ is equivalent to saying η can be obtained as the limiting distribution of convolution powers of some μ after normalization (but without re-centering). The word “strict” refers to the absence of re-centering. When T_n can be taken as the isotropic matrix $b_n Id$, $b_n \in \mathbb{R}_+$, this agrees with the definition of the strict domain of attraction.

Definition 2.5 (Definition 1.10.1 [13]). Let $\eta \in \mathcal{M}^1(\mathbb{V})$ be operator-stable. Then its strict domain of normal operator-attraction $\text{DNOA}_s(\eta)$ consists of all $\mu \in \mathcal{M}^1(\mathbb{V})$ such that

$$\eta = \lim_{n \rightarrow \infty} n^{-E}(\mu^{(n)})$$

for some $E \in \text{EXP}(\eta)$.

Example 2.3. Let U (resp. V) be a random variable on \mathbb{Z} in the domain of normal attraction of the α (resp. β) symmetric-stable law ν_α (ν_β resp.) on \mathbb{R} . The measure η on \mathbb{R}^2 $\eta(dx, dy) = \nu_\alpha(dx) \otimes \nu_\beta(dy)$ is operator stable with exponent

$$E = \begin{pmatrix} \frac{1}{\alpha} & 0 \\ 0 & \frac{1}{\beta} \end{pmatrix}.$$

It is clear that the law of $(U, V)^T$ is in $\text{DNOA}_s(\eta)$. Set

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix},$$

and let μ denote the distribution of $(X, Y)^T$. In order to obtain convergence of $\mu^{(n)}$ we need to rotate back by a rotation of angle θ then normalize component-wise. That is, setting

$$T_n = \begin{pmatrix} n^{-\frac{1}{\alpha}} & 0 \\ 0 & n^{-\frac{1}{\beta}} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix},$$

we have $\eta = \lim_{n \rightarrow \infty} T_n(\mu^{(n)})$. So, in this case, $\mu \in \text{DOA}_s(\eta)$ but does not belong to $\text{DNOA}_s(\eta)$.

Remark 2.6. Theorem 4.11.5 of [15] gives a practical criterion to show that a given measure ν belongs to the domain of normal attraction of a full operator-stable law η without Gaussian part. More precisely, the following statement is a simple modification

of [15, Theorem 4.11.5]. Let η be operator-stable, symmetric, with no Gaussian part and Lévy measure W given by

$$W(B) = \int_{\mathbb{S}} \int_0^\infty \mathbf{1}_B(r^E y) \frac{M(dy)}{r} \frac{dr}{r}.$$

Let $E \in \text{EXP}(\eta)$. A necessary and sufficient condition for a probability measure ν to be such that $n^{-E}(\nu^{(n)}) \implies \eta$ is that

$$\lim_{t \rightarrow \infty} t\nu(\{s^E x : x \in \Omega, s > t\}) = M(\Omega)$$

for any measurable $\Omega \in \mathbb{S}_d$ with $M(\partial\Omega) = 0$.

Example 2.4. For $\gamma \in (0, 2)$, let η_γ^t be symmetric stable law on \mathbb{R} with Fourier transform $e^{-t|\xi|^\gamma}$. On \mathbb{Z} , fix a doubly infinite symmetric sequence $z_k = -z_{-k}$, $k \in \mathbb{Z}$, and reals $p_k = p_{-k} \geq 0$ with $\sum p_k = 1$. Consider the probability measure

$$\mu = \sum_{k \in \mathbb{Z}} p_k \mathbf{1}_{z_k}.$$

Consider the case when $z_k = \lfloor k^\beta \rfloor$ and $p_k = c_\alpha(1 + |k|)^{-\alpha}$ with $\alpha > 1$, $\beta \geq 1$ and $\gamma = \beta/(\alpha - 1) < 2$. Then, by Remark 2.6 (in fact, in this particular case, by [15, Theorem 4.11.5]), $n^{-\gamma} \mu^{(n)} \implies \eta_\gamma^c$ for some fixed $c > 0$.

Note that if $z_k = \lfloor 2^{\beta k} \rfloor$ and $p_k = 2^{-\alpha k}$ with $\alpha, \beta > 0$ and $\gamma = \beta/\alpha$ then $t\nu(\{s^\gamma : s > t\})$ stays in a compact interval in $(0, \infty)$ but does not converges.

Notation 1. A measure $\mu \in \mathcal{M}^1(\mathbb{V})$ is said to be adapted if μ is not supported by a proper linear subspace of \mathbb{V} . Let $\mathcal{M}_a^1(\mathbb{V})$ denote the set of adapted probability measures on \mathbb{V} .

The theorem below is a characterization of strictly operator-stable laws as those adapted distributions whose domain of strict operator-attraction is non-empty.

Theorem 2.7 (Theorem 1.6.4 [13]). *For $\eta \in \mathcal{M}_a^1(\mathbb{V})$ the following assertions are equivalent:*

1. η is strictly operator-stable.
2. $\eta \in \text{DOA}_s(\eta)$.
3. $\text{DOA}_s(\eta) \neq \emptyset$.

For $\mu \in \text{DOA}_s(\eta)$, the choice of normalization sequence T_n is in general not unique. In particular, we can adjust T_n using the symmetries of the limiting distribution η and the convergence still holds.

Definition 2.8 (Definition 1.2.8. [13]). Let $\eta \in \mathcal{M}^1(\mathbb{V})$ be non-degenerate. Let $\text{Sym}(\eta)$ be the set of all $A \in GL(\mathbb{V})$ such that there exists some $a \in \mathbb{V}$ such that $A(\eta) * \delta_a = \eta$. The group $\text{Sym}(\eta)$ is called the symmetry group of η . It is a closed subgroup of $GL(\mathbb{V})$. The invariance group $\text{Inv}(\eta)$ is the set of all $A \in GL(\mathbb{V})$ such that $A(\eta) = \eta$. The group $\text{Inv}(\eta)$ is a closed subgroup of $\text{Sym}(\eta)$.

The following technical result is important for our purpose. It says that we can always adjust the normalization sequence by elements in $\text{Inv}(\eta)$, so that the new normalization sequence has nice regular variation properties.

Theorem 2.9 (Theorem 1.10.19 [13]). *Suppose μ is in the strict domain of attraction of a full operator stable law η , that is, there exists a sequence of invertible matrices $B_n \in GL(\mathbb{V})$ such that*

$$B_n^{-1}\mu^{(n)} \Longrightarrow \eta.$$

Then there exists a modified normalization sequence $\{B'_n = B_n S_n\}$, $S_n \in \text{Inv}(\eta)$, hence still fulfilling

$$(B'_n)^{-1}\mu^{(n)} \Longrightarrow \eta,$$

with the property that $\{B'_n\}$ has regular variation in the sense that

$$B'_n(B'_{[nt]})^{-1} \rightarrow t^{-E},$$

where the convergence is uniform in t on compact subsets of \mathbb{R}_+^\times .

2.3 Two more examples on \mathbb{Z}^2

In this subsection we discuss two examples on \mathbb{Z}^2 that are in the strict domain of operator-attraction of some operator-stable laws. For later use, we include the additional requirement that the inverse of the normalization sequence preserve the lattice \mathbb{Z}^2 .

Note that a key point in these examples is that they describe probability measures supported on the square lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ which implies a certain rigidity in the choice of the Euclidean structure on \mathbb{R}^2 .

Example 2.5. Let e_1, e_2 be the standard basis for \mathbb{R}^2 . Consider the two unit vectors $u_1 = \frac{1}{\sqrt{2}}(e_1 + e_2)$ and $u_2 = \frac{1}{\sqrt{1+\pi^2}}(e_1 + \pi e_2)$. Let μ_1, μ_2 be probability measures defined by

$$\begin{aligned} \mu_1(x_1, x_2) &= \frac{c_1}{(1 + |x_1|)^{\alpha+1}} \mathbf{1}_{\{x_1=x_2\}}, \\ \mu_2(x_1, x_2) &= \frac{c_2}{(1 + |x_1|)^{\beta+1}} \mathbf{1}_{\{|x_2 - \pi x_1| \leq 1\}}, \end{aligned}$$

where c_1 and c_2 are normalizing constants and $\alpha, \beta \in (0, 2)$. Take

$$\mu = \frac{1}{2}(\mu_1 + \mu_2).$$

Write $P = (u_1, u_2) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{1+\pi^2}} \\ \frac{1}{\sqrt{2}} & \frac{\pi}{\sqrt{1+\pi^2}} \end{pmatrix}$ and $E = P \begin{pmatrix} \frac{1}{\alpha} & 0 \\ 0 & \frac{1}{\beta} \end{pmatrix} P^{-1}$. Then we can check that for $\Omega \in \mathcal{B}(\mathbb{S}_2)$ with $u_1, u_2 \notin \partial\Omega$, we have

$$\lim_{t \rightarrow \infty} t\mu(\{s^E x : x \in \Omega, s > t\}) = \lambda_1 \mathbf{1}_{\{u_1\}}(\Omega) + \lambda_2 \mathbf{1}_{\{u_2\}}(\Omega),$$

where λ_1 and λ_2 are positive constants determined by μ . Consider the generalized Poisson law η with Lévy measure W given by

$$W(B) = \sum_{i=1,2} \int_0^\infty \mathbf{1}_B(r^E u_i) \frac{\lambda_i}{r} \frac{dr}{r}.$$

Note that W is supported on the union of the one-dimensional subspaces $\mathbb{R}u_1$ and $\mathbb{R}u_2$ and η is operator-stable with exponent E . The law η can be viewed as a product of two one-dimensional symmetric stable laws supported on $\mathbb{R}u_1$ and $\mathbb{R}u_2$ and of exponents α and β , respectively (the exact scale parameter for each of these stable laws

is determined by the constants λ_1, λ_2). From the convergence theorem, Theorem 2.2, $\mu \in \text{DNOA}_s(\eta)$. Set

$$B_n = \left\lfloor P \begin{pmatrix} n^{\frac{1}{\alpha}} & 0 \\ 0 & n^{\frac{1}{\beta}} \end{pmatrix} P^{-1} \right\rfloor$$

where $\lfloor \cdot \rfloor$ means take integer parts of each matrix entry. Then $n^E - B_n$ is a matrix with entries in $[0, 1)$ and it follows that $B_n \cdot n^{-E} \rightarrow I$. We conclude that $B_n^{-1} \mu^{(n)} \Rightarrow \eta$.

Example 2.6. Take $u_1, u_2, P, \alpha, \beta$ and E the same as in the previous example. Write $\delta_r(x_1, x_2) = (r^{1/\alpha} x_1, r^{1/\beta} x_2)$. Let \mathbb{S} denote the Euclidean unit circle, σ the Lebesgue measure on \mathbb{S} . Let Γ be the union of the two arcs $[0, \frac{\pi}{2}]$ and $[\pi, \frac{3\pi}{2}]$ of \mathbb{S} . Define the Lévy measure W as

$$W(B) = \int_{\Gamma} \int_0^{\infty} \mathbf{1}_B(\delta_r x) \frac{\sigma(dx) dr}{r^2}.$$

Then W is supported in the cone $\{x \in \mathbb{R}^2 : x_1 x_2 \geq 0\}$. Consider measure $PW(\cdot) = W(P^{-1}\cdot)$, that is the pushforward of measure W under linear transformation P . Let η be the generalized Poisson law with Lévy measure PW . Take a discrete approximation μ of PW supported on \mathbb{Z}^2 by setting

$$\mu(x) = PW([x_1, x_1 + 1) \times [x_2, x_2 + 1)).$$

One can check that $nB_n^{-1} \mu \rightarrow PW$ weakly with $B_n = \lfloor n^E \rfloor$. It follows from Theorem 2.2 that

$$B_n^{-1} \mu^{(n)} \Rightarrow \eta.$$

3 Functionals of the occupation time vector

Given a probability measure μ on the lattice \mathbb{Z}^d , let $(X_i)_{i \geq 0}$ be the associated random walk. Let $(l(n, x))_{x \in \mathbb{Z}^d}$ be the occupation time vector at time n where $l(n, x) = \#\{k \in \{0, \dots, n\} : X_k = x\}$. Let $F : [0, \infty) \rightarrow [0, \infty)$.

In this section we introduce basic natural hypotheses on μ and F under which we can derive the log-asymptotic behavior of

$$\mathbf{E} \left(e^{-\sum_{x \in \mathbb{Z}^d} F(l(n, x))} \right).$$

Definition 3.1 (Convergence assumption). We say that μ satisfies the convergence assumption (C- B_n) if there exists a sequence of invertible matrices $B_n \in \mathbb{Z}^{d \times d}$ and a probability distribution η such that

$$B_n^{-1} \mu^{(n)} \Rightarrow \eta. \tag{C- B_n }$$

Remark 3.2. Note that (C- B_n) requires the matrices B_n to have integer entries so that $B_n \mathbb{Z}^d \subset \mathbb{Z}^d$. Note also that the distribution η is strictly operator-stable.

Under the convergence assumption (C- B_n), [12] provides a local limit theorem that plays an important role in the proof of the uniform large deviation principle.

