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Testing for local covariate trend effects in volatility models*

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Abstract: With the large amounts of modern financial and econometric data available from disparate informational sources, it becomes increasingly critical to develop inferential tools for the impact of exogenous factors on volatility of financial time series. We develop a new Local Covariate Trend test (LOCOT) for the significance of an exogenous covariate in the autoregressive conditional heteroscedastic volatility model, where the covariate effect can be nonlinear. The new LOCOT statistic is based on an artificial high-dimensional one-way ANOVA where the number of factor levels increases with the sample size. We derive asymptotic properties of the new LOCOT statistic and show its competitive finite sample performance in a broad range of simulation studies. We illustrate utility of the new testing approach in application to volatility analysis of three major cryptoassets and their relationship with the prices of gold and the S&P500 index.

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1. Introduction

Since its inception by Engle (1982), a family of Autoregressive Conditional Heteroscedasticity (ARCH) models remains the primary tool to model volatility of financial time series (see overviews by Beyaztas et al., 2018; Brenner et al., 1996; Fabozzi et al., 2014; May and Herce, 2002, and references therein). In many real scenarios, volatility of financial processes is driven by some exogenous variables (Engle and Patton, 2001). For example, news intensity is recently shown to exhibit a noticeable impact on volatilities of stock prices and currency exchanges (Chua and Tsiaplias, 2019; Sadik et al., 2018; Sidorov et al., 2014).

A natural question then arises on whether we can test for the effects of exogenous factors in volatility. Addressing this important question allows us to assist in appropriate data and model selection as well as in improving volatility forecasting. Remarkably, despite its high importance in applications, hypothesis testing and inference for exogenous covariates in conditional heteroscedastic models remain yet a substantially under-explored area. The primary goal of this paper is to develop a hypothesis test for the significance of exogenous covariates in autoregressive conditional heteroscedastic volatility models where the effect of the covariate can be nonlinear.

Although trends in the mean due to covariate effects have been widely studied within ARCH settings, systematic analysis of covariate effects in the volatility equation of ARCH models has received noticeably less attention (see, e.g., Francq and Sucarrat, 2017; Han, 2015; Han and Kristensen, 2014, and references therein). In turn, most available results on the covariates in the family of ARCH models focus on parameter estimation algorithms and their associated asymptotic properties rather than on hypothesis testing (Chatterjee and Das, 2003; Han and Park, 2008; Hansen et al., 2012; Robert, 2002; Sucarrat et al., 2016). A notable effort in this direction has been undertaken by Francq et al. (2019) who consider a class of asymmetric GARCH models with multiple exogenous covariates with linear effects. While the primary focus of Francq et al. (2019) is still on parameter estimation rather than hypothesis testing, their approach also allows us to perform hypothesis testing for the significance of the exogenous variables using the asymptotic distribution of the estimators, which is shown to be multivariate normal under some regularity conditions. Most recently, a related approach in a form of likelihood ratio statistic for testing whether GARCH-X reduces to GARCH is proposed by Pedersen and Rahbek (2019). Nevertheless, to the best of our knowledge, no other approaches on hypothesis testing for effects of exogenous covariates in the volatility equation of ARCH models have been proposed in the literature, even for the case of linear effects.

In this paper we introduce a new Local Covariate Trend testing (LOCOT) approach for the significance of an exogenous covariate in an ARCH volatility model, where the covariate effect is a nonparametric functional. The proposed testing framework allows us to test for a broad range of nonlinear effects of an exogenous covariate on volatility dynamics. The new LOCOT statistic is based on recent new developments of high-dimensional one-way ANOVA where the number of factor levels increases with the sample size. The key idea is to measure the local effect of the covariate on the residuals of the null model in small neighborhoods of time, so that if different neighborhoods exhibit significantly different average effects, then there likely exists an overall covariate effect. Since the neighborhoods are small relatively to the sample size, both linear and nonlinear effects can be detected (Lyubchich and Gel, 2016; Lyubchich et al., 2013; Wang et al., 2008; Zambom and Akritas, 2014). The proposed new LOCOT approach is validated through extensive simulations which indicate a competitive performance of the new method both in terms of size and power of the test. Furthermore, the new approach is illustrated in application to detecting the impact of exogenous factors such as Gold prices and Standard & Poor's 500 Index on volatilities of the top three main crypto-assets, namely, Bitcoin, Litecoin, and Ethereum.

The remainder of the paper is as follows. In Section 2, we introduce the new LOCOT statistic for assessing the significance of the exogenous variable in the autoregressive conditional heteroscedastic model. The asymptotic distribution of the LOCOT statistic under the null and local alternative hypotheses with some regularity conditions are presented in Section 3. Section 4 presents a wide range of simulation results on the level and power of the test statistic compared to that of Francq et al. (2019). Finally, Section 5 illustrates an application of LOCOT to volatility analysis of three major crypto-assets.

2. The test statistic

Let ϵ_t, $t = 1, \dots, T$, be observations from

$$\begin{cases} \epsilon_t = z_t \sigma_t \\ \sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j \epsilon_{t-j}^2 + m(X_{t-1}), \end{cases} \tag{1}$$

where z_t are independent and identically distributed (i.i.d.) random variables with zero mean, unit variance and $Ez_t^4 < \infty$. We assume that z_t are independent of X_t, which are also assumed to be i.i.d.. Model (1) is called an Autoregressive Conditionally Heteroscedastic Model of order p with exogenous regressors, or ARCH(p)-$m(X)$ (see, e.g., Engle and Patton, 2001; Han, 2015, and references therein). Here X_t is an exogenous covariate such that $E\big(\alpha_0 + m(X_{t-1})\big)^2 < \infty$. Furthermore, $\alpha_0 + m(X_t) > 0$, $\alpha_j \geq 0$ for $0 \leq j \leq p$, and $\max\{1, (Ez_t^4)^{1/2}\}\big(\sum_{j=1}^{p} \alpha_j\big) \leq 1$. We assume that $E(\epsilon_t^2 | \mathcal{I}_{t-1}) = \sigma_t^2 > 0$, where $\mathcal{I}_{t-1}$ is the σ-algebra generated by $\{\epsilon_u, X_u, u < t\}$. These assumptions ensure existence of strictly stationary solution with finite forth moment for (1). Note that in the case of GARCH(p, q)-$m(X)$, the assumption $\sum_{j=1}^{p} \alpha_j \leq 1$ is to be substituted by the condition that the top Lyapunov exponent for a sequence of matrices C_{0t} associated with GARCH(p, q)-$m(X)$ is strictly negative Francq et al. (2019) (for the exact form of C_{0t} see Hamadeh and Zakoïan (2011)).

Our goal is to assess the effect of X_t on the volatility of ϵ_t and test the hypothesis

$$H_0 : m(X_t) \in S_\Theta, \tag{2}$$

where $S_\Theta = \{m(\cdot,\theta), \theta \in \Theta\}$ is a parametric family of functions, $\Theta \subset \mathbb{R}^p$, and $m(\cdot,\theta) : \theta \to \mathbb{R}$.

By setting $v_t = \epsilon_t^2 - E(\epsilon_t^2|\mathcal{I}_{t-1}) = \epsilon_t^2 - \sigma_t^2$, we can re-write (1) under H_0 as an autoregressive model of order p, or AR(p)

$$\epsilon_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j \epsilon_{t-j}^2 + m(X_{t-1},\theta) + v_t. \tag{3}$$

Note that the error term v_t in (3) is no longer independent and identically distributed, but is white noise with variance

$$\tau = Var(z_t^2) E\left[\alpha_0 + \sum_{j=1}^{p} \alpha_j \epsilon_{t-j}^2 + m(X_{t-1},\theta)\right]^2$$
$$= Var(z_t^2)\Big[\alpha_0^2 + E\left(\sum_{j=1}^{p} \alpha_j \epsilon_{t-j}^2\right)^2 + Em^2(X_{t-1},\theta) + 2\alpha_0 \sum_{j=1}^{p} \alpha_j E(\epsilon_{t-j}^2)$$
$$+2\alpha_0 E\big[m(X_{t-1},\theta)\big] + 2E[m(X_{t-1},\theta)] \sum_{j=1}^{p} \alpha_j E[\epsilon_{t-j}^2]\Big].$$

Moreover, v_t is a martingale difference since $E(v_t|\mathcal{T}_{t-1}) = 0$, where $\mathcal{T}_{t-1}$ is a σ-algebra generated by $(v_{t-1}, v_{t-2}, \ldots)$. Note that in view of the assumptions in (1), $Ev_t^2 < \infty$.

