Electron. Commun. Probab. **25** (2020), article no. 56, 1–14.
https://doi.org/10.1214/20-ECP340
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Law of the iterated logarithm for a random Dirichlet series

Marco Aymone* Susana Frómeta† Ricardo Misturini‡

Abstract

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of *i.i.d.* random variables with distribution $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$. Let $F(\sigma) = \sum_{n=1}^{\infty} X_n n^{-\sigma}$. We prove that the following holds almost surely

$$\limsup_{\sigma\to 1/2^+} \frac{F(\sigma)}{\sqrt{2\mathbb{E}F(\sigma)^2 \log\log \mathbb{E}F(\sigma)^2}} = 1.$$

Keywords: random series; law of the iterated logarithm; Dirichlet series.
MSC2020 subject classifications: Primary 60G50, Secondary 11M41.
Submitted to ECP on April 28, 2020, final version accepted on July 22, 2020.

1 Introduction

Let $\mathcal{S} = \{1 \le n_1 < n_2 < \ldots\}$ be a set of non-negative real numbers and $(a_n)_{n\in\mathcal{S}}$ be a sequence of complex numbers. A Dirichlet series is a series of the form $F(s) = \sum_{n\in\mathcal{S}} a_n n^{-s}$, where s is a complex number $s = \sigma + it$. A standard result for series of this type is that if F converges at $s = s_0$, then it converges at all $s \in \mathbb{C}$ with $Re(s) > Re(s_0)$, and F defines an analytic function in the half plane $\{s \in \mathbb{C} : Re(s) > Re(s_0)\}$. Hence, when F converges at some point s_0, the following abscissa of convergence is well defined: $\sigma_c := \inf\{\sigma \in \mathbb{R} : F(\sigma) \text{ converges }\}$.

An important example of a Dirichlet series is the Riemann ζ function:

$$\zeta(\sigma) := \sum_{n=1}^{\infty} \frac{1}{n^\sigma}.$$

It follows that $\zeta(\sigma)$ has abscissa of convergence $\sigma_c = 1$. Moreover, $\zeta(\sigma)$ has a singularity at $s = 1$. Indeed, as $\sigma \to 1^+$, $\zeta(\sigma) \sim \frac{1}{\sigma-1}$.

The study of the behavior of a Dirichlet series near its line of abscissa of convergence σ_c is classical in Analysis and in Analytic Number Theory. For instance, one can obtain the prime number Theorem – the statement that the number of primes below x is asymptotically $x/\log x$ – from the classical Wiener-Ikehara Theorem, a Tauberian result; see, for instance Chapter II.7 of [7].

Let $(X_n)_{n\in\mathbb{N}}$ be *i.i.d.* random variables with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$. In this paper we are interested in the behavior of the random Dirichlet series

$$F(\sigma) := \sum_{n=1}^{\infty} \frac{X_n}{n^\sigma} \tag{1.1}$$

*UFMG, Brazil. E-mail: aymone.marco@gmail.com
†UFRGS, Brazil. E-mail: susana.frometa@ufrgs.br
‡UFRGS, Brazil. E-mail: ricardo.misturini@ufrgs.br

near its abscissa of convergence σ_c. By the Kolmogorov's one-series Theorem, $F(\sigma)$ converges if and only if $\sigma > 1/2$, and thus $\sigma_c = 1/2$.

We say that a Dirichlet series is analytic in its abscissa of convergence if this Dirichlet series has an analytic continuation to the open set consisted of the union of the half plane $Re(s) > \sigma_c$ with an open ball with some positive radius and centered at σ_c. It is important to observe that if such analytic continuation exists, then it is unique. In this terminology, sometimes a Dirichlet series may be analytic in its abscissa of convergence σ_c, for example, the Dirichlet η function $\eta(\sigma) = \sum_{n=1}^{\infty}(-1)^{n+1}n^{-\sigma}$ that has $\sigma_c = 0$. Indeed, the Riemann ζ function has analytic continuation to $\mathbb{C} \setminus \{1\}$ with a simple pole at $s = 1$, and for $s \neq 1$ in the half plane $Re(s) > 0$ we have the formula $\eta(s) = (1 - 2^{1-s})\zeta(s)$. Since $(1 - 2^{1-s})$ is an entire function and has a zero at $s = 1$, we obtain that $\eta(s)$ has analytic continuation to $\mathbb{C}$, in particular it is analytic in an open set containing its abscissa of convergence. On the other hand, sometimes a Dirichlet series has a singularity in its abscissa of convergence, which is, for instance the case of our Random Dirichlet series $F(\sigma)$; see, for instance, Theorem 4, pg. 44 of the book of Kahane [4].

In [1], it has been shown that, with probability 1, the function F has infinitely many zeroes accumulating at $1/2$. To prove that, the following Central Limit Theorem has been established: $F(\sigma)/\sqrt{\mathbb{E}F(\sigma)^2} \to_d \mathcal{N}(0,1)$, as $\sigma \to 1/2^+$, where $\mathcal{N}(0,1)$ stands for the standard Gaussian distribution. Moreover, it has been proved that, *almost surely*,

$$\limsup_{\sigma \to 1/2^+} \frac{F(\sigma)}{\sqrt{\mathbb{E}F(\sigma)^2}} = \infty.$$

Thus, a natural question is what is the asymptotics of $F(\sigma)$ as $\sigma \to 1/2^+$. Our main result states:

Theorem 1.1. *Let $F(\sigma)$ be the random Dirichlet series defined in* (1.1). *Then*

$$\limsup_{\sigma \to 1/2^+} \frac{F(\sigma)}{\sqrt{2\mathbb{E}F(\sigma)^2 \log\log \mathbb{E}F(\sigma)^2}} = 1, \text{ almost surely.}$$

Since $F(\sigma)$ is a symmetric random variable, we have the $\liminf$ of the same quantity above equals to -1.

As $\sigma \to 1/2^+$, $\mathbb{E}F(\sigma)^2/(2\sigma - 1)^{-1} \to 1$ (see Lemma 2.1), hence Theorem 1.1 is equivalent to: Almost surely

$$\limsup_{\sigma \to 1/2^+} \frac{F(\sigma)}{\sqrt{\frac{2}{2\sigma - 1} \log\log \frac{1}{2\sigma - 1}}} = 1.$$

Theorem 1.1 is the corresponding Law of the Iterated Logarithm (LIL) for the random Dirichlet series $F(\sigma)$. For the random geometric series, $G(\beta) := \sum_{n=0}^{\infty} X_n \beta^n$, studied by Bovier and Picco in [2] and [3] , it has been established that, almost surely

$$\limsup_{\beta \to 1^-} \frac{G(\beta)}{\sqrt{2\mathbb{E}G(\beta)^2 \log\log \mathbb{E}G(\beta)^2}} = 1.$$

The main issue to obtain these results is that, in comparison with the classical LIL for the simple random walk, we do not have at our disposal a similar result to the Levy's maximal inequality:

$$\mathbb{P}\left(\max_{1 \le m \le n} \left|\sum_{k=1}^{m} X_k\right| \ge t\right) \le 3 \max_{1 \le m \le n} \mathbb{P}\left(\left|\sum_{k=1}^{m} X_k\right| \ge \frac{t}{3}\right). \tag{1.2}$$

In the classical proof of the LIL for $S(x) = \sum_{n \le x} X_n$, the size of $S(x_k)$ is controlled along a sequence $x_k \to \infty$, and the size of $S(x)$ for $x \in [x_k, x_{k+1}]$ is controlled via (1.2).

