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The tail empirical process for long memory stochastic volatility models with leverage

Clémonell Bilayi-Biakana, Gail Ivanoff and Rafał Kulik

Department of Mathematics and Statistics, University of Ottawa, 150 Louis Pasteur Private, Ottawa ON K1N 6N5

e-mail: cbilayib@uottawa.ca; givanoff@uottawa.ca; rkulik@uottawa.ca

Abstract: We consider tail empirical processes of long memory stochastic volatility models with heavy tails and leverage. We study the limiting behaviour of the tail empirical process with both fixed and random levels. We show a dichotomous behaviour for the tail empirical process with fixed levels, according to the interplay between the long memory parameter and the tail index; leverage does not play a role. On the other hand, the tail empirical process with random levels is not affected by either long memory or leverage. The tail empirical process with random levels is used to construct a family of estimators of the tail index, including the famous Hill estimator and harmonic mean estimators. The paper can be viewed as an extension of [21]; while the presence of leverage in the model creates additional theoretical problems, the limiting behaviour remains unchanged.

Keywords and phrases: Long memory, tail empirical process, Hill estimator, harmonic mean estimator, stochastic volatility, leverage.

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1. Introduction

The tail empirical process (TEP) is an important tool used in nonparametric estimation of extremal quantities, like the Hill estimator of the index of regular variation or various risk measures. In this article, we consider a long memory stochastic volatility model with leverage. The model consists of two building blocks: volatility and noise. Informally speaking, by leverage we mean that volatility and noise are not independent. This model is of interest in finance. Our goal is to study weak convergence for the tail empirical processes associated with heavy tailed long memory stochastic volatility sequences with leverage. These results are not only of theoretical interest, but are applicable to different statistical procedures based on intermediate extremes. A similar problem was studied in the case of independent, identically distributed random variables in [15], or for weakly dependent sequences in [14], [13], [26], [24]. In [21] the authors considered heavy tailed, long memory stochastic volatility (LMSV) models and obtained asymptotic results for tail empirical processes. This was extended later on to the multiparameter situation in [22].

However, in the latter two articles leverage was excluded, greatly simplifying theoretical considerations. As evidenced in [23], the presence of long memory, heavy tails and leverage may affect the limiting behaviour of relevant statistics.

It turns out that in the present setting, leverage does not affect the limiting behaviour of the tail empirical process, and hence the results are comparable to those in [21] where leverage is not present. The limiting behaviour depends only on the interplay between the tail index α and the strength of long memory. However, it should be pointed out clearly that the extension from models without leverage to those with leverage is highly nontrivial from a theoretical point of view. In [21] the authors were able to exploit independence between volatility and noise. Here this approach is not applicable and instead we use the Doob decomposition of the tail empirical process into martingale and long memory parts. This makes the proof of tightness technically very involved.

Furthermore, as in [21], for applications we must replace unobservable quantiles with appropriate order statistics. It turns out that the limiting behaviour of the resulting *TEP with random levels* is not affected by either long memory or leverage. This, through integral functionals, allows us to obtain limiting results for different estimators of the tail index, including the classical Hill estimator (see [10] for results in the i.i.d. case) or the more general class of harmonic mean estimators (see [2] again for results in the i.i.d. case). A comprehensive numerical study of different estimators of the tail index in the case of long memory and heavy tails (but no leverage) is presented in [25].

In summary, our contribution in this paper is twofold. From a theoretical point of view, our most important contribution is the proof of weak convergence of the tail empirical process (with fixed and random levels) in the presence of heavy tails, long memory and leverage. Due to the complicated dependence structure of the process, the proof is not at all straightforward. From a practical point of view, the key result is that the asymptotic behaviour of the TEP with random levels is unaffected by the presence of long memory and/or leverage in the model, and so in applications the log returns may be handled exactly as if they were i.i.d. heavy-tailed random variables. This greatly enhances the utility of the LMSV model with leverage considered here.

The rest of the paper is organized as follows. In Section 2 we introduce the model and state all relevant assumptions. In Section 3 we state our main result on convergence of the tail empirical process with fixed levels (Theorem 3.2). This theorem is complemented by the corresponding result for random levels (Theorem 3.8). In Section 4 we prove weak convergence of integral functionals of the tail empirical process, which provides a unified approach to central limit theorems for estimators of the tail index (Theorem 4.2). In Section 5, we discuss the role of the various assumptions and provide an illustrative example. We end with a brief conclusion and directions for further research in Section 6. Proofs are given in Section 7 and relevant technical details on regular variation and long memory sequences can be found in Appendix A.

2. Long memory stochastic volatility model with leverage

One of the common features of financial data is that returns are uncorrelated, but their squares, or absolute values, are (highly) correlated, a property referred to as *long memory*. A second widely accepted feature is that log-returns are heavy tailed, in the sense that some moment of the log-returns is infinite. A third important feature is *leverage*. In the financial time series context, leverage is understood as negative dependence between previous returns and future volatility; see (12). The leverage effect has been well documented in the economic literature. As stated in [16], *any mathematical model approximating the evolution of asset price should be able to generate the leverage effect*. See more details in Section 2.2 below.

Motivated by these empirical findings, one of the common modelling approaches is to represent log-returns $\{X_j\}$ as a stochastic volatility sequence $X_j = \sigma_j Z_j$, where $\{Z_j\}$ is an i.i.d. noise sequence and $\{\sigma_j^2\}$ is the conditional variance or, more generally, a certain process which stands as a proxy for the volatility. In such a process, long memory can only be modelled through the sequence $\{\sigma_j\}$, while the tails can be modelled either through the sequence $\{Z_j\}$ or through $\{\sigma_j\}$, or both. The well known GARCH processes belong to this class of models. The volatility sequence $\{\sigma_j\}$ is heavy tailed unless the distribution of Z_0 has finite support, and leverage can be present. However, long memory of squares cannot be modelled by the GARCH processes, since such models (under the appropriate technical conditions) are mixing. See [17, Chapter 3].

Consequently, the so-called long memory stochastic volatility (LMSV) model was introduced in [5]. An overview of such models is given in [11] and [12]. In the classical LMSV model, $\{Z_j\}$ is a sequence of i.i.d. standard normal random variables, independent of the volatility sequence $\{\sigma_j\}$, assumed to be of the form $\sigma_j = \exp(Y_j)$, where $\{Y_j\}$ is a long memory Gaussian sequence. However, the independence assumption excludes the possibility of modelling leverage effects. We thus consider the long memory stochastic volatility model with leverage:

$$X_j = \phi(Y_j) Z_j \ , \quad j \in \mathbb{Z}. \tag{1}$$

We make the following assumptions.

2.1. Assumptions

A(i) The sequence $\{Y_j\}$ is strictly stationary and ergodic long memory Gaussian, that is

$$Y_j = \sum_{i=1}^{\infty} a_i \epsilon_{j-i} \ ,$$

where $\{\epsilon_j\}$ is a sequence of i.i.d. standard normal random variables and

$$a_i = i^{d-1} \ell_a(i), \quad \sum_{i=1}^{\infty} a_i^2 = 1 \ .$$

As a consequence, $\gamma_Y(j) = \mathrm{Cov}(Y_0, Y_j) \sim j^{2d-1}\ell_Y(j)$. Note that ℓ_a and ℓ_Y are slowly varying functions at infinity such that:

$$\ell_Y(j) = \ell_a^2(j) B(1-2d, d),$$

where $B(a,b)$ denotes the Beta function and $0 < d < \frac{1}{2}$ is referred to as the long memory parameter (for details, see [1]). Furthermore, we assume that $\{(\epsilon_j, Z_j)\}$ is a sequence of i.i.d. random vectors. For each fixed j, ϵ_j and Z_j may be dependent, but due to the construction above, the random variables Y_j and Z_j are independent. However, there can be dependence between the sequences $\{Z_j\}$ and $\{Y_j\}$, allowing for *leverage* in the model. See more discussion in Section 2.2.

A(ii) The random variables Z_j are i.i.d. with tail distribution function $\bar{F}_Z$:

$$\bar{F}_Z(x) = c^* x^{-\alpha} \exp\left(\int_1^x \frac{\eta^*(u)}{u}\, du\right),\ x > 0, \tag{2}$$

where $\alpha, c^* > 0$ and $\eta^*(\cdot)$ is either nonnegative or nonpositive, regularly varying at infinity with index $-\kappa$, $\kappa > 0$. It is also assumed that η^* is bounded – that is, there exists $\beta > 0$ such that $\forall\ x > 0$,

$$|\eta^*(x)| \le \beta. \tag{3}$$

We shall refer to a function of the form (2) as *second-order regular varying* at infinity with parameters $-\alpha, -\kappa$ and rate function η^*. This set of functions is denoted by $2RV_\infty(-\alpha, -\kappa, \eta^*)$.

A(iii) The function ϕ is a nonnegative measurable function and $\phi(Y)$ is not equal to 0 with probability one. We denote by m the Hermite rank of ϕ^α. (For more details on Hermite rank, see [1] pg. 108.)

A(iv) Let $k_n \to \infty$ be an increasing sequence of positive integers such that $k_n/n \to 0$ and let u_n be defined by $u_n = \bar{F}_X^{-1}(k_n/n)$, where $\bar{F}_X^{-1}$ is the inverse function of the tail distribution function $\bar{F}_X$ of X. (As will be argued below, $\bar{F}_X$ is continuous). For ease of notation, we suppress dependence of k_n on n, which is the standard practice in the extreme value literature.

A(v) For all $n \ge 1$, define $\{a_{n,m}\}$ and $\{b_{n,m}\}$ as follows:

$$a_{n,m} := \left(\sqrt{n\bar{F}_Z(u_n)} + \frac{n}{b_{n,m}}\right) \mathbb{1}_{\{m(1-2d)<1\}} + \sqrt{n}\mathbb{1}_{\{m(1-2d)>1\}}\ ,$$

$$b_{n,m} := n^{1-m(\frac{1}{2}-d)} \sqrt{\frac{2m!(\ell_Y(n))^m}{[(2d-1)m+1][(2d-1)m+2]}}\ .$$

We assume that

$$a_{n,m}\eta^*\left(\bar{F}_X^{-1}(k/n)\right) = a_{n,m}\eta^*(u_n) \underset{n\to\infty}{\to} 0\ . \tag{4}$$

A(vi) For $\alpha, \beta, \kappa > 0$ as above, there exists $\epsilon > 0$ such that

$$E\left((\phi(Y))^{2\alpha+2\beta}\right) + E\left((\phi(Y))^{2\alpha-2\beta}\right) < \infty, \tag{5a}$$

$$E\left((\phi(Y))^{\alpha+\kappa+\epsilon}\right) + E\left((\phi(Y))^{\alpha+\kappa-\epsilon}\right) < \infty. \tag{5b}$$

For clarity throughout the remainder of the article, when referring to the long memory stochastic volatility model with leverage, we suppose that all the assumptions A(i) to A(vi) are satisfied. However, some results do not require all the assumptions. See more discussion in Section 5.

Remark 2.1. The elementary consequence of A(ii) is that the second-order regular variation of Z also implies that $\bar{F}_Z$ is regularly varying at infinity with index $-\alpha$ – that is

$$J_x(t) := \frac{\bar{F}_Z(xt)}{\bar{F}_Z(x)} \underset{x\to\infty}{\to} T(t) := t^{-\alpha}, \tag{6}$$

uniformly on compact subsets of $(0, \infty)$. Furthermore, by (5a), an application of Breiman's Lemma (see [6]) gives

$$\bar{F}_X(x) = E\left(\bar{F}_Z\left(x/\phi(Y_1)\right)\right) \underset{x\to\infty}{\sim} E(\phi^\alpha(Y))\bar{F}_Z(x). \tag{7}$$

Therefore, $\bar{F}_X$ is also regularly varying at infinity with index $-\alpha$ and

$$T_x(t) := \frac{\bar{F}_X(xt)}{\bar{F}_X(x)} \underset{x\to\infty}{\to} T(t) = t^{-\alpha}\ . \tag{8}$$

Moreover $\bar{F}_Z(x)/\bar{F}_X(x)$ is bounded away from zero and infinity, that is, there exists $\lambda > 0$ such that for all $x > 0$,

$$1/\lambda < \frac{\bar{F}_Z(x)}{\bar{F}_X(x)} < \lambda\ . \tag{9}$$

Furthermore, A(ii) implies that $\bar{F}_Z$ is continuous. So is $\bar{F}_X$, by (7). Finally, as will be shown by eqs. (29a) and (29b), (4) controls bias. In particular, for any $\tau_0 > 0$ we have

$$a_{n,m} \sup_{t>\tau_0} |J_{u_n}(t) - T(t)| \underset{n\to\infty}{\to} 0 \text{ and } a_{n,m} \sup_{t>\tau_0} |T_{u_n}(t) - T(t)| \underset{n\to\infty}{\to} 0\ . \tag{10}$$