Note under H_0 in (2), we have

$$E(v_t|X_{t-1}) = E(\epsilon_t^2 - E(\epsilon_t^2|\mathcal{I}_{t-1})|X_{t-1}) = E(z_t^2\sigma_t^2|X_{t-1}) - E(\epsilon_t^2|\mathcal{I}_{t-1})$$
$$= E(\alpha_0 + \sum_{j=1}^{p} \alpha_j \epsilon_{t-j}^2 + m(X_{t-1})|X_{t-1}) - \alpha_0 - \sum_{j=1}^{p} \alpha_j \epsilon_{t-j}^2 - m(X_{t-1})$$
$$= \sum_{j=1}^{p} \alpha_j (E(\epsilon_{t-j}^2) - \epsilon_{t-j}^2), \tag{4}$$

which is a function of $\epsilon_{t-1}, \epsilon_{t-2}, \ldots$ and, hence, does not depend on X_{t-1}. Hence, the key rationale behind our approach is that testing H_0 (2) is equivalent to testing for a relationship (in terms of conditional mean) between v_t and X_{t-1}.

Now, consider the residuals

$$\hat{v}_{t+p} = \epsilon_{t+p}^2 - \hat{\alpha}_0 - \sum_{j=1}^{p} \hat{\alpha}_j \epsilon_{t+p-j}^2 - m(X_{t+p-1},\hat{\theta}), t = 1,\ldots,n, n = T - p, \tag{5}$$

where $\hat{\alpha}_j$ and $\hat{\theta}$ are some suitable $\sqrt{n}$-consistent estimators of α_j, $j = 0, 1, \ldots, p$ and θ, respectively (Chatterjee and Das, 2003). Under H_0 in (2) $\hat{v}_t$ is approximately white noise similar to v_t and, as a result, we can develop a test statistic for (2) based on the strength of the relationship between $\hat{v}_t$ and X_{t-1}.

Inspired by recent advancements in the theory of high-dimensional analysis of variance when the number of factor levels goes to infinity (Akritas and Papadatos, 2004; Wang and Keilegom, 2007), we propose testing for H_0 by building a one-way ANOVA design where the factor levels are defined by n covariate observations in the following way. Let without loss of generality $(X_{t-1}, \hat{v}_t), t = 2, \ldots, n$ be arranged in such a way that $X_{t_1} < X_{t_2}$ whenever $t_1 < t_2$. Let $(X_{t-1}, \hat{v}_t)$ be the data from a balanced one-way ANOVA design with $\hat{v}_t$ being the data at "level" X_{t-1}. Since ANOVA statistics require more than one observation per factor level, we augment each level by including the $\hat{v}_\ell$ corresponding to the $(k_n - 1)$ covariate values $X_{\ell-1}$ closest to X_{t-1}, for a fixed constant k_n. Here k_n represents the number of observations in each factor level, which we call a *window size* (see Remark 1 for discussion on a strategy for automatically choosing the window size k_n). Formally, we define windows

$$W_t = \left\{ \ell : |\hat{F}_X(X_{\ell-1}) - \hat{F}_X(X_{t-1})| \leq \frac{k_n - 1}{2n} \right\}, \tag{6}$$

where $\hat{F}_X$ is the empirical cumulative distribution of X. Note that symmetric windows cannot be constructed at the edges, that is, for the lowest and highest k_n values of the covariate. In these cases, asymmetric windows can be used, which can be shown to have asymptotic negligible effects on the limiting properties of the test statistic.

Intuitively, under H_0, there should be no difference in the mean values of the observations across the factor levels defined by the windows W_t, and hence an ANOVA F-test can be used. As a result, we propose using the following Local Covariate Trend test (LOCOT) statistic

$$\begin{aligned} T_n &= MST_n - MSE_n \\ &= \frac{k_n}{n-1} \sum_{t=1}^{n} (\hat{v}_{t\cdot} - \hat{v}_{\cdot\cdot})^2 - \frac{1}{n(k_n - 1)} \sum_{t=1}^{n} \sum_{\ell \in W_t} (\hat{v}_\ell - \hat{v}_{t\cdot})^2, \end{aligned} \tag{7}$$

where $\hat{v}_{t\cdot} = (1/k_n) \sum_{\ell \in W_t} \hat{v}_\ell$ and $\hat{v}_{\cdot\cdot} = (1/nk_n) \sum_{t=1}^{n} \sum_{\ell \in W_t} \hat{v}_\ell$. Akritas and Arnold (2000); Akritas and Papadatos (2004); Boos and Brownie (1995) studied the asymptotic behavior of $\sqrt{n}(F_n - 1)$, where $F_n = MST_n/MSE_n$, when n goes to infinity. Since MSE_n tends to a constant when $n \to \infty$, by Slutsky Theorem the asymptotic distribution of $\sqrt{n}(F - 1)$ is equivalent to that of $\sqrt{n}(MST_n - MSE_n)$. In Section 3, we show that under some regularity conditions the LOCOT statistic T_n converges to a Normal distribution as n goes to infinity. Note that T_n can be written as $\mathbf{V}'A\mathbf{V}$, where $\mathbf{V}$ is the vector of nk_n augmented observations

$$\mathbf{V} = (\hat{v}_\ell, \ell \in W_1, \ldots, \hat{v}_\ell, \ell \in W_n)$$

and A is a constant matrix defined in the Appendix.

Remark. To choose an optimal window size k_n, we can employ the subsampling approach of Lyubchich and Gel (2016) based on m-out-of-n resampling (see

Bickel et al., 2012; Bickel and Sakov, 2008). Another approach for the choice of k_n is to generate B pseudo-datasets, each of which with covariate values randomly permuted, that is, under the assumption that H_0 is true. Then we select a value of k_n that yields the most accurate empirical level of the test. In our numerical experiments, we employ the permutation-based approach.

3. Asymptotic results

We now turn to asymptotic properties of the new LOCOT statistic T_n (7). Consider the following conditions:

C1: $m(x, \theta)$ is twice continuously differentiable with respect to θ and x and

$$\sup_{\theta,x}\left|\frac{\partial}{\partial\theta_j}m(x,\theta)\right| < \infty, \quad \sup_{\theta,x}\left|\frac{\partial^2}{\partial x^2}m(x,\theta)\right| < \infty,$$
$$\sup_{\theta,x}\left|\frac{\partial^2}{\partial\theta_j\partial x}m(x,\theta)\right| < \infty,\ \sup_{\theta,x}\left|\frac{\partial^2}{\partial\theta_j\partial\theta_l}m(x,\theta)\right| < \infty, j = 1,\dots,\dim(\Theta).$$

C2: Under H_0, $\hat{\theta} - \theta = O_p(n^{-1/2})$.

C3: Density $f_X(\cdot)$ of covariate X is twice continuously differentiable in a compact support $\mathcal{X}$ with $f(x) > 0$ for $x \in \mathcal{X}$.

C4: $\tau < \infty$.

Conditions C1 and C2 guarantee the convergence rate of the estimator and, hence, the convergence of the LOCOT statistic which is based on the residuals $\hat{v}_t$ in (5). Condition C3 typically holds for probability density functions with compact support, and constitutes the only restriction on the covariate X, allowing for a random design for the exogenous covariate X in ARCH-X models. Next we state a formal result on the asymptotic properties of T_n under H_0.