In our case and in the random geometric case, the supremum is taken over continuous parameters and a maximal inequality is not available.

The proof of Theorem 1.1 is divided into two main steps: an upper bound and a lower bound. For the lower bound we follow the ideas of [2] to show that for any $\gamma > 0$, there is a sequence $\sigma_k \to 1/2^+$ such that, *almost surely*,

$$\limsup_{k\to\infty} \frac{F(\sigma_k)}{\sqrt{\mathbb{E}F(\sigma_k)^2 \log\log \mathbb{E}F(\sigma_k)^2}} \geq 1-\gamma. \tag{1.3}$$

To show that, one main ingredient is to find a lower bound for

$$\mathbb{P}\left(\frac{F(\sigma_k)}{\sqrt{\mathbb{E}F(\sigma_k)^2 \log\log \mathbb{E}F(\sigma_k)^2}} \geq 1-\gamma \right) \tag{1.4}$$

using standard large deviation techniques, and this is made in Lemma 3.1. We conclude the proof of the lower bound using the second Borel-Cantelli lemma, and for that, we will construct independent events that are asymptotic equivalent to those in (1.4), as $k \to \infty$.

For the upper bound, we show that over an specific sequence $\sigma_k \to 1/2^+$,

$$\limsup_{k\to\infty} \frac{F(\sigma_k)}{\sqrt{\mathbb{E}F(\sigma_k)^2 \log\log \mathbb{E}F(\sigma_k)^2}} \leq 1+\gamma. \tag{1.5}$$

Then we control the the size of $F(\sigma)$ for $\sigma \in [\sigma_k, \sigma_{k-1}]$ by following an approach different from the one in [2], where it was used a renormalization idea that is suitable for geometric series. Here we argue as in the proof of the Kolmogorov-Čentsov Theorem; see, for instance, Chapter 2.2 of [5]. Indeed, we consider a dyadic partition of each interval $[\sigma_k, \sigma_{k-1}]$, that is, intervals of the form $[\tau_{l,n}(k), \tau_{l,n+1}(k)]$ where $\tau_{l,n}(k) = \sigma_k + \frac{n}{2^l}(\sigma_{k-1} - \sigma_k)$. Then we exploit the fact that $F(\sigma)$ is differentiable as a function of σ, and, with that, we control the size of the difference of F at consecutive elements of the dyadic partition: $|F(\tau_{l,n}(k)) - F(\tau_{l,n+1}(k))|$.

Here we present some heuristics that will give us the intuition of the bound that will be obtained in Lemma 3.4. We have by the mean value theorem that

$$|F(s) - F(t)| \leq |s-t| \max_{u\in[s,t]} |F'(u)|,$$

and this inequality is nearly optimal if F' is continuous and s and t are close to each other. On the one hand, the derivative of a Dirichlet series is an analytic function, since it is also a Dirichlet series with same abscissa σ_c: $F'(\sigma) = -\sum_{n=1}^{\infty} X_n n^{-\sigma} \log n$. On the other hand, by standard estimates, for σ close to $1/2^+$, $\mathbb{E}F'(\sigma)^2 = \sum_{n=1}^{\infty} n^{-2\sigma}(\log n)^2 \sim \frac{1}{(2\sigma-1)^3}$. Then we show that, if $s,t \in [\sigma_k, \sigma_{k-1}]$, $|F(s) - F(t)|$ is bounded above by something that behaves as

$$\begin{aligned}
\sqrt{\mathbb{E}|F(\sigma_k) - F(\sigma_{k-1})|^2} &\leq |\sigma_k - \sigma_{k-1}| \max_{u\in[\sigma_k,\sigma_{k-1}]} \sqrt{\mathbb{E}F'(u)^2} \\
&\ll \frac{|\sigma_k - \sigma_{k-1}|}{(2\sigma_k - 1)^{3/2}} \\
&\ll \frac{|2\sigma_k - 1|}{(2\sigma_k - 1)^{3/2}} \\
&= \frac{1}{(2\sigma_k - 1)^{1/2}} \\
&\leq \sqrt{\mathbb{E}F(\sigma_k)^2},
\end{aligned}$$

where, in the third line above it is used a particular property of the chosen sequence σ_k. Combining this with (1.5), we obtain the upper bound

$$\limsup_{\sigma\to 1/2^+} \frac{F(\sigma)}{\sqrt{\mathbb{E}F(\sigma)^2 \log\log \mathbb{E}F(\sigma)^2}} \leq 1+\gamma.$$

2 Preliminaries

2.1 Notation

Here we use $f(x) \ll g(x)$ whenever there exists a constant $c > 0$ such that $|f(x)| \leq c|g(x)|$, in a certain range of x – This range could be all the interval $x \in [0, \infty)$ or $x \in (a-\delta, a+\delta)$, $a \in \mathbb{R}, \delta > 0$. We say that $f(x) \sim g(x)$ if $\lim \frac{f(x)}{g(x)} = 1$.

Here, $F(\sigma) = \sum_{n=1}^{\infty} \frac{X_n}{n^\sigma}$, where X_n are *i.i.d.* random variables with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$. By the Kolmogorov's one-series Theorem, it follows that $F(\sigma)$ is convergent for all $\sigma > 1/2$ and divergent for $\sigma \leq 1/2$. Moreover, for $s = \sigma + it$, in the half plane $Re(s) > 1/2$, $F(s)$ is an analytic function; see Chapter I of [6].

2.2 Estimates for the Riemann ζ function

We begin with some standard estimates for the Riemann ζ function. These are classical, and we provide a proof here for the convenience of the reader.

Lemma 2.1. *Let $\sigma > 1$. As $\sigma \to 1$, $\zeta(\sigma)$ is of the order of $\frac{1}{\sigma-1}$, in fact we have that*

$$\frac{1}{\sigma-1} \leq \zeta(\sigma) \leq \frac{\sigma}{\sigma-1}.$$

Moreover, for any $M > 1$

$$\sum_{n=1}^{M} \frac{1}{n^\sigma} \leq \frac{1}{\sigma-1}\left(\sigma - \frac{1}{M^{\sigma-1}}\right)$$
$$\sum_{n>M} \frac{1}{n^\sigma} \leq \frac{1}{(\sigma-1)M^{\sigma-1}}.$$

Proof. Since the function $f(t) = 1/t^\sigma$ is decreasing for $t > 0$, we can compare the sum with the integral obtaining

$$\int_1^{M+1} \frac{1}{t^\sigma} dt \leq \sum_{n=1}^{M} \frac{1}{n^\sigma} \leq 1 + \int_1^M \frac{1}{t^\sigma} dt \quad \text{and} \quad \sum_{n=M+1}^{\infty} \frac{1}{n^\sigma} \leq \int_M^\infty \frac{1}{t^\sigma} dt,$$

which gives the desired estimates. □

2.3 Some basic results for $\sum_{k=1}^{\infty} a_k X_k$

Lemma 2.2. *Let $\{X_k\}_{k\geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$, and $\{a_k\}_{k\geq 1}$ a sequence of real numbers such that $\sum_{k=1}^{\infty} a_k^2 < \infty$, then*

$$\mathbb{E}\left[\exp\left(\sum_{k=1}^{\infty} a_k X_k\right)\right] = \prod_{k=1}^{\infty} \mathbb{E}\left[\exp(a_k X_k)\right] \leq \exp\left(\frac{1}{2}\sum_{k=1}^{\infty} a_k^2\right) < \infty.$$