Theorem 3.3 (Theorem 6.4 [12]). *Suppose μ is in the domain of attraction of a symmetric, adapted strictly operator-stable law η on \mathbb{R}^d with density g , that is, there exists a sequence of invertible matrices B_n such that*

$$B_n^{-1} \mu^{(n)} \Rightarrow \eta.$$

Then

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{Z}^d} |\det B_n| \left| \mu^{(n)}(x) - |\det B_n^{-1}| g(B_n^{-1}x) \right| = 0.$$

Remark 3.4. Note that the density g of an operator-stable law is always smooth. In [8] it is essentially proved, although not stated explicitly, that given the local limit theorem, the scaled occupation time measures satisfy a uniform large deviation principle in \mathcal{L}_1 . We will state and outline the proof of the large deviation principles later in this paper.

Remark 3.5. It is somewhat surprising that, in this case, the “weak limit assumption” always implies the local limit theorem. The proof in [12] relies on the Fourier transform. On the Heisenberg group, there are measures that converges to a (Heisenberg group) Gaussian law, but do not satisfy the local limit theorem.

Example 3.1. Fix $\alpha, \beta \in (0, 2)$ and consider the probability measure μ on $\mathbb{Z}^2 \subset \mathbb{R}^2$ given by

$$\mu = \frac{1}{2} \left(\sum_{x \in \mathbb{Z}} c_\alpha (1 + |x|)^{-1-\alpha} \delta_{(x,0)} + \sum_{y \in \mathbb{Z}} c_\beta (1 + |y|)^{-1-\beta} \delta_{(0,y)} \right).$$

Set $E = \begin{pmatrix} 1/\alpha & 0 \\ 0 & 1/\beta \end{pmatrix}$ and $\eta = \eta_\alpha \otimes \eta_\beta$ where η_α, η_β are (appropriately scaled) one dimensional symmetric stable laws with parameters α, β , respectively. Then condition (C- B_n) is satisfied with $B_n = \begin{pmatrix} \lfloor n^{1/\alpha} \rfloor & 0 \\ 0 & \lfloor n^{1/\beta} \rfloor \end{pmatrix}$. Theorem 3.3 provides a local limit theorem for $\mu^{(n)}((x, y))$ in the form

$$n^{1/\alpha+1/\beta} |\mu^{(n)}((x, y)) - n^{-(1/\alpha+1/\beta)} f_\alpha^{c_1}(x/n^{1/\alpha}) f_\beta^{c_2}(y/n^{1/\beta})| \rightarrow 0$$

where f_α^t is the density of the symmetric stable semigroup, i.e., has Fourier transform $e^{-t|\xi|^\alpha}$ and c_1, c_2 are appropriate constants.

Next we introduce a scaling assumption regarding the function F . It is the operator-stable analog of the scaling assumption in [3].

Definition 3.6 (Scaling assumption). Let B_n be as in condition (C- B_n). We say that a function $F : [0, \infty) \rightarrow [0, \infty)$ satisfies the scaling assumption (S- B_n - a_n) if F is concave, sub-additive, increasing with $F(0) = 0$ and there exist a non-decreasing sequence $n \rightarrow a_n \in \mathbb{N}$ and a limiting function $\tilde{F} : [0, \infty) \rightarrow [0, \infty)$, \tilde{F} not identically zero, such that for $y > 0$,

$$\lim_{n \rightarrow \infty} \frac{a_n \det(B_{a_n})}{n} F \left(\frac{n}{\det(B_{a_n})} y \right) = \tilde{F}(y), \tag{S- B_n - a_n }$$

uniformly over compact sets in $(0, \infty)$.

The following technical proposition is crucial. It is analogous to [3, Proposition 1.1]. The proof is given in the Appendix.

Proposition 3.7. Assume the convergence assumption (C- B_n) and the scaling assumption (S- B_n - a_n) as above. Then there exists $\gamma \in [0, 1]$ such that

$$\tilde{F}(y) = \tilde{F}(1)y^\gamma, \quad y > 0,$$

Moreover, there exists $\kappa > 0$ such that

$$\lim_{n \rightarrow \infty} \frac{a_{\lfloor \lambda n \rfloor}}{a_n} = \lambda^\kappa \text{ for all } \lambda \in \mathbb{R}^+ \text{ and } \lim_{n \rightarrow \infty} \frac{\log a_n}{\log n} = \kappa.$$

Definition 3.8. Following [3], given μ satisfying (C- B_n) and a function $F : [0, \infty) \rightarrow [0, \infty)$, we say that the pair $(F, (B_n))$ is in the γ -class, if there is a sequence a_n such that the scaling assumption (S- B_n - a_n) is satisfied, and the limiting function \tilde{F} is homogeneous with exponent γ .

The following statement is the main result of this paper. The proof is given in Section 5.2.

Theorem 3.9. Fix a symmetric probability measure μ on \mathbb{Z}^d and a function $F : [0, \infty) \rightarrow [0, \infty)$. Under the convergence assumption (C- B_n) and the scaling assumption (S- B_n - a_n), there exists a constant $k(\eta, \tilde{F}) \in (0, \infty)$ such that

$$\lim_{n \rightarrow \infty} \frac{a_n}{n} \log \mathbf{E} \left(e^{-\sum_{x \in \mathbb{Z}^d} F(l(n,x))} \right) = -k \left(\eta, \tilde{F} \right). \tag{3.1}$$

Further, for any $\epsilon > 0$ small enough there is $R > 1$ such that

$$\lim_{n \rightarrow \infty} \frac{a_n}{n} \log \mathbf{E} \left(e^{-\sum_{x \in \mathbb{Z}^d} F(l(n,x))} \mathbf{1}_{B(R)}(B_{a_n}^{-1}(X_n)) \right) \geq -(1 + \epsilon)k \left(\eta, \tilde{F} \right). \tag{3.2}$$

Here $B(R)$ is the ball of radius R in \mathbb{R}^d .

Example 3.2. Theorems 1.1 and 1.2 are special cases of Theorem 3.9. In both cases, let E be the diagonal matrix with i -th diagonal entry $1/\alpha_i \in (2, \infty)$. Let η be an operator-stable law with exponent E and Fourier transform $\hat{\eta} = e^{-\Theta}$. Let μ be a measure such that

$$n(1 - \hat{\mu}(n^{-E}\xi)) \rightarrow \Theta(\xi). \tag{3.3}$$

Let B_n be the diagonal matrix with i -th diagonal entry $\lfloor n^{1/\alpha_i} \rfloor$. By Theorem 2.2, (3.3) implies that condition (C- B_n) is satisfied.

To obtain Theorem 1.1, set $F(s) = \mathbf{1}_{(0,\infty)}(s)$. Define $a_n = \lfloor a'_n \rfloor$ where a'_n is given by $a'_n \det(B_{a'_n}) = n$, that is, $a'_n = n^{1/(1+\tau)}$ where $\tau = \sum 1/\alpha_i$ is the trace of E . It is easy to see that condition (S- B_n - a_n) with $\tilde{F} = \mathbf{1}_{(0,\infty)}$.

For Theorem 1.2, we simply set $F(s) = \tilde{F}(s) = s^\gamma$, $\gamma \in (0, 1)$ and $a'_n = n^{(1-\gamma)/(1+\tau(1-\gamma))}$. Condition (S- B_n - a_n) follows.

Example 3.3. Assume that $\mu \in \text{DNOA}_s(\eta)$, $B_n = \lfloor n^E \rfloor$, $\text{tr}(E) = \tau$ and $F(y) = y^\gamma \ell(y)$ where $\gamma \in [0, 1]$ and ℓ is a slow varying function (at infinity) such that $\ell(t^a \ell(t)^b) \sim c(a)\ell(t)$ for any $a > 0$ and $b \in \mathbb{R}$. (e.g., $\ell(t) = (\log t)^\beta$, $\beta \in \mathbb{R}$). Then $\tilde{F}(y) = cy^\gamma$ and a_n is determined by solving

$$a_n^{1+\tau(1-\gamma)} \ell(na_n^{-\tau}) \sim n^{1-\gamma},$$

that is

$$a_n \sim c \left(\frac{n^{1-\gamma}}{\ell(n)} \right)^{1/(1+\tau(1-\gamma))}.$$

In this case the theorem yields the existence of a constant $k \in (0, \infty)$ such that

$$\log \mathbf{E} \left(\exp \left(- \sum_{x \in \mathbb{Z}^d} F(l(n,x)) \right) \right) \sim -k \left(n^{\gamma+\tau(1-\gamma)} \ell(n) \right)^{1/(1+\tau(1-\gamma))}.$$

Example 3.4. The previous examples treat cases where μ belongs to the domain of normal attraction of η . It is worth pointing out that Theorem 3.9 does not require normal attraction. For example, consider the case where μ is supported on \mathbb{Z} and is of the form $\mu(k) = \frac{c_\phi}{\phi(|k|)}$ where $\phi : [0, \infty) \rightarrow [1, \infty)$ is continuous and regularly varying of index $1 + \alpha$, $\alpha \in (0, 2)$. By a classical result (see [10]), μ is in the domain of attraction of an α -stable law η on \mathbb{R} . The normalizing sequence b_n such that $b_n^{-1}(\mu^{(n)}) \Rightarrow \eta$ can be chosen as the solution of the equation $b_n^{-2} \mathcal{G}(b_n) = 1/n$ where $\mathcal{G}(n) = \sum_0^n k^2 \mu(|k|)$, that is $b_n \sim \kappa \psi(1/n)$ where ψ is the inverse of $s \mapsto s/\phi(s)$. Note that ψ is regularly varying of index $-1/\alpha$. Suppose that $F(s) = s^\gamma$, $\gamma \in (0, 1)$. The sequence a_n in Theorem 3.9 is then given by equation (S- B_n - a_n), that is, $n^{-1} a_n b_{a_n} (n/b_{a_n})^\gamma = 1$, equivalently, $a_n \psi(1/a_n)^{1-\gamma} = \kappa^{\gamma-1} n^{1-\gamma}$. It follows that a_n varies regularly of index $\alpha(1-\gamma)/(1+\alpha-\gamma)$. Of course, a_n can be computed more explicitly in terms of ϕ .

4 Applications to random walks on groups

This section applies the large deviation asymptotics of Theorem 3.9 to obtain precise information about the decay of the return probability of random walks on wreath products with base \mathbb{Z}^d . We treat certain classes of random walks with unbounded support on the base and we allow a large class of lamp groups.

4.1 Random walks on wreath products

First we briefly review definition of wreath products and a special type of random walks on them. Our notation follows [16]. Let H, K be two finitely generated groups. Denote the identity element of K by e_K and identity element of H by e_H . Let K_H denote the direct sum:

$$K_H = \sum_{h \in H} K_h.$$

The elements of K_H are functions $f : H \rightarrow K, h \mapsto f(h) = k_h$, which have finite support in the sense that $\{h \in H : f(h) = k_h \neq e_K\}$ is finite. Multiplication on K_H is simply coordinate-wise multiplication. The identity element of K_H is the constant function $e_K : h \mapsto e_K$ which, abusing notation, we denote by e_K . The group H acts on K_H by translation:

$$\tau_h f(h'^{-1}h'), \quad h, h' \in H.$$

The wreath product $K \wr H$ is defined to be semidirect product

$$K \wr H = K_H \rtimes_\tau H, \\ (f, h)(f', h') = (f \cdot \tau_h f', hh').$$

In the lamplighter interpretation of wreath products, H corresponds to the base on which the lamplighter lives and K corresponds to the lamp. We embed K and H naturally in $K \wr H$ via the injective homomorphisms

$$k \longmapsto \underline{k} = (\mathbf{k}_{e_H}, e_H), \quad \mathbf{k}_{e_H}(e_H) = k, \quad \mathbf{k}_{e_H}(h) = e_K \text{ if } h \neq e_H \\ h \longmapsto \underline{h} = (e_K, h).$$

Let μ and ν be probability measures on H and K respectively. Through the embedding, μ and ν can be viewed as probability measures on $K \wr H$. Consider the measure

$$q = \nu * \mu * \nu$$

on $K \wr H$. This is the switch-walk-switch measure on $K \wr H$ with switch-measure ν and walk-measure μ .

Let (X_i) be the random walk on H driven by μ , and let $l(n, h)$ denote the number of visits to h in the first n steps:

$$l(n, h) = \#\{i : 0 \leq i \leq n, X_i = h\}.$$

Set also

$$l_*^g(n, h) = \begin{cases} l(n, h) & \text{if } h \notin \{e_H, g\} \\ l(n, e_H) - 1/2 & \text{if } h = g \\ l(n, e_H) - 1 & \text{if } h = e_H. \end{cases}$$

From [16], probability that the random walk on $K \wr H$ driven by q is at $(h, g) \in K \wr H$ at time n is given by

$$q^{(n)}((f, g)) = \mathbf{E} \left(\prod_{h \in H} \nu^{(2l_*^g(n, h))}(f(h)) \mathbf{1}_{\{X_n = g\}} \right)$$

Note that \mathbf{E} stands for expectation with respect to the random walk $(X_i)_0^\infty$ on H started at e_H .

From now on we assume that ν satisfies $\nu(e_K) = \epsilon > 0$ so that

$$\epsilon \nu^{(n-1)}(e_K) \leq \nu^{(n)}(e_K) \leq \epsilon^{-1} \nu^{(n-1)}(e_K).$$

Write $f \stackrel{C}{\asymp} g$ if $C^{-1}f \leq g \leq Cf$. Under these circumstances, we have

$$q^{(n)}((e_K, g)) \stackrel{1/\epsilon^3}{\asymp} \mathbf{E} \left(\prod_{h \in H} \nu^{(2l(n,h))}(e_K) \mathbf{1}_{\{X_n=g\}} \right)$$

so that we can essentially ignore the difference between l and l_* .

