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On properties of a flow generated by an SDE with discontinuous drift*

Olga V. Aryasova† Andrey Yu. Pilipenko‡

Abstract

We consider a stochastic flow on $\mathbb{R}$ generated by an SDE with its drift being a function of bounded variation. We show that the flow is differentiable with respect to the initial conditions. Asymptotic properties of the flow are studied.

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Introduction

Consider an SDE of the form

$$\begin{cases} d\varphi_t(x) = \alpha(\varphi_t(x))dt + \sigma(\varphi_t(x))dw(t), \\ \varphi_0(x) = x, \end{cases} \tag{0.1}$$

where $x \in \mathbb{R}$, $(w(t))_{t\geq 0}$ is a one-dimensional Wiener process.

It is well known (cf. [12]) that if the coefficients of (0.1) are continuously differentiable and the derivatives are bounded and Hölder continuous then there exists a flow of diffeomorphisms for equation (0.1). Under the condition of Lipschitz continuity of the coefficients it was shown the existence of a flow of homeomorphysms (ibid.). Moreover, in the latter situation Bouleau and Hirsch [4] established the differentiability of the flow in generalized sense. Recently, the essential improvement of the results was obtained by Flandoli et al. [6]. They proved the existence of a flow of diffeomorphysms in the case of a smooth non-degenerate noise and a possibly unbounded Hölder continuous drift term.

An SDE with bounded variation drift and $\sigma \equiv 1$ was treated by Attanasio [1], who stated the existence of stochastic flow of class $C^{1,\varepsilon}$, $\varepsilon < 1/2$, under the assumption about boundedness of the positive or the negative part of the distributional derivative

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†Institute of Geophysics, National Academy of Sciences of Ukraine. E-mail: oaryasova@gmail.com
‡Institute of Mathematics, National Academy of Sciences of Ukraine. E-mail: apilip@imath.kiev.ua

of α. We consider equation (0.1) with $\sigma \equiv 1$ and α being a function of bounded variation. We have not additional assumptions about boundedness of the derivative. Besides our method is different from their one.

Note that sometimes the strong solution may exist even if α is a measure. However, in this case the flow may be discontinuous in x. For example, if $\alpha(x) = \beta\delta_0(x),\ \sigma \equiv 1$, where $\beta \in [-1, 1],\ \delta_0$ is a Dirac delta function at zero, then the corresponding strong solution of (0.1) exists and it is a skew Brownion motion [9] but the flow is discontinuous and coalescent (see Barlow et al. [2] and Burdzy and Kaspi [5]).

1 The main results

Consider an SDE

$$\begin{cases} d\varphi_t(x) = \alpha(\varphi_t(x))dt + dw(t), \\ \varphi_0(x) = x, \end{cases} \tag{1.1}$$

where $x \in \mathbb{R},\ \alpha$ is a function on $\mathbb{R}$, $(w(t))_{t\geq 0}$ is a one-dimensional Wiener process.

Later on the function α will be assumed to satisfy some of the following conditions.

(A) α has bounded variation on each compact subset of $\mathbb{R}$;

(B) for all $x \in \mathbb{R}$

$$|\alpha(x)|^2 \leq C(1 + |x|^2);$$

(C) α is a function of bounded variation on $\mathbb{R}$;

(D) there exist $a < 0, b > 0$ such that

$$\alpha(x) \to a,\ \ x \to +\infty,$$
$$\alpha(x) \to b,\ \ x \to -\infty.$$

Given $p \geq 1$, denote by $W^1_{p,loc}(\mathbb{R})$ the set of functions defined on $\mathbb{R}$ that belong to the Sobolev space $W^1_p([c, d])$ for all $\{c, d\} \subset \mathbb{R}$, $c < d$. The results about differentiability and non-coalescence of the flow generated by equation (1.1) is represented as the following statement.

Theorem 1.1. *Let α satisfy conditions (A), (B). Then*

1) For each $x \in \mathbb{R}$ there exists a unique strong solution to equation (1.1).

2) For all $t \geq 0$,

$$P\{\forall p \geq 1\ :\ \varphi_t(\cdot) \in W^1_{p,loc}(\mathbb{R})\} = 1.$$

3) For $t \geq 0$ the Sobolev derivative $\nabla\varphi_t(x)$ is of the form

$$P\left\{\nabla\varphi_t(x) = \exp\left\{\int_{-\infty}^{+\infty} L_z^{\varphi(x)}(t)d\alpha(z)\right\},\ x \in \mathbb{R}\right\} = 1. \tag{1.2}$$

where $L_z^{\varphi(x)}(t)$ is a local time of the process $(\varphi_s(x))_{s\in[0,t]}$ at the point z.

4) For all $\{x_1, x_2\} \subset \mathbb{R},\ x_1 \neq x_2$,

$$P\left\{\varphi_t(x_1) \neq \varphi_t(x_2),\ t \geq 0\right\} = 1.$$

Remark 1.2. *We define the local time of the process $(\varphi_t(x))_{t\geq 0}$ at the point $y \in \mathbb{R}$ by the formula*

$$L_y^{\varphi(x)}(t) = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{[y,y+\varepsilon)}(\varphi_s(x))ds,\ t \geq 0.$$

We prove the Theorem 1.1 in two stage. At the first one we consider α having a compact support on $\mathbb{R}$. In Sections 2-4 we obtain auxiliary results for this stage of proof. The Theorem is proved in Section 5.

In the next sections we analyze the asymptotic behavior of the flow as $t \to \infty$. To do this in Section 6 we find the stationary distribution of the process solving (1.1) under conditions (C), (D). The main result about asymptotic properties of the flow is represented in the following Theorem, proof of which can be found in Section 7. In Section 8 the example is represented.

Theorem 1.3. *Let α satisfy conditions (C), (D). Then for all $\{x_1, x_2\} \subset \mathbb{R},\ x_1 < x_2$,*

$$\frac{\ln(\varphi_t(x_2) - \varphi_t(x_1))}{t} \to \int_{-\infty}^{+\infty} \left(-\int_z^{+\infty} \alpha(y)dP_{stat}(y) \right) d\alpha(z),\ t \to \infty,\ \textit{almost surely,}$$

where P_{stat} is a stationary distribution of the process $(\varphi_t(x))_{t\geq 0}$.

Remark 1.4. *Under the conditions of Theorem the stationary distribution of the process $(\varphi_t(x))_{t\geq 0}$ does not depend on the starting point x.*

2 Approximation of the SDE by SDEs with smooth coefficients

Let α be a function of bounded variation on $\mathbb{R}$ such that it has a compact support. Then for each $x \in \mathbb{R}$ there exists a unique strong solution to (1.1) (cf. [19]).

For $n \geq 1$, let g_n be a continuously differentiable function on $\mathbb{R}$ equal to zero out of $\left(-\frac{1}{n}, \frac{1}{n}\right)$ and such that $g_n(x) \geq 0, x \in \mathbb{R}, \int_{\mathbb{R}} g_n(z)dz = 1$. Put, for $x \in \mathbb{R}$,

$$\alpha_n(x) = \int_{\mathbb{R}} g_n(x-y)\alpha(y)dy.$$

Then $\alpha_n(x) \to \alpha(x)$ as $n \to \infty$ at all points of continuity of α.

For $n \geq 1$, consider an SDE

$$\begin{cases} d\varphi_t^n(x) = \alpha_n(\varphi_t^n(x))dt + dw(t), \\ \varphi_0^n(x) = x. \end{cases} \tag{2.1}$$

Remark 2.1. *There exists $S > 0$ such that for all $n \geq 1, z \in \mathbb{R},\ |z| \geq S,\ \alpha_n(z) = 0$. Besides,*

$$\sup_{x\in\mathbb{R}} |\alpha_n(x)| \leq \sup_{x\in\mathbb{R}} |\alpha(x)|,\ n \geq 1.$$

Remark 2.2. *For each $n \geq 1$, α_n is a function of bounded variation on $\mathbb{R}$, and*

$$\operatorname*{Var}_{\mathbb{R}} \alpha_n \leq \operatorname*{Var}_{\mathbb{R}} \alpha.$$

Lemma 2.3. *For each $p \geq 1$,*

1) for all $t \geq 0$,

$$\sup_{x\in\mathbb{R}} \left(\mathbb{E}(|\varphi_t^n(x)|^p + |\varphi_t(x)|^p)\right) < \infty;$$

2) for all $x \in \mathbb{R},\ t \geq 0,$

$$\mathbb{E}|\varphi_t^n(x) - \varphi_t(x)|^p \to 0 \text{ as } n \to \infty.$$

Proof. The convergence almost surely can be shown by arguments similar to that of McKean [14], Ch.3.10a. The boundedness of the coefficients of (2.1) guarantees the uniform boundedness of the moments:

$$\sup_{n,x} \mathbb{E}|\varphi_t^n(x) - x|^p < \infty.$$

This and convergence almost surely imply the statement of Lemma. □

3 Local times

For each $x \in \mathbb{R}, n \geq 1$, the processes $(\varphi_t(x))_{t\geq 0}$ and $(\varphi_t^n(x))_{t\geq 0}$ solving equations (1.1) and (2.1) are continuous semimartingales. Then, almost surely, there exist local times of these processes defined by the formulas

$$L_y^{\varphi(x)}(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{[y,y+\varepsilon)}(\varphi_s(x)) d\langle \varphi(x), \varphi(x)\rangle_s = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{[y,y+\varepsilon)}(\varphi_s(x)) ds,$$
$$L_y^{\varphi^n(x)}(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{[y,y+\varepsilon)}(\varphi_s^n(x)) d\langle \varphi^n(x), \varphi^n(x)\rangle_s = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \mathbb{1}_{[y,y+\varepsilon)}(\varphi_s^n(x)) ds.$$

Remark 3.1. *It follows from the definition that the local times is measurable with respect to the triple* (t, x, y), $t > 0,\ x \in \mathbb{R},\ \ y \in \mathbb{R}$.