Proof. Notice that, since $\log\cosh x \leq \frac{x^2}{2}$, we have

$$\prod_{k=1}^{\infty} \mathbb{E}\left[\exp(a_k X_k)\right] = \exp\left(\sum_{k=1}^{\infty} \log\cosh a_k\right) \leq \exp\left(\frac{1}{2}\sum_{k=1}^{\infty} a_k^2\right) < \infty.$$

The inequality

$$\mathbb{E}\left[\exp\left(\sum_{k=1}^{\infty} a_k X_k\right)\right] \leq \prod_{k=1}^{\infty} \mathbb{E}\left[\exp(a_k X_k)\right]$$

follows from Fatou's Lemma. In order to prove the equality, let us define $Y_n = \prod_{k=1}^{n} e^{a_k X_k}$. We want to use the dominated convergence theorem to show that $\mathbb{E}[\lim_{n\to\infty} Y_n] =$

$\lim_{n\to\infty}\mathbb{E}[Y_n]$. Observe that Y_n is a non-negative submartingale with respect to the σ-algebra $\mathcal{F}_n$ generated by $\{X_1,\dots,X_n\}$, indeed

$$\mathbb{E}\left[Y_{n+1}|\mathcal{F}_n\right]=Y_n\mathbb{E}\left[e^{a_{n+1}X_{n+1}}\right]=Y_n\cosh a_{n+1}\geq Y_n.$$

Also notice that

$$\mathbb{E}\left[Y_n^2\right]=\exp\left(\sum_{k=1}^n\log\cosh 2a_k\right)\leq\exp\left(2\sum_{k=1}^n a_k^2\right)\leq\exp\left(2\sum_{k=1}^\infty a_k^2\right)<\infty.$$

Using Cauchy-Schwarz and then Doob's inequality, we obtain

$$\mathbb{E}\left[\max_{1\leq k\leq n}Y_k\right]\leq\mathbb{E}\left[\max_{1\leq k\leq n}Y_n^2\right]^{1/2}\leq 2\mathbb{E}\left[Y_n^2\right]^{1/2}\leq 2\exp\left(\sum_{k=1}^\infty a_k^2\right)<\infty.$$

Then, by Fatou's Lemma, $\mathbb{E}\left[\sup_{k\geq 1}Y_k\right]<\infty$. Therefore, the proof is concluded using the dominated convergence theorem. □

In the following we will recall the Hoeffding's inequality. Since in some situations we will need this result for infinitely many summands, which holds in our case, we present the proof to make clear that such generalization is possible. The case of a finite number of summands is contained in the lemma below considering a sequence $\{a_k\}$ with only a finite number of non-zero terms.

Lemma 2.3 (Hoeffding's inequality)**.** *Let $\{X_k\}_{k\geq 1}$ be a sequence of i.i.d. random variables with $\mathbb{P}(X_1=1)=\mathbb{P}(X_1=-1)=1/2$, and $\{a_k\}_{k\geq 1}$ a sequence of real numbers such that $\sum_{k=1}^\infty a_k^2<\infty$, then, for any $\lambda>0$,*

$$\mathbb{P}\left(\sum_{k=1}^\infty a_kX_k\geq\lambda\right)\leq\exp\left(-\frac{\lambda^2}{2\sum_{k=1}^\infty a_k^2}\right).$$

Proof. For $t\in\mathbb{R}$, by Markov's inequality and Lemma 2.2, we have

$$\mathbb{P}\left(\sum_{k=1}^\infty a_kX_k\geq\lambda\right)=\mathbb{P}\big(e^{t\sum_{k=1}^\infty a_kX_k}\geq e^{t\lambda}\big)\leq e^{-t\lambda}\mathbb{E}\big[e^{\sum_{k=1}^\infty ta_kX_k}\big]\leq\exp\left(-t\lambda+\frac{t^2}{2}\sum_{k=1}^\infty a_k^2\right).$$

Choosing $t=\lambda/\sum_{k=1}^\infty a_k^2$ we obtain the desired result. □

3 Proof of the main result

Let us adopt the notation

$$\bar{F}(\sigma)=\frac{F(\sigma)}{\sqrt{\mathbb{E}F(\sigma)^2}}. \tag{3.1}$$

The proof of Theorem 1.1 will be made in four steps that we will describe in the following.

Step 1. We first prove that, for all $\gamma>0$, there exists a deterministic sequence $\sigma_k\to\frac{1}{2}^+$ such that

$$\mathbb{P}\left(\limsup_{k\to\infty}\frac{\bar{F}(\sigma_k)}{\sqrt{2\log\log\mathbb{E}F(\sigma_k)^2}}\geq 1-\gamma\right)=1. \tag{3.2}$$

Step 2. Let $\epsilon>0$ be fixed and small. Then we prove that for the sequence $\sigma_k=\frac{1}{2}+\frac{1}{2\exp(k^{1-\delta})}$, with $0<\delta<\epsilon/2$, holds

$$\mathbb{P}\left(\limsup_{k\to\infty}\frac{\bar{F}(\sigma_k)}{\sqrt{2\log\log\mathbb{E}F(\sigma_k)^2}}\leq\sqrt{1+\epsilon}\right)=1. \tag{3.3}$$

Step 3. Finally we prove that if σ_k is as in the *step 2*, then there exists a set Ω^* with probability 1, such that for each $\omega \in \Omega^*$, there exists a $k_0 = k_0(\omega)$, such that for all $k \geq k_0$,

$$\max_{\sigma \in [\sigma_k, \sigma_{k-1}]} |F(\sigma) - F(\sigma_k)| \ll \sqrt{\mathbb{E}F(\sigma_k)^2}. \tag{3.4}$$

Step 4. We conclude from (3.3) and (3.4) that for any $\gamma > 0$

$$\mathbb{P}\left(\limsup_{\sigma \to 1/2^+} \frac{\bar{F}(\sigma)}{\sqrt{2 \log\log \mathbb{E}F(\sigma)^2}} \leq 1 + \gamma\right) = 1, \tag{3.5}$$

and hence, the Theorem 1.1 follows from (3.2) and (3.5).

Now let us proceed to the execution of the steps described above.

Step 1

Let us split the normalized Dirichlet series $\bar{F}(\sigma)$ in three different parts: F_1, F_2 and F_3, where

$$F_i(\sigma) = \frac{1}{\sqrt{\mathbb{E}F(\sigma)^2}} \sum_{n=N_{i-1}+1}^{N_i} \frac{X_n}{n^\sigma},$$

with $N_0 = 0$, $N_3 = \infty$. The other parameters, $N_1 = N_1(\sigma)$ and $N_2 = N_2(\sigma)$, will be determined later in order to:

$$\mathbb{P}\left(\limsup_{k \to \infty} \frac{|F_1(\sigma_k)|}{\sqrt{2 \log\log \mathbb{E}F(\sigma_k)^2}} = 0\right) = 1, \tag{3.6}$$

$$\mathbb{P}\left(\limsup_{k \to \infty} \frac{|F_3(\sigma_k)|}{\sqrt{2 \log\log \mathbb{E}F(\sigma_k)^2}} = 0\right) = 1, \tag{3.7}$$

and

$$N_1(\sigma_{k+1}) \geq N_2(\sigma_k). \tag{3.8}$$

The condition (3.8) is required in order to $\{F_2(\sigma_k)\}_{k=1}^\infty$ be a family of independent random variables.