Let $\{\mathcal{G}_j\}$ be the minimal filtration generated by the independent and identically distributed random vectors $\{(\epsilon_j, Z_j)\}$, that is

$$\mathcal{G}_j := \sigma\left(\{(\epsilon_k, Z_k) : k \le j\}\right),\ j \in \mathbb{Z}\ .$$

As a consequence, X_j is $\mathcal{G}_j$-adapted, Y_j is $\mathcal{G}_{j-1}$-measurable and we have

$$E(\mathbb{1}_{\{X_j>x\}} \| \mathcal{G}_{j-1}) = E(\mathbb{1}_{\{\phi(Y_j)Z_j>x\}} \| \mathcal{G}_{j-1}) = \bar{F}_Z(x/\phi(Y_j)),\ \ x > 0\ . \tag{11}$$

2.2. Leverage

In the financial literature, leverage is understood as an asymmetric behaviour of stock prices. More specifically, increase in volatility is (negatively) associated with a movement of stock prices. The original modelling approach to leverage is due to [19]. The authors consider the model

$$X_j = \sigma_j Z_j \ , \quad \log \sigma_j^2 = Y_j \ , \quad Y_j = \rho Y_{j-1} + \epsilon_{j-1} \ ,$$

where $\rho \in (-1,1)$, $\{(\epsilon_j, Z_j)\}$ are i.i.d. normal vectors with mean zero, unit variance and correlation ω. Writing the logarithm of the volatility as

$$Y_{j+1} = \rho Y_j + \omega \sigma_j^{-1} X_j + (\epsilon_j - \omega Z_j) \ ,$$

and noting that $E\left(\epsilon_j - \omega Z_j \| X_j, \sigma_j\right) = E(\epsilon_j - \omega Z_j \| Z_j) = 0$ we conclude

$$E\left(\log \sigma_{j+1}^2 \| X_j, \sigma_j\right) = \rho Y_j + \omega \sigma_j^{-1} X_j$$

and thus

$$E\left(\log \sigma_{j+1}^2 \| X_j\right) = \mu + \omega \nu X_j \ , \tag{12}$$

where μ, ν are constants. Thus, the expected log-volatility is a linear function of X_j whenever $\omega \neq 0$. Of course, our model extends the AR(1) assumption on the sequence $\{Y_j\}$ allowing for long memory.

Other modeling approaches to leverage can be found e.g. in [28] or [7]. The only difference lies in alternative specifications in the equation for $\log \sigma_j^2$, allowing for an additional random term.

3. Tail empirical process

In this section, we present two main results: Theorem 3.2 and Theorem 3.8. They both pertain to weak convergence of the tail empirical process of the long memory stochastic volatility model with leverage in (1). The first one deals with deterministic levels whereas the second one with random levels (order statistics). We note different limiting behaviour – in particular, while the limiting behaviour of the tail empirical process with deterministic levels could be affected by long memory, this is not the case for the process with random levels. Furthermore, neither of the tail empirical processes is affected by the presence of leverage.

Recall that $k = k_n \to \infty$ is an increasing sequence of positive integers such that $k/n \to 0$, as $n \to \infty$ and u_n is defined by $k = n\bar{F}_X(u_n)$.

3.1. Deterministic levels

Definition 3.1. *Given a sequence (k_n) as above, the empirical tail distribution function of $\{X_j\}$ is defined as:*

$$\widetilde{T}_n(t) := \frac{1}{k} \sum_{j=1}^{n} \mathbb{1}_{\{X_j > u_n t\}} = \frac{1}{k} \sum_{j=1}^{n} \mathbb{1}_{\{\phi(Y_j) Z_j > u_n t\}}, \quad t > 0.$$

In the sequel, let $V_{j,n}(t) := \mathbb{1}_{\{\phi(Y_j)Z_j > u_n t\}}$. We note that (cf. (8))

$$T_{u_n}(t) = E\left(\widetilde{T}_n(t)\right) = \frac{\bar{F}_X(u_n t)}{\bar{F}_X(u_n)} \text{ and } \lim_{n\to\infty} T_{u_n}(t) = t^{-\alpha} = T(t)\ .$$

The tail empirical process (with deterministic levels) is defined by

$$\widetilde{S}_n(t) = k\left(\widetilde{T}_n(t) - T_{u_n}(t)\right) = \sum_{j=1}^{n} \left[V_{j,n}(t) - E\left(V_{j,n}(t)\right)\right],\ t > 0\ .$$

Our goal is to determine the asymptotic behaviour of $\widetilde{S}_n$ under suitable normalizations. The structure of the model considered in (1) suggests the following martingale-long memory Doob decomposition:

$$\widetilde{S}_n(t) := M_n(t) + L_n(t), \quad t > 0, \tag{13}$$

where the summands M_n and L_n are defined as follows:

$$M_n(t) := \sum_{j=1}^{n} \left[V_{j,n}(t) - E\left(V_{j,n}(t) \| \mathcal{G}_{j-1}\right)\right]\ , \tag{14a}$$

$$L_n(t) := \sum_{j=1}^{n} \left[E\left(V_{j,n}(t) \| \mathcal{G}_{j-1}\right) - E\left(V_{j,n}(t)\right)\right]. \tag{14b}$$

We will call M_n and L_n the **martingale** and **long memory parts**, respectively. To establish weak convergence of $\widetilde{S}_n$ under suitable normalizations, we will establish weak convergence for M_n and L_n, suitably normalized. This will then determine the appropriate normalization for $\widetilde{S}_n$ (cf. Theorem 3.2). The finite dimensional convergence of M_n will be handled with a classical martingale central limit theorem (cf. Theorem 2.5 in [20]), while tightness requires tedious technical arguments. The process L_n will be handled with a limit theorem for Hermite polynomials (cf. [1, p. 223, 229]). In what follows, define

$$\mu_{\phi,\alpha}(m) = E(H_m(Y)\phi^{\alpha}(Y))\ ,$$

where H_m denotes the Hermite polynomial of order m.

Theorem 3.2. *Let $\{X_j\}$ be the long memory stochastic volatility model with leverage given in* (1).

- *If $m(1-2d) < 1$ and $\frac{b_{n,m}}{n}\sqrt{k} \to 0$ as $n \to \infty$, then*

$$\frac{\widetilde{S}_n(t)}{\sqrt{k}} \xrightarrow[n\to\infty]{d} (W \circ T)(t), \tag{15}$$

in $D(0,\infty)$ equipped with the Skorokhod J_1 topology, where $W(\cdot)$ denotes a standard Brownian motion on $(0,\infty)$.

- *If* $m(1-2d) < 1$ *and* $\frac{b_{n,m}}{n}\sqrt{k} \to \infty$ *as* $n \to \infty$, *then*

$$\frac{n}{kb_{n,m}}\widetilde{S}_n(t) \xrightarrow[n\to\infty]{d} \frac{\mu_{\phi,\alpha}(m)}{m!E(\phi^\alpha(Y))}T(t)\xi_{m,d+1/2}(1),$$

in $D(0,\infty)$ *equipped with the Skorokhod* J_1 *topology, where* $\xi_{m,d+1/2}(1)$ *is a Hermite-Rosenblatt random variable.*

- *If* $m(1-2d) > 1$, *then* (15) *holds.*

The proof of this theorem can be found in Section 7.2.

Remark 3.3. We note that leverage has no effect on the limiting distribution. Long memory affects the limiting behaviour. We have a dichotomous behaviour, according to strength of long memory (that is, the value of the parameter d that appears explicitly in the definition of $b_{n,m}$). In the long memory case, the limiting random variable, $\xi_{m,d+1/2}(1)$, is Hermite-Rosenblatt with the Hurst parameter $d + \frac{1}{2}$ (see Definition 3.24 in [1]). It is a non-Gaussian limit unless $m = 1$.

Remark 3.4. We note that thanks to Assumption A(v), the effect of bias introduced by replacing $T_{u_n}(t)$ by $T(t)$ is negligible (see (10)), and so the process

$$\widetilde{S}^*_n(t) := k\left(\widetilde{T}_n(t) - T(t)\right), \quad t > 0$$

has the same limiting behaviour as $\widetilde{S}_n$.

Remark 3.5. The upper quantile (u_n) depends on the unknown distribution F_X and so its value is not known. This means that the empirical tail distribution function $\widetilde{T}_n$ cannot be observed, and so we introduce a data-based version in the next section.

3.2. Random levels

In this section we consider the tail empirical process with random levels that is used to construct estimators of the tail index.

Let $X_1, \ldots, X_n$ be a sample from the stochastic volatility model with leverage defined in (1). Let $X_{(1)} \le \cdots \le X_{(i)} \le \cdots \le X_{(n)}$ be the corresponding order statistics. Let $F_{n,X}$ be the usual empirical distribution function and $\bar{F}_{n,X}(x) = 1 - F_{n,X}(x)$. Since $\bar{F}_X$ is continuous, $u_n = \bar{F}_X^{-1}(k/n)$ for $k = k_n$ and $\bar{F}_{n,X}^{-1}(k/n) = X_{(n-k)}$, and so it is then natural to approximate u_n with $X_{(n-k)}$.

Definition 3.6. *Let* $\{X_j\}$ *be the long memory stochastic volatility model with leverage given in* (1).

- *The empirical tail distribution function with random levels of* $\{X_j\}$ *is*

$$\widehat{T}_n(t) := \frac{1}{k}\sum_{j=1}^{n} \mathbb{1}_{\{X_j > X_{(n-k)}t\}}, \quad t > 0. \tag{16}$$

- *The tail empirical process with random levels of* $\{X_j\}$ *is*

$$\widehat{S}_n(t) := k\left(\widehat{T}_n(t) - T(t)\right). \tag{17}$$

We aim at studying the asymptotic behaviour of $\widehat{S}_n$. First, we need an extension to Theorem 3.2, the next lemma.

Lemma 3.7. *Let* $\{X_j\}$ *be the long memory stochastic volatility model with leverage given in* (1).

- *If* $m(1-2d) < 1$ *and* $\frac{b_{n,m}}{n}\sqrt{k} \to 0$, *or* $m(1-2d) > 1$, *then*

$$\left(\frac{\widetilde{S}^*_n(t)}{\sqrt{k}}\,,\sqrt{k}\left(\frac{X_{(n-k)}}{u_n} - 1\right)\right) \xrightarrow[n\to\infty]{d} \left((W\circ T)(t)\,,\frac{W(1)}{\alpha}\right). \tag{18}$$

- *If* $m(1-2d) < 1$ *and* $\frac{b_{n,m}}{n}\sqrt{k} \to \infty$ *then*

$$\left(\frac{n}{kb_{n,m}}\widetilde{S}^*_n(t)\,,\frac{n}{b_{n,m}}\left(\frac{X_{(n-k)}}{u_n} - 1\right)\right)$$
$$\xrightarrow[n\to\infty]{d} \left(\frac{\mu_{\phi,\alpha}(m)}{m!E(\phi^\alpha(Y))}T(t)\xi_{m,d+1/2}(1), \frac{\mu_{\phi,\alpha}(m)}{\alpha m!E(\phi^\alpha(Y))}\xi_{m,d+1/2}(1)\right).$$

These two joint weak convergences hold in $D(0,\infty)\times\mathbb{R}$.

Now, we state the result for random levels. Notice that by introducing random levels, the tail empirical process vanishes at 1 and ∞, which forces the limiting process to be of a bridge type. More surprisingly, the introduction of random levels appears to cause the effect of long memory to disappear. The reason for this, as will be seen in the proof, is that the limiting behaviour of $\widehat{S}_n$ follows informally from the continuous mapping theorem applied to $\widetilde{S}^*_n$ and $X_{(n-k)}/u_n$. Thanks to the degenerate structure of the limiting process for $\widetilde{S}^*_n$ (that is, a random variable scaled by a deterministic function), the long memory effect cancels out. Note however that long memory does play a role in assumption (4) which controls bias. This can affect the choice of k; see Example 5.1.

Once again, the presence of leverage does not affect the limit.

Theorem 3.8. *Let* $\{X_j\}$ *be the long memory stochastic volatility model with leverage given in* (1). *Then,*

$$\frac{\widehat{S}_n(t)}{\sqrt{k}} \xrightarrow[n\to\infty]{d} W\left(T(t)\right) - T(t)W(1), \tag{19}$$

in $D(0,\infty)$ *equipped with the Skorokhod* J_1 *topology. The limiting process* $W\left(T(\cdot)\right) - T(\cdot)W(1)$ *is a centered time-changed Brownian bridge on* $[1,\infty)$.

We refer to Section 7.3 for the proof of this result.

4. Integral functionals

The power of weak convergence theory comes from the fact that many diverse results emerge as corollaries of a basic convergence theorem. As we shall see in Theorem 4.1, our main convergence Theorem 3.8 can be extended to integral functionals of the tail empirical process with random levels. This in turn yields a unified approach to establishing weak convergence of estimators of the tail index (cf. Section 4.1). In what follows, r denotes a nonnegative integer.