Remark 3.2. *The family* $L^{\varphi(x)}, L^{\varphi^n(x)}$ *may be chosen such that the maps* $(t, y) \to L_y^{\varphi(x)}(t)$, $(t, y) \to L_y^{\varphi^n(x)}(t)$ *are continuous in* t *and cádlág in* y *(cf. [18], Ch.VI). Further we consider such modifications.*

In this section we prove the convergence in square mean and tightness of the sequence of the local times $\{L_y^{\varphi^n(x)}(t) - L_y^{\varphi(x)}(t) :\ n \geq 1\}$.

Lemma 3.3. *For all* $t \geq 0,\ \{x, y\} \subset \mathbb{R}$,

$$\mathbb{E}|L_y^{\varphi^n(x)}(t) - L_y^{\varphi(x)}(t)|^2 \to 0 \ \text{ as } n \to \infty.$$

Proof. By Tanaka's formula (see [18], p. 223)

$$L_y^{\varphi^n(x)}(t) = (\varphi_t^n(x) - y)^+ - (x - y)^+ - \int_0^t \mathbb{1}_{(y,\infty)}(\varphi_s^n(x)) dw(s)$$
$$- \int_0^t \mathbb{1}_{(y,\infty)}(\varphi_s^n(x)) \alpha_n(\varphi_s^n(x)) ds. \quad (3.1)$$

$$L_y^{\varphi(x)}(t) = (\varphi_t(x) - y)^+ - (x - y)^+ - \int_0^t \mathbb{1}_{(y,\infty)}(\varphi_s(x)) dw(s)$$
$$- \int_0^t \mathbb{1}_{(y,\infty)}(\varphi_s(x)) \alpha(\varphi_s(x)) ds. \quad (3.2)$$

Then

$$\mathbb{E}\left(L_y^{\varphi^n(x)}(t) - L_y^{\varphi(x)}(t)\right)^2 \leq K(I + II + III),$$

where K is a constant,

$$\begin{aligned} I &= \mathbb{E}\left((\varphi_t^n(x)-y)^+ - (\varphi_t(x)-y)^+\right)^2, \\ II &= \mathbb{E}\left(\int_0^t \mathbb{1}_{(y,\infty)}(\varphi_s^n(x))dw(s) - \int_0^t \mathbb{1}_{(y,\infty)}(\varphi_s(x))dw(s)\right)^2, \\ III &= \mathbb{E}\left(\int_0^t \mathbb{1}_{(y,\infty)}(\varphi_s^n(x))\alpha_n(\varphi_s^n(x))ds - \int_0^t \mathbb{1}_{(y,\infty)}(\varphi_s(x))\alpha(\varphi_s(x))ds\right)^2. \end{aligned}$$

For I the convergence follows from Lemma 2.3.

To prove the convergence of II and III to 0 we need the following statement.

Proposition 3.4. *Let $\{\xi_n : n \geq 0\}$ be a sequence of random variables. Assume that for any $n \geq 1$ the distribution of ξ_n is absolutely continuous w.r.t. a probability measure ν. Denote the corresponding density by q_n. Let $\{f_n : n \geq 0\}$ be a sequence of measurable functions. Suppose that the following conditions hold:*

1) $\xi_n \to \xi_0,\ n \to \infty$ in probability;

2) $f_n \to f_0,\ n \to \infty$ in measure ν;

3) the sequence of densities $\{q_n :\ n \geq 1\}$ is uniformly integrable w.r.t. measure ν.

Then $f_n(\xi) \to f_0(\xi_0),\ n \to \infty$, in probability.

Proof. The proof is similar to [11], Lemma 2. □

According to the Lebesgue dominated convergence theorem, to prove that $III \to 0,\ n \to \infty$, it is enough to show that

$$\mathbb{1}_{(y,\infty)}(\varphi_s^n(x))\alpha_n(\varphi_s^n(x)) \to \mathbb{1}_{(y,\infty)}(\varphi_s(x))\alpha(\varphi_s(x)),\ n \to \infty,\ \text{in probability.} \tag{3.3}$$

Apply the Proposition 3.4 in which we put $\xi_n = \varphi_s^n(x)$. Let $g_n(t,x,y),\ t \geq 0, x \in \mathbb{R}, y \in \mathbb{R}$, be the transition probability density of the process $(\varphi_t^n(x))_{t\geq 0}$. The density satisfies the inequality (cf. [17], Lemma 2.10)

$$g_n(t,x,y) \leq K\frac{1}{\sqrt{t}}e^{-\mu\frac{(y-x)^2}{t}} \tag{3.4}$$

in every domain of the form $t \in [0,T],\ x \in \mathbb{R},\ y \in \mathbb{R}$. Here $T > 0,\ \mu \in (0,1/2)$, K is a constant that depends only on $T,\ \mu$ and $\sup_{n,x}|\alpha_n(x)|$. Put

$$\rho(y) = C\exp\left\{-\mu\frac{(y-x)^2}{t}\right\},$$

and

$$\nu(dy) = \rho(y)dy,$$

where $C = \sqrt{\mu/(\pi t)}$. Then the distribution of ξ_n is absolutely continuous w.r.t. ν, and the corresponding Radon-Nikodim density is equal to

$$q_n(t,x,y) = \frac{g_n(t,x,y)}{\rho(y)}.$$

The sequence $\{q_n(t,x,y) :\ n \geq 1\}$ is uniformly bounded for fixed $t > 0,\ x \in \mathbb{R}$, and, consequently, uniformly integrable w.r.t. measure ν. Thus by Proposition 3.4 relation (3.3) is justified. The convergence of II can be shown analogously. The Lemma is proved. □

Lemma 3.5. *Let $\{c, d\} \subset \mathbb{R},\ c < d$. Then*

1) *For each pair $(t, x) \in [0, \infty) \times \mathbb{R}$, the local times $L_y^{\varphi(x)}(t),\, L_y^{\varphi^n(x)}(t),\ n \geq 1$, are continuous in y on $[c, d]$.*

2) *For each fixed pair $(t, x),\ t \geq 0,\ x \in \mathbb{R}$, the family of random elements $\{L_\cdot^{\varphi^n(x)}(t) - L_\cdot^{\varphi(x)}(t) :\ n \geq 1\}$ is tight in $C([c, d])$.*

Proof. We prove the Lemma for $c = -1,\ d = 1$. The case of arbitrary c, d can be treated similarly.

Put

$$R_y^{x,n}(t) = L_y^{\varphi^n(x)}(t) - L_y^{\varphi(x)}(t).$$

By virtue of [3], Theorem 12.3 to prove the tightness it is enough to show that

1) the sequence $\{R_0^{x,n}(t) :\ n \geq 1\}$ is tight;

2) there exist $\gamma \geq 0, \alpha > 1$, and $K > 0$, such that for all $\{y_1, y_2\} \subset [-1, 1]$

$$\mathbb{E}|R_{y_2}^{x,n}(t) - R_{y_1}^{x,n}(t)|^\gamma \leq K|y_2 - y_1|^\alpha. \tag{3.5}$$

Besides, according to [3], Th.12.4, inequality (3.5) provides the continuity of $R_y^{x,n}(t)$ with respect to y on $[-1, 1]$ for each pair (t, x) and each $n \geq 1$.

The first item follows from Lemma 3.3 since the fact that $\mathbb{E}(R_0^{x,n}(t))^2 \to 0$ as $n \to \infty$, implies $L_0^{\varphi^n(x)}(t) - L_0^{\varphi(x)}(t) \to 0$ in probability as $n \to \infty$. The convergence ensures the tightness.