We use the first Borel-Cantelli lemma to prove (3.6) and (3.7) for suitable σ_k, $N_1(\sigma_k)$ and $N_2(\sigma_k)$. We would like to find sequences λ_k and η_k such that

$$\sum_{k=1}^\infty \mathbb{P}(|F_1(\sigma_k)| \geq \lambda_k) < \infty, \quad \text{with} \quad \frac{\lambda_k}{\sqrt{\log\log \mathbb{E}F(\sigma_k)^2}} \to 0, \tag{3.9}$$

and

$$\sum_{k=1}^\infty \mathbb{P}(|F_3(\sigma_k)| \geq \eta_k) < \infty, \quad \text{with} \quad \frac{\eta_k}{\sqrt{\log\log \mathbb{E}F(\sigma_k)^2}} \to 0. \tag{3.10}$$

Using Lemma 2.3, we obtain the bounds

$$\mathbb{P}(|F_1(\sigma_k)| \geq \lambda_k) = 2\mathbb{P}(F_1(\sigma_k) \geq \lambda_k) \leq 2\exp\left(-\frac{\lambda_k^2 \mathbb{E}F(\sigma_k)^2}{2\sum_{n=1}^{N_1} n^{-2\sigma_k}}\right),$$

and

$$\mathbb{P}(|F_3(\sigma_k)| \geq \eta_k) = 2\mathbb{P}(F_3(\sigma_k) \geq \eta_k) \leq 2\exp\left(-\frac{\eta_k^2 \mathbb{E}F(\sigma_k)^2}{2\sum_{n=N_2+1}^{\infty} n^{-2\sigma_k}}\right).$$

Then, (3.9) and (3.10) will hold if we choose the sequences $\lambda_k = \sqrt{2(1+\epsilon)\alpha_k}$ and $\eta_k = \sqrt{2(1+\epsilon)\beta_k}$ and require the conditions

$$\frac{1}{\alpha_k \mathbb{E}F(\sigma_k)^2} \sum_{n=1}^{N_1} \frac{1}{n^{2\sigma_k}} \leq \frac{1}{\log k}, \quad \text{with} \quad \frac{\alpha_k}{\log\log \mathbb{E}F(\sigma_k)^2} \to 0, \tag{3.11}$$

and

$$\frac{1}{\beta_k \mathbb{E}F(\sigma_k)^2} \sum_{n=N_2+1}^{\infty} \frac{1}{n^{2\sigma_k}} \leq \frac{1}{\log k}, \quad \text{with} \quad \frac{\beta_k}{\log\log \mathbb{E}F(\sigma_k)^2} \to 0. \tag{3.12}$$

Let us consider, for $\delta > 0$, the sequence $\sigma_k \to \frac{1}{2}^+$ to be

$$\sigma_k = \frac{1}{2} + \frac{1}{2\exp\left(k^{1+\delta}\right)}. \tag{3.13}$$

Then, using Lemma 2.1, the conditions (3.11) and (3.12) will hold if we require

$$N_1(\sigma_k) \leq \left(1 + \exp(-k^{1+\delta}) - \frac{\alpha_k}{\log k}\right)^{-\exp(k^{1+\delta})}, \quad \text{with} \quad \frac{\alpha_k}{\log k} \to 0, \tag{3.14}$$

and

$$N_2(\sigma_k) \geq \left(\frac{\log k}{\beta_k}\right)^{\exp(k^{1+\delta})}, \quad \text{with} \quad \frac{\beta_k}{\log k} \to 0. \tag{3.15}$$

Let us choose N_1 and N_2 assuming equality in (3.14) and (3.15), and $\alpha_k = \sqrt{\log k}$. Recall that we are also looking for N_1 and N_2 satisfying (3.8), and, for that, we should have

$$\beta_k \geq \log k \left(1 + \exp(-(k+1)^{1+\delta}) - (\log(k+1))^{-1/2}\right)^{\exp\left((k+1)^{1+\delta} - k^{1+\delta}\right)}.$$

Such choice of β_k will be possible if

$$\lim_{k\to\infty} \left(1 + \exp(-(k+1)^{1+\delta}) - (\log(k+1))^{-1/2}\right)^{\exp\left((k+1)^{1+\delta} - k^{1+\delta}\right)} = 0,$$

which can be checked to be true by using L'Hôpital rule. For this limit, the necessity of the condition $\delta > 0$ is crucial.

We have just found sequences σ_k, $N_1(\sigma_k)$ and $N_2(\sigma_k)$ satisfying (3.6), (3.7) and (3.8). To complete the proof of (3.2) we need to show that

$$\mathbb{P}\left(\limsup_{k\to\infty} F_2(\sigma_k) \geq (1-\gamma)\sqrt{2\log\log \mathbb{E}F(\sigma_k)^2}\right) = 1.$$

Since $N_1(\sigma_{k+1}) \geq N_2(\sigma_k)$, we have the required independence needed for the second Borel-Cantelli lemma. Therefore, we must prove that the series

$$\sum_{k=1}^{\infty} \mathbb{P}\left(F_2(\sigma_k) \geq (1-\gamma)\sqrt{2\log\log \mathbb{E}F(\sigma_k)^2}\right) \tag{3.16}$$

diverges.

The next paragraphs will be devoted to find a lower estimate for the probability in (3.16). Let us recall from (3.1) that $\bar{F}(\sigma)$ denotes the normalized version of $F(\sigma)$. Since the terms $F_1(\sigma_k)$ and $F_3(\sigma_k)$ are irrelevant owing to the $(2\log\log \mathbb{E}F(\sigma_k)^2)^{-1/2}$ term, we will use a lower bound as the one stated in the following:

Lemma 3.1. *Let $f(\sigma)$ be a function that goes to $+\infty$ as $\sigma \to \frac{1}{2}^+$ and satisfies the condition*

$$\lim_{\sigma \to \frac{1}{2}^+} \frac{f(\sigma)}{\sqrt{\mathbb{E}F(\sigma)^2}} = 0. \tag{3.17}$$

Then, for all $\delta, \lambda, \epsilon > 0$, there exists $\delta_1 > 0$ such that for $\sigma - \frac{1}{2} \leq \delta_1$, we have

$$\mathbb{P}(\bar{F}(\sigma) \geq \delta f(\sigma)) \geq \left(\frac{1}{2} - \epsilon\right) \exp\left(-\frac{1}{2}\delta^2(1+\lambda)^2 f(\sigma)^2\right).$$

The bound in Lemma 3.1 will be used for the law of the iterated logarithm with the function $f(\sigma) = \sqrt{2\log\log \mathbb{E}F(\sigma)^2}$.