Theorem 4.1. *Let $\{X_j\}$ be the long memory stochastic volatility model with leverage given in (1). If $\alpha > 2(1-r)$, then*

$$\frac{1}{\sqrt{k}}\int_1^\infty \frac{\widehat{S}_n(t)}{t^r}\,dt \xrightarrow[n\to\infty]{d} \int_1^\infty \frac{W(T(t)) - W(1)T(t)}{t^r}\,dt. \tag{20}$$

The proof (see Section 7.3) is not trivial due to the fact that the infinite integral is not a continuous functional.

4.1. Tail index estimation

We consider the long memory stochastic volatility model with leverage defined in (1). Since the tail distribution of X is regularly varying with index $-\alpha$, then this raises the question of estimating the index of regular variation α. Therefore, we restrict our attention to the *harmonic moment estimators (*HME*)* $\widehat{\gamma}_{r,k}$ of order r of $\gamma := 1/\alpha$. We aim at studying their asymptotic normality. For this purpose, we begin with their construction. Recalling that $T(t) = t^{-\alpha}$, we have for $r \geq 0$,

$$\zeta_r := \int_1^\infty \frac{T(t)}{t^r}\,dt = \frac{1}{\alpha + r - 1}\cdot \tag{21}$$

If $\widehat{\zeta}_{r,k}$ denotes an estimator of ζ_r, then the plug-in method and (16) yield

$$\widehat{\zeta}_{r,k} = \int_1^\infty \frac{\widehat{T}_n(t)}{t^r}\,dt = \int_1^\infty \frac{1}{k}\sum_{j=1}^n \mathbb{1}_{\{X_{(j)} > X_{(n-k)}t\}}\frac{dt}{t^r} = \frac{1}{k}\sum_{j=1}^n \int_1^\infty \mathbb{1}_{\left\{\frac{X_{(j)}}{X_{(n-k)}} > t\right\}}\frac{dt}{t^r}\cdot$$

Furthermore, since $t \geq 1$, we have

$$\begin{aligned}
\widehat{\zeta}_{r,k} &= \frac{1}{k}\sum_{j=1}^k \int_1^{\frac{X_{(n-j+1)}}{X_{(n-k)}}} \frac{dt}{t^r} \\
&= \begin{cases} \frac{1}{r-1}\left(1 - \frac{1}{k}\sum_{j=1}^k \left(\frac{X_{(n-k)}}{X_{(n-j+1)}}\right)^{r-1}\right) & \text{if } r \neq 1, \\ \frac{1}{k}\sum_{j=1}^k \ln\left(\frac{X_{(n-j+1)}}{X_{(n-k)}}\right) & \text{if } r = 1. \end{cases}
\end{aligned}$$

To derive the estimators of $\gamma = 1/\alpha$, we solve for $1/\alpha$ in (21) and obtain

$$\zeta_r = \frac{1}{\alpha + r - 1} \Rightarrow \frac{1}{\alpha} = \frac{\zeta_r}{1 + (1-r)\zeta_r} \cdot$$

Thanks to the plug-in method, we derive the HMEs below:

$$\widehat{\gamma}_{r,k} = \frac{\widehat{\zeta}_{r,k}}{1 + (1-r)\widehat{\zeta}_{r,k}} \tag{22}$$

$$= \begin{cases} \frac{1}{r-1}\left(\left(\frac{1}{k}\sum_{j=1}^{k}\left(\frac{X_{(n-k)}}{X_{(n-j+1)}}\right)^{r-1}\right)^{-1} - 1\right) & \text{if } r \neq 1\ , \\ \frac{1}{k}\sum_{j=1}^{k}\ln\left(\frac{X_{(n-j+1)}}{X_{(n-k)}}\right) & \text{if } r = 1\ . \end{cases} \tag{23}$$

- The HME that corresponds to $r = 1$ is the Hill estimator of $\gamma = 1/\alpha$.
- The HME that corresponds to $r = 2$ is the t-Hill estimator of γ, that is

$$\widehat{\gamma}_{2,k} = \left(\frac{1}{k}\sum_{j=1}^{k}\frac{X_{(n-k)}}{X_{(n-j+1)}}\right)^{-1} - 1\ .$$

The main result of this section, the asymptotic normality of $\widehat{\gamma}_{r,k}$, is a simple application of the delta method and Theorem 4.1.

Theorem 4.2. *Let $\{X_j\}$ be the long memory stochastic volatility model with leverage in* (1). *If $\alpha > 2(1-r)$, then*

$$\sqrt{k}\,(\widehat{\gamma}_{r,k} - \gamma) \xrightarrow[n\to\infty]{d} \frac{(\alpha + r - 1)}{(\alpha^3(\alpha + 2r - 2))^{1/2}}\mathcal{N}\ ,$$

where $\mathcal{N}$ is a standard normal random variable.

5. Comments and example

- The Gaussian assumption on Y_j can be easily replaced by assuming that Y_j is an infinite order moving average process. Instead of using Hermite polynomials, convergence of the long memory part will be concluded using tools such as Appell polynomials or a version of martingale approximation. See [1, Section 4.2.5].
- We excluded the case of $d = 0$ which yields short memory. It is justified in [8] that in the case of short memory, the stochastic volatility sequence $\{X_j\}$ is mixing and limiting results for tail empirical processes can be concluded from [26]. For the tail empirical process with deterministic levels, instead of A(ii), only regular variation is needed, while the moment conditions (5a)-(5b) can be replaced with a weaker assumption, $E\left((\phi(Y))^{\alpha+\epsilon}\right) < \infty$

for some $\epsilon > 0$, in order to guarantee that the tail function $\bar{F}_X$ is regularly varying. For the tail empirical process with random levels and for the Hill estimator, a version of second-order regular variation is needed.
Our method of proof uses second-order regular variation for the tail empirical process with deterministic levels. Possibly, with another method of proof, this could be avoided, although second-order regular variation will likely still be needed for the tail empirical process with random levels.

- More specifically, for finite dimensional convergence of the martingale part (Proposition 7.1) the moment condition (5a) is not needed, but second-order regular variation plays a crucial role in the proof. The bias condition (4) is not used.
- Lemma 7.2 does not require any distributional assumption on Z. Also, the moment conditions are not needed. Lemma 7.3 requires the moment assumption (5a). Only regular variation of Z is needed, second-order regular variation is not required. Lemma 7.4 again requires only (5a). In summary, tightness of the martingale part (Proposition 7.5) requires only regular variation and the moment condition (5a).
- Thus, weak convergence of the martingale part requires all assumptions except for (4).
- Weak convergence of the long memory part (Proposition 7.6) needs (5b) and second order regular variation with (4).

Example 5.1. Let the tail distribution function, $\bar{F}_Z$, be of the form:

$$\bar{F}_Z(x) = \begin{cases} \frac{1}{2}\left(x^{-\alpha} + x^{-\alpha\delta}\right) \ , & x \geq 1 \ , \\ \qquad 1 \ , & 0 < x < 1 \ , \end{cases}$$

where $\alpha > 0$, $\delta > 1$.

1. Notice that $\bar{F}_Z$ fulfills (2) and (3). In fact, for all $x \geq 1$,

$$\bar{F}_Z(x) = x^{-\alpha} \exp\left(\int_1^x \frac{\alpha(\delta-1)t^{-\alpha(\delta-1)-1}dt}{1+t^{-\alpha(\delta-1)}}\right) .$$

   Therefore, $\bar{F}_Z \in 2RV_\infty(-\alpha, -\alpha(\delta-1), \eta^*)$, where the rate function is defined by

$$\eta^*(x) = x^{-\alpha(\delta-1)} \frac{\alpha(\delta-1)}{1+x^{-\alpha(\delta-1)}} = \frac{\alpha(\delta-1)}{1+x^{\alpha(\delta-1)}} \underset{x\to\infty}{\sim} \alpha(\delta-1)x^{-\alpha(\delta-1)} .$$

   Note that η^* is nonnegative, regularly varying at infinity with index $-\alpha(\delta-1)$ and is bounded on $[1,\infty)$ by $\beta = \alpha(\delta-1)$.
2. Finally to get a sense of (4), denote by c a generic, nonnegative constant that can be different at each appearance. We will assume for simplicity that $m = 1$ and we will ignore the slowly varying function in the definition of $b_{n,1}$. Recall also that Z and X are tail equivalent and $k = n\bar{F}_X(u_n)$. Thus, in order to verify (4) is suffices to show that

$$a_{n,1}\eta^*(\bar{F}_X^{-1}(k/n)) \to 0 \ ,$$

or equivalently

$$\{\sqrt{k} + n^{1/2-d}\}\eta^*(\bar{F}_X^{-1}(k/n)) \to 0 \; .$$

Since $\bar{F}_X^{-1}(y) \sim cy^{-1/\alpha}$ as $y \to 0$ and η^* is regularly varying with index $-\alpha(\delta - 1)$, this in turn is equivalent to

$$\{\sqrt{k} + n^{1/2-d}\} \left(\frac{k}{n}\right)^{\delta-1} \to 0 \; .$$

Thus we get the following restrictions on the choice of k:

$$k = o\left(n^{1-\frac{1}{2\delta-1}}\right) \; , \quad k = o\left(n^{1-\frac{1/2-d}{\delta-1}}\right) \; .$$

The first restriction is the same as in i.i.d. case, while the second condition (stemming from long memory) may be more restrictive for small d. In summary, if $k = o(n^{1-\frac{1}{2\delta-2}})$, then bias is negligible.

6. Conclusion

In this article, we have considered the heavy-tailed long memory stochastic volatility model with leverage given in (1). We have studied the limiting behaviour of the tail empirical process with both fixed and random levels (Theorems 3.2 and 3.8). We have shown a dichotomous behaviour for the tail empirical process with fixed levels, according to the interplay between the long memory parameter d and the tail index α; leverage does not play a role in the limiting results, but makes the proofs technically involved. On the other hand, the tail empirical process with random levels is unaffected by either long memory or leverage. Further, we have proven the weak convergence of the corresponding integral functionals (Theorem 4.1). The tail empirical process with random levels is used to construct a family of estimators of the tail index, including the famous Hill estimator and harmonic mean estimators. Consequently, all HMEs of the tail index of $\{X_j\}$ remain valid for this model and have the same asymptotic behaviour as in the case of i.i.d. observations (Theorem 4.2).

There are several directions for further research on long memory stochastic volatility models with leverage. For example, the asymptotic behaviour of estimators of risk measures will be affected by long memory, since the scaled intermediate order statistics are affected by long memory (cf. Lemma 3.7). Other topics of interest include: detection of changes in the model and bootstrap techniques for the model.

7. Proofs

In this section, we gather the proofs of all our results. Recall that $k = n\bar{F}_X(u_n)$. Throughout the subsequent proofs, we define

$$\rho_n := \frac{X_{(n-k)}}{u_n} \cdot$$

7.1. Some technical results

We start with some technical results used in the proofs.

Potter's bounds Regular variation of $\bar{F}_Z$ and $\bar{F}_X$ yields different versions of Potter's bounds (cf. [27], [4, p. 25]). We state them the way they are used in this paper. First, for all $\epsilon > 0$, there exists $D(\epsilon) > 1$ such that $\forall\ x \geq 1$, $t > 0$,

$$J_x(t) \leq D(\epsilon) \max\left(1, t^{-(\alpha+\epsilon)}\right) . \tag{24}$$

Further $\forall\ C > 1$, $\epsilon > 0$, there exists $\delta = \delta(C;\epsilon) \geq 0$ such that for $x \geq \delta$, $t > 0$,

$$T_x(t) \leq C\left(t^{-(\alpha+\varepsilon)} \vee t^{-(\alpha-\varepsilon)}\right). \tag{25}$$

Notice that (9) in conjunction with (25) yield

$$\frac{\bar{F}_Z(xt)}{\bar{F}_X(x)} \leq \lambda C\left(t^{-(\alpha+\varepsilon)} \vee t^{-(\alpha-\varepsilon)}\right).$$

Derivatives It follows from (2) and (3) that there exists $M > 0$ such that for all $x \geq 1$ and $t > 0$,

$$J'_x(t) \underset{x\to\infty}{\to} T'(t) = -\alpha t^{-\alpha-1} \tag{26a}$$

$$|J'_x(t)| \leq M\left(t^{-(\alpha+\beta+1)} \vee t^{-(\alpha-\beta+1)}\right) . \tag{26b}$$

As a consequence, for all $x \geq 1$ and $t > 0$,

$$\frac{d}{dt}E\left(J_x(t/\phi(Y))\right) = E\left(\frac{d}{dt}J_x(t/\phi(Y))\right). \tag{27}$$

The bound in (26b) is transferred to $T_x(t)$. Indeed, we have

$$T_x(t) = \frac{\bar{F}_Z(x)}{\bar{F}_X(x)}E\left(\frac{\bar{F}_Z\left(xt/\phi(Y)\right)}{\bar{F}_Z(x)}\right) = \frac{\bar{F}_Z(x)}{\bar{F}_X(x)}E\left(J_x\left(t/\phi(Y)\right)\right).$$