The proof of the second item is standard enough. We give necessary calculations though. Assume that $y_1 < y_2$ and represent $L_{y_2}^{\varphi^n(x)}(t) - L_{y_1}^{\varphi^n(x)}(t)$ in the form

$$L_{y_2}^{\varphi^n(x)}(t) - L_{y_1}^{\varphi^n(x)}(t) = I - II - III - IV, \tag{3.6}$$

where

$$\begin{aligned}
I &= (\varphi_t^n(x) - y_2)^+ - (\varphi_t^n(x) - y_1)^+,\\
II &= (x - y_2)^+ - (x - y_1)^+,\\
III &= \int_0^t \mathbb{1}_{(y_1, y_2]}(\varphi_s^n(x))dw(s),\\
IV &= \int_0^t \mathbb{1}_{(y_1, y_2]}(\varphi_s^n(x))\alpha_n(\varphi_s^n(x))ds.
\end{aligned}$$

It is easy to see that

$$\mathbb{E}I^2 \leq (y_2 - y_1)^2,\ \mathbb{E}II^2 \leq (y_2 - y_1)^2, \tag{3.7}$$

Making use of Burkholder's inequality (cf. [10], Ch.3, Th. 3.1) we obtain that for each fixed $T > 0$,

$$\begin{aligned}
\mathbb{E}\max_{0 \leq t \leq T} III^4 &\leq C\mathbb{E}\left(\int_0^T \mathbb{1}_{(y_1, y_2]}(\varphi_s^n(x))ds\right)^2\\
&\leq 2C\mathbb{E}\left(\int_0^T \mathbb{1}_{(y_1, y_2]}(\varphi_s^n(x))ds \int_s^T \mathbb{1}_{(y_1, y_2]}(\varphi_u^n(x))du\right)\\
&= 2C\int_0^T ds \int_s^T \mathbb{E}\left(\mathbb{1}_{(y_1, y_2]}(\varphi_s^n(x))\mathbb{1}_{(y_1, y_2]}(\varphi_u^n(x))\right)du\\
&= 2C\int_0^T ds \int_s^T du \int_{y_1}^{y_2} dy \int_{y_1}^{y_2} g_n(s, x, y)g_n(u - s, y, z)dz,
\end{aligned}$$

where $g_n(t,x,y),\ t\geq 0,\ x\in\mathbb{R},\ y\in\mathbb{R}$, is the transition probability density of the process $(\varphi_t^n(x))_{t\geq 0}$. So we have (see (3.4))

$$\mathbb{E}\max_{0\leq t\leq T} III^4 \leq 2CK^2\int_0^T ds\int_s^T du\int_{y_1}^{y_2}dy\int_{y_1}^{y_2}\frac{e^{-\mu\frac{(y-x)^2}{s}}}{\sqrt{s}}\frac{e^{-\mu\frac{(z-y)^2}{u-s}}}{\sqrt{u-s}}dz$$
$$\leq \tilde{K}(y_2-y_1)^2\int_0^t\int_s^t s^{-1/2}(u-s)^{-1/2}du \leq \tilde{K}(y_2-y_1)^2T. \quad (3.8)$$

Here we denote by $\tilde{K}$ different constants.

Using Hölder inequality and Remark 2.2 we obtain

$$\mathbb{E}IV^4 \leq \mathbb{E}\left[\left(\int_0^t \mathbb{1}_{(y_1,y_2]}(\varphi_s^n(x))ds\right)^2\left(\int_0^t \alpha_n^2(\varphi_s^n(x))ds\right)^2\right]$$
$$\leq (||\alpha||t)^2\,\mathbb{E}\left(\int_0^t \mathbb{1}_{(y_1,y_2]}(\varphi_s^n(x))ds\right)^2.$$

Then from estimate (3.8) we obtain

$$\mathbb{E}IV^4 \leq \tilde{K}t^3(y_2-y_1)^2. \quad (3.9)$$

So each summand in the right-hand side of (3.6) satisfies the second condition of Theorem 12.3 of [3]. Then the left-hand side of (3.6) is continuous with respect to y and tight in $C([-1,1])$. Note that the estimates similar to (3.7)–(3.9) hold for the process $(\varphi_t(x))_{t\geq 0}$. This fact guarantees the continuity of $L_{\cdot}^{\varphi(x)}(t)$ with respect to y on $[-1,1]$. So $\{L_{\cdot}^{\varphi^n(x)}(t) - L_{\cdot}^{\varphi(x)}(t)\ :\ n\geq 1\}$ is a tight sequence of random elements in $C([-1,1])$. The Lemma is proved. □

4 Differential properties of the flow $\varphi_t(x)$

Denote by $\psi_t^n(x)$ the derivative of the function $\varphi_t^n(x)$ with respect to x, i.e.

$$\psi_t^n(x) = (\varphi_t^n(x))'_x\,.$$

Then $\psi_t^n(x)$ is a solution to the following differential equation

$$d\psi_t^n(x) = \alpha_n'(\varphi_t^n(x))\psi_t^n(x)dt.$$

Solving this equation we get

$$\psi_t^n(x) = \exp\left\{\int_0^t \alpha_n'(\varphi_s^n(x))ds\right\}. \quad (4.1)$$

Lemma 4.1. *For all* $t\geq 0,\ x\in\mathbb{R}$,

$$\int_0^t \alpha_n'(\varphi_s^n(x))ds \to \int_{-\infty}^{+\infty} L_z^{\varphi(x)}(t)d\alpha(z),\ n\to\infty,$$

in probability.

Proof. For each pair $(t,x),\ t\geq 0,\ x\in\mathbb{R}$, according to the occupation times formula (see [18], Ch.VI, Corollary 1.6) we have, almost surely,

$$\int_0^t \alpha_n'(\varphi_s^n(x))ds = \int_{\mathbb{R}}\alpha_n'(z)L_z^{\varphi^n(x)}(t)dz$$
$$= \int_{\mathbb{R}}\alpha_n'(z)(L_z^{\varphi^n(x)}(t) - L_z^{\varphi(x)}(t))dz + \int_{\mathbb{R}}\alpha_n'(z)L_z^{\varphi(x)}(t)dz$$
$$= \int_{\mathbb{R}}(L_z^{\varphi^n(x)}(t) - L_z^{\varphi(x)}(t))d\alpha_n(z) + \int_{\mathbb{R}}L_z^{\varphi(x)}(t)d\alpha_n(z) = I + II. \quad (4.2)$$

Remark 2.1 and the continuity of the processes $(L_z^{\varphi(x)}(t))_{t\ge 0}$, $(L_z^{\varphi^n(x)}(t))_{t\ge 0}$ in z entail the existence of the integrals in the right-hand side of (4.2). Besides, this leads to the relation

$$II \to \int_{-\infty}^{+\infty} L_z^{\varphi(x)}(t)d\alpha(z),\ n\to\infty,\ \text{almost surely.}$$

To prove the Lemma it remains to show that

$$I \to 0,\ n\to\infty,\ \text{in probability.}$$

Lemma 3.5 together with Prokhorov's theorem (cf. [3], Th.6.1) show that the family $\left\{L_\cdot^{\varphi^n(x)}(t) - L_\cdot^{\varphi(x)}(t):\ n\ge 1\right\}$ is relatively compact in $C([-S,S])$ (here S is a constant defined in Remark 2.1). By Lemma 3.3, all the finite-dimensional distributions of $L_\cdot^{\varphi^n(x)}(t) - L_\cdot^{\varphi(x)}(t)$ converge to that of the random element in $C([-S,S])$ identically equal to 0. Therefore the sequence of random elements $\left\{L_\cdot^{\varphi^n(x)}(t) - L_\cdot^{\varphi(x)}(t):\ n\ge 1\right\}$ converge in distribution to 0 in $C([-S,S])$ (see [3], Theorem 8.1). Then for all $\varepsilon>0$,

$$P\left\{\sup_{y\in[-S,S]}\left|L_y^{\varphi^n(x)}(t) - L_y^{\varphi(x)}(t)\right| > \varepsilon\right\} \to 0,\ n\to\infty.$$

We have (remind that for all $n\ge 1$, $\operatorname{supp}\alpha_n \in [-S,S]$)

$$\begin{aligned}
P&\left\{\left|\int_{\mathbb{R}}\left(L_z^{\varphi^n(x)}(t) - L_z^{\varphi(x)}(t)\right)d\alpha_n(z)\right| > \varepsilon\right\}\\
&\qquad\le P\left\{\sup_{y\in[-S,S]}\left|L_y^{\varphi^n(x)}(t) - L_y^{\varphi(x)}(t)\right|\cdot \operatorname*{Var}_{\mathbb{R}}\alpha_n > \varepsilon\right\}\\
&\qquad\quad\le P\left\{\sup_{y\in[-S,S]}\left|L_y^{\varphi^n(x)}(t) - L_y^{\varphi(x)}(t)\right| > \frac{\varepsilon}{\operatorname{Var}_{\mathbb{R}}\alpha}\right\} \to 0, n\to\infty.
\end{aligned}$$

The assertion of the Lemma follows immediately. □

5 Proof of Theorem 1.1

Proof. **Stage 1.** Let α be a function of bounded variation on $\mathbb{R}$ having a compact support.

Lemma 4.1 guarantees that for each $t\ge 0,\ x\in\mathbb{R}$,

$$\psi_t^n(x) = \exp\left\{\int_0^t \alpha_n'(\varphi_s^n(x))ds\right\} \to \exp\left\{\int_{\mathbb{R}} L_z^{\varphi(s)}(t)d\alpha(z)\right\} =: \psi_t(x),\ n\to\infty, \tag{5.1}$$

in probability. Let us estimate the pth moment of the process $(\psi_t^n(x))_{t\ge 0}$.