Proof. For all $\bar{\lambda} > 0$, let us consider the event $A = A(\sigma, \delta, \bar{\lambda})$ in which $\bar{F}(\sigma) \in [\delta f(\sigma), \delta(1+\bar{\lambda})f(\sigma)]$. Then

$$\mathbb{P}(\bar{F}(\sigma) \geq \delta f(\sigma)) \geq \mathbb{P}(A).$$

For each n, define the probability measure

$$\widetilde{\mathbb{P}}_{t_0}(dX_n) = \frac{\exp\left(\frac{t_0 X_n}{n^\sigma \sqrt{\mathbb{E}F(\sigma)^2}}\right)}{\cosh\left(\frac{t_0}{n^\sigma \sqrt{\mathbb{E}F(\sigma)^2}}\right)} \mathbb{P}(dX_n),$$

where $t_0 > 0$ will be chosen later. The introduction of this Radon-Nikodym factor is a classical tool in the proof of the lower bound in large deviation theory. Let $\widetilde{\mathbb{P}}(dX)$ be the probability measure consisted in the product measure of each $\widetilde{\mathbb{P}}(dX_n)$, $n \geq 1$.

We have

$$\begin{aligned}
\mathbb{P}(\bar{F}(\sigma) \geq \delta f(\sigma)) &\geq \int_A \mathbb{P}(dX) \\
&= \exp\left(\sum_{n=1}^{\infty} \log\cosh \frac{t_0}{n^\sigma \sqrt{\mathbb{E}F(\sigma)^2}}\right) \int_A \exp\left(-\sum_{n=1}^{\infty} \frac{t_0 X_n}{n^\sigma \sqrt{\mathbb{E}F(\sigma)^2}}\right) \widetilde{\mathbb{P}}_{t_0}(dX).
\end{aligned}$$

Since, on the event A,

$$-\sum_{n=1}^{\infty} \frac{t_0 X_n}{n^\sigma \sqrt{\mathbb{E}F(\sigma)^2}} \geq -t_0\delta(1+\bar{\lambda})f(\sigma),$$

we obtain

$$\mathbb{P}(\bar{F}(\sigma) \geq \delta f(\sigma)) \geq \exp\left(-t_0\delta(1+\bar{\lambda})f(\sigma) + \sum_{n=1}^{\infty} \log\cosh \frac{t_0}{n^\sigma \sqrt{\mathbb{E}F(\sigma)^2}}\right) \widetilde{\mathbb{P}}_{t_0}(A). \tag{3.18}$$

Let us denote by $h(t)$ the function

$$h(t) = \sum_{n=1}^{\infty} \tanh\left(\frac{t}{n^\sigma \sqrt{\mathbb{E}F(\sigma)^2}}\right) \frac{1}{n^\sigma \sqrt{\mathbb{E}F(\sigma)^2}}. \tag{3.19}$$

Observe $h(t)$ is an increasing function of t. We chose t_0 as the (unique) solution of the equation

$$\delta f(\sigma) = h(t_0). \tag{3.20}$$

The following lemma states some properties of t_0. The proof will be postponed to the end of this subsection.

Lemma 3.2. *If t_0 is the solution of* (3.20)*, then for any $\lambda > 0$, there exists a $\delta_1 > 0$ such that, if $\sigma - \frac{1}{2} \le \delta_1$, we have*

$$\delta f(\sigma) \le t_0 \le \delta(1+\lambda) f(\sigma). \tag{3.21}$$

Moreover,

$$-t_0\delta(1+\lambda)f(\sigma) + \sum_{n=1}^{\infty} \log\cosh \frac{t_0}{n^\sigma \sqrt{\mathbb{E}F(\sigma)^2}} \ge -\frac{1}{2}\delta^2(1+2\lambda)^2 f(\sigma)^2. \tag{3.22}$$

Using Lemma 3.2 in (3.18) with $\bar{\lambda} = \lambda/2$, in order to conclude the proof of Lemma 3.1 we only need to show that for all $\epsilon > 0$, exists $\delta_1 > 0$ such that for $\sigma - \frac{1}{2} \le \delta_1$, we have

$$\widetilde{\mathbb{P}}_{t_0}(\bar{F}(\sigma) \in [\delta f(\sigma), \delta(1+\bar{\lambda})f(\sigma)]) \ge \frac{1}{2} - \epsilon.$$

It is sufficient to prove

$$1 - \widetilde{\mathbb{P}}_{t_0}(\bar{F}(\sigma) < \delta f(\sigma)) \ge \frac{1}{2} - \frac{\epsilon}{2} \tag{3.23}$$

and

$$\widetilde{\mathbb{P}}_{t_0}(\bar{F}(\sigma) > \delta(1+\bar{\lambda})f(\sigma)) \le \frac{\epsilon}{2}. \tag{3.24}$$

We will show that $\bar{F}(\sigma) - \delta f(\sigma)$ converge in law, under $\widetilde{\mathbb{P}}_{t_0}$ to a standard Gaussian random variable, as $\sigma \to \frac{1}{2}^+$. For that, we will prove the convergence of the corresponding moment generating functions.

Observing that $M_n = \exp\left(t\sum_{k=1}^n \frac{X_k}{k^\sigma\sqrt{\mathbb{E}F(\sigma)^2}}\right) \Big/ b_n$, where $b_n = \prod_{k=1}^n \frac{\cosh \frac{t+t_0}{k^\sigma\sqrt{\mathbb{E}F(\sigma)^2}}}{\cosh \frac{t_0}{k^\sigma\sqrt{\mathbb{E}F(\sigma)^2}}}$, is a martingale under $\widetilde{\mathbb{P}}_{t_0}$, with respect to the σ-algebra $\mathcal{F}_n$ generated by $\{X_1, \dots, X_n\}$, we can reproduce Lemma 2.2 for $\widetilde{\mathbb{E}}_{t_0}$. Then

$$\widetilde{\mathbb{E}}_{t_0}\left[e^{t\bar{F}(\sigma)}\right] = \frac{\exp\left(\sum_{n=1}^{\infty} \log\cosh \frac{t+t_0}{n^\sigma\sqrt{\mathbb{E}F(\sigma)^2}}\right)}{\exp\left(\sum_{n=1}^{\infty} \log\cosh \frac{t_0}{n^\sigma\sqrt{\mathbb{E}F(\sigma)^2}}\right)} = \frac{\mathbb{E}\left[e^{(t+t_0)\bar{F}(\sigma)}\right]}{\mathbb{E}\left[e^{t_0\bar{F}(\sigma)}\right]}. \tag{3.25}$$

Using the estimates

$$\frac{x^2}{2} - \frac{x^4}{8} \le \log\cosh x \le \frac{x^2}{2}, \text{ for all } x \in \mathbb{R}, \tag{3.26}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{4\sigma}} \le \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{3.27}$$

and Lemma 2.2, we have

$$\exp\left(\frac{t^2}{2} - \frac{t^4\pi^2}{48(\mathbb{E}F(\sigma)^2)^2}\right) \le \mathbb{E}\left[e^{t\bar{F}(\sigma)}\right] \le \exp\left(\frac{t^2}{2}\right). \tag{3.28}$$

Note that, in particular, (3.28) gives us $\lim_{\sigma\to\frac{1}{2}^+} \mathbb{E}\left[e^{t\bar{F}(\sigma)}\right] = e^{\frac{t^2}{2}}$, which yields an alternative proof of the Central Limit Theorem for $\bar{F}(\sigma)$ that was proved in [1] using the convergence of characteristic functions.