By taking the derivative with respect to x and using (27), we obtain

$$T'_x(t) = \frac{\bar{F}_Z(x)}{\bar{F}_X(x)}\frac{d}{dt}E\left(J_x\left(t/\phi(Y)\right)\right) = \frac{\bar{F}_Z(x)}{\bar{F}_X(x)}E\left((1/\phi(Y))J'_x\left(t/\phi(Y)\right)\right).$$

Therefore, (26b) in conjunction with (9) yield that

$$|T'_x(t)| \leq \lambda M\left(t^{-(\alpha+\beta+1)} \vee t^{-(\alpha-\beta+1)}\right)\left(E\left([\phi(Y)]^{\alpha+\beta}\right) + E\left([\phi(Y)]^{\alpha-\beta}\right)\right).$$

Thus by (5a), $K_0 := \lambda M\left(E\left([\phi(Y)]^{\alpha+\beta}\right) + E\left([\phi(Y)]^{\alpha-\beta}\right)\right) < \infty$ and hence

$$|T'_x(t)| \leq K_0\left(t^{-(\alpha+\beta+1)} \vee t^{-(\alpha-\beta+1)}\right).$$

Second-order regular variation of X By (7), regular variation of Z is transferred to X. The same applies to second-order regular variation [21, p.117]. Indeed, we have $\bar{F}_X \in 2RV_\infty(-\alpha, -\kappa, \widetilde{\eta})$, with the rate function $\tilde{\eta}$ defined as

$$\tilde{\eta}(x) = \frac{E\left(\phi^\alpha(Y)\eta^*(x/\phi(Y))\ell^*(x/\phi(Y))\right)}{E\left(\phi^\alpha(Y)\ell^*(x/\phi(Y))\right)} \underset{x\to\infty}{\sim} \frac{E\left(\phi^{\alpha+\kappa}(Y)\right)}{E\left(\phi^\alpha(Y)\right)}\eta^*(x). \tag{28}$$

Bias Convergence in (6) or (8) induces bias in statistical inference. This bias is controlled with help of second-order regular variation. From [21] we have that for all $\epsilon > 0$, there exist $C(\epsilon) > 0$, $C_1(\epsilon) > 0$ such that for all $x \geq 1$, $t > 0$,

$$|J_x(t) - T(t)| \leq C(\epsilon)\left(t^{-(\alpha+\kappa+\epsilon)} \vee t^{-(\alpha+\kappa-\epsilon)}\right)|\eta^*(x)|\;, \tag{29a}$$

$$|T_x(t) - T(t)| \leq C_1(\epsilon)\left(t^{-(\alpha+\kappa+\epsilon)} \vee t^{-(\alpha+\kappa-\epsilon)}\right)|\tilde{\eta}(x)|\;. \tag{29b}$$

Thus, (4) and (29a) imply that for all $\tau_0 > 0$,

$$a_{n,m} \sup_{t>\tau_0} |J_{u_n}(t) - T(t)| \underset{n\to\infty}{\to} 0\;. \tag{30}$$

By (28) and (29b), the above bound also holds when J_{u_n} is replaced with T_{u_n} and $\bar{F}_Z$ with $\bar{F}_X$. Therefore, (10) is justified.

7.2. Proof of Theorem 3.2: TEP with deterministic levels

Recall that the proof of Theorem 3.2 is based on the martingale-long memory Doob decomposition (13). Therefore, Propositions 7.1, 7.5 and 7.6 below will prove Theorem 3.2.

7.2.1. Weak convergence of the martingale part

In this subsection, we state and prove in Proposition 7.1 finite dimensional convergence of $\{M_n\}$ defined in (14a), its tightness in Proposition 7.5. The proof of fidi convergence is straightforward, but tightness requires tedious and technical calculations due to dependence between $\{Y_j\}$ and $\{Z_j\}$.

Finite dimensional convergence

Proposition 7.1. *Let $\{X_j\}$ be the long memory stochastic volatility model with leverage given in (1). Then for any set of points $t_1, \dots, t_k > 0$,*

$$\left(\frac{M_n(t_i)}{\sqrt{k}}\right)_{1\leq i\leq k} \xrightarrow[n\to\infty]{d} (W \circ T(t_i))_{1\leq i\leq k}\,, \tag{31}$$

where W is a standard Brownian motion.

Proof. By the Cramér-Wold device, proving (31) is equivalent to showing:

$$\sum_{i=1}^{m} \frac{a_i M_n(t_i)}{\sqrt{k}} \xrightarrow[n\to\infty]{d} \sum_{i=1}^{m} a_i W \circ T(t_i), \tag{32}$$

for all $a_1, \ldots, a_m \in \mathbb{R}$ and $t_1, \ldots, t_m > 0$. We have

$$\begin{aligned}\sum_{i=1}^{m} a_i M_n(t_i) &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_i \triangle_j M_n(t_i) = \sum_{j=1}^{n}\sum_{i=1}^{m} a_i \triangle_j M_n(t_i) \\ &=: \sum_{j=1}^{n} \triangle_j M_n(t_1, \ldots, t_k),\end{aligned}$$

where $\triangle_j M_n(t) := V_{j,n}(t) - E(V_{j,n}(t) \| \mathcal{G}_{j-1})$ and $\triangle_j M_n^*(t) = \triangle_j M_n(t)/\sqrt{k}$. It is clear that $\{\triangle_j M_n\}$ is a martingale difference sequence. Now by Theorem 2.5 in [20], to prove (32) it is sufficient to show that for all $t_1, \ldots, t_m, \epsilon > 0$,

$$\sum_{j=1}^{n} E\left((\triangle_j M_n^*(t_1, \ldots, t_m))^2 \| \mathcal{G}_{j-1}\right) \xrightarrow[n\to\infty]{p} \sum_{i=1}^{m} a_i^2 t_i^{-\alpha} + 2\sum_{i<q} a_i a_q (t_i \vee t_q)^{-\alpha}, \tag{33a}$$

$$\sum_{j=1}^{n} E\left((\triangle_j M_n^*(t_1, \ldots, t_m))^2 \mathbb{1}_{\{|\triangle_j M_n^*(t_1,\ldots,t_m)| > \epsilon \| \mathcal{G}_{j-1}\}}\right) \xrightarrow[n\to\infty]{p} 0. \tag{33b}$$

To prove (33a), it is enough to show that for all t_i, $t_q > 0$,

$$\sum_{j=1}^{n} E\left(\triangle_j M_n^*(t_i) \triangle_j M_n^*(t_q) \| \mathcal{G}_{j-1}\right) \xrightarrow[n\to\infty]{p} (t_i \vee t_q)^{-\alpha}\ . \tag{34}$$

It follows from the definition of $\triangle_j M_n^*$ that

$$\begin{aligned}E\left(\triangle_j M_n^*(t_i) \triangle_j M_n^*(t_q) \| \mathcal{G}_{j-1}\right) &= \frac{1}{n\bar{F}_X(u_n)} \mathrm{Cov}\left(V_{j,n}(t_i), V_{j,n}(t_q) \| \mathcal{G}_{j-1}\right) \\ = \frac{E\left(V_{j,n}(t_i) V_{j,n}(t_q) \| \mathcal{G}_{j-1}\right)}{n\bar{F}_X(u_n)} &- \frac{E\left(V_{j,n}(t_i) \| \mathcal{G}_{j-1}\right) E\left(V_{j,n}(t_q) \| \mathcal{G}_{j-1}\right)}{n\bar{F}_X(u_n)}.\end{aligned}$$

Consequently, to establish (34) it is sufficient to prove

$$\frac{1}{n\bar{F}_X(u_n)} \sum_{j=1}^{n} E\left(V_{j,n}(t_i) V_{j,n}(t_q) \| \mathcal{G}_{j-1}\right) \xrightarrow[n\to\infty]{p} (t_i \vee t_q)^{-\alpha}, \tag{35a}$$

$$\frac{1}{n\bar{F}_X(u_n)} \sum_{j=1}^{n} E\left(V_{j,n}(t_i) \| \mathcal{G}_{j-1}\right) E\left(V_{j,n}(t_q) \| \mathcal{G}_{j-1}\right) \xrightarrow[n\to\infty]{p} 0. \tag{35b}$$

We start with (35a). Note that (11) in conjunction with (6) yield

$$\frac{1}{n\bar{F}_X(u_n)} \sum_{j=1}^{n} E(V_{j,n}(t_i) V_{j,n}(t_q) \| \mathcal{G}_{j-1}) = \frac{1}{n\bar{F}_X(u_n)} \sum_{j=1}^{n} \bar{F}_Z\left(u_n [\phi(Y_j)]^{-1} (t_i \vee t_q)\right)$$

$$= \underbrace{\frac{\bar{F}_Z(u_n)}{n\bar{F}_X(u_n)} \sum_{j=1}^{n} \left(J_{u_n}\left(t_i \vee t_q/\phi(Y_j)\right) - (t_i \vee t_q)^{-\alpha} \phi^{\alpha}(Y_j)\right)}_{B_n}$$

$$+ \underbrace{\frac{\bar{F}_Z(u_n)}{n(t_i \vee t_q)^{\alpha} \bar{F}_X(u_n)} \sum_{j=1}^{n} \phi^{\alpha}(Y_j)}_{A_n}.$$

Ergodicity, Slutsky's Theorem and (7) yield $A_n \underset{n\to\infty}{\to} (t_i \vee t_q)^{-\alpha}$, w.p.1.

On the other hand, by (7), showing that $B_n \xrightarrow[n\to\infty]{p} 0$, is equivalent to proving

$$B_n^* := \frac{1}{n} \sum_{j=1}^{n} \left(J_{u_n}\left(t_i \vee t_q/\phi(Y_j)\right) - T\left(t_i \vee t_q/\phi(Y_j)\right)\right) \xrightarrow[n\to\infty]{p} 0.$$

Stationarity of $\{Y_j\}$ and (29a) yield

$$\begin{aligned} E(|B_n^*|) &\leq E\left(|J_{u_n}\left(t_i \vee t_q/\phi(Y)\right) - T\left(t_i \vee t_q/\phi(Y_j)\right)|\right) \\ &\leq D(\epsilon)(t_i \vee t_q)^{-(\kappa+\alpha+\epsilon)} \vee (t_i \vee t_q)^{-(\kappa+\alpha-\epsilon)} |\eta^*(u_n)| \underset{n\to\infty}{\to} 0, \end{aligned}$$

where $D(\epsilon) = C(\epsilon)\left(E\left((\phi(Y))^{\kappa+\alpha+\epsilon}\right) + E\left((\phi(Y))^{\kappa+\alpha-\epsilon}\right)\right)$ is a constant depending on ϵ but not on n. Thus, $B_n^* \xrightarrow[n\to\infty]{p} 0$. Hence (35a) is proven.

Now, we deal with (35b). By (11), (6) and (24), we have $\forall\ \delta > 0$,

$$\begin{aligned} &\frac{1}{n\bar{F}_X(u_n)} \sum_{j=1}^{n} E\left(V_{j,n}(t_i) \| \mathcal{G}_{j-1}\right) E\left(V_{j,n}(t_q) \| \mathcal{G}_{j-1}\right) \\ &= \frac{\bar{F}_Z^2(u_n)}{n\bar{F}_X(u_n)} \sum_{j=1}^{n} J_{u_n}\left(t_i/\phi(Y_j)\right) J_{u_n}\left(t_q/\phi(Y_j)\right) \\ &\leq \frac{C(\delta)B(\delta)\bar{F}_Z^2(u_n)}{n\bar{F}_X(u_n)} \sum_{j=1}^{n} \left(1 \vee t_i^{-(\delta+\alpha)} \phi^{(\delta+\alpha)}(Y_j)\right) \left(1 \vee t_q^{-(\delta+\alpha)} \phi^{(\delta+\alpha)}(Y_j)\right) \\ &\leq \frac{C(\delta)B(\delta)\bar{F}_Z^2(u_n)}{n\bar{F}_X(u_n)} \\ &\quad \times \sum_{j=1}^{n} \left(1 + \left(t_i^{-(\alpha+\delta)} + t_q^{-(\alpha+\delta)}\right) \phi^{(\alpha+\delta)}(Y_j) + (t_i t_q)^{-(\alpha+\delta)} \phi^{2(\alpha+\delta)}(Y_j)\right) \\ &= \frac{\bar{F}_Z^2(u_n)}{\bar{F}_X(u_n)} \Bigg(R(\delta) + \frac{K(t_i,\delta) + G(t_q,\delta)}{n} \sum_{j=1}^{n} \phi^{(\delta+\alpha)}(Y_j) \\ &\quad + \frac{I(t_i,t_q,\delta)}{n} \sum_{j=1}^{n} \phi^{2(\delta+\alpha)}(Y_j) \Bigg), \end{aligned}$$

where $C(\delta)$, $B(\delta)$, $R(\delta)$, $K(t_i, \delta)$, $G(t_q, \delta)$ and $I(t_i, t_q, \delta)$ are constants depending on t_i, t_q and δ but not on n. By regular variation of $(X_j)_j$ and $(Z_j)_j$, ergodicity and Slutsky's Theorem, we conclude that (35b) holds. This concludes the proof of (33a). It remains to show (33b). We observe that

$$|\triangle_j M_n^*(t)| = \frac{|\triangle_j M_n^*(t)|}{\sqrt{k}} \leq \frac{1}{\sqrt{k}} \underset{n\to\infty}{\to} 0.$$

So, for arbitrary $\epsilon > 0, \mathbb{1}_{\{|\triangle_j M_n^*(t)|>\epsilon\}} = 0$, for all n sufficiently large. Thus, (33b) is proven. □

Tightness We consider for ease of notation the following setup:

$$V_{j,n}(s,t) := V_{j,n}(s) - V_{j,n}(t) = \mathbb{1}_{\{u_n s<\phi(Y_j)Z_j<u_n t\}}, \tag{36a}$$

$$\triangle_j M_n(s,t) := V_{j,n}(s,t) - E\left(V_{j,n}(s,t) \| \mathcal{G}_{j-1}\right), \tag{36b}$$

$$M_n(s,t) := M_n(s) - M_n(t) = \sum_{j=1}^{n} \triangle_j M_n(s,t), \tag{36c}$$

$$\triangle_j M_n^*(s,t) := \frac{\triangle_j M_n(s,t)}{\sqrt{k}} \text{ and } M_n^*(s,t) := \frac{M_n(s,t)}{\sqrt{k}}\ , \tag{36d}$$

for all $0 < s < t < \infty$. We state and prove Lemmas 7.2 to 7.4. These results will serve as ingredients for the proof of tightness in Proposition 7.5.