For all $p\ge 1,\ t\ge 0,\ x\in\mathbb{R}$ by occupation times formula, we have

$$\mathbb{E}|\psi_t^n(x)|^p = \mathbb{E}\exp\left\{p\int_0^t \alpha_n'(\varphi_s^n(x))ds\right\} = \mathbb{E}\exp\left\{p\int_{-\infty}^{+\infty} L_z^{\varphi^n(x)}(t)d\alpha_n(z)\right\}.$$

Let $p_1,\dots,p_4$ be such that $p_k>1,\ k=\overline{1,4}$, and $\sum_{k=1}^4 \frac{1}{p_k} = 1$. Using Hölder's inequality and Tanaka's formula we get

$$\mathbb{E}|\psi_t^n(x)|^p \le \prod_{k=1}^4\left(\mathbb{E}\exp\left\{pp_k\int_{-\infty}^{+\infty} f_k(t,x,z)d\alpha_n(z)\right\}\right)^{1/p_k},$$

where

$$f_1(t,x,z) = (\varphi_t^n(x) - z)^+,$$
$$f_2(t,x,z) = -(x-z)^+,$$
$$f_3(t,x,z) = -\int_0^t \mathbb{1}_{(z,+\infty)}(\varphi_s^n(x))dw(s),$$
$$f_4(t,x,z) = -\int_0^t \mathbb{1}_{(z,+\infty)}(\varphi_s^n(x))\alpha_n(\varphi_s^n(x))ds.$$

Then we have

$$\begin{aligned}\mathbb{E}\exp&\left\{pp_1\int_{-\infty}^{+\infty} f_1(t,x,z)d\alpha_n(z)\right\}\\ &\leq \mathbb{E}\exp\left\{pp_1\int_{-\infty}^{+\infty}\left(|x-z| + \int_0^t |\alpha_n(\varphi_s(x))|ds + |w(t)|\right)d\alpha_n(z)\right\}\\ &\leq \exp\left\{Cpp_1 \operatorname*{Var}_{\mathbb{R}}\alpha + ||\alpha||t\right\}\mathbb{E}e^{|w(t)|}, \quad (5.2)\end{aligned}$$

$$\mathbb{E}\exp\left\{pp_2\int_{-\infty}^{+\infty} f_2(t,x,z)d\alpha_n(z)\right\} \leq \mathbb{E}\exp\left\{Cpp_2\operatorname*{Var}_{\mathbb{R}}\alpha\right\}, \quad (5.3)$$

where C is some positive constant,

$$\mathbb{E}\exp\left\{pp_4\int_{-\infty}^{+\infty} f_4(t,x,z)d\alpha_n(z)\right\} \leq \mathbb{E}\exp\left\{pp_4\operatorname*{Var}_{\mathbb{R}}\alpha\cdot||\alpha||\right\}, \quad (5.4)$$

Consider f_3. Using Jensen's inequality, we get

$$\begin{aligned}\mathbb{E}\exp&\left\{pp_3\int_{-\infty}^{+\infty} f_3(t,x,z)d\alpha_n(z)\right\}\\ &\leq \frac{1}{\operatorname{Var}_{\mathbb{R}}\alpha_n}\int_{-\infty}^{+\infty}\mathbb{E}\exp\left\{pp_3\operatorname*{Var}_{\mathbb{R}}\alpha_n f_3(t,x,z)\right\}d\alpha_n(z)\\ &\leq \frac{1}{\operatorname{Var}_{\mathbb{R}}\alpha_n}\int_{-\infty}^{+\infty}\sup_{v\in\mathbb{R}}\mathbb{E}\exp\left\{pp_3\operatorname*{Var}_{\mathbb{R}}\alpha_n f_3(t,x,v)\right\}d\alpha_n(z)\\ &= \sup_{v\in\mathbb{R}}\mathbb{E}\exp\left\{pp_3\operatorname*{Var}_{\mathbb{R}}\alpha_n f_3(t,x,v)\right\}. \quad (5.5)\end{aligned}$$

Let v be fixed. By [10], Th. II.7.2$'$, for each pair (x,v), the process $M_t(x,v) := -\int_0^t \mathbb{1}_{(v,+\infty)}(\varphi_s^n(x))dw(s)$, $t\geq 0$, is a local square integrable martingale that can be represented as follows

$$M_t(x,v) = W^{x,v}(\tau_t(x,v)),$$

where $(W^{x,v}(t))_{t\geq 0}$ is a standard Wiener process, $\tau_t(x,v) = \int_0^t \mathbb{1}_{(v,+\infty)}(\varphi_s^n(x))ds$.

Note that for all $\{x,z\}\subset\mathbb{R}$, $\tau_t(x,v)\leq t$. Then

$$\mathbb{E}\exp\left\{pp_3\operatorname*{Var}_{\mathbb{R}}\alpha_n f_3(t,x,v)\right\} \leq \mathbb{E}\exp\left\{pp_3\operatorname*{Var}_{\mathbb{R}}\alpha\sup_{s\in[0,t]}|W^{x,v}(s)|\right\} = C,$$

where C is a constant independent of x and v. This and (5.5) imply the estimate

$$\mathbb{E}\exp\left\{pp_3\int_{-\infty}^{+\infty} f_3(t,x,z)d\alpha_n(z)\right\} \leq C, \quad (5.6)$$

where C ia some constant. Now the uniform boundedness of the pth moment follows from inequalities (5.2)-(5.6). This and (5.1) imply that for all $t \geq 0,\ p \geq 1$,

$$\mathbb{E}|\psi_t^n(x) - \psi_t(x)|^p \to 0,\ n \to \infty.$$

Since

$$\sup_{n,x} \mathbb{E}\left(|\psi_t^n(x)|^p + |\psi_t(x)|^p\right) < \infty,$$

by the dominated convergence theorem, we get the relation

$$\mathbb{E}\int_c^d |\psi_t^n(x) - \psi_t(x)|^p\, dx \to 0,\ n \to \infty,$$

valid for all $\{c, d\} \in \mathbb{R},\ c < d,\ p \geq 1$. So there exists a subsequence $\{n_k :\ k \geq 1\}$ such that

$$\int_c^d |\psi_t^{n_k}(x) - \psi_t(x)|^p\, dx \to 0 \text{ a.s. as } n \to \infty.$$

Without loss of generality we can suppose that

$$\int_c^d |\psi_t^n(x) - \psi_t(x)|^p\, dx \to 0 \text{ a.s. as } n \to \infty, \tag{5.7}$$

and (see Lemma 2.3)

$$\int_c^d |\varphi_t^n(x) - \varphi_t(x)|^p\, dx \to 0 \text{ a.s. as } n \to \infty. \tag{5.8}$$

This imply that, almost surely, the function $\varphi_t(x),\ t \geq 0,\ x \in \mathbb{R}$, has a weak derivative in the Sobolev sense with respect to x in any interval $[c, d]$ (cf. [16], §19.5), and this derivative is equal to $\psi_t(x) = \exp\left\{\int_{-\infty}^{+\infty} L_z^{\varphi(x)}(t) d\alpha(z)\right\},\ x \in [c, d]$. Besides, for all $\{x_1, x_2\} \subset \mathbb{R}$, $x_1 < x_2$, the equality

$$\varphi_t(x_2) - \varphi_t(x_1) = \int_{x_1}^{x_2} \psi_t(y) dy = \int_{x_1}^{x_2} \exp\left\{\int_{-\infty}^{+\infty} L_z^{\varphi(y)}(t) d\alpha(z)\right\} dy \tag{5.9}$$

holds true almost surely. Note that generally the exceptional set depends on t.

Fix $T > 0$. Since $L_z^{\varphi(y)}(t)$ is continuous in t and z (see Remark 3.2), monotonic in t, and $\operatorname{supp} \alpha \subset [-S, S]$, we have

$$\begin{aligned} \forall y \in [x_1, x_2]\ \ P\Bigg\{\int_{-\infty}^{+\infty} L_z^{\varphi(y)}(t) d\alpha(z) &\geq -\sup_{z\in[-S,S]} L_z^{\varphi(y)}(t) \cdot ||\alpha|| \\ &\geq -\sup_{z\in[-S,S]} L_z^{\varphi(y)}(T) \cdot ||\alpha|| > -\infty,\ t \in [0, T]\Bigg\} = 1. \end{aligned}$$

Put $M_T(y) = \sup_{z\in[-S,S]} L_z^{\varphi(y)}(T) \cdot ||\alpha||$. Then by the continuity of $L_z^{\varphi(y)}(t)$ in t and Fubini's theorem,

$$P\left\{\inf_{t\in[0,T]} \int_{-\infty}^{+\infty} L_z^{\varphi(y)}(t) d\alpha(z) \geq -M_T(y) > -\infty \text{ for almost all } y \in [x_1, x_2]\right\} = 1$$

This implies that for all $T > 0,\ x_1 < x_2$,

$$\begin{aligned} P\left\{\inf_{t\in[0,T]} (\varphi_t(x_2) - \varphi_t(x_1)) > 0\right\} = P\left\{\inf_{t\in[0,T]} \int_{x_1}^{x_2} \exp\left\{\int_{-\infty}^{+\infty} L_z^{\varphi(y)}(t) d\alpha(z)\right\} dy\right\} \geq \\ P\left\{\inf_{t\in[0,T]} \int_{x_1}^{x_2} \exp\left\{-M_T(y)\right\} dy > 0\right\} = 1. \end{aligned}$$