Theorem 1. *Assume conditions C1–C4 hold. Then, under H_0 in (2), as $n \to \infty$, $k_n \to \infty$ and k_n is such that $k_n^{5/2}/n^{1/2} \to 0$,*

$$\left(\frac{n}{k_n}\right)^{1/2} T_n \xrightarrow{d} N(0, 4\tau^2/3).$$

Note that the covariance structure of the volatility defined via the parameters α_j, $j = 1, \dots, p$ (see (1) and (3)) does not affect the asymptotic distribution of the LOCOT statistic due to the filtering of $\hat{v}$. We estimate the variance τ of v_t by applying Rice (1984)'s estimator to the filtered data

$$\hat{\tau} = \frac{1}{2(n-1)}\sum_{i=2}^{n}(\hat{v}_i - \hat{v}_{i-1})^2.$$

In view of the $\sqrt{n}$-consistency of $\hat{\alpha}_j$ and $\hat{\theta}$, $\hat{\tau}$ is a consistent estimator of τ.

In order to obtain an insight on the power of the proposed LOCOT statistic, we consider a sequence of local linear alternatives of the following form

$$H_a : m(x, \theta) = \theta_0 + \theta_1 x + (nk_n)^{-1/4}h(x). \tag{8}$$

Theorem 2 shows that the proposed LOCOT statistic converges to a Normal distribution under local linear alternatives as in (8), detecting effects at the rate of $(nk_n)^{-1/4}$. The shift in the alternative distribution compared to that of the null hypothesis is seen in the asymptotic mean, which shows that the power of the test comes from the variability in the local function $h(\cdot)$ departing from the null model. Similar results can be shown for other parametric forms.

Theorem 2. *Assume conditions C1–C4 hold and let $h(x)$ be Lipschitz continuous. Then, under H_a in (8), as $n \to \infty$, $k_n \to \infty$ and k_n is such that $k_n^{5/2}/n^{1/2} \to 0$,*

$$\left(\frac{n}{k_n}\right)^{1/2} T_n \xrightarrow{d} N(Var(h(X)), 4\tau^2/3).$$

4. Simulation study

We now assess the finite sample performance of the proposed LOCOT test for the impact of exogenous covariates on the volatility in ARCH-X models using Monte Carlo simulations [1]. We consider different trend functions of the exogenous variable and three probability distributions for the noise z_t in (1), with a varying degree of heavy-tailedness. As a competing approach, we consider the parametric asymptotic χ^2-test introduced by Francq et al. (2019) which we refer to, throughout the text, as the Francq-Thieu test.

We generate a sample of 2,000 observations from the following eight models: $\epsilon_t = z_t\sigma_t$ with

$$\begin{aligned}
M_1 &: ARCH(1) - X^2 : && \sigma_t^2 = 0.2 + 0.4\epsilon_{t-1}^2 + \theta/3(X_{t-1} - 2.5)^2 \\
M_2 &: ARCH(2) - X^2 : && \sigma_t^2 = 0.2 + 0.3\epsilon_{t-1}^2 + 0.2\epsilon_{t-2}^2 + \theta/3(X_{t-1} - 2.5)^2 \\
M_3 &: ARCH(1) - Xsin(X) : && \sigma_t^2 = 10.2 + 0.4\epsilon_{t-1}^2 + \theta X_{t-1}\sin(\pi X_{t-1}/2) \\
M_4 &: ARCH(2) - Xsin(X) : && \sigma_t^2 = 10.2 + 0.3\epsilon_{t-1}^2 + 0.2\epsilon_{t-2}^2 \\
& && \qquad + \theta X_{t-1}\sin(\pi X_{t-1}/2) \\
M_5 &: ARCH(1) - X : && \sigma_t^2 = 0.2 + 0.4\epsilon_{t-1}^2 + \theta X_{t-1} \\
M_6 &: ARCH(2) - X : && \sigma_t^2 = 0.1 + 0.3\epsilon_{t-1}^2 + 0.2\epsilon_{t-2}^2 + \theta X_{t-1}
\end{aligned}$$

and the following two GARCH(1,1)-$m(X)$ models

$$\begin{aligned}
M_7 &: GARCH(1,1) - X^2 : \sigma_t^2 = 0.2 + 0.4\epsilon_{t-1}^2 + 0.3\sigma_{t-1}^2 + \theta/3(X_{t-1} - 2.5)^2 \\
M_8 &: GARCH(1,1) - XSin(X) : \sigma_t^2 = 20.2 + 0.3\epsilon_{t-1}^2 + 0.3\sigma_{t-1}^2 \\
& \qquad + \theta X_{t-1}\sin(\pi X_{t-1}/2)
\end{aligned}$$

where z_t are independent and identically distributed random variables, following a standard Normal N(0,1), Laplace, and t-Student distribution with 7 degrees of freedom. The exogenous covariate X_t is independent and identically distributed

[1]The code is to be available from the R package *funtimes* and from the authors.

following a uniform distribution $U(0, 5)$. The number of Monte Carlo simulations is 1,000. The first four ARCH models (i.e., M_1–M_4) and the two GARCH models (i.e, M_7 and M_8) have nonlinear functions of the exogenous covariate in the volatility component, while the ARCH(1)-X models M_5 and M_6 have linear functions of covariate. To estimate model parameters, we use the quasi-maximum likelihood method of Francq et al. (2019).

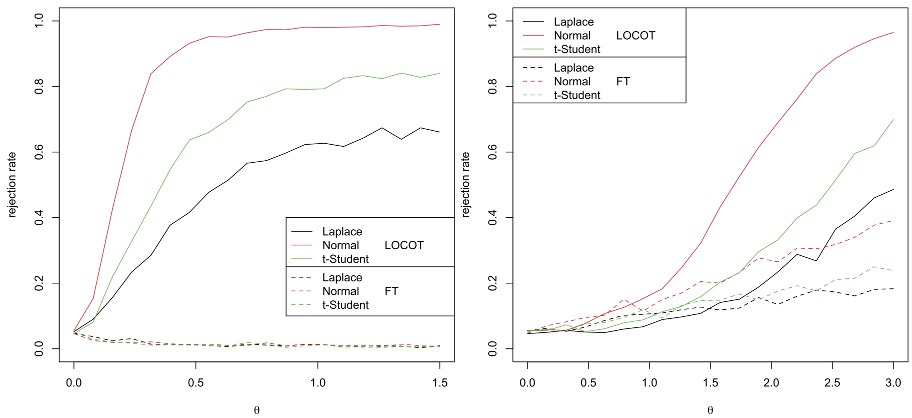
Table 1 shows the results for the empirical levels of the tests under H_0 : $m(X, \theta) = 0$ (i.e., all considered ARCH-X and GARCH-X models with $\theta = 0$). As Table 1 indicates, both the proposed LOCOT and Francq-Thieu statistics achieve empirical levels very close to a nominal level α of 0.05 for all eight models and all three error distributions.

TABLE 1
Rejection rates for the LOCOT and Francq-Thieu statistics under H_0 for nominal level α of 0.05.

	LOCOT			Francq-Thieu test		
	Laplace	Normal	t-Student	Laplace	Normal	t-Student
ARCH(1)-X^2	.052	.055	.046	.047	.046	.051
ARCH(1)-$XSin(X)$	.046	.055	.056	.055	.047	.052
ARCH(2)-X^2	.044	.056	.049	.051	.057	.036
ARCH(2)-$XSin(X)$	.053	.047	.044	.054	.055	.050
ARCH(1)-X	.055	.055	.046	.066	.046	.051
ARCH(2)-X	.047	.050	.051	.044	.056	.052
GARCH(1,1)-X^2	.051	.060	.045	.052	.047	.049
GARCH(1,1)-$XSin(X)$	.050	.045	.051	.051	.049	.043