Set

$$F_K(n) := -\log \nu^{(2n)}(e_K)$$

so that, for any $g \in H$,

$$q^{(n)}((e_K, g)) \simeq \mathbf{E} \left(e^{-\sum_H F_K(l(n,h))} \mathbf{1}_{\{X_n=g\}} \right). \tag{4.1}$$

Definition 4.1 (weak scaling assumption). We say that ν satisfies the upper weak scaling assumption (US- B_n - a_n) if there exist a constant $c_0 > 0$ and a function $F : [0, \infty) \rightarrow [0, \infty)$ satisfying (S- B_n - a_n) and such that

$$\forall n \in \mathbb{N}, \quad c_0 F(n) \leq F_K(n). \tag{US- B_n - a_n)}$$

We say that ν satisfies the lower weak scaling assumption (LS- B_n - a_n) if there exist a constant $C_0 < \infty$ and a function $F : [0, \infty) \rightarrow [0, \infty)$ satisfying (B_n - a_n) and such that

$$\forall n \in \mathbb{N}, \quad F_K(n) \leq C_0 F(n) \tag{LS- B_n - a_n)}$$

If F_K satisfies both the upper and lower conditions,

$$\forall n \in \mathbb{N}, \quad c_0 F(n) \leq F_K(n) \leq C_0 F(n) \tag{WS- B_n - a_n)}$$

then we say it satisfies the weak scaling assumption (WS- B_n - a_n).

We can now use the large deviation asymptotics to estimate the return probability on wreath product $K \wr \mathbb{Z}^d$.

Theorem 4.2. Let μ be a symmetric probability measure on \mathbb{Z}^d which satisfies the convergence assumption (C- B_n). Let ν be a symmetric probability measure on K with $\nu(e_K) > 0$.

- Assume that ν satisfies (US- B_n - a_n). Then the switch-walk-switch measure $q = \nu * \mu * \nu$ on $K \wr \mathbb{Z}^d$ satisfies

$$\limsup_{n \rightarrow \infty} \frac{a_n}{n} \log q^{(n)}(e) \leq -k \left(\eta, c_0 \tilde{F} \right).$$

- Assume instead that ν satisfies (LS- B_n - a_n). Then we have

$$\liminf_{n \rightarrow \infty} \frac{a_{2n}}{2n} \log q^{(2n)}(e) \geq -k \left(\eta, C_0 \tilde{F} \right).$$

Remark 4.3. Roughly speaking, this theorem says the following: Assume we know how to normalize $\mu^{(n)}$ on the base \mathbb{Z}^d via a transformation B_n so that it converges to a limiting distribution η . Assume we know the behavior of the probability of return of the random walk on K driven by ν in the sense that $\log(\nu^{(2n)}(e_K)) \simeq -F(n)$. Then $q^{(n)}(e) \simeq \exp(-\frac{n}{a_n})$, where a_n can be computed from the scaling relation

$$\frac{a_n \det(B_{a_n})}{n} F\left(\frac{n}{\det(B_{a_n})}\right) \simeq 1.$$

Proof. The first statement follows immediately from (3.1) in Theorem 3.9. The second statement is deduced from (3.2) as follows. Since q is symmetric, we have $q^{(2n)}(e_H) \geq q^{(2n)}(g)$ for any $g \in K \wr \mathbb{Z}^d$. In particular, if $B(r) = B_{\mathbb{Z}^d}(r)$ is the ball of radius r in the lattice \mathbb{Z}^d then, by (4.1),

$$\begin{aligned} \#B(r)q^{(2n)}(e) &\geq c \sum_{h \in B(r)} q^{(2n)}(e_K, h) \\ &\simeq \mathbf{E}\left(e^{-\sum_H F_K(\ell(n,h))} \mathbf{1}_{B(r)}(X_{2n})\right). \end{aligned}$$

Picking $r = Ra_{2n}$ and using the fact that a_n has regular variation of order $\kappa > 0$ (see Proposition 3.7), one easily deduces from (3.2) that

$$\liminf_{n \rightarrow \infty} \frac{a_{2n}}{2n} \log q^{(2n)}(e) \geq -k\left(\eta, C_0 \tilde{F}\right).$$

as desired. □

Example 4.1 ($K \wr \mathbb{Z}^d$). (See Example 3.3) Let μ be a symmetric probability measure on \mathbb{Z}^d , $\mu \in \text{DNOA}_s(\eta)$, $B_n = \lfloor n^E \rfloor$, $\text{tr}(E) = \tau$ (this implies $\tau \geq d/2$). Let ν be a symmetric probability measure on K with $\nu(e_K) > 0$ and such that

$$\log \nu^{(2n)}(e_K) \simeq -F(n).$$

Assume that F is of the form $F(y) = y^\gamma \ell(y)$ where $\gamma \in [0, 1]$ and ℓ is a slow varying function (at infinity) such that $\ell(t^a \ell(t)^b) \sim c(a) \ell(t)$ for any $a > 0$ and $b \in \mathbb{R}$. (e.g., $\ell(t) = (\log t)^\beta$, $\beta \in \mathbb{R}$). Let q be the switch-walk-switch measure on $K \wr \mathbb{Z}^d$ associated with μ and ν . Then

$$\log q^{(2n)}(e) \simeq -\left(n^{\gamma+\tau(1-\gamma)} \ell(n)\right)^{1/(1+\tau(1-\gamma))}.$$

For a concrete example on $(\mathbb{Z} \wr \mathbb{Z}) \wr \mathbb{Z}^d$, let μ be the uniform probability on

$$\{0, \pm s_1, \dots, \pm s_d\} \subset \mathbb{Z}^d$$

where s_1, \dots, s_d are the unit vectors generating the square lattice \mathbb{Z}^d . Obviously, μ is in domain of normal attraction of the Gaussian measure and $\tau = d/2$. On $K = \mathbb{Z} \wr \mathbb{Z}$, let ν be the switch-walk-switch measure on $\mathbb{Z} \wr \mathbb{Z}$ where both the switch-measure and walk-measure are simple random walk on \mathbb{Z} with holding. In this case, $F(y) = y^{1/3} (\log y)^{2/3}$ and $\gamma = 1/3$ (see, e.g., [16]). Hence the measure $q = \nu * \mu * \nu$ on $K \wr \mathbb{Z}^d$ satisfies

$$\log q^{(2n)}(e) \simeq -n^{(1+d)/(3+d)} (\log n)^{2/(3+d)}.$$

We note that this result can also be obtained from Erschler's results [9]. Finally, keeping K and ν as above, we replace the measure μ on \mathbb{Z}^d by the measure $\mu_\alpha(x) = c(1 + \|x\|)^{-d-\alpha}$, $\alpha \in (0, 2)$, $\|x\| = (\sum_1^d |x_i|^2)^{1/2}$. Note that $\mu_\alpha \in \text{DNOA}_s(\eta_\alpha)$ where η_α is the rotationally symmetric α -stable law on \mathbb{R}^d and $\tau = d/\alpha$. If we set $q_\alpha = \nu * \mu_\alpha * \nu$ then we obtain

$$\log q_\alpha^{(2n)}(e) \simeq -n^{(1+2d/\alpha)/(3+2d/\alpha)} (\log n)^{2/(3+2d/\alpha)}.$$

The next theorem captures the fact that a better understanding of the return probability on the lamp-group K leads to a more precise asymptotics for $q^{(n)}(e)$.

Theorem 4.4. *Let μ be a symmetric probability measure on \mathbb{Z}^d which satisfies the convergence assumption (C- B_n). Let ν be a symmetric probability measure on K with $\nu(e_K) > 0$. Assume that the function $F_K(n) = -\log \nu^{(2n)}(e_K)$ satisfies the scaling assumption (S- $B_n^{-a_n}$). Then the measure $q = \nu * \mu * \nu$ on $K \wr \mathbb{Z}^d$ satisfies*

$$\lim_{n \rightarrow \infty} \frac{a_{2n}}{2n} \log q^{(2n)}(e) = -k \left(\eta, \tilde{F}_K \right).$$

Example 4.2. Referring to the setting of Theorem 4.4, assume that $\nu^{(2n)}(e_K)$ satisfies $\nu^{(2n)}(e_K) \simeq n^{-\theta}$ so that $F_K(n) \sim \theta \log n$. Assume μ is in the domain of normal attraction of η . Let $E \in \text{EXP}(\eta)$ such that $n^{-E}(\mu^{(n)}) \implies \eta$. Set $B_n = \lfloor n^E \rfloor$ (take integer values of all entries). Let $\tau = \text{tr}(E)$ be the trace of E . Solving for t in the scaling equation

$$\frac{t^{1+\tau}}{n} \log \left(\frac{n}{t^\tau} \right) = 1,$$

yields

$$a_n = \lfloor t \rfloor \sim \left(\frac{n}{\log n} \right)^{\frac{1}{1+\tau}}.$$

Then F_K satisfies the scaling assumption

$$\lim_{n \rightarrow \infty} \frac{a_n \det(B_{a_n})}{n} F_K \left(\frac{n}{\det(B_{a_n})} y \right) = \theta, \text{ for } y > 0.$$

Hence Theorem 4.4 yields

$$\lim_{n \rightarrow \infty} \frac{1}{(2n)^{\frac{\tau}{1+\tau}} (\log 2n)^{\frac{1}{1+\tau}}} \log q^{(2n)}(e) = -k \left(\eta, \tilde{F} \right),$$

where the limiting function \tilde{F} is given by $\tilde{F}(y) = \theta \cdot \mathbf{1}_{\{y>0\}}$.

4.2 Assorted examples

In this section we describe a number of explicit applications of Theorems 4.2 and 4.4.

Example 4.3. Let \mathbb{Z}^d be equipped with the canonical generating d -tuple $S = (s_1, \dots, s_d)$ and fix $a = (\alpha_1, \dots, \alpha_d) \in (0, 2)^d$. Consider the probability measure μ_a given by

$$\mu_a(x) = \frac{1}{d} \sum_{i=1}^d \sum_{n \in \mathbb{Z}} \frac{c(\alpha_i)}{(1 + |n|)^{1+\alpha_i}} \mathbf{1}_{\{n\}}(x_i), \quad x = (x_1, \dots, x_d). \tag{4.2}$$

This measure is quite obviously in the domain of normal operator attraction of $\eta_a = \otimes_1^d \eta_{\alpha_i}$ where η_{α_i} is a measure on \mathbb{R} which is symmetric and α_i -stable. In particular, the diagonal $d \times d$ matrix E_a with i -th diagonal entry $1/\alpha_i$ is in $\text{EXP}(\eta_a)$. The Dirichlet form \mathcal{E}_{η_a} associated to the limit law η_a is best described via Fourier transform as $\mathcal{E}_{\eta_a}(f, f) = \sum_1^d c_i \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 |\xi_i|^{2\alpha_i} d\xi$, $c_i > 0$, $i = 1, \dots, d$ (the scale parameters c_i are related but not equal to $c(\alpha_i)$).

Theorem 4.5. *On $H = \mathbb{Z}^d$, consider the measures μ_a defined above, $a \in (0, 2)^d$. Define $\alpha \in (0, 2)$ by*

$$\frac{1}{\alpha} = \frac{1}{d} \sum_1^d \frac{1}{\alpha_i}.$$

1. Let K be a finite group and let ν be the uniform measure on K . On $K \wr \mathbb{Z}^d$, let $q_a = \nu * \mu_a * \nu$. Then there exists a constant $k = k(d, a, |K|)$ such that

$$\log q_a^{(n)}(e) \sim -kn^{d/(d+\alpha)}.$$

2. Let $K = \mathbb{Z}^D$ and ν be a symmetric probability measure on \mathbb{Z}^D with $\nu(e_K) > 0$ which is in the domain of normal attraction of an adapted strictly operator-stable law η . On $\mathbb{Z}^D \wr \mathbb{Z}^d$, let $q_a = \nu * \mu_a * \nu$. Then there exists a constant $k = k(d, a, D, \nu)$ such that

$$\log q_a^{(n)}(e) \sim -kn^{d/(d+\alpha)}(\log n)^{\alpha/(d+\alpha)}.$$

Example 4.4. Set $H = \mathbb{Z}^d$, $K = \mathbb{Z}^D$, $G = K \wr H = \mathbb{Z}^D \wr \mathbb{Z}^d$. One natural set of generators of G is obtained by joining the canonical generators of $H = \mathbb{Z}^d$ and $K = \mathbb{Z}^D$ as follows. Let $(s_i^H)_1^d$ and $(s_i^K)_1^D$ be the canonical generators of H and K , respectively. Let $S = (s_i)_{1}^{d+D}$ be the generating tuple of G given by

$$s_i = (e_K, s_i^H) \text{ for } i \in \{1, \dots, d\} \text{ and } s_i = (s_i^K, e_H) \text{ for } i \in \{d+1, \dots, d+D\}.$$

Of course, $e_K = 0$ in \mathbb{Z}^D and $e_H = 0$ in \mathbb{Z}^d . Let

$$a = (\alpha_1, \dots, \alpha_{d+D}) \in (0, 2)^{d+D}$$

be a $(d+D)$ -tuple. Let $b = b(a) = (\beta_i)_1^d$ and $c = c(a) = (\gamma_i)_1^D$ with $\beta_i = \alpha_i$, $i = 1, \dots, d$ and $\gamma_i = \alpha_{d+i}$, $i = 1, \dots, D$. Let μ_b^H, μ_c^K be the probability measures on $H = \mathbb{Z}^d, K = \mathbb{Z}^D$, respectively, defined at (4.2). Let q be the switch-walk-switch measure on $G = K \wr H$ given by $q = \mu_c^K * \mu_b^H * \mu_c^K$. The theorem stated above applies and yields

$$\log q^{(n)}(e) \sim -k(d, D, a)n^{d/(d+\beta)}(\log n)^{\beta/(d+\beta)}, \quad \frac{1}{\beta} = \frac{1}{d} \sum_1^d \frac{1}{\beta_i}.$$

For S and a as defined above, let $\mu_{S,a}$ the the probability measure on $G = K \wr H$ defined by

$$\mu_{S,a}(g) = \frac{1}{k} \sum_i \sum_{n \in \mathbb{Z}} \mathbf{1}_{\{s_i^n\}}(g) \mu_i(n), \quad \mu_i(n) = c_i(1 + |n|)^{-1-\alpha_i}. \quad (4.3)$$

In words, this walk takes steps along the (discrete) one parameter groups $\langle s_i \rangle = \{s_i^n, n \in \mathbb{Z}\} \subset G$ and the steps along $\langle s_i \rangle$ are distributed according to a symmetric stable-like power law with exponent α_i . These measures $\mu_{S,a}$ are very natural from an algebraic point of view and one expects that the properties of the associated random walks depend in interesting way on the structure of the group G , the generating k -tuple S and the choice of the k -dimensional parameter a .