Lemma 7.2. *Let $\{X_j\}$ be the long memory stochastic volatility model with leverage given in (1). Then,*

$$\begin{aligned} E\left((M_n^*(s,t))^4\right) \quad &\leq \quad 2\lambda^2 C_4 E\left(\left(J_{u_n}\left(s/\phi(Y)\right) - J_{u_n}\left(t/\phi(Y)\right)\right)^2\right) \\ &\quad + \frac{16 C_4}{k}\left|T_{u_n}(s) - T_{u_n}(t)\right|\ , \end{aligned} \tag{37}$$

where C_4 is a constant defined in Rosenthal's inequality (cf. [18, p. 23-24]) and λ is as in (9).

Proof. Since $\{\triangle_j M_n\}$ is a martingale difference sequence, then Rosenthal's inequality ([18, p. 23-24]) holds and there exists a constant C_4 such that

$$\begin{aligned} &E\left((M_n^*(s,t))^4\right) \\ &\leq C_4\left(\sum_{j=1}^{n} E\left((\triangle_j M_n^*(s,t))^4\right) + E\left(\left(\sum_{j=1}^{n} E\left((\triangle_j M_n^*(s,t))^2 \|\mathcal{G}_{j-1}\right)\right)^2\right)\right). \end{aligned}$$

Note that if V is a nonnegative random variable, then for any σ-field $\mathcal{F}$, we have

$$E\left((V - E(V\|\mathcal{F}))^4\right) \leq 8E\left(V^4\right). \tag{38}$$

This along with stationarity of $\left(\triangle_j M_n^*(s,t)\right)_j$ yield

$$\sum_{j=1}^n E\left(\left(\triangle_j M_n^*(s,t)\right)^4\right) \le \frac{8E\left(\left(V_{1,n}(t,s)\right)^4\right)}{n\left(\bar{F}_X(u_n)\right)^2} \le \frac{8\left|T_{u_n}(s)-T_{u_n}(t)\right|}{n\bar{F}_X(u_n)}. \tag{39}$$

On the other hand, we have

$$\begin{aligned}
&E\left(\left(\sum_{j=1}^n E\left(\left(\triangle_j M_n^*(s,t)\right)^2 \|\mathcal{G}_{j-1}\right)\right)^2\right)\\
&= E\left(\sum_{j=1}^n \left(E\left(\left(\triangle_j M_n^*(s,t)\right)^2 \|\mathcal{G}_{j-1}\right)\right)^2\right)\\
&+ 2E\left(\sum_{i<j}^n E\left(\left(\triangle_i M_n^*(s,t)\right)^2 \|\mathcal{G}_{i-1}\right) E\left(\left(\triangle_j M_n^*(s,t)\right)^2 \|\mathcal{G}_{j-1}\right)\right).
\end{aligned}$$

First, stationarity of $\left(\triangle_j M_n^*(s,t)\right)_j$, Jensen's inequality, and (38) yield

$$\begin{aligned}
&E\left(\sum_{j=1}^n \left(E\left(\left(\triangle_j M_n^*(s,t)\right)^2 \|\mathcal{G}_{j-1}\right)\right)^2\right) = nE\left(\left(E\left(\left(\triangle_1 M_n^*(t,s)\right)^2 \|\mathcal{G}_0\right)\right)^2\right)\\
&\le nE\left(\left(\triangle_1 M_n^*(s,t)\right)^4\right) \le \frac{8E\left(V_{1,n}(t,s)\right)}{n\left(\bar{F}_X(u_n)\right)^2} = \frac{8\left|T_{u_n}(s)-T_{u_n}(t)\right|}{n\bar{F}_X(u_n)}.
\end{aligned}$$

Second, Cauchy-Schwartz's inequality and stationarity of $\left(\triangle_j M_n^*(s,t)\right)_j$ yield

$$\begin{aligned}
&2E\left(\sum_{i<j}^n E\left(\left(\triangle_i M_n^*(s,t)\right)^2 \|\mathcal{G}_{i-1}\right) E\left(\left(\triangle_j M_n^*(s,t)\right)^2 \|\mathcal{G}_{j-1}\right)\right)\\
&\le \frac{2n(n-1)}{\left(n\bar{F}_X(u_n)\right)^2} E\left(\left(E\left(\left(\triangle_1 M_n(s,t)\right)^2 \|\mathcal{G}_0\right)\right)^2\right)\\
&\le 2\left(\frac{\bar{F}_Z(u_n)}{\bar{F}_X(u_n)}\right)^2 E\left(\left(\frac{\bar{F}_Z\left(u_n s/\phi(Y)\right)}{\bar{F}_Z(u_n)} - \frac{\bar{F}_Z\left(u_n t/\phi(Y)\right)}{\bar{F}_Z(u_n)}\right)^2\right).
\end{aligned}$$

Therefore, using (9), it follows that for $n \ge 1$,

$$\begin{aligned}
&E\left(\left(\sum_{j=1}^n E\left(\left(\triangle_j M_n^*(s,t)\right)^2 \|\mathcal{G}_{j-1}\right)\right)^2\right) \le \frac{8\left|T_{u_n}(s)-T_{u_n}(t)\right|}{n\bar{F}_X(u_n)}\\
&+ 2\lambda^2 E\left(\left(J_{u_n}\left(s/\phi(Y)\right) - J_{u_n}\left(t/\phi(Y)\right)\right)^2\right).
\end{aligned} \tag{40}$$

Thus, (39) and (40) imply that (37) holds. $\square$

Lemma 7.3. *Let $\{X_j\}$ be the long memory stochastic volatility model with leverage given in (1). Then for all $0 < a \le s, t \le b < \infty$, there exists a positive constant $C_{a,b,4}$ such that*

$$E\left((M_n^*(s,t))^4\right) \le C_{a,b,4}\left(\frac{|s-t|}{n\bar{F}_X(u_n)} + (s-t)^2\right). \tag{41}$$

Proof. The mean value theorem in conjunction with (26b) yield that there exists $s < \tau = \tau(\omega) < t$, such that:

$$\begin{aligned}
&E\left((J_{u_n}(s/\phi(Y)) - J_{u_n}(t/\phi(Y)))^2\right)\\
&= (s-t)^2 E\left(\left(J'_{u_n}\left([\phi(Y)]^{-1}\tau\right)[\phi(Y)]^{-1}\right)^2\right)\\
&\le (M(s-t))^2\, E\left(\left(\left([\phi(Y)]^{-1}\tau\right)^{-(\alpha+\beta+1)} \vee \left([\phi(Y)]^{-1}\tau\right)^{-(\alpha-\beta+1)}[\phi(Y)]^{-1}\right)^2\right)\\
&\le (M(s-t))^2\, E\left(\left(\frac{[\phi(Y)]^{\alpha+\beta}}{\tau^{\alpha+\beta+1}} \vee \frac{[\phi(Y)]^{\alpha-\beta}}{\tau^{\alpha-\beta+1}}\right)^2\right) \le C_{a,b}\,(s-t)^2\ ,
\end{aligned}$$

where $C_{a,b} = \left(M\max\left(a^{-\alpha-1}\left(a^{-\beta}\vee b^{\beta}\right)\right)\right)^2 E\left(\left([\phi(Y)]^{\alpha+\beta} + [\phi(Y)]^{\alpha-\beta}\right)^2\right)$. The constant is finite by (5a). Hence, (37) becomes:

$$E\left((M_n^*(s,t))^4\right) \quad \le \quad \frac{16C_4}{n\bar{F}_X(u_n)}\,|T_{u_n}(s) - T_{u_n}(t)| + 2\lambda^2 C_4 C_{a,b}(s-t)^2\ .$$

Again by the mean value theorem, there exists $\tau^* \in (s,t)$ such that:

$$\begin{aligned}
E\left((M_n^*(s,t))^4\right) &\le \frac{16C_4}{n\bar{F}_X(u_n)}|s-t|T'_{u_n}(\tau^*) + 2\lambda^2 C_4 C_{a,b}(s-t)^2\\
&\le \frac{16C_4K_0}{n\bar{F}_X(u_n)}\left(a^{-\alpha-1}\left(a^{-\beta}\vee b^{\beta}\right)\right)|s-t| + 2\lambda^2 C_4 C_{a,b}(s-t)^2.
\end{aligned}$$

Hence, (41) follows by taking

$$C_{a,b,4} = \max\left(16C_4K_0\left(a^{-\alpha-1}\left(a^{-\beta}\vee b^{\beta}\right)\right), 2\lambda^2 C_4 C_{a,b}\right)\ . \qquad \square$$

Note now that the process M_n^* defined in (36d) satisfies

$$M_n^*(t) = M_n^\circ(t) - M_n^{\circ\circ}(t),$$

where M_n° and $M_n^{\circ\circ}$ are two non-increasing processes such that $\forall\ t > 0$,

$$M_n^\circ(t) = \frac{1}{\sqrt{k}}\sum_{j=1}^n V_{j,n}(t)\ ,\ M_n^{\circ\circ}(t) = \frac{1}{\sqrt{k}}\sum_{j=1}^n \bar{F}_Z\left(u_n t/\phi(Y_j)\right).$$

Lemma 7.4. *Let $\{X_j\}$ be the long memory stochastic volatility model with leverage in (1). Then*

$$\max_{0\le i\le l_n}\left|M_n^{\circ\circ}\left(a+t_{i+1,n}\right) - M_n^{\circ\circ}\left(a+t_{i,n}\right)\right| \xrightarrow[n\to\infty]{p} 0 \tag{42}$$

where $l_n = [(b-a)k]$, $t_{i,n} := i/k$ and $t_{l_n+1} := b-a$. Note that $[x]$ stands for the integer part of the real number x.

Proof. It follows from (6) that

$$M_n^{\circ\circ}(t) := \frac{1}{\sqrt{k}} \sum_{j=1}^{n} \bar{F}_Z\left(u_n t/\phi(Y_j)\right) = \sqrt{k}\frac{\bar{F}_Z(u_n)}{n\bar{F}_X(u_n)} \sum_{j=1}^{n} J_{u_n}\left(t/\phi(Y_j)\right).$$

Let $\Delta_i = M_n^{\circ\circ}(a+t_{i+1,n}) - M_n^{\circ\circ}(a+t_{i,n})$. It follows that for $0 \le i \le l_{n-1}$,

$$\begin{aligned}
&\left|M_n^{\circ\circ}(a+t_{i+1,n}) - M_n^{\circ\circ}(a+t_{i,n})\right| \\
&= \left|M_n^{\circ\circ}\left(a+\frac{i+1}{n\bar{F}_X(u_n)}\right) - M_n^{\circ\circ}\left(a+\frac{i}{n\bar{F}_X(u_n)}\right)\right| \\
&= \sqrt{k}\frac{\bar{F}_Z(u_n)}{n\bar{F}_X(u_n)} \sum_{j=1}^{n} \left|J_{u_n}\left(\frac{a+t_{i+1,n}}{\phi(Y_j)}\right) - J_{u_n}\left(\frac{a+t_{i,n}}{\phi(Y_j)}\right)\right| \\
&\le \frac{\lambda}{n} \sum_{j=1}^{n} \left|J_{u_n}\left(\frac{a+t_{i+1,n}}{\phi(Y_j)}\right) - J_{u_n}\left(\frac{a+t_{i,n}}{\phi(Y_j)}\right)\right| \sqrt{k},
\end{aligned}$$

where the last inequality holds by (9). The mean value theorem and (26b) yield that there exists $\tau_{i,n,j} = \tau_{i,n,j}(\omega) \in (t_{i,n}, t_{i,n+1})$ such that if $i \le l_{n-1}$,

$$\begin{aligned}
&\left|M_n^{\circ\circ}(a+t_{i+1,n}) - M_n^{\circ\circ}(a+t_{i,n})\right| \\
&\le \lambda \sum_{j=1}^{n} \frac{1}{n\bar{F}_X(u_n)\phi(Y_j)} \left|J'_{u_n}\left(\frac{a+\tau_{i,n,j}}{\phi(Y_j)}\right)\right| \sqrt{k} \\
&\le \frac{\lambda M}{n\sqrt{k}} \sum_{j=1}^{n} \frac{1}{\phi(Y_j)} \left(\left(\frac{a+\tau_{i,n,j}}{\phi(Y_j)}\right)^{-\alpha-\beta-1} \vee \left(\frac{a+\tau_{i,n,j}}{\phi(Y_j)}\right)^{-\alpha+\beta-1}\right) \\
&\le \frac{\lambda M}{\sqrt{k}} \max\left(a^{-\alpha-1}\left(a^{-\beta} \vee b^{\beta}\right)\right) \frac{1}{n} \sum_{j=1}^{n} \left(\phi^{\alpha+\beta}(Y_j) + \phi^{\alpha-\beta}(Y_j)\right) .
\end{aligned}$$