Using (3.28) in (3.25) we provide the following upper and lower bounds:

$$\widetilde{\mathbb{E}}_{t_0}\left[e^{t\bar{F}(\sigma)}\right] e^{-t\delta f(\sigma)} \le \exp\left(\frac{t^2}{2} + t(t_0 - \delta f(\sigma)) + \frac{t_0^4\pi^2}{48(\mathbb{E}F(\sigma)^2)^2}\right)$$

and

$$\widetilde{\mathbb{E}}_{t_0}\left[e^{t\bar{F}(\sigma)}\right]e^{-t\delta f(\sigma)} \geq \exp\left(\frac{t^2}{2} + t(t_0 - \delta f(\sigma)) - \frac{(t+t_0)^4\pi^2}{48(\mathbb{E}F(\sigma)^2)^2}\right).$$

Thus, by (3.21) and the condition (3.17), we obtain

$$\lim_{\sigma\to\frac{1}{2}^+} \widetilde{\mathbb{E}}_{t_0}\left[e^{t\bar{F}(\sigma)}\right]\cdot e^{-t\delta f(\sigma)} = e^{\frac{t^2}{2}},$$

which gives us the convergence (under the law of $\tilde{\mathbb{P}}_{t_0}$) to the standard Gaussian variable, therefore

$$\lim_{\sigma\to\frac{1}{2}^+} \widetilde{\mathbb{P}}_{t_0}(\bar{F}(\sigma) - \delta f(\sigma) > 0) = \frac{1}{2},$$

which proves (3.23).

It remains to prove (3.24). Since $f(\sigma)$ explodes as $\sigma \to \frac{1}{2}^+$, for any fixed $a > 0$, we have

$$\begin{aligned}
&\limsup_{\sigma\to\frac{1}{2}^+} \widetilde{\mathbb{P}}_{t_0}(\bar{F}(\sigma) > \delta(1+\bar{\lambda})f(\sigma)) \\
&\qquad = \limsup_{\sigma\to\frac{1}{2}^+} \widetilde{\mathbb{P}}_{t_0}(\bar{F}(\sigma) - \delta f(\sigma) > \delta\bar{\lambda}f(\sigma)) \\
&\qquad \leq \limsup_{\sigma\to\frac{1}{2}^+} \widetilde{\mathbb{P}}_{t_0}(\bar{F}(\sigma) - \delta f(\sigma) > a) \\
&\qquad = \int_a^\infty \frac{e^{-x^2/2}}{\sqrt{2\pi}}dx,
\end{aligned}$$

which goes to zero as $a \to \infty$.

This shows that $\widetilde{\mathbb{P}}_{t_0}(\bar{F}(\sigma) > \delta(1+\bar{\lambda})f(\sigma))$ can be arbitrarily small as $\sigma \to \frac{1}{2}^+$, which gives us (3.24). And this completes the proof of Lemma 3.1. □

Proof of Lemma 3.2. Using that $\tanh x \leq x$, for $x \geq 0$, we obtain

$$\delta f(\sigma) = h(t_0) \leq \frac{t_0}{\mathbb{E}F(\sigma)^2}\sum_{n=1}^\infty \frac{1}{n^{2\sigma}} = t_0,$$

which proves the lower bound of t_0 stated in (3.21).

For the upper bound we use that $\tanh x \geq x - \frac{x^3}{2}$ and (3.27). We obtain

$$\delta f(\sigma) \geq t_0 - t_0^3\frac{\pi^2}{12(\mathbb{E}F(\sigma)^2)^2} =: g(t_0).$$

The cubic function $g(t)$ hits its maximum at $\hat{t} = \frac{2}{\pi}\mathbb{E}F(\sigma)^2$, and $g(\hat{t}) = \frac{2}{3}\hat{t}$.

Now, since f satisfies (3.17), we have, for σ close enough to $\frac{1}{2}$, that

$$\delta f(\sigma) \leq \epsilon\sqrt{\mathbb{E}F(\sigma)^2} < \frac{2}{3}\hat{t}.$$

Then, since the increasing function h satisfies $g(t) \leq h(t) \leq t$, the solution t_0 of (3.20) must satisfies $t_0 < \hat{t}$, which implies $g(t_0) \geq \frac{2}{3}t_0$.

This implies $t_0 \leq \frac{3}{2}g(t_0) \leq \frac{3}{2}\delta f(\sigma)$. Notice that this already gives us an upper bound for t_0, however, this bound can be improved. Indeed,

$$t_0 \leq \delta f(\sigma) + t_0^3\frac{\pi^2}{12(\mathbb{E}F(\sigma)^2)^2} \leq \delta f(\sigma)\left(1 + \frac{27\delta^2\pi^2}{96}\frac{f(\sigma)^2}{(\mathbb{E}F(\sigma)^2)^2}\right).$$

Again, by (3.17), there exists $\delta_1 > 0$ such that, if $\sigma - \frac{1}{2} \le \delta_1$, we have the upper bound stated in (3.21).

Now we will prove (3.22). Using (3.26), (3.27) and (3.21), we obtain

$$\begin{aligned} -t_0\delta(1+\lambda)f(\sigma) &+ \sum_{n=1}^{\infty} \log\cosh \frac{t_0}{n^\sigma\sqrt{\mathbb{E}F(\sigma)^2}} \\ &\ge -t_0\delta(1+\lambda)f(\sigma) + \frac{t_0^2}{2} - \frac{\pi^2}{48}\frac{t_0^4}{(\mathbb{E}F(\sigma)^2)^2} \\ &\ge -\frac{1}{2}\delta^2 f(\sigma)^2\left(2(1+\lambda)^2 - 1 + \frac{\pi^2\delta^2(1+\lambda)^4}{24}\frac{f(\sigma)^2}{(\mathbb{E}F(\sigma)^2)^2}\right). \end{aligned} \tag{3.29}$$

Using (3.17) again, we have that, for σ close to $\frac{1}{2}^+$,

$$\left(\frac{f(\sigma)}{\mathbb{E}F(\sigma)^2}\right)^2 \le \frac{24}{\pi^2\delta^2(1+\lambda)^4}\cdot 2\lambda^2.$$

Then, the expression in (3.29) is bounded below by $-\frac{1}{2}\delta^2 f(\sigma)^2(1+2\lambda)^2$, which proves (3.22). $\square$

Now, by Lemma 3.1, and considering σ_k as in (3.13), we have that, if k is big enough,

$$\mathbb{P}\left(F_2(\sigma_k) \ge (1-\gamma)\sqrt{2\log\log\mathbb{E}F(\sigma_k)^2}\right) \ge \left(\frac{1}{2} - \epsilon\right)\frac{1}{k^{(1-\gamma)^2(1+\lambda)^2(1+\delta)}}.$$

Therefore, for any $\gamma > 0$, a suitable choice of the parameters λ, δ gives us the divergence of the series (3.16). Thus, the proof of *step 1* is completed.