The Dirichlet forms $\mathcal{E}_{\mu_{S,a}}$ and \mathcal{E}_q associated with the measures $\mu_{S,a}$ and q on G satisfy

$$\mathcal{E}_{\mu_{S,a}} \simeq \mathcal{E}_q.$$

Hence it follows from [17] that

$$\log \mu_{S,a}^{(n)}(e) \simeq -n^{d/(d+\beta)}(\log n)^{\beta/(d+\beta)}$$

where β is as above. Note that β depends only on the first d coordinates of the parameter $a = (\alpha_i)_1^{d+D}$. In this sense, the random walks associated with the collection of the measures $\mu_{S,a}$ when a varies can distinguish among the $d+D$ generators s_i , $1 \leq i \leq d+D$ of $K \wr H$ between those which come from H and those which come from K .

Example 4.5. Consider the iterated wreath product

$$(\dots (\mathbb{Z}_2 \wr \mathbb{Z}^{d_1}) \wr \mathbb{Z}^{d_2}) \wr \dots) \wr \mathbb{Z}^{d_k}.$$

Note that \wr is not associative so that this iterated wreath product is different from the iterated wreath product $\mathbb{Z}_2 \wr (\dots \wr (\mathbb{Z} \wr \mathbb{Z}) \dots)$ considered in [9]. Here we are iterating the lamps while in [9] the base is iterated. Set

$$\gamma_i = \sum_1^i \frac{d_j}{\alpha_j}, \quad i = 1, \dots, k.$$

For each $i = 1, \dots, k$, fix $\alpha_i \in (0, 2)$ and a probability measure μ_i on \mathbb{Z}^{d_i} which is symmetric, satisfies $\mu_i(0) > 0$ and is in the domain of normal attraction of the rotationally α_i -stable law η_i on \mathbb{R}^{d_i} . Let q_0 be the uniform measure on $\mathbb{Z}_2 = \{0, 1\}$. Iteratively, define the switch-walk-switch probability measure

$$q_i = q_{i-1} * \mu_i * q_{i-1}$$

on $(\dots (\mathbb{Z}_2 \wr \mathbb{Z}^{d_1}) \wr \mathbb{Z}^{d_2}) \wr \dots) \wr \mathbb{Z}^{d_i}$.

Applying Corollary 4.4 iteratively, we obtain

$$\lim_{n \rightarrow \infty} n^{-\frac{\gamma_k}{1+\gamma_k}} \log q_k^{(n)}(e) = -c_k$$

where the constant c_k can be obtained as follows. The constant c_1 is given by [8] whereas, for $2 \leq i \leq k$ and referring to (3.1)-(3.2), $c_i = k \left(v_i, \tilde{F}_i \right)$ where

$$\tilde{F}_i(y) = c_{i-1} y^{\frac{\gamma_i-1}{1+\gamma_i-1}}.$$

Similarly, we can consider the iterated wreath product

$$(\dots (\mathbb{Z}^{d_0} \wr \mathbb{Z}^{d_1}) \wr \mathbb{Z}^{d_2}) \wr \dots) \wr \mathbb{Z}^{d_k},$$

starting with lamp group \mathbb{Z}^{d_0} instead of \mathbb{Z}_2 and $q_0 = \mu_0$ in the domain of normal attraction of the rotationally symmetric α_0 -stable distribution on \mathbb{R}^{d_0} . In this case, we obtain

$$\lim_{n \rightarrow \infty} [n^{\gamma_k/(1+\gamma_k)} (\log n)^{1/(1+\gamma_k)}]^{-1} \log q^{(n)}(e) = -c_k.$$

The constant c_k can be obtained iteratively with $c_1 = k \left(\eta, \frac{d_0}{\alpha_0} \mathbf{1}_{\{y>0\}} \right)$ and $c_i = k \left(v_i, \tilde{F}_i \right)$, with \tilde{F}_i as above for $1 < i \leq k$.

4.3 Application to fastest decay under moment conditions

This section describes applications of Theorem 4.2 to the computation of the group invariants $\tilde{\Phi}_{G,\rho}$ introduced in [2]. Recall that [17] introduce a group invariant Φ_G which is a decreasing function of n (defined up to the equivalence relation \simeq) such that

$$\phi^{(2n)}(e) \simeq \Phi_G(n)$$

for all finitely supported symmetric probability measure ϕ with generating support.

Let ρ be a function

$$\rho : G \rightarrow [1, \infty).$$

The weak ρ -moment of the probability measure μ is defined as

$$W(\rho, \mu) := \sup_{s>0} s \mu(x : \rho(x) > s).$$

Definition 4.6 (Definition 2.1 [2]: Fastest decay under weak ρ -moment). Let G be a locally compact unimodular group. Fix a compact symmetric neighborhood Ω of e . Let $\tilde{\mathcal{S}}_{G,\rho}^{\Omega,K}$ be the set of all symmetric continuous probability densities ϕ on G with the properties that $\|\phi\|_\infty \leq K$ and $W(\rho, \phi d\lambda) \leq K \sup_{\Omega^2} \{\rho\}$. Set

$$\tilde{\Phi}_{G,\rho}^{\Omega,K}(n) := \inf\{\phi^{(2n)}(e) : \phi \in \tilde{\mathcal{S}}_{G,\rho}^{\Omega,K}\}.$$

Here we will only consider the case when G is finitely generated and ρ is one of the power function $\rho_\alpha(x) = (1 + |x|)^\alpha$, $\alpha \in (0, 2)$ where $|\cdot|$ is the word distance on a fixed Cayley graph of G . We are concerned with the decay of $\tilde{\Phi}_{G,\rho_\alpha}^{\Omega,K}$ when n is large. By Proposition 1.2 [2], we can drop the reference to Ω and K . Lower bounds on $\tilde{\Phi}_{G,\rho}$ follow from general comparison and subordination results, see [2]. Here, we are interested in obtaining upper bounds on $\tilde{\Phi}_{G,\rho}$.

By definition, for any probability measure ϕ on G which satisfies the weak ρ -moment condition, $n \mapsto \phi^{(2n)}(e)$ provides an upper bound for $\tilde{\Phi}_{G,\rho}$. When G is a wreath product $G = K \wr \mathbb{Z}^d$, we can use measures of the form $\phi = \nu * \mu * \nu$ and apply Theorem 4.2 to estimate $\phi^{(2n)}(e)$. Also because of the natural embedding of K and \mathbb{Z}^d in the wreath product $K \wr \mathbb{Z}^d$, it's not hard to estimate the needed weak ρ -moment of ϕ . We shall see that, in certain cases, the measures ϕ of this type actually achieve the fastest decay rate given by $\tilde{\Phi}_{G,\rho}$, up to the equivalence relation \simeq . This technique was already used in [2, Theorem 5.1] to determine $\tilde{\Phi}_{\mathbb{Z}_2 \wr \mathbb{Z}^d, \rho_\alpha}$. In this case, the classical result of Donsker and Varadhan [8] is all one needs. In the examples below, we use Theorem 4.2 to obtain precise upper bounds on $\Phi_{K \wr \mathbb{Z}^d}$ in some other cases.

Example 4.6. In this example we consider $G = K \wr \mathbb{Z}^d$ when K is either finite or has polynomial growth or has exponential volume growth and $\Phi_K(n) \simeq \exp(-n^{1/3})$. The first case is already treated in [2]. We note that these three cases exhaust all possibilities when K is a polycyclic group. The third case also covers the situations when K is the Baumslag-Solitar group or the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$.

Theorem 4.7. Fix $\alpha \in (0, 2)$. Let G be the group $K \wr \mathbb{Z}^d$.

1. Assume that K is finite. Then

$$\log \tilde{\Phi}_{G,\rho_\alpha}(n) \simeq -n^{d/(d+\alpha)}.$$

2. Assume that K has polynomial volume growth. Then

$$\log \tilde{\Phi}_{G,\rho_\alpha}(n) \simeq -n^{d/(d+\alpha)} (\log n)^{\alpha/(d+\alpha)}.$$

3. Assume that K has exponential growth and satisfies $\Phi_K(n) \simeq \exp(-n^{1/3})$. Then

$$\log \tilde{\Phi}_{G,\rho_\alpha}(n) \simeq -n^{(d+1)/(d+1+\alpha)}.$$

Proof. The lower bounds can be obtained by applying [2, Theorem 3.3]. For this purpose, one needs to compute the function Φ_G . For $q = \nu * \mu * \nu$ on $G = K \wr \mathbb{Z}^d$, where μ and ν are associated with simple random walk (with holding) on \mathbb{Z}^d and K respectively, we can apply Theorem 4.2 to obtain $q^{(2n)}(e) \simeq \Phi_G(n)$. The case when K is finite is already treated in [16, 2]. When K has polynomial volume growth, then $\nu^{(2n)}(o) \simeq n^{-\frac{d}{2}}$ and, as in Example 4.2,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\frac{d}{d+2}} (\log n)^{\frac{2}{d+2}}} \log q^{(2n)}(e) = -c_q.$$

If K is such that $\Phi_K(n) \simeq \exp(-n^{1/3})$ then Example 4.1 yields

$$\log q^{(2n)}(e) \simeq \log \Phi_G(n) \simeq -n^{\frac{d+1}{d+3}}.$$

These estimates on Φ_G allow us to appeal to [2, Theorem 3.3] to obtain the stated lower bounds for Φ_{G,ρ_α} .

To prove the stated upper bounds, it suffices to exhibit a probability measure measure in $\tilde{\mathcal{S}}_{G,\rho_\alpha}$ that has the proper decay. On \mathbb{Z}^d , set

$$\mu_\alpha(x) = \frac{c_\alpha}{(1 + \|x\|)^{\alpha+1}}.$$

Then μ_α is in the domain of normal attraction of the rotationally symmetric α -stable distribution on \mathbb{R}^d and it has a finite weak α -moment.

In the case when K is of polynomial volume growth, take $q_\alpha = \nu * \mu_\alpha * \nu$, where ν is simple random walk on K . Then $\nu * \mu_\alpha * \nu$ has weak α -moment and, by Example 4.2,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\frac{d}{d+\alpha}} (\log n)^{\frac{\alpha}{d+\alpha}}} \log q_\alpha^{(2n)}(e) = -c_{q_\alpha}.$$

Therefore in this case

$$\log \tilde{\Phi}_{G,\rho_\alpha}(n) \leq -cn^{\frac{d}{d+\alpha}} (\log n)^{\frac{\alpha}{d+\alpha}}.$$

This matches the previously proved lower bound.

In the second case, when K has exponential growth, let U be a symmetric generating set of K . As in [2, Theorem 4.10], pick $p_i = c_\alpha 4^{-i\alpha}$ with $\sum_1^\infty p_i = 1$ and set

$$\nu_\alpha = \sum_{i=1}^\infty \frac{p_i}{|U^{4^i}|} \mathbf{1}_{U^{4^i}}.$$

Then ν_α has weak α -moment on K and, by [2, Theorem 4.1]

$$\nu_\alpha^{(n)}(e_K) \leq \exp(-cn^{\frac{1}{1+\alpha}}).$$

Then $\nu_\alpha * \mu_\alpha * \nu_\alpha$ has weak α -moment on G . Applying Theorem 4.2 and the computations of Example 4.1 to $q_\alpha = \nu_\alpha * \mu_\alpha * \nu_\alpha$, we obtain

$$\limsup_{n \rightarrow \infty} \frac{1}{n^{\frac{d+1}{d+\alpha+1}}} \log q_\alpha^{(2n)}(e) \leq -c_{q_\alpha}.$$

This gives the desired upper bound on Φ_{G,ρ_α} . □

Example 4.7. Consider the iterated wreath product

$$G = (\dots (K \wr \mathbb{Z}^{d_1}) \wr \dots) \wr \mathbb{Z}^{d_r}, \quad d_i \in \mathbb{N}_+.$$

Set

$$d = \sum_1^r d_i.$$

Fix $\alpha \in (0, 2)$. If K is finite then we we have

$$\log \tilde{\Phi}_{G,\rho_\alpha}(n) \simeq -n^{\frac{d}{\alpha+d}}.$$

If K has polynomial volume growth, then

$$\log \tilde{\Phi}_{G,\rho_\alpha}(n) \simeq -cn^{\frac{d}{\alpha+d}} (\log n)^{\frac{\alpha}{\alpha+d}}.$$

These are the results stated as Theorem 1.7 in the introduction.