Consequently, we have

$$\max_{0 \le i \le l_{n-1}} |\Delta_i| \le \frac{\lambda M}{\sqrt{k}} \max\left(a^{-\alpha-1}\left(a^{-\beta} \vee b^{\beta}\right)\right) \left(\frac{1}{n} \sum_{j=1}^{n} \left(\phi^{\alpha+\beta}(Y_j) + \phi^{\alpha-\beta}(Y_j)\right)\right)$$

and the latter expression converges to zero in probability, by the Law of Large Numbers and (5a). For $i = l_n$, since $M_n^{\circ\circ}$ is monotone and $b < a + \frac{l_n+1}{n\bar{F}_X(u_n)}$, we obtain

$$\begin{aligned}
\left|M_n^{\circ\circ}\left(a+t_{l_{n+1},n}\right) - M_n^{\circ\circ}\left(a+t_{l_n,n}\right)\right| &= \left|M_n^{\circ\circ}(b) - M_n^{\circ\circ}\left(a+t_{l_n,n}\right)\right| \\
&\le \left|M_n^{\circ\circ}\left(a+\frac{l_n+1}{n\bar{F}_X(u_n)}\right) - M_n^{\circ\circ}\left(a+\frac{l_n}{n\bar{F}_X(u_n)}\right)\right|.
\end{aligned}$$

By the same argument as above the last term converges to zero in probability. □

We wrap up this subsection with the statement and proof of Proposition 7.5.

Proposition 7.5. *Let $\{X_j\}$ be the long memory stochastic volatility model with leverage in (1). The process M_n is tight in $D(0,\infty)$.*

Proof. The proof consists of verifying all three conditions of Theorem A.1, which is a restatement of Theorem 1.1 in [9, p. 2-3]. For this, let $\gamma = 4,\ \delta = 2,\ c_n = 1/n\bar{F}_X(u_n)$ and $\xi_n = M_n^*$. First, letting $s = a$ and $t \to \infty$ in the statement of Lemma 7.2 we obtain via (24)

$$\begin{aligned} E\left((M_n^*(a))^4\right) &\leq 2\lambda^2 C_4 E\left((J_{u_n}(a/\phi(Y)))^2\right) + \frac{16C_4}{n\bar{F}_X(u_n)} T_{u_n}(a) \\ &\leq 2\lambda^2 C_4 C^2(\beta)\left(1 + a^{-2(\alpha+\beta)} E\Big(\phi^{2(\alpha+\beta)}(Y)\Big)\right) + o(1). \end{aligned}$$

This proves (55). Second, from Lemma 7.3, if $|s-t| \geq c_n = 1/k$, then

$$E\left((M_n^*(s,t))^4\right) \leq C_{a,b,4}\left(\frac{|s-t|}{n\bar{F}_X(u_n)} + (s-t)^2\right) \leq 2C_{a,b,4}(s-t)^2\,.$$

This proves (56). Third, Lemma 7.4 yields (42) on the interval $[a,b]$. Hence, by Theorem A.1 and the remark following, the process M_n^* is tight in $D[a,b]$, where $0 < a < b < \infty$. Since a, b are arbitrary this implies tightness on $D(0,\infty)$. □

7.2.2. *Long memory part*

This subsection deals with the weak convergence of the long memory process L_n defined in (14b).

Proposition 7.6. *Let $\{X_j\}$ be the long memory stochastic volatility model with leverage given in (1).*

- *If $m(1-2d) < 1$, then*

$$\frac{n}{kb_{n,m}} L_n(t) \xrightarrow[n\to\infty]{d} \frac{\mu_{\phi,\alpha}(m)}{m!E\left(\phi^\alpha(Y)\right)} T(t)\xi_{m,d+1/2}(1) \ \text{ in } D(0,\infty). \tag{43}$$

- *If $m(1-2d) > 1$, then*

$$\frac{\sqrt{n}}{k} L_n(t) \xrightarrow[n\to\infty]{d} t^{-\alpha} \frac{\sigma\mathcal{N}}{E\left(\phi^\alpha(Y)\right)} \quad \text{in } D(0,\infty), \tag{44}$$

where the limiting variance is defined as follows:

$$\sigma^2 = \sum_{i=m}^{\infty} \frac{\mu^2(i)}{i!}\sigma_i^2 < \infty \ \text{ and } \ \sigma_i^2 = \lim_{n\to\infty} \operatorname{Var}\left(\frac{1}{\sqrt{n}}\sum_{j=1}^{n} H_i(Y_j)\right) \in (0,\infty).$$

Proof. Notice that $E\left(V_{1,n}(t)\right) = E\left(\bar{F}_Z(u_n t/\phi(Y))\right)$. By (14b) and (6), we have

$$\frac{L_n(t)}{\bar{F}_Z(u_n)} = \sum_{j=1}^{n} \left[J_{u_n}\left(t/\phi(Y_j)\right) - E\left(J_{u_n}\left(t/\phi(Y_j)\right)\right)\right] = \sum_{j=1}^{3} L_{n,j}(t), \tag{45}$$

where the summands $L_{n,j}$'s are respectively defined as follows:

$$L_{n,1}(t) := \sum_{j=1}^{n} \left[J_{u_n}\left(t/\phi(Y_j)\right) - T\left(t/\phi(Y_j)\right)\right], \tag{46a}$$

$$L_{n,2}(t) := \sum_{j=1}^{n} \left[T\left(t/\phi(Y_j)\right) - E\left(T\left(t/\phi(Y_j)\right)\right)\right], \tag{46b}$$

$$L_{n,3}(t) := \sum_{j=1}^{n} \left[E\left(T\left(t/\phi(Y_j)\right)\right) - E\left(J_{u_n}\left(t/\phi(Y_j)\right)\right)\right]. \tag{46c}$$

We start by establishing weak convergence of $L_{n,2}$. Note that (6) yields

$$L_{n,2}(t) = t^{-\alpha} \sum_{j=1}^{n} \left(\phi^{\alpha}(Y_j) - E(\phi^{\alpha}(Y_j))\right) = t^{-\alpha} \sum_{j=1}^{n} \psi_{\alpha}(Y_j),$$

where $\psi_{\alpha}(\cdot) := \phi^{\alpha}(\cdot) - E(\phi^{\alpha}(\cdot))$. This function is of Hermite rank m. Indeed,

$$\mu_{\phi,\alpha}(m) = E\left(H_m(Y)\psi_{\alpha}(Y)\right) = E\left(H_m(Y)\phi^{\alpha}(Y)\right).$$

By [1, p. 223, 229], if $m(1-2d) < 1$, then for $t > t_0 > 0$,

$$\frac{L_{n,2}(t)}{b_{n,m}} = \frac{t^{-\alpha}}{b_{n,m}} \sum_{j=1}^{n} \psi_{\alpha}(Y_j) \xrightarrow[n\to\infty]{d} \frac{\mu_{\phi,\alpha}(m)}{m!} T(t) \xi_{m,d+1/2}(1),$$

in the uniform topology on every compact subset of $(0,\infty)$. It remains to show that $L_{n,1} + L_{n,3}$ is negligible, when divided by $b_{n,m}$. By stationarity of $\{Y_j\}$, (29a), (4) and (5b), we have for every $t_0 > 0$,

$$\begin{aligned}
E&\left(\sup_{t>t_0} \frac{|L_{n,1}(t)|}{b_{n,m}}\right) \leq \frac{n}{b_{n,m}} E\left(\sup_{t>t_0} \left|J_{u_n}\left(t/\phi(Y)\right) - T\left(t/\phi(Y)\right)\right|\right) \\
&\leq C(\epsilon) E\left(\sup_{t>t_0} \left(\left(t/\phi(Y)\right)^{-(\kappa+\alpha+\epsilon)} \vee \left(t/\phi(Y)\right)^{-(\kappa+\alpha-\epsilon)}\right)\right) \frac{n}{b_{n,m}} |\eta^*(u_n)| \\
&\leq H(\epsilon) \sup_{t>t_0} \left(t^{-(\kappa+\alpha+\epsilon)} \vee t^{-(\kappa+\alpha-\epsilon)}\right) \frac{n}{b_{n,m}} |\eta^*(u_n)| \underset{n\to\infty}{\to} 0\,,
\end{aligned}$$

where $H(\epsilon) = C(\epsilon)\left(E\left(\phi^{\alpha+\kappa+\epsilon}(Y)\right) + E\left(\phi^{\alpha+\kappa-\epsilon}(Y)\right)\right) < \infty$ and does not depend on n. This allows us to conclude that $E\left(\sup_{t>t_0} \frac{|L_{n,1}(t)|}{b_{n,m}}\right) = o(1)$.

Finally, we deal with the third summand $L_{n,3}$. Notice that

$$\sup_{t>t_0} \frac{|L_{n,3}(t)|}{b_{n,m}} \leq E\left(\sup_{t>t_0} \frac{|L_{n,1}(t)|}{b_{n,m}}\right) \underset{n\to\infty}{\to} 0.$$

All in all, $(L_{n,1} + L_{n,3})/b_{n,m}$ is negligible on compact subsets of $(0, \infty)$. So,

$$\frac{L_n(t)}{b_{n,m}\bar{F}_Z(u_n)E\left(\phi^\alpha(Y)\right)} \xrightarrow[n\to\infty]{d} \frac{\mu_{\phi,\alpha}(m)}{m!E\left(\phi^\alpha(Y)\right)}T(t)\xi_{m,d+1/2}(1) \text{ in } D(0,\infty).$$

Slutsky's lemma and (7) complete the proof of (43) for the case $m(1-2d) < 1$. Now, assume that $m(1-2d) > 1$. We keep the same notation and the decompositions as for the previous case. By [1, p. 223, 229], if $m(1-2d) > 1$, then

$$\frac{L_{n,2}(t)}{\sqrt{n}} \xrightarrow[n\to\infty]{d} \sigma t^{-\alpha}\mathcal{N} \text{ in } D(0,\infty),$$

with $\sigma \in (0, \infty)$. Furthermore, it follows from (4) that

$$E\left(\sup_{t>t_0}\frac{|L_{n,1}(t)|}{\sqrt{n}}\right) \le K(\epsilon)\sup_{t>t_0}\left(t^{-(\rho+\alpha+\epsilon)} \vee t^{-(\rho+\alpha-\epsilon)}\right)\frac{n}{\sqrt{n}}|\eta^*(u_n)| \underset{n\to\infty}{\to} 0.$$

The corresponding argument applies to $L_{n,3}$. Thus,

$$\frac{L_n(t)}{\sqrt{n}\bar{F}_Z(u_n)E\left(\phi^\alpha(Y)\right)} \xrightarrow[n\to\infty]{d} t^{-\alpha}\frac{\sigma\mathcal{N}}{E\left(\phi^\alpha(Y)\right)} \text{ in } D(0,\infty).$$

Since $\bar{F}_X(u_n) \sim E\left(\phi^\alpha(Y)\right)\bar{F}_Z(u_n)$ (by Breiman lemma) and $\bar{F}_X(u_n) = k/n$, Slutsky's lemma finishes the proof of (44) for $m(1-2d) > 1$. □

7.2.3. Conclusion of proof of Theorem 3.2

Propositions 7.1 and 7.5 imply weak convergence of the martingale part, Proposition 7.6 gives weak convergence of the long memory part. The final statement of Theorem 3.2 comes from comparing the rates of convergence of the martingale and long memory parts. □

7.3. Proof of Theorems 3.8 and 4.1: TEP with random levels

In this section we present the proofs of Theorems 3.8 and 4.1.