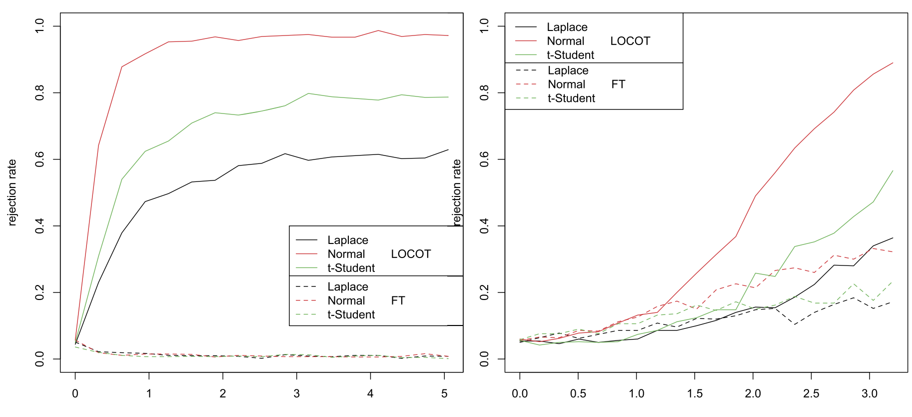
The empirical power functions of the LOCOT and Francq-Thieu statistics are depicted in Figs. 1-4. Note that since the signal-to-noise ratio is dictated by parameter θ, higher values of θ are associated with stronger deviations from H_0 and, hence, are expected to imply higher power of the tests. Figures 1-4 suggest that the new LOCOT method delivers higher power to detect nonlinear alternatives for all models under all error distributions. More specifically, when the models contain a quadratic function of the exogenous variable (i.e., Figs. 1a, 2a and 3a), LOCOT exhibits a rapidly increasing power function as θ increases; in contrast, the Francq-Thieu statistic fails to detect any deviations from H_0 for all three considered error distributions. For the models with the exogenous variable in the form of $XSin(X)$ (i.e., Figs. 1b, 2b and 3b), while both the LOCOT and Francq-Thieu statistics deliver increasing power functions, LOCOT tends to substantially outperform the Francq-Thieu approach, especially when the error distribution is Normal. Since the Francq-Thieu statistic is designed to detect linear alternatives (Francq et al., 2019), as expected, the Francq-Thieu approach yields higher rate of power increase than LOCOT for the ARCH-X models M_5 and M_6 with linear effects of the exogenous covariate (see Fig. 4a and 4b, respectively).

Overall, our simulation studies indicate that the LOCOT statistic is a highly competitive approach to detect nonlinear effects of the exogenous covariate on volatility within the ARCH framework.



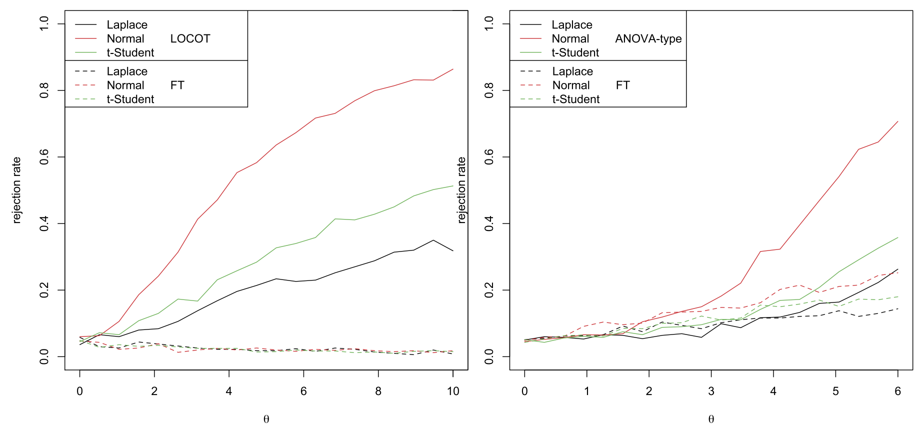
(a) M_1: ARCH(1)-X^2 (b) M_2: ARCH(1)-$XSinX$

FIG 1. *Empirical power of the LOCOT and Francq-Thieu (FT) statistics to detect nonlinear exogenous effects in ARCH(1) models in respect to varying θ. Empirical power curve with respect to the strength of the non-additive signal θ. Left: ARCH(1)-X^2, right: ARCH(1)-$XSinX$.*



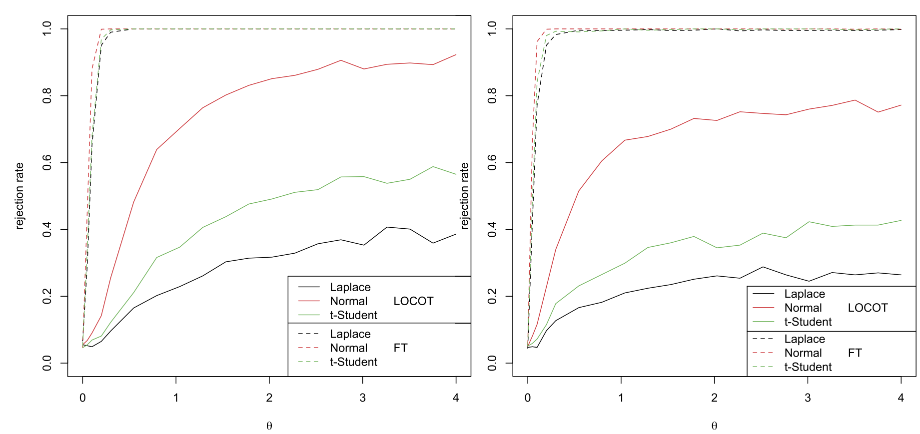
(a) M_2: ARCH(2)-$XSinX$ (b) M_4: ARCH(2)-X^2

FIG 2. *Empirical power of the LOCOT and Francq-Thieu (FT) statistics to detect nonlinear exogenous effects in ARCH(2) models in respect to varying θ.*



(a) M_7: GARCH(1,1)-X^2 (b) M_8: GARCH(1,1)-$XSinX$

Fig 3. *Empirical power of the LOCOT and Francq-Thieu (FT) statistics to detect nonlinear exogenous effects in GARCH models in respect to varying θ.*



(a) M_5: ARCH(1)-X (b) M_6: ARCH(2)-X

Fig 4. *Empirical power of the LOCOT and Francq-Thieu (FT) statistics to detect linear exogenous effects in ARCH(1) models in respect to varying θ.*

5. Trends in volatility of cryptocurrencies

With the introduction of Bitcoin in 2009 (Nakamoto, 2008), blockchain technology have witnessed an ever increasing boom of interest in areas as diverse as food regulatory compliance, secure management of electronic health records, and financial cryptomarket. While the range of blockchain applications continues to grow, cryptocurrencies remain one of the primary spots of interest. Remarkably, not only did the recent sharp raises and falls of Bitcoin not diminish attention to digital assets but in contrast led to a gain of interest – primarily due to unparalleled opportunities to raise a fast fortune in the yet unregulated Wild West of digital instruments. However, the unprecedented investment opportunities in cryptocurrencies go hand in hand with a high risk of losing everything overnight.

In this paper we aim to enhance our understanding of the potential factors behind dynamics of cryptocurrency volatility. In particular, we are interested in whether exogenous variables such as Gold prices and Standard & Poor's 500 Index (S&P 500) exhibit an impact on volatilities of one of the top three main crypto assets, namely, Bitcoin, Litecoin, and Ethereum. While some recent studies assess dynamics of Bitcoin Blau (2018); Cermak (2017); Chu et al. (2017); Dyhrberg (2016), volatility of Litecoin and particularly Ethereum remain largely understudied.

We collect Bitcoin, Litecoin, and Ethereum daily data from their respective blockchains using their official software. We have installed the core wallets of Bitcoin (2018) and Litecoin (2018) that downloaded the entire Bitcoin and Litecoin blockchain data from September, 2016 to June, 2018. In turn, we use the EthR library (Collier, 2019) to query Ethereum blocks through the Go Ethereum Client (i.e., Geth), and our set contains all the Ethereum data from September, 2016 to June, 2018. Gold prices were obtained from the Perth Mint website (PerthMint, 2019), and the S&P 500 Index data from the Yahoo Finance website (Yahoo, 2019).

To assess volatility of cryptoassets, we consider a time series of log-returns Y_t, i.e., $Y_t = \log(Coinprice_t) - \log(Coinprice_{t-1})$, where $Coinprice_t$ is price of the cryptoasser (i.e., Bitcoin, Litecoin, or Ethereum) on day t. In turn, X_t denotes the exogenous information, that is, either daily records of Gold price or S&P500, on day t.