Proof. As in the previous example, the lower bounds follows from [2, Theorem 3.3] and a lower bound on $\log \Phi_G$. By example 4.5,

$$\log \Phi_G(n) \simeq n^{\frac{d}{2+d}} (\log n)^{\frac{2}{2+d}}.$$

Hence [2, Theorem 3.3] gives

$$\log \tilde{\Phi}_{G, \rho_\alpha}(n) \geq -C_\alpha n^{\frac{d}{\alpha+d}} (\log n)^{\frac{\alpha}{\alpha+d}}.$$

For the upper bound, let $\mu_{\alpha, i}$ be a symmetric α -stable like probability measure on \mathbb{Z}^d . Let $q_{\alpha, 1} = \mu_{\alpha, 0} * \mu_{\alpha, 1} * \mu_{\alpha, 0}$, and iteratively define $q_{\alpha, i+1} = q_{\alpha, i} * \mu_{\alpha, i+1} * q_{\alpha, i}$. Then it's clear that $q_{\alpha, r}$ has a finite weak α -moment on G and, as in example 4.5,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{\frac{d}{\alpha+d}} (\log n)^{\frac{\alpha}{\alpha+d}}} \log q_{\alpha, r}^{(2n)}(e) = -c_{\alpha, r}.$$

Therefore

$$\log \tilde{\Phi}_{G, \rho_\alpha}(n) \leq -cn^{\frac{d}{\alpha+d}} (\log n)^{\frac{\alpha}{\alpha+d}}.$$

□

5 Donsker and Varadhan type large deviations

The goal of this section is to outline the proof of Theorem 3.9, the key result of this article. The proof follows [8] closely. Several other classical sources are also needed to put together the necessary details.

5.1 Statement of the large deviation principle in L^1

On \mathbb{Z}^d , we fix a symmetric probability measure μ and an operator-stable law η such that the convergence assumption (C- B_n) is satisfied.

We need to introduce some notation from [8] in order to state the results. Let π be the projection map $\pi : \mathbb{R}^d \rightarrow \mathbb{R}^d / \mathbb{Z}^d$, and let \mathbb{T} denote the d -dimensional torus which we also identify with the fundamental domain $[-\frac{1}{2}, \frac{1}{2}]^d$.

For $\lambda > 0$, set

$$\mathcal{L}_\lambda^{(n)} = \pi \left(B_{[\lambda a_n]}^{-1}(\mathbb{Z}^d) \right).$$

That is, we take the image of the original lattice \mathbb{Z}^d under the transformation $B_{[\lambda a_n]}^{-1}$, and project it to the torus \mathbb{T} . Then $\mathcal{L}_\lambda^{(n)}$ is a cocompact lattice on \mathbb{T} and the volume of the fundamental domain $\mathbb{T} / \mathcal{L}_\lambda^{(n)}$ is $|\det B_{[\lambda a_n]}^{-1}|$. This is the case because we assume that the matrices B_m , $m = 1, 2, \dots$, have integer entries so that $B_m \mathbb{Z}^d \subset \mathbb{Z}^d$.

In what follows, symbols decorated with \sim are always used to describe quantities associated with the projected random walk on the torus. Note that the construction depends on the choice of sequence a_n and parameter λ , for simplicity we will drop reference to a_n and λ when no confusion arises.

Under the projection map π , we can push forward the measure $B_{[\lambda a_n]}^{-1} \mu$ on $B_{[\lambda a_n]}^{-1}(\mathbb{Z}^d)$ to a measure $\tilde{\mu}_{n, \lambda}$ on $\mathcal{L}_\lambda^{(n)}$, that is

$$\tilde{\mu}_{n, \lambda}(y) = \sum_{x \in \mathbb{Z}^d : \pi(B_{[\lambda a_n]}^{-1}(x)) = y} \mu(x).$$

Let $\tilde{S}_k^{(n)}$ be the random walk on $\mathcal{L}_\lambda^{(n)}$ associated with $\tilde{\mu}_{n, \lambda}$, starting at 0. It's easy to check that

$$\tilde{S}_k^{(n)} \stackrel{\text{law}}{=} \pi \left(B_{[\lambda a_n]}^{-1}(S_k) \right).$$

Consider the occupation time measure $\tilde{L}_k^{(n)}$ defined as

$$\tilde{L}_k^{(n)}(A) = \frac{1}{k} \sum_{j=1}^k \chi_A \left(\tilde{S}_k^{(n)} \right),$$

for any Borel set A in \mathbb{T} .

For $\mathbf{T} = \mathbb{T}$ or $\mathbf{T} = \bar{\Omega}$ with Ω an open set in \mathbb{R}^d , let $\mathcal{M}_1(\mathbf{T})$ be the space of probability measures on \mathbf{T} endowed with the weak topology. Let $\mathcal{L}_1(\mathbf{T})$ the space of all probability densities on \mathbf{T} endowed with the L^1 -topology.

Let $P_k^{(n)}$ be the distribution of $\tilde{L}_k^{(n)}$ in $\mathcal{M}_1(\mathbb{T})$, a measure on measures. Define the scaled indicator function $\chi_n : [-\frac{1}{2}, \frac{1}{2}]^d \rightarrow \mathbb{R}$ by setting

$$\chi_n(x) = |\det B_{\lfloor \lambda a_n \rfloor}|^{-1} \chi_{B_{\lfloor \lambda a_n \rfloor}^{-1}([-\frac{1}{2}, \frac{1}{2}]^d)}.$$

Define

$$\tilde{L}_k^n = \tilde{L}_k^{(n)} * \chi_n.$$

Let \tilde{P}_k^n be the distribution of \tilde{L}_k^n in $\mathcal{M}_1(\mathbb{T})$. With this mollification, \tilde{L}_k^n is absolutely continuous with respect to Lebesgue measure on \mathbb{T} . Let $\tilde{f}_{k,\lambda}^{(n)}$ denote the density of \tilde{P}_k^n with respect to Lebesgue measure. Let $Q_{k,\lambda}^{(n)}$ be the distribution of $\tilde{f}_{k,\lambda}^{(n)}$ in $\mathcal{L}_1(\mathbb{T})$.

We will use the following function spaces (this notation is consistent with [7, 8]):

$$\begin{aligned} \mathcal{U} &= \{u \in C^\infty(\mathbb{R}^d), \inf u > 0, \sup u < \infty\}, \\ \mathcal{U}_{\mathbb{T}} &= \{u \in C^\infty(\mathbb{T}), u > 0\}, \\ \mathcal{F}_{\mathbb{T}} &= \{f \in C^\infty(\mathbb{T}), f \geq 0, \|f\|_1 = 1\}, \\ \mathcal{F}_{\Omega} &= \{f \in C_c^\infty(\Omega), f \geq 0, \|f\|_1 = 1\}, \end{aligned}$$

where Ω is an open subset of \mathbb{R}^d .

Theorem 5.1 (Large deviation principle in $\mathcal{L}_1(\mathbb{T})$). *Assume that the convergence assumption (C- B_n) is satisfied. Let a_n to be any sequence of positive integers increasing to infinity and satisfying $a_n |\det B_{a_n}| \leq n$. Let $Q_{n,\lambda}^{(n)}$ be the distribution of $\tilde{f}_{n,\lambda}^{(n)}$ on $\mathcal{L}_1(\mathbb{T})$. Then we have the large deviation principle in the strong $\mathcal{L}_1(\mathbb{T})$ topology. Namely, for any Borel set D in $\mathcal{L}_1(\mathbb{T})$,*

$$\begin{aligned} -\lambda^{-1} \inf_{f \in D^\circ} I_{L_{\tilde{\eta}}}(f) &\leq \liminf_{n \rightarrow \infty} \frac{1}{n/a_n} \log Q_{n,\lambda}^{(n)}(D) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n/a_n} \log Q_{n,\lambda}^{(n)}(D) \leq -\lambda^{-1} \inf_{f \in \bar{D}} I_{L_{\tilde{\eta}}}(f), \end{aligned}$$

and the rate function is given by

$$I_{L_{\tilde{\eta}}}(f) = - \inf_{u \in \mathcal{U}_{\mathbb{T}}} \int_{\mathbb{T}} \frac{L_{\tilde{\eta}} u}{u}(x) f(x) dx = \mathcal{E}_{\tilde{\eta}}(\sqrt{f}, \sqrt{f}).$$

This result will be useful in the upper bound direction. To obtain a lower bound, we need to have a version with Dirichlet boundary condition.

Let $L_k^{(n)}$ be the occupation time measure of the random walk $S_k^{(n)} = B_{a_n}^{-1}(S_k)$. Perform the same mollification as above but on \mathbb{R}^d , setting

$$L_k^n = L_k^{(n)} * \chi_n.$$

Then L_k^n is absolutely continuous with respect to Lebesgue measure. Let $f_k^{(n)}$ denotes the corresponding density. Let \mathcal{G} be the collection of all bounded domain Ω in \mathbb{R}^d such that $0 \in \Omega$ and $\partial\Omega$ has Lebesgue measure 0. For any Borel set $A \subset \mathcal{L}_1(\bar{\Omega})$, define

$$Q_{k,\Omega}^{(n)}(A) := P \left(f_k^{(n)} \in A \right).$$

That is, $Q_{k,\Omega}^{(n)}$ is the distribution of the occupation time measure of $S_j^{(n)}$ at time k with Dirichlet boundary on $\partial\Omega$. As in the case of the projected version, we have the following large deviation principle.

Theorem 5.2 (Large deviation principle in $\mathcal{L}_1(\bar{\Omega})$). *Under the convergence assumption (C- B_n), let a_n be any sequence of positive integers increasing to infinity satisfying $a_n |\det B_{a_n}| \leq n$. Let $Q_{n,\Omega}^{(n)}$ be the distribution of $f_n^{(n)}$ in $\mathcal{L}_1(\bar{\Omega})$. Then we have large deviation principle in the strong $\mathcal{L}_1(\bar{\Omega})$ topology. Namely, for any Borel set $A \subset \mathcal{L}_1(\bar{\Omega})$,*

$$\begin{aligned} - \inf_{f \in A^\circ} I_{L_\eta}(f) &\leq \liminf_{n \rightarrow \infty} \frac{1}{n/a_n} \log Q_{n,\Omega}^{(n)}(A) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n/a_n} \log Q_{n,\Omega}^{(n)}(A) \leq - \inf_{f \in A} I_{L_\eta}(\eta), \end{aligned}$$

and

$$I_{L_\eta}(f) = - \inf_{u \in \mathcal{U}} \int_{\Omega} \frac{L_\eta u}{u}(x) f(x) dx = \mathcal{E}_\eta(\sqrt{f}, \sqrt{f}).$$

The outline of the proof of these results is given in Section 5.3. It follows [8] closely.

5.2 Asymptotics of functional expressions

Throughout this short section, we fix a symmetric probability measure μ on \mathbb{Z}^d and an operator-stable law η on \mathbb{R}^d such that the convergence assumption (C- B_n) and scaling assumption (S- B_n - a_n) of Definitions 3.1-3.6 are satisfied. In particular, in what follows, (a_n) is the non-decreasing and regularly varying sequence of integers provided by Definition 3.6 (see also Proposition 3.7). The functions F and \tilde{F} are as in Definition 3.6. Let $(l(n, x))_{x \in \mathbb{Z}^d}$ be the occupation time vector up to time n for the random walk driven μ . The goal of this subsection is to use the large deviation principles in L^1 to prove Theorem 3.9.