Step 2

Lemma 3.3. *Let $\epsilon > 0$ be small and $\delta = \epsilon/2$. Let $\sigma_k = \frac{1}{2} + \frac{1}{2\exp(k^{1-\delta})}$. Then it a.s. holds that*

$$\limsup_{k\to\infty} \frac{\bar{F}(\sigma_k)}{\sqrt{2\log\log\mathbb{E}F(\sigma_k)^2}} \le \sqrt{1+\epsilon}.$$

Proof. We have, by the Hoeffding inequality that

$$\mathbb{P}\left(\bar{F}(\sigma_k) \ge \sqrt{2(1+\epsilon)\log\log\mathbb{E}F(\sigma_k)^2}\right) \le \exp\left(-(1+\epsilon)\log\log\mathbb{E}F(\sigma_k)^2\right).$$

By Lemma 2.1, we have $\log\log\mathbb{E}F(\sigma_k)^2 \ge \log\log\frac{1}{2\sigma_k - 1} = (1-\delta)\log k$. We also have $(1+\epsilon)(1-\delta) = 1+\gamma$, where $\gamma = \epsilon/2 - \epsilon^2/2 > 0$, provided that $\epsilon > 0$ is small. Thus

$$\mathbb{P}\left(\bar{F}(\sigma_k) \ge \sqrt{2(1+\epsilon)\log\log\mathbb{E}F(\sigma_k)^2}\right) \le \exp(-(1+\gamma)\log k) = \frac{1}{k^{1+\gamma}}.$$

Hence,

$$\sum_{k=1}^{\infty}\mathbb{P}(F(\sigma_k) \ge \sqrt{2(1+\epsilon)\mathbb{E}F(\sigma_k)^2\log\log\mathbb{E}F(\sigma_k)^2}) < \infty.$$

The Borel-Cantelli Lemma completes the proof. $\square$

Step 3

Lemma 3.4. *Let $\sigma_k = \frac{1}{2} + \frac{1}{2\exp(k^{1-\delta})}$, where $\delta > 0$ is a fixed small constant. For $\mathbb{P}$ almost all $\omega \in \Omega$, there exists $k_0 = k_0(\omega)$ such that for $k \geq k_0$, we have that*

$$\max_{\sigma \in [\sigma_k, \sigma_{k-1}]} |F(\sigma) - F(\sigma_k)| \ll \sqrt{\mathbb{E}F(\sigma_k)^2}.$$

Proof. For a non negative integer l, we define $\tau_{l,0} = \tau_{l,0}(k) = \sigma_k$ and $\tau_{l,n} = \tau_{l,n}(k) = \sigma_k + \frac{n}{2^l}(\sigma_{k-1} - \sigma_k)$, where $0 \leq n \leq 2^l$. Let $\lambda_{k,l}$ be a constant to be chosen later and consider the event

$$\mathcal{A}_{l,k} = [\max_{0 \leq n \leq 2^l - 1} |F(\tau_{l,n+1}) - F(\tau_{l,n})| \geq \lambda_{k,l}].$$

Let

$$U_k(\omega) = \min\left\{u \in \mathbb{N} : \omega \in \bigcap_{l=u}^{\infty} \mathcal{A}_{l,k}^c\right\}.$$

One can check that $[U_k \leq L] = \bigcap_{l=L}^{\infty} \mathcal{A}_{l,k}^c$. Thus

$$\mathbb{P}(U_k > L) \leq \sum_{l=L}^{\infty} \mathbb{P}(\mathcal{A}_{l,k}) \leq \sum_{l=L}^{\infty} \sum_{n=0}^{2^l - 1} \mathbb{P}(|F(\tau_{l,n+1}) - F(\tau_{l,n})| \geq \lambda_{k,l}).$$

Next, we will estimate each probability in the inner sum above. We have, by the mean value theorem, that

$$F(\tau_{l,n+1}) - F(\tau_{l,n}) = (\tau_{l,n+1} - \tau_{l,n}) \sum_{m=1}^{\infty} -X_m m^{-\theta_{l,n,m}} \log m,$$

where $\theta_{l,n,m} \in (\tau_{l,n}, \tau_{l,n+1})$. Thus,

$$\begin{aligned}
\mathbb{E}|F(\tau_{l,n+1}) - F(\tau_{l,n})|^2 &= (\tau_{l,n+1} - \tau_{l,n})^2 \sum_{m=1}^{\infty} m^{-2\theta_{l,n,m}} \log^2 m \\
&\leq \frac{(\sigma_k - \sigma_{k-1})^2}{4^l} \sum_{m=1}^{\infty} m^{-2\sigma_k} \log^2 m \\
&\ll \frac{(\sigma_k - \sigma_{k-1})^2}{4^l} \int_1^{\infty} t^{-2\sigma_k} \log^2 t dt \\
&= \frac{(\sigma_k - \sigma_{k-1})^2}{4^l} \int_0^{\infty} t^2 \exp(-(2\sigma_k - 1)t) dt \\
&= \frac{(\sigma_k - \sigma_{k-1})^2}{4^l} \frac{2}{(2\sigma_k - 1)^3} \\
&\ll \frac{(\exp(-k^{1-\delta})/k^{\delta})^2}{4^l} \frac{1}{(\exp(-k^{1-\delta}))^3} \\
&= \frac{\exp(k^{1-\delta})}{4^l k^{2\delta}}.
\end{aligned}$$

Thus, by the Hoeffding inequality, for some constant c_0, we have that

$$\mathbb{P}(|F(\tau_{l,n+1}) - F(\tau_{l,n})| \geq \lambda_{k,l}) \leq \exp\left(-c_0 \frac{\lambda_{k,l}^2}{2} \frac{4^l k^{2\delta}}{\exp(k^{1-\delta})}\right),$$

and hence

$$\begin{aligned}\mathbb{P}(U_k > 1) &\le \sum_{l=1}^{\infty} 2^l \exp\Big(- \frac{c_0 \lambda_{k,l}^2}{2} \frac{4^l k^{2\delta}}{\exp(k^{1-\delta})}\Big) \\ &= \sum_{l=1}^{\infty} \exp\Big(- \frac{c_0 \lambda_{k,l}^2}{2} \frac{4^l k^{2\delta}}{\exp(k^{1-\delta})} + l \log 2\Big).\end{aligned}$$

Choose

$$\lambda_{k,l}^2 = \frac{2}{c_0} \frac{\exp(k^{1-\delta})}{4^l} l.$$

Hence,

$$\mathbb{P}(U_k > 1) \le \sum_{l=1}^{\infty} \exp((-k^{2\delta} + \log 2) l) \ll \exp(-k^{2\delta}).$$

Thus, $\sum_{k=1}^{\infty} \mathbb{P}(U_k > 1) < \infty$, and hence, by the Borel-Cantelli Lemma, there exists a set Ω^* of probability 1 such that for all $\omega \in \Omega^*$, $U_k(\omega) = 1$, for $k \ge k_0(\omega)$.