Our primary interest is to assess whether conventional market indicators such as Gold price and S&P500 exhibit an impact on the dynamics of crypto-market volatility. That is, given

$$\begin{cases} Y_t = z_t\sigma_t \\ \sigma_t^2 = \alpha_0 + \alpha_1 Y_{t-1}^2 + m(X_{t-1}), \end{cases} \tag{9}$$

we are interested in testing the hypothesis

$$\begin{aligned} &H_0 : m(X_t) \in S_\Theta, \\ &H_1 : m(X_t) \not\subset S_\Theta. \end{aligned} \tag{10}$$

Here we consider three different tests: $H_0^1 : m_1(X_t) = X_t$, $H_0^2 : m_2(X_t) = [(X_t - \mu_X)/\sigma_X]^2$, and $H_0^3 : m_3(X_t) = \log(X_t)$, where μ_X and σ_X^2 are the mean and variance of $X_1, \ldots, X_t$ respectively.

We address the hypotheses (10) using our proposed nonparametric LOCOT statistic and the Francq-Thieu test, starting from a family of linear trends. Table 2 presents the summary of our results. As the new LOCOT approach suggests, Gold prices appear to show no effect on the volatility of Bitcoin and Ethereum but exhibit a statistical significant trend in the volatilities of Litecoin. In turn, S&P500 yields a marginal result on the border of significance (i.e., p-value of 0.09) for daily Bitcoin volatility.

While we cannot affirmatively attribute these findings to one cause, there are a number of potential explanations to these findings. First, around 30% of illicit trade is done using cryptocurrencies, and the usage of particularly Litecoin in such criminal activities has recently increased. As a result, Litecoin may be used in money laundering schemes to convert gold (Mcquad, 2017). Second, in some countries, e.g., in Russia, Litecoin is used to hide assets. While Bitcoin is too big and is less likely to be sensitive to Gold transactions in such activities, trading Gold may still impact Litecoin price (Roberts, 2018). Third, Litecoin appears to be less widely used for consumer tech purchases than Bitcoin and Ethereum and more as a speculative instrument, hence Litecoin may have more intrinsic ties to the financial market. Forth, historically lower transaction costs of Litecoin over Bitcoin and Ethereum may draw hedgefunds and investment managers to buy and sell large quantities of Litecoin in portfolios mixed with safe haven assets like gold, in order to boost returns, while holding gold.

Remarkably, the Francq-Thieu test does not detect any trends in the volatilities of Bitcoin, Litecoin and Ethereum associated with Gold prices and S&P500 for any of the functions considered.

An interesting next research step in this direction is to assess potential trends and their causes in volatility dynamics of blockchain transaction graph characteristics such as chainlets (Dey et al., 2020), topological signatures (Abay et al., 2019; Gidea et al., 2020; Li et al., 2020) and more conventional network summaries (Kurbucz, 2019). Such graph summaries may serve a early warning signal of the crypto-asset health and as such be an important investment indicator (Dixon et al., 2019).

6. Discussion and future work

Despite the increasing interest to quantifying the impact of exogenous factors in financial time series, analysis of exogenous covariates in volatility equations of the ARCH family yet remains largely under-studied. In this paper we have developed a new nonparametric Local Covariate Trend test (LOCOT) statistic for assessing the significance of an exogenous covariate as a volatility component in ARCH models, based on the high-dimensional one-way ANOVA F-test. We have derived asymptotic properties of the new LOCOT statistic under the null and alternative hypotheses. Our extensive simulation studies have indicated

TABLE 2

Summary of the trend in volatility tests for Bitcoin, Litecoin, or Ethereum, driven by the exogenous factors Gold price and S&P500 Index. The daily data are recorded from September, 2016 to September, 2018.

			p-values	
	Method	Coin	Gold Prices	S&P500
$m_1(X_t)$		Bitcoin	0.221	0.093
	LOCOT	Litecoin	0.014	0.391
		Ethereum	0.667	0.669
		Bitcoin	0.766	0.313
	Franc & Thieu	Litecoin	0.805	0.594
		Ethereum	0.923	0.450
$m_2(X_t)$		Bitcoin	0.112	0.759
	LOCOT	Litecoin	0.038	0.273
		Ethereum	0.882	0.661
		Bitcoin	0.478	0.999
	Franc & Thieu	Litecoin	0.999	0.998
		Ethereum	0.998	0.998
$m_3(X_t)$		Bitcoin	0.221	0.093
	LOCOT	Litecoin	0.014	0.391
		Ethereum	0.667	0.669
		Bitcoin	0.998	0.999
	Franc & Thieu	Litecoin	0.899	0.878
		Ethereum	0.999	0.844

high competitiveness of the LOCOT approach for detection of a broad range of nonlinear exogenous effects in volatility of conditionally heteroscedastic time series. Furthermore, we have illustrated the new approach in application to analysis of volatility of three major cryptocurrencies and their relationship with such exogenous factors as gold and S&P prices.

In the future, we plan to advance the proposed LOCOT approach to a multivariate case, as well as to integrate LOCOT with variable selection criteria.

Appendix

Proof of Theorem 1. The proof of is given for $p = 1$ and θ univariate. The case where $p > 1$ and multivariate θ can be shown using similar steps.

Write the null hypothesis residuals as

$$\begin{aligned}
\hat{v}_t &= \epsilon_t^2 - \hat{\alpha}_0 - \hat{\alpha}_1\epsilon_{t-1}^2 - m(X_{t-1}, \hat{\theta}) \\
&= \epsilon_t^2 - \alpha_0 - \alpha_1\epsilon_{t-1}^2 - m(X_{t-1}, \theta) - (\hat{\alpha}_0 - \alpha_0) - (\hat{\alpha}_1 - \alpha_1)\epsilon_{t-1}^2 \\
&\quad -(m(X_{t-1}, \hat{\theta}) - m(X_{t-1}, \theta)) \\
&= v_t - (\hat{\alpha}_0 - \alpha_0) - (\hat{\alpha}_1 - \alpha_1)\epsilon_{t-1}^2 - (m(X_{t-1}, \hat{\theta}) - m(X_{t-1}, \theta)).
\end{aligned}$$

Let $\mathbf{V} = (\hat{v}_\ell, \ell \in W_1, \ldots, \hat{v}_\ell, \ell \in W_n)$, $\alpha_V = \Big((\hat{\alpha}_1 - \alpha_1)\epsilon_{\ell-1}^2, \ell \in W_1, \ldots, (\hat{\alpha}_1 - \alpha_1)\epsilon_{\ell-1}^2, \ell \in W_n\Big)$, $\mathbf{v}_V = (v_\ell, \ell \in W_1, \ldots, v_\ell, \ell \in W_n)$, and $\mathbf{m}_V = ((m(X_{\ell-1}, \hat{\theta}) - m(X_{\ell-1}, \theta)), \ell \in W_1, \ldots, (m(X_{\ell-1}, \hat{\theta}) - m(X_{\ell-1}, \theta)), \ell \in W_n)$. The statistic $T_n =$

$MST_n - MSE_n$ can be written as

$$\begin{aligned} T = \mathbf{V}'\mathbf{A}\mathbf{V} \quad = \quad & \mathbf{v}_V'\mathbf{A}\mathbf{v}_V + (\hat{\alpha}_0 - \alpha_0)^2\mathbf{e}_1'\mathbf{A}\mathbf{e}_1 + \alpha_V'\mathbf{A}\alpha_V + \mathbf{m}_V'\mathbf{A}\mathbf{m}_V \\ & +2\mathbf{v}_V'\mathbf{A}\mathbf{e}_1(\hat{\alpha}_0 - \alpha_0) + 2\mathbf{v}_V'\mathbf{A}\alpha_V + 2\mathbf{v}_V'\mathbf{A}\mathbf{m}_V \\ & +2(\hat{\alpha}_0 - \alpha_0)\mathbf{e}_1'\mathbf{A}\alpha_V + 2(\hat{\alpha}_0 - \alpha_0)\mathbf{e}_1'\mathbf{A}\mathbf{m}_V + 2\alpha_V'\mathbf{A}\mathbf{m}_V, \end{aligned} \tag{11}$$

where $\mathbf{e}_1 = (1, 1, \dots, 1)$ is a n by 1 vector of ones and

$$A = \frac{nk_n - 1}{n(n-1)k_n(k_n-1)} \oplus_{i=1}^n J_{k_n} - \frac{1}{n(n-1)k_n} J_{nk_n} - \frac{1}{n(k_n-1)} I_{nk_n}, \tag{12}$$

where I_r is a identity matrix of dimension r, J_r is a $r \times r$ matrix of 1's and $\oplus$ is the Kronecker sum or direct sum.