Passing to the limit as T tends to $+\infty$, we arrive at the relation

$$P\left\{\varphi_t(x_2)-\varphi_t(x_1)>0,\ t\geq 0\right\}=1.$$

Stage 2. Let α be an arbitrary function on $\mathbb{R}$ satisfying conditions (A), (B). For $n\geq 1$, let h_n be a smooth function on $\mathbb{R}$ such that $0\leq h_n(x)\leq 1,\ x\in\mathbb{R}$; $h_n(x)=1,\ x\in[-n,n]$; $h_n(x)=0,\ |x|>n+1$. Put

$$\alpha_n(x)=\alpha(x)h_n(x), x\in\mathbb{R}.$$

Suppose $(\varphi_t^n(x))_{t\geq 0}$ is a solution of equation (2.1). Put $\tau_n=\sup\left\{t:\ \sup_{0\leq s\leq t}|\varphi_t^n(x)|\leq n\right\}$. As $\alpha(x)=\alpha_n(x)$ on $[-n,n]$, we have $\varphi_t(x)=\varphi_t^n(x)$ on $[0,\tau_t]$ almost surely. To prove the existence and uniqueness of a strong solution to equation (1.1) we need to show that $\tau_n\to+\infty,\ n\to\infty$, almost surely. By Chebyshev's inequality and condition (B) for $T>0$,

$$\begin{aligned}P\{\tau_n<T\}=P\{\sup_{0\leq t\leq T}|\varphi_t^n(x)|>n\}&\leq\frac{1}{n^2}\mathbb{E}\left(\sup_{0\leq t\leq T}(\varphi_t^n(x))^2\right)\\&\leq\frac{C}{n^2}\left((\varphi_t^n(0))+T\mathbb{E}\sup_{0\leq t\leq T}\int_0^t(1+(\varphi_t^n(x))^2)ds+T\right)\\&\qquad\leq\frac{C}{n^2}\left(K(1+T+T^2)+T\int_0^T\mathbb{E}\sup_{0\leq s\leq T}(\varphi_s^n(x))^2ds\right),\end{aligned}$$

where C,K are some positive constants. The Gronwall-Bellman inequality implies

$$\mathbb{E}\sup_{0\leq t\leq T}(\varphi_t^n(x))^2\leq C_1,$$

where C_1 is a constant depending only on T and x. This fact and monotonicity of the sequence $\{\tau_n:\ n\geq 1\}$ give $\tau_n\to+\infty$ as $n\to\infty$ almost surely. Hence there exists a unique strong solution to equation (1.1).

To prove the differentiability of the flow let us consider an arbitrary interval $[x_1,x_2]$. By comparison theorem (cf. [15], Th. 2.1) $\varphi_t(x_1)\leq\varphi_t(x)\leq\varphi_t(x_2)$. Denote

$$M_t=\max_{s\in[0,t]}\left(|\varphi_s(x_1)|\vee|\varphi_s(x_2)|\right).$$

There exists $N>0$ such that $M_t<N$. Then $\varphi_s(x)=\varphi_s^n(x)$ for all $x\in[x_1,x_2],\ s\in[0,t]$, and $n>N$, almost surely. Consequently, for all $n>N$, the local times and the derivatives of the processes $\varphi_s(x),\varphi_s^n(x)$ coincide on $x\in[x_1,x_2],\ s\in[0,t]$. This entails assertions 2)-4) of the Theorem. □

6 Stationary distribution

Assume that a function α satisfies conditions (C), (D). In this section we prove the existence of a stationary distribution for the process $(\varphi_t(x))_{t\geq 0}$ provided that conditions (C), (D) are justified. Apply Theorem 3 of [8], §18 to equation (1.1). Put

$$s(x)=\int_0^x\exp\left\{-2\int_0^z\alpha(y)dy\right\}dz,\ x\in\mathbb{R}.$$

By (D),

$$s(x)\to+\infty,\ x\to+\infty,$$
$$s(x)\to-\infty,\ x\to-\infty.$$

Besides, s has a continuous positive derivative

$$s'(x)=\exp\{-2\int_0^x\alpha(z)dz\},\ x\in\mathbb{R}.$$

Let $q(\cdot) = s^{-1}(\cdot)$ be a continuously differentiable function on $\mathbb{R}$ inverse to $s(\cdot)$. The function $\eta_t(x) = s(\varphi_t(x))$ is a solution of the SDE

$$\begin{cases} d\eta_t(x) = \sigma(\eta_t(x))dw(t), \\ \eta_0(x) = s(x), \end{cases}$$

where $\sigma(y) = s'(q(y)) = \exp\{-2\int_0^{q(y)} \alpha(z)dz\},\ y \in \mathbb{R}$. Using (D) it is easy to see that

$$\int_{-\infty}^{+\infty} \frac{1}{\sigma^2(y)} < \infty. \tag{6.1}$$

The continuity of q and boundedness of α provide that the function σ is locally Lipschitz continuous. Let us see that σ is globally Lipschitz continuous function on $\mathbb{R}$. As a locally Lipschitz continuous function it has a derivative at almost all points $x \in \mathbb{R}$, and the derivative is as follows

$$\sigma'(y) = -2\alpha(q(y))q'(y)\exp\left\{-\int_0^{q(y)} \alpha(z)dz\right\}.$$

Taking into account that

$$q'(y) = \frac{1}{s'(q(y))} = \exp\left\{2\int_0^{q(y)} \alpha(z)dz\right\},$$

we arrive at the formula

$$\sigma'(y) = -2\alpha(q(y))$$

valid for almost all $y \in \mathbb{R}$. Then according to the Newton-Leibniz formula for locally absolutely continuous functions, for all $\{x_1, x_2\} \subset \mathbb{R}$,

$$|\sigma(x_2) - \sigma(x_1)| = \left|\int_{x_1}^{x_2} 2\alpha(q(y))dy\right| \le 2||\alpha|| \cdot |x_2 - x_1|.$$

So σ is Lipschitz continuous, and the conditions of [8], §18, Theorem 3 are fulfilled. Let $\Phi_{t,x}(y) <\ y \in \mathbb{R}$, be the distribution function of the random variable $\varphi_t(x)$, i.e.

$$\Phi_{t,x}(y) = P\{\varphi_t(x) < y\}.$$

The Theorem implies the existence of a stationary distribution $P_{stat}(y),\ y \in \mathbb{R}$, and for all $\{x, y\} \subset \mathbb{R}$,

$$P_{stat}(y) = \lim_{t\to\infty} \Phi_{t,x}(y).$$

7 Proof of Theorem 1.3

Heuristically the asymptotic behavior of the difference $\varphi_t(x_2)-\varphi_t(x_1)$ can be guessed as follows. If we represent the local time from (5.9) by Tanaka's formula (3.2), then by the ergodic theorem, the last integral in the right-hand side of (3.2) is equivalent to $t\int_y^{+\infty} \alpha(z)dP_{stat}(z)$ as t tends to ∞. The first member is bounded in probability because $\varphi_t(x)$ converges weakly to the stationary distribution. The stochastic integral in the right-hand side of (3.2) is a continuous martingale with its characteristics being less than or equal to t. Therefore it is naturally to expect that $L_y^{\varphi(x)}(t) \sim t\int_y^{+\infty} \alpha(z)dP_{stat}(z),\ t \to \infty$, and, respectively,

$$\ln\left(\varphi_t(x_2) - \varphi_t(x_1)\right) \sim t\int_{-\infty}^{+\infty}\left(-\int_z^{+\infty} \alpha(y)dP_{stat}(y)\right)d\alpha(z),\ t \to \infty.$$

Below we give the rigorous proof of this fact.

Proof. In this proof we will use the representation of the function α in the form

$$\alpha(x) = \alpha_1(x) - \alpha_2(x),\ x \in \mathbb{R},$$

where α_1, α_2 are nondecreasing functions on $\mathbb{R}$. Using Jensen's inequality we get the lower bound for $\frac{\ln(\varphi_t(x_2)-\varphi_t(x_1))}{t}$ as follows

$$\begin{aligned}\frac{\ln(\varphi_t(x_2)-\varphi_t(x_1))}{t} &= \frac{\ln\left(\int_{x_1}^{x_2}\exp\left\{\int_{-\infty}^{+\infty} L_z^{\varphi(x)}(t)d\alpha(z)\right\}dx\right)}{t}\\ &\geq \frac{\int_{x_1}^{x_2}\ln\left((x_2-x_1)\exp\left\{\int_{-\infty}^{+\infty} L_z^{\varphi(x)}(t)d\alpha(z)\right\}\right)dx}{t(x_2-x_1)}\\ &= \frac{1}{t}\ln(x_2-x_1) + \frac{1}{t}\int_{x_1}^{x_2}\frac{\int_{-\infty}^{+\infty} L_z^{\varphi(x)}(t)d\alpha(z)}{x_2-x_1}dx. \qquad (7.1)\end{aligned}$$

On the other hand, let p_1, p_2, p_3 be grater than 1 and such that $\sum_k \frac{1}{p_k} = 1$. Then by Hölder's inequality we obtain

$$\begin{aligned}\frac{\ln(\varphi_t(x_2)-\varphi_t(x_1))}{t} &= \frac{\ln\left(\int_{x_1}^{x_2}\exp\left\{\int_{-\infty}^{+\infty} L_z^{\varphi(x)}(t)d\alpha(z)\right\}dx\right)}{t}\\ &= \frac{\ln\left(\int_{x_1}^{x_2}\exp\left\{\sum_{k=1}^3\int_{-\infty}^{+\infty} f_k(t,x,z)d\alpha(z)\right\}dx\right)}{t}\\ &\leq \frac{\ln\left(\prod_{k=1}^3\left(\int_{x_1}^{x_2}\exp\left\{p_k\int_{-\infty}^{+\infty} f_k(t,x,z)d\alpha(z)\right\}dx\right)^{1/p_k}\right)}{t}\\ &= \frac{\sum_{k=1}^3\frac{1}{p_k}\ln\left(\int_{x_1}^{x_2}\exp\left\{p_k\int_{-\infty}^{+\infty} f_k(t,x,z)d\alpha(z)\right\}dx\right)}{t}, \qquad (7.2)\end{aligned}$$

where

$$\begin{aligned}f_1(t,x,z) &= (\varphi_t(x)-z)^+ - (x-z)^+,\\ f_2(t,x,z) &= -\int_0^t \mathbb{1}_{(z,\infty)}(\varphi_s(x))dw(s),\\ f_3(t,x,z) &= -\int_0^t \mathbb{1}_{(z,\infty)}(\varphi_s(x))\alpha(\varphi_s(x))ds.\end{aligned}$$

Let us show that the right-hand side of (7.2) converges to $\int_{-\infty}^{+\infty}\left(-\int_z^{+\infty}\alpha(y)dP_{stat}(y)\right)d\alpha(z)$ almost surely. The same relation for the right-hand side of (7.1) can be proved similarly.