Proposition 5.3. *Under the above hypotheses, we have the lower bound*

$$\begin{aligned} \liminf_{n \rightarrow \infty} \frac{a_n}{n} \log E \left(\exp \left(- \sum_{x \in \mathbb{Z}^d} F(l(n, x)) \right) \mathbf{1}_{\{\text{supp}(L_n^{(n)}) \subset \Omega\}} \right) \\ \geq - \inf_{f \in \mathcal{F}_\Omega} \left\{ \mathcal{E}_\eta(\sqrt{f}, \sqrt{f}) + \int_{\Omega} \tilde{F}(f(x)) dx \right\}. \end{aligned}$$

Proof. The proof is essentially the same as for [3, Lemma 4.2]. Use the lower bound in Theorem 5.2 and Varadhan’s lemma (see [21, Theorems 2.2, 2.3]). \square

Proposition 5.4. *Under the above hypotheses, we have upper bound*

$$\begin{aligned} \limsup_{n \rightarrow \infty} \frac{a_n}{n} \log E \left(\exp \left(- \sum_{x \in \mathbb{Z}^d} F(l(n, x)) \right) \right) \\ \leq - \sup_{\lambda > 0} \inf_{f \in \mathcal{F}_\mathbb{T}} \left\{ \lambda^{-1} \mathcal{E}_{\tilde{\eta}}(\sqrt{f}, \sqrt{f}) + c_0 \lambda^{(1-\gamma) \text{tr } E} \int_{\mathbb{T}} \tilde{F}(f(x)) dx \right\}. \end{aligned}$$

Proof. First, since F is sub-additive, write

$$\begin{aligned} E \left(\exp \left(- \sum_{x \in \mathbb{Z}^d} F(l(n, x)) \right) \right) &\leq E \left(\exp \left(- \sum_{y \in \mathcal{L}_\lambda^{(n)}} F(\tilde{l}(n, y)) \right) \right) \\ &= E_{Q_{n,\lambda}^{(n)}} \left(\exp \left(- \det(B_{\lambda a_n}) \int_{\mathbb{T}} F \left(\frac{n}{\det(B_{\lambda a_n})} f(x) \right) dx \right) \right). \end{aligned}$$

Next, follow the line of reasoning used to prove the Corollary of Theorem 6 in [8], using the large deviation upper bound in $\mathcal{L}_1(\mathbb{T})$ and Varadhan’s lemma. From the (lower bound part of) the scaling assumption and the regular variation property of $\det B_{a_n}$, we have for any parameter $\lambda > 0$,

$$\liminf_{n \rightarrow \infty} \frac{a_n \det(B_{\lambda a_n})}{n} F\left(\frac{n}{\det(B_{\lambda a_n})} y\right) \geq \lambda^{(1-\gamma) \operatorname{tr} E} \tilde{F}(y), \quad y > 0.$$

Setting $\mathbf{D}_n = \det(B_{\lambda a_n})$, we obtain

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{a_n}{n} \log E \left(\exp \left(- \sum_{x \in \mathbb{Z}^d} F(l(n, x)) \right) \right) \\ & \leq \limsup_{n \rightarrow \infty} \frac{a_n}{n} \log E_{Q_{n, \lambda}^{(n)}} \left(\exp \left(- \mathbf{D}_n \int_{\mathbb{T}} F\left(\frac{n}{\mathbf{D}_n} f(x)\right) dx \right) \right) \\ & = \limsup_{n \rightarrow \infty} \frac{a_n}{n} \log E_{Q_{n, \lambda}^{(n)}} \left(\exp \left(- \frac{n}{a_n} \int_{\mathbb{T}} \frac{a_n \mathbf{D}_n}{n} F\left(\frac{n}{\mathbf{D}_n} f(x)\right) dx \right) \right) \\ & \leq - \inf_{f \in \mathcal{F}_{\mathbb{T}}} \left\{ \lambda^{-1} \mathcal{E}_{\tilde{\eta}}(\sqrt{f}, \sqrt{f}) + \lambda^{(1-\gamma) \operatorname{tr} E} \int_{\mathbb{T}} \tilde{F}(f(x)) dx \right\}. \end{aligned}$$

The last step comes from Varadhan’s lemma. Since the choice of parameter λ is arbitrary, we can optimize over all $\lambda > 0$. □

The following lemma is proved in the appendix. It shows that the constants appearing in the upper and lower bounds actually match up. In particular, since this constant appears as both a sup and an inf of some nonnegative quantities, it follows clearly that the constant $k(\eta, \tilde{F})$ defined below takes value in $(0, \infty)$.

Lemma 5.5. *Suppose \tilde{F} is a homogeneous function with exponent $\gamma \in [0, 1]$, that is $\tilde{F}(0) = 0, \tilde{F}(y) = \tilde{F}(1)y^\gamma$ for $y > 0$; and η is a full operator-stable law with exponent E . Then there exists a constant $k(\eta, \tilde{F}) \in (0, \infty)$ such that*

$$\begin{aligned} k(\eta, \tilde{F}) &= \sup_{\lambda > 0} \inf_{f \in \mathcal{F}_{\mathbb{T}}} \left\{ \lambda^{-1} \mathcal{E}_{\tilde{\eta}}(\sqrt{f}, \sqrt{f}) + \lambda^{(1-\gamma) \operatorname{tr} E} \int_{\mathbb{T}} \tilde{F}(f(x)) dx \right\} \\ &= \inf_{\Omega \in \mathcal{G}} \inf_{f \in \mathcal{F}_{\Omega}} \left\{ \mathcal{E}_{\eta}(\sqrt{f}, \sqrt{f}) + \int_{\Omega} \tilde{F}(f(x)) dx \right\}. \end{aligned}$$

5.3 Proof of the large deviation principle in L^1

In this section we indicate how to adapt [8] to prove the large deviation principles as stated in Theorems 5.1 and 5.2. For this purpose we first develop a large deviation principle in the weak topology following Lemma 3.1 and Appendix A in [11].

Throughout this subsection we assume

$$B_n^{-1} \mu^{(n)} \implies \eta. \tag{C- B_n }$$

First we establish asymptotics for exponential moment generating functions. Compare with [11, Lemma A.1] which treats simple random walk on \mathbb{Z}^d .

Proposition 5.6. *For the projected occupation measure, for any $f \in C(\mathbb{T})$ and any sequence (a_n) satisfying $a_n \rightarrow \infty$ and $a_n = o(n)$ as $n \rightarrow \infty$,*

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n/a_n} \log E \left(\exp \left(\frac{n}{a_n} \langle f, \tilde{L}_n^{(n)} \rangle \right) \right) \\ & = \sup_{g \in \mathcal{F}_{\mathbb{T}}} \left\{ \int_{\mathbb{T}} f(x) g(x) dx - \lambda^{-1} \mathcal{E}_{\tilde{\eta}}(\sqrt{g}, \sqrt{g}) \right\}. \end{aligned}$$

For the occupation measure with Dirichlet boundary condition, for any function $f \in C_0(\Omega)$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{1}{n/a_n} \log E \left(\exp \left(\frac{n}{a_n} \langle f, L_n^{(n)} \rangle \right) \mathbf{1}_{\{\text{supp}(L_n^{(n)}) \subset \Omega\}} \right) \\ &= \sup_{g \in \mathcal{F}_\Omega} \left\{ \int_\Omega f(x)g(x)dx - \mathcal{E}_\eta(\sqrt{g}, \sqrt{g}) \right\}. \end{aligned}$$

Proof. In the lower bound direction, we have the following Feynman-Kac estimates as consequences of functional limit theorem. For the proof, adapt the arguments given in [4, Theorem 7.1]. For any sequence (a_n) satisfying $a_n \rightarrow \infty$ and $a_n = o(n)$ as $n \rightarrow \infty$, and any $f \in C(\mathbb{T})$,

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{1}{n/a_n} \log E \left(\exp \left(\frac{1}{a_n} \sum_{k=1}^n f(\tilde{S}_k^{(n)}) \right) \right) \\ & \geq \sup_{g \in \mathcal{F}_\mathbb{T}} \left\{ \int_\mathbb{T} f(x)g(x)dx - \lambda^{-1} \mathcal{E}_{\tilde{\eta}}(\sqrt{g}, \sqrt{g}) \right\}. \end{aligned}$$

Similarly, for $f \in C_0(\Omega)$,

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \frac{1}{n/a_n} \log E \left(\exp \left(\frac{1}{a_n} \sum_{k=1}^n f(S_k^{(n)}) \right) \mathbf{1}_{\{\text{supp}(L_n^{(n)}) \subset \Omega\}} \right) \\ & \geq \sup_{g \in \mathcal{F}_\Omega} \left\{ \int_\Omega f(x)g(x)dx - \mathcal{E}_\eta(\sqrt{g}, \sqrt{g}) \right\}. \end{aligned}$$

In the upper bound direction, as a consequence of the convergence assumption, we can adapt the proof of [8, Theorem 3] to have the following large deviation upper bound. Let C be a closed of $\mathcal{M}_1(\mathbb{T})$. Then

$$\limsup_{n \rightarrow \infty} \frac{1}{n/a_n} \log \tilde{P}_{n,\lambda}^{(n)}(C) \leq -\lambda^{-1} \inf_{\nu \in C} I_{L_{\tilde{\eta}}}(\nu),$$

where

$$I_{L_{\tilde{\eta}}}(\nu) = - \inf_{u \in \mathcal{U}_\mathbb{T}} \int_\mathbb{T} \frac{L_{\tilde{\eta}}u}{u}(x) d\nu(x).$$

Similarly, if C is compact in $\mathcal{M}_1(\bar{\Omega})$,

$$\limsup_{n \rightarrow \infty} \frac{1}{n/a_n} \log P_{n,\Omega}^{(n)}(C) \leq - \inf_{\nu \in C} I_{L_\eta}(\nu),$$

where

$$I_{L_\eta}(\nu) = - \inf_{u \in \mathcal{U}_\Omega} \int_\Omega \frac{L_\eta u}{u}(x) d\nu(x).$$

Either on \mathbb{T} or in Ω , apply Varadhan’s lemma ([21, Theorem 2.2]) to the large deviation upper bound to obtain the upper bounds needed for Proposition 5.6. \square

By the Gartner-Ellis theorem (e.g., [5, Theorem 4.5.20]), we obtain the large deviation principle in the weak topology stated in the following Theorem. Compare with [11, Lemma 3.1].

Theorem 5.7. For any Borel set B in $\mathcal{M}_1(\mathbb{T})$ and any sequence (a_n) satisfying $a_n \rightarrow \infty$ and $a_n = o(n)$ as $n \rightarrow \infty$,

$$\begin{aligned} -\lambda^{-1} \inf_{f \in B^\circ} I_{L_{\tilde{\eta}}}(f) & \leq \liminf_{n \rightarrow \infty} \frac{1}{n/a_n} \log \tilde{P}_{n,\lambda}^{(n)}(B) \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{n/a_n} \log \tilde{P}_{n,\lambda}^{(n)}(B) \leq -\lambda^{-1} \inf_{f \in \bar{B}} I_{L_{\tilde{\eta}}}(f). \end{aligned}$$

Similarly, for any Borel set A in $\mathcal{M}_1(\bar{\Omega})$,

$$\begin{aligned} - \inf_{f \in A^\circ} I_{L_\eta}(f) &\leq \liminf_{n \rightarrow \infty} \frac{1}{n/a_n} \log P_{n,\Omega}^{(n)}(A) \\ &\leq \limsup_{n \rightarrow \infty} \frac{1}{n/a_n} \log P_{n,\Omega}^{(n)}(A) \leq - \inf_{f \in \bar{A}} I_{L_\eta}(\eta). \end{aligned}$$

Next, following [8], we use the local limit theorem to upgrade the large deviation principle in the weak topology to a result in the strong L^1 -topology. This is a rather technical task. As shown in [8, Theorem 6], the key point is to obtain a super-exponential estimate on the L^1 -distance of the density function to its smooth mollification. This, in turn, requires uniform properties of the transition probabilities that are provided by the local limit theorem.

Let $\{\psi_\epsilon\}$, $\epsilon \rightarrow 0$, be an approximation of the identity on \mathbb{R}^d with ψ_ϵ is smooth, symmetric, compactly supported inside $(-\frac{\epsilon}{2}, \frac{\epsilon}{2})^d$. Thinking of ψ_ϵ also as a function on \mathbb{T} , set $K_\epsilon : L^1(\mathbb{T}) \rightarrow L^1(\mathbb{T})$ as

$$K_\epsilon f(x) = \int_{\mathbb{T}} f(y) \psi_\epsilon(x - y) dy.$$

Theorem 5.8. *For every $\delta > 0$, $\lambda > 0$ and sequence a_n tending to infinity such that $a_n |\det B_{a_n}| \leq n$, we have*

$$\limsup_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{n/a_n} \log Q_{n,\lambda}^{(n)} \left(f : \int_{\mathbb{T}} |K_\epsilon f(x) - f(x)| dx \geq \delta \right) = -\infty.$$

We follow step by step the proof of [8, Theorem 5]. To this end, we adapt to our situation the sequence of lemmas in [8] that are used to prove this theorem. The first lemma provides an elementary way to select a δ -net of functions. Recall that

$$\chi_n(x) = |\det B_{\lfloor \lambda a_n \rfloor}|^{-1} \chi_{B_{\lfloor \lambda a_n \rfloor}^{-1}([- \frac{1}{2}, \frac{1}{2}]^d)}.$$

Lemma 5.9. *Let $M_{n,\epsilon} \subset C(\mathbb{T})$ be the set of functions*

$$M_{n,\epsilon} = \{V = (K_\epsilon - I)\chi_n g : g \in C(\mathbb{T}), \|g\|_\infty \leq 1\}.$$

For any $\delta > 0$, there exist functions V_1, \dots, V_J such that for any $V \in M_{n,\epsilon}$,

$$\inf_{1 \leq i \leq J} \sup_{x \in \mathcal{L}^{(n)}} |V(x) - V_i(x)| \leq \frac{\delta}{2},$$

and

$$J = J(n, \epsilon, \delta) \leq \left(\frac{8}{\delta} + 1 \right)^{|\det B_{\lfloor \lambda a_n \rfloor}|}.$$

Proof. The proof is identical to [8, Lemma 4.1]. □

The second lemma is similar to [8, Lemma 4.2] and concerns the uniform control of the transition probabilities. Such uniform control appears as Assumption (U) in [6, Section 4.1] and [5, Section 6.3] to obtain the large deviation principle in \mathcal{L}_1 .

Lemma 5.10. *There exists $n_0 \in \mathbb{N}$ and constant $c < \infty$ such that for all $n \geq n_0$,*

$$\sup_{x \in \mathcal{L}^{(n)}} \tilde{\mu}_n^{*a_n}(x) \leq c \inf_{x \in \mathcal{L}^{(n)}} \tilde{\mu}_n^{*a_n}(x).$$

Proof. Recall the local limit theorem in [12],

$$\limsup_{n \rightarrow \infty} \sup_{x \in \mathbb{Z}^d} |\det B_n| \cdot \left| \mu^{(n)}(x) - |\det B_n^{-1}| g(B_n^{-1}x) \right| = 0.$$

Applying this local limit result along the sequence $\{\lfloor \lambda a_n \rfloor\}$ and projecting onto \mathbb{T} , we have

$$\limsup_{n \rightarrow \infty} \sup_{y \in \mathcal{L}^{(n)}} \left| |\det B_{a_n}| \tilde{\mu}_n^{*a_n}(y) - \tilde{g}_{\lambda^{-1}}(y) \right| = 0.$$

Since the density g is continuous, the desired result follows. \square

We have the following uniform estimate with respect to the starting point.