In what follows, using steps similar to those in Wang and Keilegom (2007), we show that asymptotic distribution of $n^{1/2}k_n^{-1/2}\mathbf{v}_V'\mathbf{A}\mathbf{v}_V$ converges to a normal distribution and all other terms are $o_p(n^{-1/2}k_n^{1/2})$. First, note that

$$\mathbf{v}_V'\mathbf{A}\mathbf{v}_V = \frac{1}{n(k_n-1)} \sum_{i=1}^n \sum_{t_1 \neq t_2}^n v_{t_1} v_{t_2} I(t_1, t_2 \in W_i)$$

and since v_t is white noise $E(\mathbf{v}_V'\mathbf{A}\mathbf{v}_V) = 0$. Hence, variance of $\mathbf{v}_V'\mathbf{A}\mathbf{v}_V$ can then be computed as

$$\begin{aligned} & E(\mathbf{v}_V'\mathbf{A}\mathbf{v}_V)^2 \\ = \; & \frac{1}{n^2(k_n-1)^2} \sum_{i_1=1}^n \sum_{i_2=1}^n \sum_{t_1 \neq \ell_1}^n \sum_{t_2 \neq \ell_2}^n E(v_{t_1} v_{\ell_1} v_{t_2} v_{\ell_2}) I(t_1, \ell_1 \in W_{i_1}, t_2, \ell_2 \in W_{i_2}) \\ = \; & \frac{1}{n^2(k_n-1)^2} \sum_{i_1=1}^n \sum_{i_2=1}^n \sum_{t \neq \ell}^n \tau^2 I(t, \ell \in W_{i_1} \cap W_{i_2}) \\ & \to \begin{cases} 4\tau^2/3 & \text{if } k_n \to \infty. \\ \frac{2(2k_n-1)}{3(k_n-1)}\tau^2 & \text{if } k_n \text{ is fixed.} \end{cases} \end{aligned}$$

To show asymptotic normality of $\mathbf{v}_V'\mathbf{A}\mathbf{v}_V$, we express $\mathbf{v}_V'\mathbf{A}\mathbf{v}_V$ as follows

$$\mathbf{v}_V'\mathbf{A}\mathbf{v}_V = \frac{1}{n} \sum_{i=1}^n A_i = \frac{1}{n} S_n,$$

where $A_i = (1/(k_n-1)) \sum_{t_1 \neq t_2} v_{t_1} v_{t_2} I(t_1, t_2 \in W_i)$, and define

$$\begin{aligned} U_{ni} &= A_{(i-1)(b_n+l_n)+1} + \dots + A_{(i-1)(b_n+l_n)+b_n} \\ V_{ni} &= A_{(i-1)(b_n+l_n)+b_n+1} + \dots + A_{i(b_n+l_n)}, \end{aligned}$$

$i = 1, \dots, r_n$, where $b_n \sim n^{2/3}k_n^{1/3}, l_n \sim k_n, r_n \sim n/b_n = n^{1/3}k_n^{-1/3}$, so that

$$S_n = \sum_{i=1}^{r_n} U_{ni} + \sum_{i=1}^{r_n} V_{ni}.$$

Using steps mimicking those in Wang et al. (2008), we can show that $\sum_{i=1}^{r_n} V_{ni} = o_p((nk_n)^{-1/2})$. Since U_{ni} are uncorrelated, normality U_{ni} is established using the ρ-mixing theorem in (Peligrad, 1987).

For the third term on the right hand side of (11), note that

$$\sqrt{\frac{n}{k_n}}\alpha_V'\mathbf{A}\alpha_V = \sqrt{\frac{n}{k_n}}\frac{(nk_n-1)}{n(n-1)k_n(k_n-1)}(\hat{\alpha}_1-\alpha_1)^2\sum_{i=1}^{n}\left(\sum_{\ell\in W_i}\epsilon_{\ell-1}^2\right)^2 \quad (13)$$
$$-\sqrt{\frac{n}{k_n}}\frac{k_n}{n(n-1)}(\hat{\alpha}_1-\alpha_1)^2\left(\sum_{i=1}^{n}\epsilon_{i-1}^2\right)^2-\sqrt{\frac{n}{k_n}}\frac{k_n}{n(k_n-1)}(\hat{\alpha}_1-\alpha_1)^2\sum_{i=1}^{n}\epsilon_{i-1}^4.$$

In view of the ARCH model in equation (2.1) of the main paper and given that ϵ_t^2 can be represented as an AR(1) process in equation (2.3) in the main paper, we find that $E(\epsilon_t^2) = \alpha_0/(1-\alpha_1)$ and $E(\epsilon_t^4) < \infty$ by the assumptions of the model. Hence, using the fact that $(\hat{\alpha}_1 - \alpha_1) = O_p(n^{-1/2})$, the third therm on the right hand side of (13) is $O_p(n^{-1/2}k_n^{-1/2}n^{-1}n) = o_p(1)$. The second term on the right hand side of (13) is $O_p(n^{-3/2}k_n^{1/2}n^{-1}n^2) = O_p(n^{-1/2}k_n^{1/2}) = o_p(1)$. Finally, the first term on the right hand side of (13) is $O_p(n^{-1/2}k_n^{-3/2}n^{-1}nk_n^2) = O_p(n^{-1/2}k_n^{1/2}) = o_p(1)$. Similarly, using 1's instead of ϵ_t in (13), the second term on the right hand side of (11) is shown to be of order $o_p(1)$.

For the sixth term on the right hand side of (11), note that

$$\sqrt{\frac{n}{k_n}}\mathbf{v}_V'\mathbf{A}\alpha_V = \sqrt{\frac{n}{k_n}}\frac{(nk_n-1)}{n(n-1)k_n(k_n-1)}(\hat{\alpha}_1-\alpha_1)\sum_{i=1}^{n}\sum_{k\in W_i}v_k\sum_{\ell\in W_i}\epsilon_{\ell-1}^2 \quad (14)$$
$$-\sqrt{\frac{n}{k_n}}\frac{k_n}{n(n-1)}(\hat{\alpha}_1-\alpha_1)\sum_{i=1}^{n}v_i\sum_{\ell=1}^{n}\epsilon_{\ell-1}^2-\sqrt{\frac{n}{k_n}}\frac{k_n}{n(k_n-1)}(\hat{\alpha}_1-\alpha_1)\sum_{i=1}^{n}v_i\epsilon_{i-1}^2.$$

Using the fact that $\epsilon_t^2 = \alpha_0 + \alpha_1\epsilon_{t-1}^2 + v_t$ and v_t is white noise, it is easy to see that $E(v_t\epsilon_{t-1}^2) = 0$, and $E(\sum_{t=1}^n v_t\epsilon_{t-1}^2)^2 = \sum_{t=1}^n E(v_t^2\epsilon_{t-1}^4)$ $+2\sum_{t<\ell}E(v_t\epsilon_{t-1}^2v_\ell\epsilon_{\ell-1}^2) = n(\alpha_0^2\tau+\tau^2)/(1-\alpha_1^2-2\alpha_0\alpha_1\tau\alpha_0/(1-\alpha_1))+0 = O(n)$. Hence, the last term on the right hand side of (14) is of order $O_p(n^{-1/2}k_n^{-1/2}n^{-1/2}n^{1/2}) = o_p(1)$. By the Marcinkiewicz-Zygmund inequality for weakly dependent processes (Dedecker et al. (2007) – Theorem 4.1), $\sum_{i=1}^n \epsilon_{t-1}^2 = O_p(n^{1/2})$, and by the Martingale CLT $\sum_{i=1}^n v_i = O_p(n^{1/2})$, so that the second term on the right hand side of (14) is of order $O_p(n^{-3/2}k_n^{1/2}n^{-1/2}n^{1/2}n^{1/2}) = o_p(1)$. Similarly, the first term on the right hand side of (14) is of order $O_p(n^{-1/2}k_n^{-3/2}n^{-1/2}nk_n^{1/2}k_n^{1/2}) = o_p(1)$.