Consider the summand with $f_1(t,x,z)$. It is easy to see that for all $\{x,z\}\subset\mathbb{R},\ t\geq 0$,

$$|(\varphi_t(x)-z)^+ - (x-z)^+| \leq |\varphi_t(x)-x|.$$

By the comparison theorem (cf. [15], Th. 2.1) for all $x\in[x_1,x_2]$,

$$|\varphi_t(x)-x| \leq |\varphi_t(x_2)-x_1|.$$

Then

$$\begin{aligned}&\frac{\ln\left(\int_{x_1}^{x_2}\exp p_1\int_{-\infty}^{+\infty} f_1(t,x,z)d\alpha(z)dx\right)}{t}\\ &\qquad\leq \frac{\ln\left(\int_{x_1}^{x_2}\exp\left\{p_1|\varphi_t(x_2)-x_1|\cdot\operatorname{Var}\alpha\right\}dx\right)}{t}\\ &\qquad\leq \frac{\ln\left((x_2-x_1)\operatorname{Var}\alpha\right)}{t} + \frac{p_1|\varphi_t(x_2)|}{t} + \frac{p_1|x_1|}{t}. \qquad (7.3)\end{aligned}$$

The first and the third summands obviously tend to 0 as $t \to \infty$. Let us show that the same assertion is true for the second summand.

Put $c_1 = a/2,\ c_2 = b/2$. Fix $\varepsilon \in (0, 1/2\min(-a, b))$. There exist $N_1 > 0$ and $N_2 < 0$ such that

$$\begin{aligned} &\alpha(x) < a + \varepsilon < c_1,\ x \geq N_1, \\ &\alpha(x) > b - \varepsilon > c_2,\ x \leq N_2. \end{aligned} \tag{7.4}$$

Consider the following stochastic differential equations

$$\chi_t^1(x) = x + c_1 t + w(t) + L_{N_1+1}^{\chi^1(x)}(t), \tag{7.5}$$

$$\chi_t^2(x) = x + c_2 t + w(t) - L_{N_2-1}^{\chi^1(x)}(t), \tag{7.6}$$

where $\left(L_{N_1+1}^{\chi^1(x)}(t)\right)_{t\geq 0}$, $\left(L_{N_2-1}^{\chi^2(x)}(t)\right)_{t\geq 0}$ are local times of the processes $\left(\chi_t^1(x)\right)_{t\geq 0}$, $\left(\chi_t^2(x)\right)_{t\geq 0}$ at the points N_1+1, N_2-1 respectively.

There exist solutions of these equations (see ([13])). Starting from $x > N_1 + 1$, the solution to the former equation is a diffusion process taking values on $[N_1+1, +\infty)$ with instantaneous reflection at the point N_1+1. For $x < N_2 - 1$, the solution of the latter equation is a diffusion process taking values on $(-\infty, N_2-1]$ with instantaneous reflection at the point $N_2 - 1$.

Given $x > N_1 + 1$, then

$$P\{\varphi_t(x) \leq \chi_t^1(x),\ t \geq 0\} = 1.$$

Indeed, let $t_{N_1+1} = \inf\{t : \chi_t^1(x) = N_1 + 1\}$. Then for all $t \in (0, t_{N_1+1})$, by (7.4)

$$\chi_t^1(t) - \varphi_t(x) = \int_0^t (c_1 - \alpha(\varphi_s(x)))ds > 0.$$

Consequently, if there exists a point $r_0 \geq t_{N_1+1}$ such that $\chi_{r_0}^1(x) < \varphi_{r_0}(x)$, then there exists a point $r_1 \in [t_{N_1+1}, r_0)$ at which $\chi_{r_1}^1(x) = \varphi_{r_1}(x)$. Moreover $\varphi_{r_1}(x) \geq N_1 + 1$. Choose $\delta > 0$ such that for all $s \in [r_1, r_1 + \delta]$, $\varphi_s(x) \geq N_1$. Then

$$\chi_s^1(x) - \varphi_s(x) = \int_{r_1}^s (c_1 - \alpha(\varphi_s(x)))ds + L_{N_1+1}^{\chi^1(x)}(s) - L_{N_1+1}^{\chi^1(x)}(r_1),\ s \in [r_1, r_1 + \delta]. \tag{7.7}$$

But the right-hand side of (7.7) is non-negative. This implies that for each $x > N_1 + 1$, and all $t \geq 0$, $\chi_t^1(x) \geq \varphi_t(x)$. By the comparison theorem (see [15], Th. 3.1) $\chi_t^1(x) \leq B_t^1(x),\ t \geq 0$, where $(B_t^1(x))_{t\geq 0}$ is a one-dimensional Brownian motion with reflection at the point $N_1 + 1$, which is a solution to the following SDE

$$B_t^1(x) = x + w(t) + L_{N_1+1}^{B(x)}.$$

Thus for all $x > N_1 + 1$,

$$\varphi_t(x) \leq B_t^1(x),\ t \geq 0. \tag{7.8}$$

Involving $(\chi_t^2(x))_{t\geq 0}$ and arguing in the same way we get the inequality

$$\varphi_t(x) \geq B_t^2(x),\ t \geq 0, \tag{7.9}$$

valid for all $x < N_2 - 1$, where $(B_t^2(x))_{t\geq 0}$ is a Brownian motion with reflection at the point $N_2 - 1$ solving the following SDE

$$B_t^2(x) = x + w(t) - L_{N_2-1}^{B(x)}.$$

It is known that for all $x \in \mathbb{R}$,

$$\begin{aligned} \frac{B_t^1(x)}{t} &\to 0,\ t \to \infty \\ \frac{B_t^2(x)}{t} &\to 0,\ t \to \infty. \end{aligned} \tag{7.10}$$

The fact that $\frac{\varphi_t(x)}{t} \to 0,\ t \to \infty$, follows now from relations (7.10), inequalities (7.8), (7.9) and assertion that for all $\{d_1, x, d_2\} \subset \mathbb{R},\ d_1 < x < d_2$,

$$\tau_x[d_1, d_2] < \infty \text{ a.s.}, \tag{7.11}$$

where

$$\tau_x[d_1, d_2] = \inf\{t \geq 0 : \varphi_t(x) = d_1 \text{ or } \varphi_t(x) = d_2\}.$$

Inequality (7.11) is a consequence of (6.1) (cf. [8], §18).

Thus we have proved that the second term in the right-hand side of (7.3) tends to zero as t tends to ∞.

Examine the third item in the right-hand side of (7.2). We have

$$\int_{-\infty}^{+\infty} \left(-\int_0^t 1\!\!1_{\varphi_s(x)>z} \alpha(\varphi_s(x)) ds \right) d\alpha(z) = \sum_{i,j=1}^{2} \mathrm{I}_{ij},$$

where

$$\mathrm{I}_{ij} = (-1)^{i+j} \int_{-\infty}^{+\infty} \left(-\int_0^t 1\!\!1_{\varphi_s(x)>z} \alpha_i(\varphi_s(x)) ds \right) d\alpha_j(z).$$

Consider I_{11}. By the comparison theorem for all $t \geq 0,\ x \in [x_1, x_2]$,

$$\begin{aligned} \int_{-\infty}^{+\infty} \left(-\int_0^t 1\!\!1_{\varphi_s(x_2)>z} \alpha_1(\varphi_s(x_2)) ds \right) d\alpha_1(z) &\leq \mathrm{I}_{11} \\ &\leq \int_{-\infty}^{+\infty} \left(-\int_0^t 1\!\!1_{\varphi_s(x_1)>z} \alpha_1(\varphi_s(x_1)) ds \right) d\alpha_1(z). \end{aligned} \tag{7.12}$$