Lemma 5.11. *Let x and y be any two points in $\mathcal{L}_\lambda^{(n)}$. There exists an integer n_0 such that for any $n \geq n_0$ and any $\theta > 0$,*

$$E_y \left[\exp \left(\theta \sum_{k=1}^n V(\tilde{S}_k^{(n)}) \right) \right] \leq C_{n,\theta} E_x \left[\exp \left(\theta \sum_{k=1}^n V(\tilde{S}_k^{(n)}) \right) \right],$$

where $C_{n,\theta} = c \exp(4\theta a_n)$ and c and n_0 are as in Lemma 4.2.

Proof. This follows from Lemma 5.10 as in the proof of [8, Lemma 4.3]. \square

Lemma 5.12. *There exists an integer n_0 such that for any $n \geq n_0$ and any $\theta > 0$,*

$$E_0 \left[\exp \left(\theta \sum_{k=1}^n V(\tilde{S}_k^{(n)}) \right) \right] \leq C_{n,\theta} \exp \left(n \tilde{I}_n^*(\theta V) \right),$$

where $C_{n,\theta}$ is as in Lemma 5.11 and \tilde{I}_n^* is Legendre transform of

$$\tilde{I}_n(\eta) = - \inf_{u \in \mathcal{U}_T} \int_{\mathbb{T}} \log \frac{\tilde{\mu}_n u}{u} d\eta,$$

that is,

$$\tilde{I}_n^*(\theta V) = \sup_{\eta \in \mathcal{M}_1(\mathbb{T})} \left\{ \int_{\mathbb{T}} \theta V d\eta - \tilde{I}_n(\eta) \right\}.$$

Proof. Follow [8, Lemma 4.4]. \square

Finally, we need the following technical lemma that controls error terms as $n \rightarrow \infty$.

Lemma 5.13. *Let η be a probability measure on $\mathcal{L}^{(n)}$ such that $\tilde{I}_n(\eta) \leq \frac{\sigma}{a_n}$, where $\sigma > 0$. Let $V = (K_\epsilon - I)\chi_n g$ where $g \in C(\mathbb{T})$, $\|g\|_\infty \leq B$. Then for any $t > 0$,*

$$\int_{\mathbb{T}} V d\eta \leq B[2h(t\sigma) + 2\Delta_t(n) + k_t(\epsilon)],$$

where

$$h(l) := 2 \inf_{a>0} \frac{l + a - \log(1 + a)}{a}$$

and

$$\Delta_t(n) = \int_{\mathbb{T}} \left| \chi_n * \tilde{\mu}_n^{(\lfloor ta_n \rfloor)}(x) - \tilde{g}_{t/\lambda}(x) \right| dx,$$

$$k_t(\epsilon) = \sup_{y \in (-\frac{\epsilon}{2}, \frac{\epsilon}{2})^d} \int_{\mathbb{T}} \left| \tilde{g}_{t/\lambda}(x - y) - \tilde{g}_{t/\lambda}(x) \right| dx.$$

Moreover, we have that $h(l) \rightarrow 0$ as $l \rightarrow 0$ and, for any fixed $t > 0$, $\Delta_t(n) \rightarrow 0$ as $n \rightarrow \infty$ and $k_t(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$.

Proof. See [8, Lemma 4.5] □

With these five lemmas, the line of reasoning used in [8, Theorem 5] gives us Theorem 5.8.

6 Appendix

6.1 Proof of regular variation properties

In this subsection we deduce from the regular variation of the sequence (B_n) the properties of \tilde{F} stated in the technical proposition 3.7. We follow [3] closely.

Proof of Proposition 3.7. By Theorem 2.9 (i.e., [13, Theorem 1.10.19]), under the convergence assumption $B_n^{-1}\mu^{(n)} \implies \eta$, there exists a modified normalization sequence (B'_n) with $B'_n = B_n S_n$, $S_n \in \text{Ivn}(\eta)$, such that (B'_n) has the regular variation property

$$B'_n(B'_{[nt]})^{-1} \rightarrow t^{-E}$$

where the convergence is uniform in t on compact subsets of \mathbb{R}_+^\times . Since $S_n \in \text{Ivn}(\eta)$ we have

$$(B'_n)^{-1}\mu^{(n)} \implies \eta.$$

Further, since $\text{Inv}(\eta)$ is a compact group, we must have $\det S_n = 1$, $\det B'_n = \det B_n$. Hence we can replace B_n in the scaling assumption by B'_n and we have

$$\lim_{n \rightarrow \infty} \frac{a_n \det(B'_{a_n})}{n} F\left(\frac{n}{\det(B'_{a_n})}y\right) = \tilde{F}(y),$$

uniformly over compact sets in $(0, \infty)$.

Set

$$\tilde{F}_n(y) := \frac{a_n \det(B'_{a_n})}{n} F\left(\frac{n}{\det(B'_{a_n})}y\right).$$

The scaling assumption now reads $\lim_{n \rightarrow \infty} \tilde{F}_n(y) = \tilde{F}(y)$. Note that by assumption, $F : [0, \infty) \rightarrow [0, \infty)$ is a concave, increasing function with $F(0) = 0$, therefore both \tilde{F}_n and \tilde{F} are concave, non-decreasing, not identically zero with value 0 at 0. Hence \tilde{F}_n and \tilde{F} are continuous and strictly positive in $(0, \infty)$, and by concavity, $y \rightarrow \frac{\tilde{F}_n(y)}{y}$ and $y \rightarrow \frac{\tilde{F}(y)}{y}$ are both non-increasing functions.

Now we show that for any $\lambda \in (0, 1)$, $\frac{a_{[\lambda n]}}{a_n}$ tends to a finite non-zero limit as $n \rightarrow \infty$. Fix a $y > 0$ and write

$$\begin{aligned} \tilde{F}_{[\lambda n]}(y) &= \frac{a_{[\lambda n]} \det(B'_{a_{[\lambda n]}})}{[\lambda n]} F\left(\frac{[\lambda n]}{\det(B'_{a_{[\lambda n]}})}y\right) \\ &= \frac{a_{[\lambda n]} \det(B'_{a_{[\lambda n]}})}{[\lambda n]} F\left(\frac{n}{\det(B'_{a_n})} \frac{[\lambda n]/n}{\det(B'_{a_{[\lambda n]}})/\det(B'_{a_n})}y\right) \\ &= \frac{a_{[\lambda n]} \det(B'_{a_{[\lambda n]}})/\det(B'_{a_n})}{a_n \frac{[\lambda n]/n}{\det(B'_{a_{[\lambda n]}})/\det(B'_{a_n})}} \tilde{F}_n\left(\frac{[\lambda n]/n}{\det(B'_{a_{[\lambda n]}})/\det(B'_{a_n})}y\right). \end{aligned}$$

As (a_n) is an increasing sequence and $\det(B_n)$ is non-decreasing with respect to n , we have $\det(B'_{a_{[\lambda n]}})/\det(B'_{a_n}) \leq 1$. Since $y \rightarrow \frac{\tilde{F}_n(y)}{y}$ is non-increasing, we have

$$\frac{\tilde{F}_n\left(\frac{[\lambda n]/n}{\det(B'_{a_{[\lambda n]}})/\det(B'_{a_n})}y\right)}{\frac{[\lambda n]/n}{\det(B'_{a_{[\lambda n]}})/\det(B'_{a_n})}y} \leq \frac{\tilde{F}_n\left(\frac{[\lambda n]}{n}y\right)}{\frac{[\lambda n]}{n}y}.$$

Therefore

$$\tilde{F}_{\lfloor \lambda n \rfloor}(y) \leq \frac{a_{\lfloor \lambda n \rfloor}}{a_n} \frac{\tilde{F}_n\left(\frac{\lfloor \lambda n \rfloor}{n}y\right)}{\frac{\lfloor \lambda n \rfloor}{n}}.$$

Letting $n \rightarrow \infty$ on both sides, the scaling assumption yields

$$\tilde{F}(y) \leq \liminf_{n \rightarrow \infty} \frac{a_{\lfloor \lambda n \rfloor}}{a_n} \frac{1}{\lambda} \tilde{F}(\lambda y).$$

Therefore

$$\liminf_{n \rightarrow \infty} \frac{a_{\lfloor \lambda n \rfloor}}{a_n} \geq \frac{\lambda \tilde{F}(y)}{\tilde{F}(\lambda y)}.$$

Since (a_n) is an increasing sequence by assumption, we have

$$\frac{\lambda \tilde{F}(y)}{\tilde{F}(\lambda y)} \leq \liminf_{n \rightarrow \infty} \frac{a_{\lfloor \lambda n \rfloor}}{a_n} \leq \limsup_{n \rightarrow \infty} \frac{a_{\lfloor \lambda n \rfloor}}{a_n} \leq 1.$$

Replacing λ by $\frac{1}{\lambda}$, we conclude that for all $\lambda \in (0, \infty)$, $\frac{a_{\lfloor \lambda n \rfloor}}{a_n}$ is uniformly bounded away from 0 and uniformly bounded from above.

Let $\phi(\lambda)$ be defined for each $\lambda \in (0, \infty)$ as a sub-sequential limit of $\frac{a_{\lfloor \lambda n \rfloor}}{a_n}$. Namely, choose some (λ -dependent) sub-sequence $t_n \rightarrow \infty$ and set $\phi(\lambda) = \lim_{n \rightarrow \infty} \frac{a_{\lfloor \lambda t_n \rfloor}}{a_{t_n}}$. From the above reasoning we know that $\phi(\lambda) \in (0, \infty)$. Consider the equation

$$\tilde{F}_{\lfloor \lambda n \rfloor}(y) = \frac{a_{\lfloor \lambda n \rfloor}}{a_n} \frac{\det(B'_{a_{\lfloor \lambda n \rfloor}})/\det(B'_{a_n})}{\lfloor \lambda n \rfloor/n} \tilde{F}_n\left(\frac{\lfloor \lambda n \rfloor/n}{\det(B'_{a_{\lfloor \lambda n \rfloor}})/\det(B'_{a_n})}y\right),$$

and take the limit along the sub-sequence (t_n) . We can indeed take the limit on the right hand side of the equation because \tilde{F} is continuous, and the convergences $\tilde{F}_n(y) \rightarrow \tilde{F}(y)$ and $B'_n(B'_{\lfloor nt \rfloor})^{-1} \rightarrow t^{-E}$ are uniform over compact sets. This yields

$$\tilde{F}(y) = \phi(\lambda) \frac{\phi(\lambda)^{\text{tr } E}}{\lambda} \tilde{F}\left(\frac{\lambda}{\phi(\lambda)^{\text{tr } E}}y\right).$$

Note that the function

$$z \rightarrow \frac{z^{\text{tr } E}}{\lambda} \tilde{F}\left(\frac{\lambda}{z^{\text{tr } E}}y\right)$$

is non-decreasing because $y \rightarrow \frac{\tilde{F}(y)}{y}$ is non-increasing. As $z \rightarrow \frac{\tilde{F}(y)}{z}$ is strictly decreasing, the solution $z_0 = z(\lambda, y)$ to

$$\frac{\tilde{F}(y)}{z} = \frac{z^{\text{tr } E}}{\lambda} \cdot \tilde{F}\left(\frac{\lambda}{z^{\text{tr } E}}y\right)$$

is unique. Hence the limit $\phi(\lambda) = \lim_{n \rightarrow \infty} \frac{a_{\lfloor \lambda n \rfloor}}{a_n}$ exists in $(0, \infty)$ for all $\lambda \in (0, \infty)$.