Let $D_{l,k} = \{\tau_{l,n} : 0 \le n \le 2^l\}$ and put $D_k = \bigcup_{l=0}^{\infty} D_{l,k}$. We shall fix $\omega \in \Omega^*$ and $m \ge n \ge 1$ where $k \ge k_0(\omega)$ and show that for $0 < |s-t| < \frac{|\sigma_k - \sigma_{k-1}|}{2^n}$, $|F(s) - F(t)| \le 2\sqrt{\frac{2}{c_0}} \exp(k^{1-\delta}/2) \sum_{j=n+1}^{m} \frac{\sqrt{j}}{2^j}$, for all $t, s \in D_{m,k}$. Indeed, for $m = n+1$, we can only have that $|s-t| = |\tau_{m,n} - \tau_{m,n+1}|$, and hence $|F(s) - F(t)| \le \lambda_{k,m} = \sqrt{\frac{2}{c_0}} \exp(k^{1-\delta}/2) \frac{\sqrt{m}}{2^m}$. Suppose now that the claim is true for $m = n+1, ..., M-1$ and consider $m = M$. Let $s, t \in D_{M,k}$ with $0 < |s-t| < \frac{|\sigma_k - \sigma_{k-1}|}{2^n}$. Consider $t' = \max\{u \le t : u \in D_{M-1,k}\}$ and $s' = \min\{u \ge s : u \in D_{M-1,k}\}$. Thus

$$\begin{aligned}|F(s) - F(t)| &\le |F(s) - F(s')| + |F(t) - F(t')| + |F(s') - F(t')| \\ &\le 2\lambda_{k,M} + 2\sqrt{\frac{2}{c_0}} \exp(k^{1-\delta}/2) \sum_{j=n+1}^{M-1} \frac{\sqrt{j}}{2^j} \\ &= 2\sqrt{\frac{2}{c_0}} \exp(k^{1-\delta}/2) \sum_{j=n+1}^{M} \frac{\sqrt{j}}{2^j}.\end{aligned}$$

Now, for any $s, t \in D_k$ with $|s-t| \le \frac{|\sigma_k - \sigma_{k-1}|}{2}$, select n such that $\frac{|\sigma_k - \sigma_{k-1}|}{2^{n+1}} \le |s-t| < \frac{|\sigma_k - \sigma_{k-1}|}{2^n}$. Thus

$$|F(s) - F(t)| \le 2\sqrt{\frac{2}{c_0}} \exp(k^{1-\delta}/2) \sum_{j=n+1}^{\infty} \frac{\sqrt{j}}{2^j} \ll \exp(k^{1-\delta}/2).$$

As D_k is dense in the interval $[\sigma_k, \sigma_{k-1}]$ and F is analytic, in particular it is continuous, we conclude that $|F(s) - F(t)| \ll \exp(k^{1-\delta}/2)$, for all $s, t \in [\sigma_k, \sigma_{k-1}]$ with $|s-t| \le \frac{|\sigma_k - \sigma_{k-1}|}{2}$. Finally, observe that $\exp(k^{1-\delta}/2) = \frac{1}{\sqrt{2\sigma_k - 1}}$, and that for $\sigma \in [\sigma_k, \sigma_{k-1}]$, $|\sigma - (\sigma + \sigma_k)/2| = |\sigma_k - (\sigma + \sigma_k)/2| \le \frac{|\sigma_k - \sigma_{k-1}|}{2}$, and hence

$$\begin{aligned}|F(\sigma) - F(\sigma_k)| &\le |F(\sigma) - F((\sigma_k + \sigma)/2)| + |F(\sigma_k) - F((\sigma_k + \sigma)/2)| \\ &\ll \frac{1}{\sqrt{2\sigma_k - 1}}.\end{aligned}$$

Since $\frac{1}{2\sigma_k - 1} \le \mathbb{E} F(\sigma_k)^2$ (see Lemma 2.1), the proof is completed. $\square$

Step 4

Lemma 3.5. *We have that*

$$\mathbb{P}\left(\limsup_{\sigma\to 1/2^+}\frac{\bar{F}(\sigma)}{\sqrt{2\log\log \mathbb{E}F(\sigma)^2}}\leq 1+\gamma\right)=1.$$

Proof. Let $k_0=k_0(\omega)$ be as in Lemma 3.4, and $1/2<\sigma<\sigma_{k_0}$. By Lemma 2.1, we have that for all k, holds

$$\exp\left(k^{1-\delta}\right)\leq \mathbb{E}F(\sigma_k)^2\leq 1+\exp\left(k^{1-\delta}\right). \tag{3.30}$$

Lets us assume that $\sigma\in[\sigma_k,\sigma_{k-1}]$ and write

$$\frac{\bar{F}(\sigma)}{\sqrt{2\log\log \mathbb{E}F(\sigma)^2}}=\frac{F(\sigma_k)}{\sqrt{2\mathbb{E}F(\sigma)^2\log\log \mathbb{E}F(\sigma)^2}}+\frac{F(\sigma)-F(\sigma_k)}{\sqrt{2\mathbb{E}F(\sigma)^2\log\log \mathbb{E}F(\sigma)^2}}.$$

By Lemma 3.3 and (3.30), we have

$$\begin{aligned}\frac{F(\sigma_k)}{\sqrt{2\mathbb{E}F(\sigma)^2\log\log \mathbb{E}F(\sigma)^2}}&\leq\sqrt{1+\epsilon}\,\frac{\sqrt{\mathbb{E}F(\sigma_k)^2\log\log \mathbb{E}F(\sigma_k)^2}}{\sqrt{\mathbb{E}F(\sigma_{k-1})^2\log\log \mathbb{E}F(\sigma_{k-1})^2}}\\&\leq\sqrt{1+\epsilon}\,(1+r_\delta(k)),\end{aligned}$$

for a function $r_\delta(k)$ satisfying $\lim_{k\to\infty}r_\delta(k)=0$.

Now, by Lemma 3.4, and using again (3.30), we have that there exists a constant c_0 that does not depend on k such that

$$\begin{aligned}\frac{F(\sigma)-F(\sigma_k)}{\sqrt{2\mathbb{E}F(\sigma)^2\log\log \mathbb{E}F(\sigma)^2}}&\leq\frac{c_0\sqrt{\mathbb{E}F(\sigma_k)^2}}{\sqrt{\mathbb{E}F(\sigma_{k-1})^2\log\log \mathbb{E}F(\sigma_{k-1})^2}}\\&\leq s_\delta(k),\end{aligned}$$

for a function $s_\delta(k)$ satisfying $\lim_{k\to\infty}s_\delta(k)=0$.

Sending $k\to\infty$ we conclude the proof of Lemma 3.5. □

References

[1] M. Aymone, *Real zeros of random Dirichlet series*, Electron. Commun. Probab., 24 (2019), Paper No. 54, 8. MR-4003128

[2] A. Bovier and P. Picco, *Limit theorems for Bernoulli convolutions*, in Disordered systems (Temuco, 1991/1992), vol. 53 of Travaux en Cours, Hermann, Paris, 1996, pp. 135–158. MR-1601234

[3] A. Bovier and P. Picco, *A law of the iterated logarithm for random geometric series*, Ann. Probab., 21 (1993), pp. 168–184. MR-1207221

[4] J.-P. Kahane, *Some random series of functions*, vol. 5 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, second ed., 1985. MR-0833073

[5] I. Karatzas and S. E. Shreve, *Brownian motion and stochastic calculus*, vol. 113 of Graduate Texts in Mathematics, Springer-Verlag, New York, second ed., 1991. MR-1121940

[6] H. L. Montgomery and R. C. Vaughan, *Multiplicative number theory. I. Classical theory*, vol. 97 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2007. MR-2378655

[7] G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, vol. 46 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1995. Translated from the second French edition (1995) by C. B. Thomas. MR-1342300

Acknowledgments. We would like to thank the anonymous referees for a careful reading of the paper and for useful suggestions and corrections.