Using the similar estimates for $\mathrm{I}_{12}, \mathrm{I}_{21}, \mathrm{I}_{22}$ we get

$$\begin{aligned} \frac{1}{p_3 t} \ln \left(\int_{x_1}^{x_2} \exp \left\{ p_3 \int_{-\infty}^{+\infty} f_3(t, x, z) d\alpha(z) \right\} dx \right) \geq \frac{1}{t} \Bigg[\frac{\ln(x_2 - x_1)}{p_3} \\ + \int_{-\infty}^{+\infty} \left(-\int_0^t 1\!\!1_{\varphi_s(x_2)>z} \alpha_1(\varphi_s(x_2)) ds \right) d\alpha_1(z) \\ + \int_{-\infty}^{+\infty} \left(-\int_0^t 1\!\!1_{\varphi_s(x_1)>z} \alpha_2(\varphi_s(x_1)) ds \right) d\alpha_1(z) \\ + \int_{-\infty}^{+\infty} \left(-\int_0^t 1\!\!1_{\varphi_s(x_1)>z} \alpha_1(\varphi_s(x_1)) ds \right) d\alpha_2(z) \\ + \int_{-\infty}^{+\infty} \left(-\int_0^t 1\!\!1_{\varphi_s(x_2)>z} \alpha_2(\varphi_s(x_2)) ds \right) d\alpha_2(z) \Bigg]. \end{aligned} \tag{7.13}$$

Obviously, the first summand in the right-hand side of (7.13) tends to 0 as t tends to ∞. By the ergodic theorem (see Theorem 3, §18 of [8]) for all $x \in \mathbb{R}, i = 1, 2$, we get

$$\frac{1}{t} \int_0^t 1\!\!1_{\varphi_s(x)>z} \alpha_i(\varphi_s(x)) ds \to \int_z^{+\infty} \alpha_i(y) dP_{stat}(y).$$

Making use of the dominated convergence theorem and collecting the members, we see that the expression in the right-hand side of (7.13) tends to

$$\int_{-\infty}^{+\infty} \left(-\int_z^{+\infty} \alpha(y) dP_{stat}(y) \right) d\alpha(z)$$

almost surely as t tends to ∞. Using the upper estimates for $\mathrm{I}_{ij},\ \{i,j\}\subset\{1,2\}$, similarly we get that

$$\frac{1}{p_3 t}\ln\left(\int_{x_1}^{x_2}\exp\left\{p_3\int_{-\infty}^{+\infty} f_3(t,x,z)d\alpha(z)\right\}dx\right)\to \int_{-\infty}^{+\infty}\left(-\int_z^{+\infty}\alpha(y)dP_{stat}(y)\right)d\alpha(z),\ t\to\infty,\ \text{almost surely.}$$

It is left to prove that the second member in the right-hand side of (7.2) converges to 0 as t tends to ∞ almost surely. It can be represented in the form

$$\frac{\ln\left(\int_{x_1}^{x_2}\exp\left\{p_2\int_{-\infty}^{+\infty}\left(-\int_0^t \mathbb{1}_{\varphi_s(x)>z}dw(s)\right)d\alpha(z)\right\}dx\right)}{t}$$
$$=\frac{\ln(x_2-x_1)}{t}+\frac{p_2\int_{-\infty}^{+\infty}\left(-\int_0^t \mathbb{1}_{\varphi_s(x_1)>z}dw(s)\right)d\alpha(z)}{t}$$
$$+\frac{\ln\left(\int_{x_1}^{x_2}\exp\left\{p_2\int_{-\infty}^{+\infty}\left(-\int_0^t\left(\mathbb{1}_{\varphi_s(x)>z}-\mathbb{1}_{\varphi_s(x_1)>z}\right)dw(s)\right)d\alpha(z)\right\}dx\right)}{t}=I+II+III.$$

Consider II. By a martingale inequality (cf. [10]), ineq. (6.16) of Ch. 1),

$$\mathbb{E}\sup_{r\in[0,t]}\left(\int_{-\infty}^{+\infty}\left(-\int_0^r \mathbb{1}_{\varphi_s(x_1)>z}dw(s)\right)d\alpha(z)\right)^2\le$$
$$=\mathbb{E}\sup_{r\in[0,t]}\left(\int_0^\tau\left(\int_{-\infty}^{+\infty}\mathbb{1}_{\varphi_s(x_1)>z}d\alpha(z)\right)dw(s)\right)^2\le$$
$$4\mathbb{E}\int_0^t\left(\int_{-\infty}^{+\infty}\mathbb{1}_{\varphi_s(x_1)>z}d\alpha(z)\right)^2 ds\le 4(\operatorname*{Var}_{\mathbb{R}}\alpha)^2 t.$$

Then by monotone convergence theorem

$$\mathbb{E}\sum_{n=1}^\infty\sup_{r\in[2^n,2^{n+1}]}\left(\frac{\int_{-\infty}^{+\infty}\left(-\int_0^r \mathbb{1}_{\varphi_s(x_1)>z}dw(s)\right)d\alpha(z)}{r}\right)^2$$
$$\le\sum_{n=1}^\infty\frac{\mathbb{E}\sup_{r\in[0,2^{n+1}]}\left(\int_{-\infty}^{+\infty}\left(-\int_0^r \mathbb{1}_{\varphi_s(x_1)>z}dw(s)\right)d\alpha(z)\right)^2}{2^{2n}}$$
$$\le\left(\operatorname*{Var}_{\mathbb{R}}\alpha\right)^2\sum_{n=1}^\infty\frac{4\cdot 2^{n+1}}{2^{2n}}=\left(\operatorname*{Var}_{\mathbb{R}}\alpha\right)^2\sum_{n=1}^\infty\frac{8}{2^n}<\infty.$$

This implies that

$$\sup_{\tau\in[2^n,2^{n+1}]}\left(\frac{\int_{-\infty}^{+\infty}\left(-\int_0^\tau \mathbb{1}_{\varphi_s(x_1)>z}dw(s)\right)d\alpha(z)}{r}\right)^2\to 0,\ n\to\infty,\ \text{almost surely.}$$

Consequently,

$$\lim_{t\to\infty}\frac{\int_{-\infty}^{+\infty}\left(-\int_0^t \mathbb{1}_{\varphi_s(x_1)>z}dw(s)\right)d\alpha(z)}{t}=0\ \text{almost surely.}\tag{7.14}$$

Note that

$$III\le\frac{(x_2-x_1)}{t}-\frac{p_2\sup_{x\in[x_1,x_2]}\sup_{r\in[0,t]}\left|\int_0^r\left(\int_{-\infty}^{+\infty}\left(\mathbb{1}_{\varphi_s(x)>z}-\mathbb{1}_{\varphi_s(x_1)>z}\right)d\alpha(z)\right)dw(s)\right|}{t}.\tag{7.15}$$

To prove that $III \to 0$ as $t \to \infty$ it is sufficient to show that

$$\frac{\sup_{x\in[x_1,x_2]}\sup_{r\in[0,t]}\left|\int_0^r\left(\int_{-\infty}^{+\infty}\left(\mathbb{1}_{\varphi_s(x)>z}-\mathbb{1}_{\varphi_s(x_1)>z}\right)d\alpha(z)\right)dw(s)\right|}{t}\to 0,\ t\to\infty,\ \text{almost surely.} \tag{7.16}$$

Put $\xi_t(x)=\int_0^t\left(\int_{-\infty}^{+\infty}\mathbb{1}_{\varphi_s(x)>z}d\alpha(z)\right)dw(s)$. According to the Garsia-Rodemich-Rumsey inequality [7] for all $t\geq 0,\ x\in[x_1,x_2],\ q>1,\ \alpha\in(\frac{1}{q},1]$, there exists $c(\alpha,q)>0$ such that

$$|\xi_t(x)-\xi_t(x_1)|^q\leq c(\alpha,q)|x-x_1|^{q\alpha-1}\iint_{[x_1,x_2]^2}\frac{|\xi_t(u)-\xi_t(v)|^q}{|u-v|^{q\alpha+1}}dudv. \tag{7.17}$$

Then for $q=4$,

$$\begin{aligned}\mathbb{E}\sup_{x\in[x_1,x_2]}\sup_{t\in[0,T]}&|\xi_t(x)-\xi_t(x_1)|^4\\ &\leq c(\alpha,4)|x_2-x_1|^{4\alpha-1}\iint_{[x_1,x_2]^2}\frac{\mathbb{E}\sup_{t\in[0,T]}|\xi_t(u)-\xi_t(v)|^4}{|u-v|^{4\alpha+1}}dudv\end{aligned} \tag{7.18}$$

Let us estimate the expectation $\mathbb{E}\sup_{t\in[0,T]}|\xi_t(u)-\xi_t(v)|^4$. According to Burkholder's inequality (cf. [10], Ch.3, Th. 3.1) we get

$$\begin{aligned}\mathbb{E}\sup_{t\in[0,T]}&|\xi_t(u)-\xi_t(v)|^4\\ &\leq C\mathbb{E}\left(\int_0^T\left(\int_{-\infty}^{+\infty}\left(\mathbb{1}_{\varphi_s(u)>z}-\mathbb{1}_{\varphi_s(v)>z}\right)d\alpha(z)\right)^2ds\right)^2.\end{aligned}$$

Consider the case of $u<v$. Making use of Hölder's inequality and applying the comparison theorem we arrive at the inequality

$$\begin{aligned}\left(\int_{-\infty}^{+\infty}\left(\mathbb{1}_{\varphi_s(u)>z}-\mathbb{1}_{\varphi_s(v)>z}\right)d\alpha(z)\right)^2&\\ &\leq 2\operatorname{Var}\alpha_1\int_{-\infty}^{+\infty}\left(\mathbb{1}_{\varphi_s(u)>z}-\mathbb{1}_{\varphi_s(v)>z}\right)^2d\alpha_1(z)\\ &+2\operatorname{Var}\alpha_2\int_{-\infty}^{+\infty}\left(\mathbb{1}_{\varphi_s(u)>z}-\mathbb{1}_{\varphi_s(v)>z}\right)^2d\alpha_2(z)\\ &\leq C\int_{-\infty}^{+\infty}\left(\mathbb{1}_{\varphi_s(u)>z}-\mathbb{1}_{\varphi_s(v)>z}\right)d\bar{\alpha}(z),\end{aligned}$$

where for $z\in\mathbb{R}$, $\bar{\alpha}(z)=\alpha_1(z)+\alpha_2(z)$, C is a constant.