Proof of Theorem 3.8. Recall the notation from Definition 3.1. The process $\widehat{S}_n$ defined in (17) can be decomposed as follows:

$$\widehat{S}_n(t) = \widehat{S}_{n,1}(t) + \widehat{S}_{n,2}(t) + \widehat{S}_{n,3}(t),$$

where the summands $\widehat{S}_{n,j}$'s are respectively defined as follows for $j = 1, 2, 3$:

$$\widehat{S}_{n,1}(t) = n\bar{F}_X(u_n)\left(\widetilde{T}_n\left(\rho_n t\right) - T_{u_n}\left(\rho_n t\right)\right), \tag{47a}$$

$$\widehat{S}_{n,2}(t) = n\bar{F}_X(u_n)\left(T_{u_n}\left(\rho_n t\right) - T\left(\rho_n t\right)\right), \tag{47b}$$

$$\widehat{S}_{n,3}(t) = n\bar{F}_X(u_n)\left(T\left(\rho_n t\right) - T(t)\right). \tag{47c}$$

Since $\widehat{S}_{n,1}(t) = \widetilde{S}_n(\rho_n t)$, $\widetilde{T}_n(\rho_n) = 1$ and $T(\rho_n t) = T(\rho_n)T(t)$, then

$$\begin{aligned}\widehat{S}_n(t) &= \left(\widetilde{S}_n(\rho_n t) - T(t)\widetilde{S}_n(\rho_n)\right) + T(t)\widetilde{S}_n(\rho_n) + \widehat{S}_{n,2}(t) + \widehat{S}_{n,3}(t) \qquad (48)\\ &= \left(\widetilde{S}_n(\rho_n t) - T(t)\widetilde{S}_n(\rho_n)\right) - n\bar{F}_X(u_n)T(t)\left(T_{u_n}(\rho_n) - T(\rho_n)\right) + \widehat{S}_{n,2}(t).\end{aligned}$$

The "martingale-long memory Doob decomposition" in (13) yields

$$\begin{aligned}&\frac{\widetilde{S}_n(\rho_n t) - T(t)\widetilde{S}_n(\rho_n)}{\sqrt{k}}\\ &\quad = \left(\frac{M_n(\rho_n t) - T(t)M_n(\rho_n)}{\sqrt{k}}\right) + \left(\frac{L_n(\rho_n t) - T(t)L_n(\rho_n)}{\sqrt{k}}\right).\end{aligned}$$

Since weak convergence to a continuous limit implies uniform convergence on compact sets, then by virtue of (31) and (18), we conclude that

$$\frac{M_n(\rho_n t) - T(t)M_n(\rho_n)}{\sqrt{k}} \xrightarrow[n\to\infty]{d} W(T(t)) - T(t)W(1) \text{ in } D(0,\infty),$$

since $M_n(\rho_n t)/\sqrt{k} \xrightarrow[n\to\infty]{d} W(T(t))$ in $D(0,\infty)$ and $M_n(\rho_n)/\sqrt{k} \xrightarrow[n\to\infty]{d} W(T(1))$.

It remains to show negligibility of the second term, that is

$$\sup_{t\ge t_0}\left(\frac{L_n(\rho_n t) - T(t)L_n(\rho_n)}{\sqrt{k}}\right) = o_P(1)\ .$$

For this purpose, recall the decomposition (45) and (46a), (46b) (46c).

$$\frac{L_n(\rho_n t) - T(t)L_n(\rho_n)}{\sqrt{k}} = \frac{\bar{F}_Z(u_n)}{\sqrt{k}}\sum_{j=1}^{3}\left(L_{n,j}(\rho_n t) - T(t)L_{n,j}(\rho_n)\right).$$

Notice that $L_{n,2}(\rho_n t) - T(t)L_{n,2}(\rho_n) = 0$. In what follows, set

$$\Theta_{n,j} := \sup_{t\ge t_0}\left(\frac{\bar{F}_Z(u_n)\,|L_{n,j}(\rho_n t) - T(t)L_{n,j}(\rho_n)|}{\sqrt{k}}\right),\ j = 1,3$$

We start dealing with negligibility of $\Theta_{n,1}$. We have

$$\begin{aligned}&|L_{n,1}(\rho_n t) - T(t)L_{n,1}(\rho_n)| \le \sum_{j=1}^{n}|J_{u_n}(\rho_n t/\phi(Y_j)) - T(\rho_n t/\phi(Y_j))|\\ &+ T(t)\sum_{j=1}^{n}|J_{u_n}(\rho_n/\phi(Y_j)) - T(\rho_n/\phi(Y_j))|\\ &\le nC(\epsilon)\Lambda(t)|\eta^*(u_n)| \qquad (50)\\ &\times \underbrace{\left(\rho_n^{-(\alpha+\kappa+\epsilon)}\left(\frac{1}{n}\sum_{j=1}^{n}\phi^{\delta_1}(Y_j)\right) + \rho_n^{-(\alpha+\kappa-\epsilon)}\left(\frac{1}{n}\sum_{j=1}^{n}\phi^{\delta_2}(Y_j)\right)\right)}_{\zeta_n},\end{aligned}$$

where $\delta_1 = \alpha + \kappa - \epsilon$, $\delta_2 = \alpha + \kappa + \epsilon$ and $\Lambda(t) := t^{-(\alpha+\kappa+\epsilon)} \vee t^{-(\alpha+\kappa-\epsilon)} + t^{-\alpha}$. Therefore using ergodicity, we obtain for all $t_0 > 0$,

$$\Theta_{n,1} \leq C(\epsilon) \sup_{t \geq t_0} \Lambda(t) \sqrt{\frac{\bar{F}_Z(u_n)}{\bar{F}_X(u_n)}} \sqrt{n\bar{F}_Z(u_n)} |\eta^*(u_n)| \zeta_n = o_P(1).$$

Furthermore since $\rho_n = 1 + o_P(1)$, so are $\rho_n^{-(\alpha+\kappa+\epsilon)}$ and $\rho_n^{-(\alpha+\kappa-\epsilon)}$, by the Continuous Mapping Theorem. Hence, $\zeta_n = O_P(1)$. On account of (7) and (4),

$$\sqrt{\frac{\bar{F}_Z(u_n)}{\bar{F}_X(u_n)}} \sqrt{n\bar{F}_Z(u_n)} |\eta^*(u_n)| = o(1).$$

As for the third term, we use (29a) and Jensen's inequality to get

$$|L_{n,3}(t)| \leq \lambda_0 n |\eta^*(u_n)| \left(t^{-(\alpha+\kappa+\epsilon)} \vee t^{-(\alpha+\kappa-\epsilon)} \right),$$

for some finite constant λ_0. Therefore, we have

$$|L_{n,3}(\rho_n t) - T(t) L_{n,3}(\rho_n)| \leq \lambda_0 n |\eta^*(u_n)| \Lambda(t) \left(\rho_n^{-(\alpha+\kappa+\epsilon)} \vee \rho_n^{-(\alpha+\kappa-\epsilon)} \right). \quad (51)$$

The same argument as the one used for $\Theta_{n,1}$ yields $\forall\ t_0 > 0$, $\Theta_{n,3} = o_P(1)$. □

Proof of Theorem 4.1. To prove (20), we need to check the assumptions of Theorem 25.5 in [3, p. 332]. As such, let $M \geq 1$. We have

$$\frac{1}{\sqrt{k}} \int_1^\infty \frac{\widehat{S}_n(t)}{t^r}\, dt = \frac{1}{\sqrt{k}} \int_1^M \frac{\widehat{S}_n(t)}{t^r}\, dt + \frac{1}{\sqrt{k}} \int_M^\infty \frac{\widehat{S}_n(t)}{t^r}\, dt\, .$$

Since the integral functionals are continuous only over compact intervals, then the continuous mapping theorem and (19) yield

$$\frac{1}{\sqrt{k}} \int_1^M \frac{\widehat{S}_n(t)}{t^r}\, dt \xrightarrow[n\to\infty]{d} \int_1^M \frac{W(T(t)) - W(1)T(t)}{t^r}\, dt.$$

Since $\alpha > 2(1-r)$, then we have

$$\begin{aligned} Var\left(\int_1^\infty \frac{W(T(t)) - W(1)T(t)}{t^r}\, dt\right) &= Var\left(\int_1^\infty \frac{W(T(t))}{t^r}\, dt\right) + \frac{Var(W(1))}{(\alpha+r-1)^2} \\ - \frac{2}{\alpha+r-1} Cov\left(W(1), \int_1^\infty \frac{W(T(t))}{t^r}\, dt\right) &= \frac{\alpha}{(\alpha+r-1)^2(\alpha+2r-2)} < \infty. \end{aligned}$$

This implies that the limiting process in (20) is Gaussian since it is a continuous linear functional of a Gaussian process. In addition, we have

$$\lim_{M\to\infty} Var\left(\int_M^\infty \frac{W(T(t)) - W(1)T(t)}{t^r}\, dt\right) = 0.$$

Therefore, we conclude the following

$$\int_1^M \frac{W\left(T(t)\right) - W(1)T(t)}{t^r}\, dt \xrightarrow[M\to\infty]{d} \int_1^\infty \frac{W\left(T(t)\right) - W(1)T(t)}{t^r}\, dt.$$

Thus, Theorem 25.5 in [3, p. 332] suggests that it remains to show $\forall\ \epsilon \geq 0$,

$$\lim_{M\to\infty} \limsup_{n\to\infty} P\left(\left|\frac{1}{\sqrt{k}} \int_M^\infty \frac{\widehat{S}_n(t)}{t^r}\, dt\right| \geq \epsilon\right) = 0. \tag{52}$$

For this purpose, we consider (48) and (47b). We have for all $\varepsilon \geq 0$,

$$\begin{aligned}
&\lim_{M\to\infty} \limsup_{n\to\infty} P\left(\left|\frac{1}{\sqrt{k}} \int_M^\infty \frac{\widehat{S}_n(t)}{t^r}\, dt\right| \geq \epsilon\right) \\
\leq\ &\lim_{M\to\infty} \limsup_{n\to\infty} P\left(\left|\int_M^\infty \frac{\widetilde{S}_n\left(\rho_n t\right) - T(t)\widetilde{S}_n\left(\rho_n\right)}{t^r\sqrt{k}}\, dt\right| \geq \frac{\varepsilon}{3}\right) \\
&+ \lim_{M\to\infty} \limsup_{n\to\infty} P\left(\left|\int_M^\infty \frac{n\bar{F}_X(u_n)T(t)\left(T_{u_n}\left(\rho_n\right) - T\left(\rho_n\right)\right)}{t^r\sqrt{k}}\, dt\right| \geq \frac{\varepsilon}{3}\right) \\
&+ \lim_{M\to\infty} \limsup_{n\to\infty} P\left(\left|\int_M^\infty \frac{\widehat{S}_{n,2}(t)}{t^r\sqrt{k}}\, dt\right| \geq \frac{\varepsilon}{3}\right)
\end{aligned}$$

So, (52) holds if these three upper bounds vanish. For ease of notation, let

$$\begin{aligned}
A_n^M :\ &= P\left(\left|\int_M^\infty \frac{\widetilde{S}_n(\rho_n t) - T(t)\widetilde{S}_n(\rho_n)}{t^r\sqrt{k}}\, dt\right| \geq \varepsilon\right) \\
&= P\left(|\rho_n|^{r-1}\left|\int_{M\rho_n}^\infty \frac{\widetilde{S}_n(v) - T\left(v/\rho_n\right)\widetilde{S}_n(\rho_n)}{v^r\sqrt{k}}\, dv\right| \geq \varepsilon\right)\ .
\end{aligned}$$

Since $\rho_n = 1 + o_P(1)$, then it suffices to deal with

$$\widetilde{A}_n^M = P\left(\left|\int_{M\rho_n}^\infty \frac{\widetilde{S}_n(v) - T\left(v/\rho_n\right)\widetilde{S}_n(\rho_n)}{v^r\sqrt{k}}\, dv\right| \geq \varepsilon\right).$$

We have $\widetilde{A}_n^M \leq \widetilde{A}_n^{M,1} + \widetilde{A}_n^{M,2}$, where $\widetilde{A}_n^{M,j}$, $j = 1, 2$ are defined by:

$$\widetilde{A}_n^{M,1} = P\left(\left|\int_{M\rho_n}^M \frac{\widetilde{S}_n(v) - T\left(v/\rho_n\right)\widetilde{S}_n(\rho_n)}{v^r\sqrt{k}}\, dv\right| \geq \frac{\varepsilon}{2}\right), \tag{53a}$$

$$\widetilde{A}_n^{M,2} = P\left(\left|\int_M^\infty \frac{\widetilde{S}_n(v) - T\left(v/\rho_n\right)\widetilde{S}_n(\rho_n)}{v^r\sqrt{k}}\, dv\right| \geq \frac{\varepsilon}{2}\right). \tag{53b}$$

We start with the summand defined in (53a). Let $\delta \geq 0$. We have

$$\widetilde{A}_n^{M,1} \leq P\left(\left|\int_{M\rho_n}^M \frac{\widetilde{S}_n(v) - T\left(v/\rho_n\right)\widetilde{S}_n(\rho_n)}{v^r\sqrt{k}}\, dv\right| \geq \frac{\varepsilon}{2},\ \ |\rho_n - 1| < \delta\right)$$

$$+ P\left(\left|\int_{M\rho_n}^{M} \frac{\widetilde{S}_n(v) - T\left(v/\rho_n\right)\widetilde{S}_n(\rho_n)}{v^r\sqrt{k}}\,dv\right| \geq \frac{\varepsilon}{2},\ \ |\rho_n - 1| \geq \delta\right)$$

$$\leq P\left(\int_{M(1-\delta)}^{M(1+\delta)} \left|\frac{\widetilde{S}_n(v) - T\left(v/\rho_n\right)\widetilde{S}_n(\rho_n)}{v^r\sqrt{k}}\right|\,dv \geq \frac{\varepsilon}{2}\right) + P\left(|\rho_n - 1| \geq \delta\right).$$

Since $\rho_n = 1 + o_P(1)$, then $\limsup_{n\to\infty} P\left(|\rho_n - 1| \geq \delta\right) = 0$. Therefore, we get

$$\lim_{M\to\infty}\limsup_{n\to\infty} \widetilde{A}_n^{M,1} \leq \frac{2}{\varepsilon}\lim_{M\to\infty}\int_{M(1-\delta)}^{M(1+\delta)} \frac{E\left(|W(T(v)) - T(v)W(1)|\right)}{v^r}\,dv$$

$$\leq \frac{2}{\varepsilon}\lim_{M\to\infty}\sqrt{2/\pi}\int_{M(1-\delta)}^{M(1+\delta)} \left(v^{-(\alpha/2+r)} + v^{-(\alpha+r)}\right)\,dv = 0,$$

as long as $\alpha > 2(1-r)$. Note that these two facts have been used in the last inequality: $W(T(v)) \stackrel{d}{=} v^{-\alpha/2}\mathcal{N}$ and $E|\mathcal{N}| = \sqrt{2/\pi}$. Thus, $\widetilde{A}_n^{M,1}$ is negligible.