Observe that

$$\begin{aligned} \phi(\lambda_1 \lambda_2) &= \lim_{n \rightarrow \infty} \frac{a_{\lfloor \lambda_1 \lambda_2 n \rfloor}}{a_n} \\ &= \lim_{n \rightarrow \infty} \frac{a_{\lfloor \lambda_1 \lambda_2 n \rfloor}}{a_{\lfloor \lambda_2 n \rfloor}} \cdot \frac{a_{\lfloor \lambda_2 n \rfloor}}{a_n} \\ &= \phi(\lambda_1) \phi(\lambda_2). \end{aligned}$$

Therefore ϕ is multiplicative and $\phi(\lambda) = \lambda^\kappa$ with $\kappa = \log_2 \phi(2)$. Plugging this back in

$$\tilde{F}(y) = \phi(\lambda) \cdot \frac{\phi(\lambda)^{\text{tr } E}}{\lambda} \cdot \tilde{F}\left(\frac{\lambda}{\phi(\lambda)^{\text{tr } E}} y\right),$$

we have

$$\tilde{F}(y) = \lambda^\kappa \cdot \frac{\lambda^{\kappa \text{tr } E}}{\lambda} \cdot \tilde{F}\left(\frac{\lambda}{\lambda^{\kappa \text{tr } E}} y\right).$$

Setting $y = 1$ gives

$$\tilde{F}(1) = \lambda^{\kappa + \kappa \text{tr } E - 1} \cdot \tilde{F}(\lambda^{1 - \kappa \text{tr } E}),$$

so that

$$\tilde{F}(y) = \tilde{F}(1) y^{\frac{1 - \kappa \text{tr } E - \kappa}{1 - \kappa \text{tr } E}}.$$

The fact that

$$\lim_{n \rightarrow \infty} \frac{\log a_n}{\log n} = \kappa,$$

follows exactly from the reasoning in [3]. □

6.2 Discussion of the constant $k(\eta, \tilde{F})$ of Lemma 5.5

In this subsection, we follow the truncation argument in [7] to prove Lemma 5.5. With the notation of Section 5, let

$$J := \sup_{\lambda > 0} \inf_{f \in \mathcal{F}_{\mathbb{T}}} \left\{ \lambda^{-1} \mathcal{E}_{\tilde{\eta}}(\sqrt{f}, \sqrt{f}) + \lambda^{(1-\gamma) \text{tr } E} \int_{\mathbb{T}} \tilde{F}(f(x)) dx \right\}$$

be the upper bound appearing in Proposition 5.4. Let $\epsilon > 0$ be an arbitrary small number. To prove Lemma 5.5, it suffices to find $\Omega \in \mathcal{G}$ and $g \in \mathcal{F}_{\Omega}$ such that

$$\mathcal{E}_{\tilde{\eta}}(\sqrt{g}, \sqrt{g}) + \int_{\Omega} \tilde{F}(g(x)) dx \leq J + \epsilon.$$

For any λ (we will choose λ large enough later on), by the definition of J , there exists $f \in \mathcal{F}_{\mathbb{T}}$ such that

$$\lambda^{-1} \mathcal{E}_{\tilde{\eta}}(\sqrt{f}, \sqrt{f}) + \lambda^{(1-\gamma) \text{tr } E} \int_{\mathbb{T}} \tilde{F}(f(x)) dx < J + \frac{\epsilon}{2}.$$

We can think of functions on \mathbb{T} also as functions on the fundamental domain $[0, 1)^d$. Following [7, Lemma 3.4], let

$$E^\lambda = \bigcup_{i=1}^d \left(\left\{ 0 \leq x_i \leq \frac{1}{\sqrt[4]{\lambda}} \right\} \cup \left\{ 1 - \frac{1}{\sqrt[4]{\lambda}} \leq x_i < 1 \right\} \right).$$

Note that there exists $a \in \mathbb{T}$ such that the translated function $f_a(x) = f(x - a)$ satisfies

$$\int_{E^\lambda} f_a dx \leq \frac{2d}{\sqrt[4]{\lambda}}.$$

Because of translation invariance of the expression on the torus, we can replace f by f_a in the expression without changing the value. Therefore we may assume that $f \in \mathcal{F}_{\mathbb{T}}$ satisfies

$$\lambda^{-1} \mathcal{E}_{\tilde{\eta}}(\sqrt{f}, \sqrt{f}) + \lambda^{(1-\gamma) \text{tr } E} \int_{\mathbb{T}} \tilde{F}(f(x)) dx < J + \frac{\epsilon}{2}, \tag{6.1}$$

and

$$\int_{E^\lambda} f dx \leq \frac{2d}{\sqrt[4]{\lambda}}.$$

Consider a smooth bump function ϕ_0 on \mathbb{R} such that $\phi_0 = 1$ on $[\frac{1}{\sqrt[4]{\lambda}}, 1 - \frac{1}{\sqrt[4]{\lambda}}]$, it vanishes outside $(\frac{1}{2\sqrt[4]{\lambda}}, 1 - \frac{1}{2\sqrt[4]{\lambda}})$, and $|\nabla\phi_0| \leq 3\sqrt[4]{\lambda}$. Let $\widetilde{\phi}_0(x_1, \dots, x_d) = \phi_0(x_1)\dots\phi_0(x_d)$ and $\psi(x) = \widetilde{\phi}_0(x)^2$. Then $\|\nabla\widetilde{\phi}_0\| \leq 3\sqrt{d} \cdot \sqrt[4]{\lambda}$.

Let $T_\lambda := \lambda^E([0, 1]^d)$, that is the image of the fundamental domain $[0, 1]^d$ under the transformation λ^E . Given a function h defined on $[0, 1]^d$, let h_λ be the function on T_λ defined by

$$h_\lambda(x) := |\det(\lambda^{-E})| h(\lambda^{-E}x).$$

Now, set $\Omega = \lambda^E(0, 1)^d$ where λ is sufficiently large and

$$g(x) := \frac{(f\psi)_\lambda(x)}{\int_{\mathbb{R}^d} (f\psi)_\lambda(x) dx}.$$

Then, we claim that

$$\mathcal{E}_\eta(\sqrt{g}, \sqrt{g}) + \int_\Omega \widetilde{F}(g(x)) dx \leq J + \epsilon.$$

To see this, first note that g is supported on T_λ and that, by the scaling properties $t^E(W) = t \cdot W$ of the Lévy measure, we have

$$\mathcal{E}_\eta(\sqrt{g}, \sqrt{g}) \leq \lambda^{-1} \mathcal{E}_{\widetilde{\eta}}(\sqrt{g_{1/\lambda}}, \sqrt{g_{1/\lambda}}).$$

Also, since $\widetilde{F}(y) = \widetilde{F}(1)y^\gamma$, we have

$$\int_\Omega \widetilde{F}(g(x)) dx = \lambda^{(1-\gamma)\text{tr} E} \int_{\mathbb{T}} \widetilde{F}(g_{1/\lambda}(x)) dx.$$

Hence we obtain

$$\begin{aligned} & \mathcal{E}_\eta(\sqrt{g}, \sqrt{g}) + \int_\Omega \widetilde{F}(g(x)) dx \\ & \leq \lambda^{-1} \mathcal{E}_{\widetilde{\eta}}(\sqrt{g_{1/\lambda}}, \sqrt{g_{1/\lambda}}) + \lambda^{(1-\gamma)\text{tr} E} \int_{\mathbb{T}} \widetilde{F}(g_{1/\lambda}(x)) dx. \end{aligned}$$

Since f satisfies (6.1), the choice of the bump function and the fact that real part of the eigenvalues of E are all $\geq \frac{1}{2}$ guarantee that for λ sufficiently large,

$$\lambda^{-1} \mathcal{E}_{\widetilde{\eta}}(\sqrt{g_{1/\lambda}}, \sqrt{g_{1/\lambda}}) + \lambda^{(1-\gamma)\text{tr} E} \int_{\mathbb{T}} \widetilde{F}(g_{1/\lambda}(x)) dx < J + \epsilon.$$

6.3 Explicit computation of constants

In this section, we illustrate Theorem 4.4 concerning switch-walk-switch random walks on certain wreath products $K \wr H$ and give some indications concerning exact the computation of the constant $k(\eta, \widetilde{F}_K)$. Let ν denote a symmetric probability measure on the lamp-group K and let $F : (0, \infty) \rightarrow (0, \infty)$ be such that $F(n) = -\log \nu^{(2n)}(e_K)$. Assume that μ is a symmetric measure on the base-group $H = \mathbb{Z}^d$ satisfying the convergence assumption

$$n^{-E} \mu^{(n)} \implies \eta,$$

that is, μ is in the domain of normal attraction of η where η is an operator-stable law with exponent E . Set

$$\hat{\eta} = e^{-\Theta} \quad \text{and} \quad \widehat{L}_\Theta f = \Theta \hat{f}.$$

We will treat cases where F satisfies the scaling assumption $(S-n^{E-a_n})$ for some sequence a_n . Let \widetilde{F} be the corresponding limit function and γ be the associated scaling exponent. See Definition 3.6 and Proposition 3.7.

Let q be the switch-walk-switch measure on $G = K \wr \mathbb{Z}^d$ given by $q = \nu * \mu * \nu$.

Example 6.1 (Power decay on K , $\gamma = 0$). This is the continuation of Example 4.2. Namely, assume $cn^{-\theta} \leq \nu^{(2n)}(e_K) \leq Cn^{-\theta}$. This means that $|F(y) - \theta \log y| \leq C'$ and $\tilde{F}(y) = \theta \mathbf{1}_{(0,\infty)}(y)$, that is, $\gamma = 0$. Under this hypothesis, Theorem 4.4 yields

$$\frac{1}{(2n)^{\text{tr } E / (\text{tr } E + 1)} (\log 2n)^{1 / (\text{tr } E + 1)}} \log q^{(2n)}(e) = -k(\eta, \theta \mathbf{1}_{(0,\infty)}).$$

A simple scaling argument as in [7, 8] shows that the constant $k(\eta, \theta \mathbf{1}_{(0,\infty)})$ as in Lemma 5.5 can be written as

$$k(\eta, \theta \mathbf{1}_{(0,\infty)}) = \theta^{1 / (\text{tr } E + 1)} (\text{tr } E + 1) \left(\frac{\lambda_1(\Theta)}{\text{tr } E} \right)^{\text{tr } E / (\text{tr } E + 1)}. \tag{6.2}$$

Here

$$\lambda_1(\Theta) = \inf_{B: |B|=1} \lambda_1(\Theta, B)$$

where the infimum is taken over all bounded open sets $B \subset \mathbb{R}^d$ such that the Lebesgue measure $|B| = 1$, $|\partial B| = 0$, and $\lambda_1(\Theta, B)$ is the principle eigenvalue for L_Θ with Dirichlet boundary on B .

Example 6.2 (Nonamenable K , $\gamma = 1$). Suppose the lamp group K is nonamenable and let ρ denote the spectral radius of ν so that

$$\left(\nu^{(2n)}(e_K) \right)^{\frac{1}{2n}} \rightarrow \rho.$$

By [16, Theorem 3.16], the switch-walk-switch measure $q = \nu * \mu * \nu$ on $G = K \wr \mathbb{Z}^d$ has spectral radius ρ^2 , namely,

$$\left(q^{(2n)}(\mathbf{e}) \right)^{\frac{1}{2n}} \rightarrow \rho^2.$$

We can recover this result using Theorem 4.4 (actually, a rather trivial special case). We have

$$F(y) = (\log \rho^2)y + o(y) \text{ and } \tilde{F} = (\log \rho^2)y.$$

The variational problem giving the constant $k(\eta, \tilde{F})$ becomes

$$\begin{aligned} k(\eta, \tilde{F}) &= \log \rho^2 + \inf \{ \mathcal{E}_\eta(f, f) : f \geq 0, \|f\|_2 = 1 \} \\ &= \log \rho^2. \end{aligned}$$

Further, if a more precise local limit theorem is known for the random walk driven by ν on K , we can derive a log-limit for $\rho^{-4n} q^{(2n)}(\mathbf{e})$. For example, assume that

$$\nu^{(2n)}(e_K) \sim c(\nu)n^{-\theta} \rho^{2n},$$

for some $\theta > 0$. Then we have

$$\log \left(\rho^{-4n} q^{(2n)}(\mathbf{e}) \right) \sim -k(\eta, \theta \mathbf{1}_{(0,\infty)}) (2n)^{\text{tr } E / (\text{tr } E + 1)} (\log 2n)^{1 / (\text{tr } E + 1)}$$

where the constant $k(\eta, \theta \mathbf{1}_{(0,\infty)})$ is given by (6.2).

Example 6.3 (Cases when $\gamma \in (0, 1)$). The case when $\log \nu^{(2n)}(e_K) \sim -cn^\gamma$, $\gamma \in (0, 1)$, presents two different difficulties. First, there are very few examples of group K for which such asymptotics is known (even so, we do produce such examples above). Second, the corresponding variational problem describing $k(\eta, s \mapsto cs^\gamma)$ is not as well studied and explicit solutions are not known except for certain cases. See [18] where is γ half of our γ .

Assume that K, ν are such that $\log \nu^{(2n)}(e_K) \sim -cn^\gamma$, $\gamma \in (0, 1)$ so that $\tilde{F}(y) = cy^\gamma$. Assume further that $H = \mathbb{Z}$ and $\mathcal{E}_\eta(f, f) = a \int_{\mathbb{R}} |\nabla f|^2 dx$. In this case, [18, Proposition 5.1] provides an explicit solution for the variational problem describing the constant $k(\eta, \tilde{F})$ and one obtains

$$\log q^{(2n)}(\mathbf{e}) \sim -k(\eta, \tilde{F})(2n)^{\frac{1+\gamma}{3-\gamma}}$$

where

$$k(\eta, \tilde{F}) = c^{\frac{2}{3-\gamma}} (2a)^{\frac{1-\gamma}{3-\gamma}} \left(\frac{3-\gamma}{1+\gamma} \right) \left(\frac{\sqrt{\pi} \Gamma\left(\frac{3-\gamma}{2-2\gamma}\right)}{\Gamma\left(\frac{1}{1-\gamma}\right)} \right)^{\frac{2-2\gamma}{3-\gamma}},$$

and the minimizer is the function $(\cos |x|)^{\frac{1}{1-\gamma}} \mathbf{1}_{[0, \pi/2]}(|x|)$, properly dilated and normalized.

When $H = \mathbb{Z}^d$, $\mathcal{E}_\eta(f, f) = a \int_{\mathbb{R}^d} |\nabla f|^2 dx$, and γ is $1/2$ — for example, this is achieved by the switch-walk-switch random walk on the wreath product $\mathbb{Z}_2 \wr \mathbb{Z}^2$ — the minimizer is given by a Bessel function and the constant $k(\eta, s \mapsto cs^{1/2})$ is explicitly computable, see [18, Proposition 5.2].

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