Then

$$\mathbb{E}\sup_{t\in[0,T]}|\xi_t(u)-\xi_t(v)|^4 \le C\mathbb{E}\left(\int_0^T\left(\int_{-\infty}^{+\infty}\left(\mathbb{1}_{\varphi_s(u)>z}-\mathbb{1}_{\varphi_s(v)>z}\right)d\bar{\alpha}(z)\right)ds\right)^2$$

$$= C\int_{\mathbb{R}}d\bar{\alpha}(z)\int_{\mathbb{R}}d\bar{\alpha}(y)\mathbb{E}\left[\int_0^T\left(\left(\mathbb{1}_{\varphi_s(u)>z}-\mathbb{1}_{\varphi_s(v)>z}\right)\int_s^T\left(\mathbb{1}_{\varphi_r(u)>y}-\mathbb{1}_{\varphi_r(v)>y}\right)dr\right)ds\right]$$

$$\le C\int_{\mathbb{R}}d\bar{\alpha}(z)\int_{\mathbb{R}}d\bar{\alpha}(y)$$

$$\times\,\mathbb{E}\left[\int_0^T\left(\left(\mathbb{1}_{\varphi_s(u)>z}-\mathbb{1}_{\varphi_s(v)>z}\right)\mathbb{E}\left(\int_s^T\left(\mathbb{1}_{\varphi_r(u)>y}-\mathbb{1}_{\varphi_r(v)>y}\right)dr/\mathfrak{F}_s\right)\right)ds\right]$$

$$\le C\int_{\mathbb{R}}d\bar{\alpha}(z)\int_{\mathbb{R}}d\bar{\alpha}(y)$$

$$\times\,\mathbb{E}\left[\int_0^T\left(\left(\mathbb{1}_{\varphi_s(u)>z}-\mathbb{1}_{\varphi_s(v)>z}\right)\mathbb{E}\left(\int_0^{T-s}\left(\mathbb{1}_{\varphi_r(u)>y}-\mathbb{1}_{\varphi_r(v)>y}\right)dr\right)\right)ds\right].$$

valid for all $T>0,\ u\in\mathbb{R},\ v\in\mathbb{R},\ u<v$, with some constant C.

By arguments similar to that in [8], §18, Remark 1 we have

$$\mathbb{E}\int_0^T(\mathbb{1}_{\varphi_s(u)>z}-\mathbb{1}_{\varphi_s(v)>z})ds \le H(u-v),\ z\in\mathbb{R},$$

where H is some positive constant. This implies

$$\mathbb{E}\sup_{x\in[x_1,x_2]}\sup_{t\in[0,T]}|\xi_t(x)-\xi_t(x_1)|^4 \le (\operatorname{Var}\bar{\alpha})^2H^2(u-v)^2.$$

The case of $u\ge v$ can be treated analogously. Thus the inequality

$$\mathbb{E}\sup_{x\in[x_1,x_2]}\sup_{t\in[0,T]}|\xi_t(x)-\xi_t(x_1)|^4$$
$$\le c(\alpha,4)|x_2-x_1|^{4\alpha-1}\iint_{[x_1,x_2]^2}\frac{C(u-v)^2}{|u-v|^{4\alpha+1}}dudv \quad (7.19)$$

holds true for all $T>0,\{x_1,x_2\}\subset\mathbb{R},\ x_1<x_2$. To provide the finiteness of the integral in the right-hand side of (7.19) we choose α such that $1-4\alpha>-1$, i.e. $\alpha\in(\frac{1}{4},\frac{1}{2})$. Finally, calculating the integral we get

$$\mathbb{E}\sup_{x\in[x_1,x_2]}\sup_{t\in[0,T]}|\xi_t(x)-\xi_t(x_1)|^4\le C(x_2-x_1)^2,$$

where C is a constant.

This inequality implies that the convergence in (7.16) holds in probability. The almost surely convergence can be justified by arguments similar to that used in the proof of formula (7.14). So we checked that $III\to 0$ as $t\to\infty$. This completes the proof of the fact that

$$\varlimsup_{t\to\infty}\frac{\ln(\varphi_t(x_2)-\varphi_t(x_1))}{t}\le\int_{-\infty}^{+\infty}\left(-\int_z^{+\infty}\alpha(y)dP_{stat}(y)\right)d\alpha(z)\ \ \text{(see (7.2))}.$$

Treating (7.1) analogously we get

$$\int_{-\infty}^{+\infty}\left(-\int_z^{+\infty}\alpha(y)dP_{stat}(y)\right)d\alpha(z)\le\varliminf_{t\to\infty}\frac{\ln(\varphi_t(x_2)-\varphi_t(x_1))}{t}$$

The Theorem 2 is proved. □

8 Example

Let $\alpha(x) = a\mathbb{1}_{x\geq 0} + b\mathbb{1}_{x<0}$, where $a < 0, b > 0$. Given $\{x_1, x_2\} \subset \mathbb{R}$, the processes $(\varphi_t(x_1))_{t\geq 0}$, $(\varphi_t(x_2))_{t\geq 0}$ move parallel to each other while being on the same semiaxis. Theorem 1.1 holds true for the solution $(\varphi_t(x))_{t\geq 0}$ of corresponding SDE. The Sobolev derivative has the form (see (1.2))

$$\nabla\varphi_t(x) = \exp\left\{(a-b)L_0^{\varphi(x)}(t)\right\},$$

where $L_0^{\varphi(x)}(t)$ is a local time of the process $(\varphi_t(x))_{t\geq 0}$ at the point zero.

Let us find stationary distribution for the process $(\varphi_t(x))_{t\geq 0}$ (see Section 6). We have

$$s(x) = \begin{cases} -\frac{1}{2a}\left(e^{-2ax} - 1\right), & x \geq 0, \\ -\frac{1}{2b}\left(e^{-2bx} - 1\right), & x < 0, \end{cases}$$

$$s'(x) = \begin{cases} e^{-2ax}, & x \geq 0, \\ e^{-2bx}, & x < 0, \end{cases}$$

and

$$\sigma(y) = s'(q(y)) = \begin{cases} -1 - 2ay, & y \geq 0, \\ -1 - 2by, & y < 0, \end{cases}$$

where

$$q(y) = \begin{cases} -\frac{1}{2a}\ln(1-2ay), & y \geq 0, \\ -\frac{1}{2b}\ln(1-2by), & y < 0, \end{cases}$$

is a continuously differentiable inverse function to $s(\cdot)$.

Put $\eta_t(x) = s(\varphi_t(x))$. Then (see Section 6) it is a solution of the SDE

$$\begin{cases} d\eta_t(x) = \sigma(\eta_t(x))dw(t), \\ \eta_0(x) = s(x). \end{cases}$$

Let $F_{t,x}(y) = P\{\eta_t(x) < y\}$ be the distribution function of the random variable $\eta_t(x)$. Then by Theorem 3 of [8], §18, for all $x \in \mathbb{R}$,

$$\lim_{t\to\infty} F_{t,x}(y) = \frac{\int_{-\infty}^{y} \frac{dz}{\sigma^2(z)}}{\int_{-\infty}^{+\infty} \frac{dz}{\sigma^2(z)}} = \begin{cases} 1 + \frac{b}{a-b}\frac{1}{1-2ay}, & y \geq 0, \\ \frac{a}{a-b}\frac{1}{1-2by}, & y < 0. \end{cases} \tag{8.1}$$

Let $\Phi_{t,x}(y),\ y \in \mathbb{R}$, be the distribution function of the random variable $\varphi_t(x)$. From (8.1) for all $x \in \mathbb{R}$, we have

$$\begin{aligned} P_{stat}(y) = \lim_{t\to\infty}\Phi_{t,x}(y) = \lim_{t\to\infty} P\{\varphi_t(x) < y\} = \lim_{t\to\infty} P\{\eta_t(x) < s(y)\} \\ = \lim_{t\to\infty} F_{t,x}(s(y)) = \begin{cases} 1 + \frac{b}{a-b}e^{2ay}, & y \geq 0, \\ \frac{a}{a-b}e^{2by}, & y < 0. \end{cases} \end{aligned}$$

The stationary distribution function $P_{stat}(y)$ has a density of the form

$$p_{stat}(y) = \begin{cases} \frac{2ab}{a-b}e^{2ay}, & y \geq 0, \\ \frac{2ab}{a-b}e^{2by}, & y < 0. \end{cases} \tag{8.2}$$

Theorem 2 now is as follows. For all $\{x_1, x_2\} \subset \mathbb{R},\ x_1 < x_2$,

$$\lim_{t\to+\infty} \frac{\ln(\varphi_t(x_2) - \varphi_t(x_1))}{t} = (b-a)\int_0^{+\infty} \frac{2a^2 b}{a-b} e^{2ay}\mathbb{1}_{y\geq 0}dy = ab \text{ almost surely.}$$

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