First note that using Lemma 1.0.2 in Zambom and Akritas (2014), we have that $|x_{i+1}-x_{j+1}| = O_p(k_nn^{-1/2})$. Then, by assumptions C2 and C3, $|m(x_i,\hat{\theta}) - m(x_j,\theta)| \le \sup_{x,\theta}\left|(\partial^2/\partial\theta\partial x)m(x,\theta)\right||\hat{\theta}-\theta||x_{i+1}-x_{j+1}| = O_p(n^{-1/2}n^{-1/2}k_n)$ uniformly in $1 \le i \le n$ and $j \in W_i$. Now express the seventh term on the right hand side of (11) $\sqrt{\frac{n}{k_n}}\mathbf{v}_V'\mathbf{A}\mathbf{m}_V$ as

$$
\begin{aligned}
&\sqrt{\frac{n}{k_n}}\frac{(nk_n-1)}{n(n-1)k_n(k_n-1)}\sum_{i=1}^{n}\sum_{k\in W_i} v_k \sum_{\ell\in W_i}(m(X_{\ell-1},\hat{\theta})-m(X_{\ell-1},\theta))\\
&-\sqrt{\frac{n}{k_n}}\frac{k_n}{n(n-1)}\sum_{i=1}^{n} v_i \sum_{\ell=1}^{n}(m(X_{\ell-1},\hat{\theta})-m(X_{\ell-1},\theta))\\
&\quad-\sqrt{\frac{n}{k_n}}\frac{k_n}{n(k_n-1)}\sum_{i=1}^{n} v_i(m(X_{i-1},\hat{\theta})-m(\mathbf{X}_{i-1},\theta))\\
=&\quad\sqrt{\frac{n}{k_n}}\frac{(nk_n-1)}{n(n-1)k_n(k_n-1)}\sum_{i=1}^{n}(m(X_{i-1},\hat{\theta})-m(X_{i-1},\theta))\sum_{k\in W_i} v_k\\
&\quad-\sqrt{\frac{n}{k_n}}\frac{k_n}{n(n-1)}\sum_{i=1}^{n} v_i \sum_{\ell=1}^{n}(m(X_{\ell-1},\hat{\theta})-m(X_{\ell-1},\theta))\\
&-\sqrt{\frac{n}{k_n}}\frac{k_n}{n(k_n-1)}\sum_{i=1}^{n} v_i(m(X_{i-1},\hat{\theta})-m(\mathbf{X}_{i-1},\theta))\\
&\quad+\sqrt{\frac{n}{k_n}}\frac{(nk_n-1)}{n(n-1)k_n(k_n-1)}O_p(n^{-1}k_n^2)\sum_{i=1}^{n}\left|\sum_{k\in W_i} v_k\right|. \qquad (15)
\end{aligned}
$$

Since v_t is white noise, using the CLT and the Mean Value Theorem it follows that (15) is equal to

$$
\begin{aligned}
&\sqrt{\frac{n}{k_n}}\frac{(nk_n-1)}{n(n-1)k_n(k_n-1)}O_p(n^{-1}k_n^3)\sum_{i=1}^{n}|v_i|\\
&-\sqrt{\frac{n}{k_n}}\frac{k_n}{n(n-1)}\sum_{i=1}^{n} v_i \sum_{\ell=1}^{n}(m(X_{\ell-1},\hat{\theta})-m(X_{\ell-1},\theta))\\
&-\sqrt{\frac{n}{k_n}}\frac{k_n}{(n-1)}\sum_{i=1}^{n} v_i(m(X_{i-1},\hat{\theta})-m(\mathbf{X}_{i-1},\theta))\\
=&\quad\sqrt{\frac{n}{k_n}}\frac{k_n}{n-1}(\hat{\theta}-\theta)\sum_{k=1}^{n}\left(\frac{\partial}{\partial\theta}m(X_{k-1},\theta)-\frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial\theta}m(X_{i-1},\theta)\right)v_k\\
&+\sqrt{\frac{n}{k_n}}\frac{k_n}{2(n-1)}(\hat{\theta}-\theta)^2\sum_{k=1}^{n}\left(\frac{\partial^2}{\partial\theta^2}m(X_{k-1},\tilde{\theta})-\frac{1}{n}\sum_{i=1}^{n}\frac{\partial^2}{\partial\theta^2}m(X_{i-1},\tilde{\theta})\right)v_k\\
&+\sqrt{\frac{n}{k_n}}O_p(n^{-1}k_n^3)=O_p(n^{-1/2}k_n^{1/2})+O_p(n^{-1/2}k_n^{5/2}).
\end{aligned}
$$

Now re-write the fourth term in (11) $\mathbf{m}_V'\mathbf{A}\mathbf{m}_V$ as

$$\begin{aligned}
&\sqrt{\frac{n}{k_n}}\mathbf{m}_V'\mathbf{A}\mathbf{m}_V \\
= &\sqrt{\frac{n}{k_n}}\frac{(nk_n-1)}{n(n-1)k_n(k_n-1)}\sum_{i=1}^{n}\left(\sum_{\ell\in W_i}(m(X_{\ell-1},\hat{\theta})-m(X_{\ell-1},\theta))\right)^2 \\
&-\sqrt{\frac{n}{k_n}}\frac{k_n}{n(n-1)}\left(\sum_{i=1}^{n}(m(X_{i-1},\hat{\theta})-m(X_{i-1},\theta))\right)^2 \\
&-\sqrt{\frac{n}{k_n}}\frac{k_n}{n(k_n-1)}\sum_{i=1}^{n}(m(X_{i-1},\hat{\theta})-m(X_{i-1},\theta))^2. \qquad (16)
\end{aligned}$$

Hence, the last term on the right hand side of (16) is of order $O_p(n^{-1/2}k_n^{-1/2}nn^{-1/2}) = o_p(1)$, while the second term is of order $O_p(n^{-3/2}k_n^{1/2}(nn^{-1/2})^2) = o_p(1)$ and the first term is of order $O_p(n^{-1/2}k_n^{-3/2}n(k_nn^{-1/2})^2) = o_p(1)$.

Now consider the last term on the right hand side of (11)

$$\begin{aligned}
\sqrt{\frac{n}{k_n}}\alpha_V'\mathbf{A}\mathbf{m}_V = &\sqrt{\frac{n}{k_n}}\frac{(nk_n-1)}{n(n-1)k_n(k_n-1)}(\hat{\alpha}_1-\alpha_1) \qquad (17)\\
&\times\sum_{i=1}^{n}\sum_{k\in W_i}\epsilon_{k-1}^2\sum_{\ell\in W_i}(m(X_{\ell-1},\hat{\theta})-m(X_{\ell-1},\theta)) \\
&-\sqrt{\frac{n}{k_n}}\frac{k_n}{n(n-1)}(\hat{\alpha}_1-\alpha_1)\sum_{i=1}^{n}\epsilon_{i-1}^2\sum_{\ell=1}^{n}(m(X_{\ell-1},\hat{\theta})-m(X_{\ell-1},\theta)) \\
&-\sqrt{\frac{n}{k_n}}\frac{k_n}{n(k_n-1)}(\hat{\alpha}_1-\alpha_1)\sum_{i=1}^{n}\epsilon_{i-1}^2(m(X_{i-1},\hat{\theta})-m(\mathbf{X}_{i-1},\theta)).
\end{aligned}$$

The second term is of order $O_p(n^{-3/2}k_n^{1/2}n^{-1/2}n^{1/2}nn^{-1/2}) = o_p(1)$. Since v_i is white noise, the third term is of order $O_p(n^{-1/2}k_n^{-1/2}n^{-1/2}n^{1/2}n^{-1/2}) = o_p(1)$. The first term is of order $O_p(n^{-1/2}k_n^{-3/2}n^{-1/2}nk_n^{1/2}k_nn^{-1/2}) = o_p(1)$.

That the remainder of the terms on the right hand side of (11) converge in probability to zero can be shown using similar steps. □

Proof of Theorem 2. Under the alternative hypothesis in (3.7) of the main paper, the residuals are

$$\begin{aligned}
\hat{v}_t &= \epsilon_t^2-\hat{\alpha}_0-\hat{\alpha}_1\epsilon_{t-1}^2-\left(\hat{\theta}_0+\hat{\theta}_1X_{t-1}\right) \\
&= \epsilon_t^2-\alpha_0-\alpha_1\epsilon_{t-1}^2-\left(\theta_0+\theta_1X_{t-1}+(nk_n)^{-1/4}h(X_{t-1})\right)-(\hat{\alpha}_0-\alpha_0) \\
&\quad-(\hat{\alpha}_1-\alpha_1)\epsilon_{t-1}^2-\left(\hat{\theta}_0+\hat{\theta}_1x-\left[\theta_0+\theta_1X_{t-1}+(nk_n)^{-1/4}h(X_{t-1})\right]\right)
\end{aligned}$$

$$= \quad v_t - (\hat{\alpha}_0 - \alpha_0) - (\hat{\alpha}_1 - \alpha_1)\epsilon_{t-1}^2 - (\hat{\theta}_0 - \theta_0)$$
$$-(\hat{\theta}_1 - \theta_1)X_{t-1} + (nk_n)^{-1/4}h(X_{t-1}).$$

Then, the statistic $T_n = MST_n - MSE_n$ can be written as

$$\begin{aligned}
T &= \mathbf{V}'\mathbf{A}\mathbf{V} = \mathbf{v}_V'\mathbf{A}\mathbf{v}_V + (\hat{\alpha}_0 - \alpha_0)^2\mathbf{e}_1'\mathbf{A}\mathbf{e}_1 + \alpha_V'\mathbf{A}\alpha_V \\
&+(\hat{\theta}_0 - \theta_0)^2\mathbf{e}_1'\mathbf{A}\mathbf{e}_1 + \theta_V'\mathbf{A}\theta_V + (nk_k)^{-1/2}\mathbf{h}_V'\mathbf{A}\mathbf{h}_V + 2\mathbf{v}_V'\mathbf{A}\mathbf{e}_1(\hat{\alpha}_0 - \alpha_0) \\
&+2\mathbf{v}_V'\mathbf{A}\alpha_V + 2\mathbf{v}_V'\mathbf{A}\mathbf{e}_1(\hat{\theta}_0 - \theta_0) + 2\mathbf{v}_V'\mathbf{A}\theta_V + 2(nk_n)^{-1/4}\mathbf{v}_V'\mathbf{A}\mathbf{h}_V \\
&+2(\hat{\alpha}_0 - \alpha_0)\mathbf{e}_1'\mathbf{A}\alpha_V + 2(\hat{\alpha}_0 - \alpha_0)\mathbf{e}_1'\mathbf{A}\mathbf{e}_1(\hat{\theta}_0 - \theta_0) + 2(\hat{\alpha}_0 - \alpha_0)\mathbf{e}_1'\mathbf{A}\theta_V \\
&+2(nk_n)^{-1/4}(\hat{\alpha}_0 - \alpha_0)\mathbf{e}_1'\mathbf{A}\mathbf{h}_V + 2\alpha_V'\mathbf{A}\mathbf{e}_1(\hat{\theta}_0 - \theta_0) + 2\alpha_V'\mathbf{A}\theta_V \\
&+2(nk_n)^{-1/4}\alpha_V'\mathbf{A}\mathbf{h}_V + 2(\hat{\theta}_0 - \theta_0)\mathbf{e}_1'\mathbf{A}\theta_V + 2(nk_n)^{-1/4}(\hat{\theta}_0 - \theta_0)\mathbf{e}_1'\mathbf{A}\mathbf{h}_V \\
&+(nk_n)^{-1/4}\theta_V'\mathbf{A}\mathbf{h}_V
\end{aligned} \tag{18}$$

where θ_V and $\mathbf{h}_V$ are defined as α_V but with $(\hat{\theta}_1 - \theta_1)X_{\ell-1}$ and $h(X_{\ell-1})$ respectively instead of $(\hat{\alpha}_1 - \alpha_1)\epsilon_{\ell-1}^2$

By Theorem 1, $n^{1/2}k_n^{-1/2}\mathbf{v}_V'\mathbf{A}\mathbf{v}_V$ converges in distribution to a $N(0, 4\tau^2/3)$ and the 2, 3, 7, 8, and 12th terms are $o_p(n^{1/2}k_n^{-1/2})$. Using similar steps, it can be shown that the 4, 5, 9, 10, 13, 14, 16, 17, and 19th terms are $o_p(n^{1/2}k_n^{-1/2})$.

Consider now the 11th term in (18) and note that

$$\begin{aligned}
&n^{1/2}k_n^{-1/2}(nk_n)^{-1/4}\mathbf{v}_V'\mathbf{A}\mathbf{h}_V \\
&= \frac{n^{1/4}}{k_n^{-3/4}}\frac{nk_n - 1}{n(n-1)k_n(k_n-1)}\sum_{i=1}^n\sum_{k\in W_i} v_k \sum_{\ell\in W_i} h(X_{\ell-1}) \\
&\quad + \frac{n^{1/4}}{k_n^{-3/4}}\frac{k_n}{n(n-1)}\sum_{i=1}^n v_i \sum_{\ell=1}^n h(X_{\ell-1}) + \frac{n^{1/4}}{k_n^{-3/4}}\frac{k_n}{n(k_n-1)}\sum_{i=1}^n v_i h(X_{i-1})
\end{aligned} \tag{19}$$

Using Lemma 1.0.2 in Zambom and Akritas (2014) and the Lipschitz continuity of $h(\cdot)$, the sum in the first term in (19) is

$$k_n\sum_{i=1}^n\left[h(X_{i-1}) + O_p\left(\frac{k_n}{\sqrt{n}}\right)\right]\sum_{k\in W_i} v_k \le k_n\sum_{i=1}^n\left[\sum_{k\in W_i} h(X_{k-1})\right]v_i$$
$$+k_n^2 O_p\left(\frac{k_n}{\sqrt{n}}\right)\sum_{i=1}^n |v_i| = k_n^2\sum_{i=1}^n v_i h(X_{i-1}) + O_p(k_n^3 n^{1/2}).$$

Furthermore, we find

$$\begin{aligned}
&n^{1/2}k_n^{-1/2}(nk_n)^{-1/4}\mathbf{v}_V'\mathbf{A}\mathbf{h}_V \\
&= \frac{n^{1/4}}{k_n^{-3/4}}\frac{nk_n}{n-1}\left[\left(\frac{1}{n}\sum_{i=1}^n v_i h(X_{i-1})\right) - \left(\frac{1}{n}\sum_{i=1}^n h(X_{i-1})\right)\left(\frac{1}{n}\sum_{i=1}^n v_i\right)\right] \\
&+\frac{n^{1/4}}{k_n^{-3/4}}O_p\left(\frac{k_n^2}{n^{1/2}}\right),
\end{aligned}$$

which goes to 0 in probability, since $E(L) = 0$ and $Var(L) = O(n^{-1})$. Hence, $L = O_p(n^{-1/2})$, where

$$L = \left(n^{-1}\sum_{i=1}^{n} v_i h(X_{i-1})\right) - \left(n^{-1}\sum_{i=1}^{n} h(X_{i-1})\right)\left(n^{-1}\sum_{i=1}^{n} v_i\right).$$

The 15, 18, 20 and 21st terms in (18) can be shown to be $o_p(1)$ using similar steps.

Now, the 6th term in (18) can be re-written as

$$\begin{aligned} & n^{1/2}k_n^{-1/2}(nk_k)^{-1/2}\mathbf{h}_V'\mathbf{A}\mathbf{h}_V \\ = & \frac{n}{n-1}\left[\left(\frac{1}{n}\sum_{i=1}^{n} h^2(X_{i-1})\right) - \left(\frac{1}{n}\sum_{i=1}^{n} h(X_{i-1})\right)^2\right] + O_p\left(\frac{k_n^2}{n^{1/2}}\right) \\ & \xrightarrow{p} Var(h(X)). \end{aligned}$$

□

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