Now, we deal with the summand defined in (53b). By virtue of (13), we have

$$\widetilde{A}_n^{M,2} \leq P\left(\left|\int_M^\infty \frac{M_n(v) - T\left(v/\rho_n\right)M_n(\rho_n)}{v^r\sqrt{k}}\,dv\right| \geq \frac{\varepsilon}{4}\right) \tag{54a}$$

$$+P\left(\left|\int_M^\infty \frac{L_n(v) - T\left(v/\rho_n\right)L_n(\rho_n)}{v^r\sqrt{k}}\,dv\right| \geq \frac{\varepsilon}{4}\right). \tag{54b}$$

We deal with the martingale part (54a) first. We have

$$P\left(\left|\int_M^\infty \frac{M_n(v) - T\left(v/\rho_n\right)M_n(\rho_n)}{v^r\sqrt{k}}\,dv\right| \geq \frac{\varepsilon}{4}\right) \quad\leq\quad B_n^{M,1} + B_n^{M,2},$$

where the summands $B_n^{M,j}$'s are respectively defined as follows for $j = 1, 2$:

$$B_n^{M,1} = P\left(\left|\int_M^\infty \frac{M_n(v)}{v^r\sqrt{k}}\,dv\right| \geq \frac{\varepsilon}{8}\right),$$

$$B_n^{M,2} = P\left(\left|\frac{M_n(\rho_n)}{\sqrt{k}}\right|\rho_n^\alpha\int_M^\infty \frac{dv}{v^{\alpha+r}} \geq \frac{\varepsilon}{8}\right).$$

Since $\{\Delta_j M_n\}$ is a stationary martingale difference sequence, then

$$B_n^{M,1} \leq \frac{64}{\varepsilon^2 k}\sum_{j=1}^n Var\left(\int_M^\infty \frac{\Delta_j M_n(v)}{v^r}\,dv\right) \leq \frac{64n}{\varepsilon^2 k}E\left(\left(\int_M^\infty \frac{\Delta_1 M_n(v)}{v^r}\,dv\right)^2\right)$$

$$\leq \frac{128}{\varepsilon^2}\int_M^\infty\left(\int_s^\infty \frac{E\left(V_{1,n}(t)\right)}{t^r\bar{F}_X(u_n)}\,dt\right)\frac{ds}{s^r}$$

$$= \frac{128}{\varepsilon^2}\int_M^\infty\left(\int_s^\infty \frac{E\left(\bar{F}_Z(u_n t/\phi(Y))\right)}{t^r\bar{F}_X(u_n)}\,dt\right)\frac{ds}{s^r}$$

$$\leq \frac{128\lambda Ck}{k\varepsilon^2}E\left(\int_M^\infty\int_s^\infty \frac{t^{-\alpha+\epsilon}\phi^{\alpha-\epsilon}(Y) \vee t^{-\alpha-\epsilon}\phi^{\alpha+\epsilon}(Y)}{t^r}\,dt\right)\frac{ds}{s^r}$$

$$\leq \frac{128\lambda C}{\varepsilon^2} E\left(\phi^{\alpha-\epsilon}(Y) + \phi^{\alpha+\epsilon}(Y)\right) \left(\int_M^\infty \int_s^\infty \frac{t^{-\alpha+\epsilon}}{t^r}\, dt\right) \frac{ds}{s^r}$$
$$= \frac{1}{\varepsilon^2} O(M^{-\alpha+\epsilon-2r+2}) \,.$$

As $M \to \infty$, the latter expression vanishes whenever $\alpha > 2(1-r)$ and $\varepsilon < \alpha - 2(1-r)$. Since $B_n^{M,1}$ decreases as ε increases, it is negligible for all $\varepsilon > 0$.

We can omit ρ_n^α when dealing with $B_n^{M,2}$ since $\rho_n = 1 + o_P(1)$. So,

$$\lim_{M\to\infty} \limsup_{n\to\infty} P\left(\left|\frac{M_n(\rho_n)}{\sqrt{k}}\right| > (\alpha+r-1)M^{\alpha+r-1}\varepsilon/8\right)$$
$$= \lim_{M\to\infty} P\left(|W(1)| > (\alpha+r-1)M^{\alpha+r-1}\varepsilon/8\right) = 0,$$

since $\alpha > 2(1-r) > 0$. So, the term $B_n^{M,2}$ is negligible.

Now, we deal with the long memory part, (54b). Recall (46a)-(46c). Then,

$$P\left(\left|\int_M^\infty \frac{L_n(v) - T\left(v/\rho_n\right) L_n(\rho_n)}{v^r\sqrt{k}}\, dv\right| \geq \frac{\varepsilon}{4}\right) \leq \sum_{j=1}^{3} I_{n,j}^M,$$

where the summands $I_{n,j}$, $j = 1, 2, 3$ are defined by:

$$I_{n,j}^M := P\left(\bar{F}_Z(u_n) \int_M^\infty \frac{|L_{n,j}(v) - T\left(v/\rho_n\right) L_{n,j}(\rho_n)|}{v^r\sqrt{k}}\, dv \geq \frac{\varepsilon}{12}\right).$$

Since $|L_{n,2}(\rho_n t) - T(t) L_{n,2}(\rho_n)| = 0$, then $I_{n,2}^M = 0$. Now consider $I_{n,1}^M$.

$$I_{n,1}^M \leq P\left(\frac{C(\varepsilon) n \bar{F}_Z(u_n) |\eta^*(u_n)|}{\sqrt{k}} \zeta_n \int_M^\infty \frac{\Lambda(v/\rho_n)}{v^r}\, dv \geq \frac{\varepsilon}{12}\right).$$

By ergodicity, $\zeta_n = O_P(1)$. Notice also that

$$\int_M^\infty \frac{\Lambda(v/\rho_n)}{v^r}\, dv \leq \rho_n^\alpha \int_M^\infty v^{-(\alpha+r)}\, dv + \rho_n^{\kappa+\alpha+\epsilon} \int_M^\infty v^{-(\kappa+\alpha+\epsilon+r)}\, dv$$
$$+ \rho_n^{\kappa+\alpha-\epsilon} \int_M^\infty v^{-(\kappa+\alpha-\epsilon+r)}\, dv = O_P(1) O(M^{-(\alpha+r)+\epsilon+1}) \,.$$

Since $\sqrt{k}|\eta^*(u_n)| \to 0$, as $n \to \infty$ and (7) holds, then we conclude that

$$\frac{C(\epsilon) n \bar{F}_Z(u_n) |\eta^*(u_n)|}{\sqrt{k}} \left(\int_M^\infty \frac{\Lambda(v/\rho_n)}{v^r}\, dv\right) \zeta_n \xrightarrow[n\to\infty]{p} 0.$$

This shows that $\lim_{n\to\infty} I_{n,1}^M = 0$, for all $M \geq 1$. Finally, consider $I_{n,3}^M$. By (51),

$$I_{n,3}^M = P\left(\bar{F}_Z(u_n) \int_M^\infty \frac{|L_{n,3}(v) - T\left(v/\rho_n\right) L_{n,3}(\rho_n)|}{v^r k} \geq \frac{\varepsilon}{12}\right)$$
$$\leq P\left(\frac{n\lambda_0 \bar{F}_Z(u_n)|\eta^*(u_n)|}{\sqrt{k}} \left(\rho_n^{-(\kappa+\alpha+\epsilon)} \vee \rho_n^{-(\kappa+\alpha-\epsilon)}\right) \int_M^\infty \frac{\Lambda(v/\rho_n)}{v^r}\, dv \geq \frac{\varepsilon}{12}\right).$$

Hence, $I^M_{n,3}$ is negligible by the same argument as for $I^M_{n,1}$. This means that (54b) is negligible. In summary, $\widetilde{A}^2_{n,M}$ is negligible.

To complete the proof, we are left to deal with the two remaining upper bounds of (52). Notice that $\sqrt{k}\,\{T_{u_n}(\rho_n) - T(\rho_n)\} = o_P(1)$, by (30). In addition, using the fact that $\int_M^\infty t^{-(\alpha+r)}dt < \infty$, we obtain

$$\lim_{M\to\infty} \limsup_{n\to\infty} P\left(\left|\int_M^\infty \frac{kT(t)\left(T_{u_n}(\rho_n) - T(\rho_n)\right)}{t^r\sqrt{k}}\, dt\right| \geq \frac{\varepsilon}{3}\right) = 0.$$

Recall (47b). The change of variable $z = \rho_n t$ and (29b) yield

$$\begin{aligned}\int_M^\infty \frac{\widehat{S}_{n,2}(t)}{t^r\sqrt{k}}\, dt &= \rho_n^{r-1}\sqrt{k}\int_{\rho_n M}^\infty \left|\frac{T_{u_n}(z) - T(z)}{z^r}\right|\, dz \\ &\leq C_1\rho_n^{r-1}\sqrt{k}|\tilde{\eta}(u_n)|\int_{\rho_n M}^\infty \left(z^{-\alpha-\kappa-r+\epsilon} \vee z^{-\alpha-\kappa-r-\epsilon}\right)\, dv \\ &= C_1\rho_n^{r-1}\sqrt{k}|\tilde{\eta}(u_n)|\frac{(\rho_n M)^{-(\alpha+\kappa+r-\epsilon)+1}}{\alpha+\kappa+r-\epsilon-1}.\end{aligned}$$

Since $\alpha > 2(1-r)$, $\alpha+\kappa+r-1 > 0$, $\rho_n = 1 + o_P(1)$ and $\sqrt{k}|\tilde{\eta}(u_n)| \to 0$, as $n \to \infty$, by (28) and (4), then we conclude that

$$\lim_{M\to\infty} \limsup_{n\to\infty} P\left(\left|\int_M^\infty \frac{\widehat{S}_{n,2}(t)}{t^r\sqrt{k}}\, dt\right| \geq \frac{\varepsilon}{3}\right) = 0. \qquad \square$$

Appendix A

Theorem A.1. *[9, p.2] Let $(\xi_n)_{n\geq 1}$ be real valued stochastic processes defined on $[0,1]$ and whose paths are in the Skorokhod space $D[0,1]$ almost surely. Furthermore, let all the finite dimensional distributions of $(\xi_n)_n$ converge, as $n \to \infty$, to the corresponding ones of a process ξ. Assume that there are constants $1 < \delta \leq \gamma$, $c > 0$, and a nonnegative sequence $c_n \to 0$, as $n \to \infty$ such that, for all $n \geq 1$, we have*

$$E\left(|\xi_n(0)|^\gamma\right) \leq c \tag{55}$$

$$E\left(|\xi_n(t) - \xi_n(s)|^\gamma\right) \leq c|t-s|^\delta, \tag{56}$$

whenever $|t-s| \geq c_n$. Furthermore, assume that the processes $(\xi_n)_n$ can be written as the differences of nondecreasing processes $(\xi_n^\circ)_n$ and $(\xi_n^{\circ\circ})_n$, and let the processes $(\xi_n^{\circ\circ})_n$ be such that:

$$\max_{j=1,\ldots,l_n} |\xi_n^{\circ\circ}(t_{j+1}) - \xi_n^{\circ\circ}(t_j)| \xrightarrow[n\to\infty]{p} 0,$$

where $t_j = jc_n$, for all $j = 0, 1, \ldots, l_n$ with $l_n := [1/c_n]$ and $t_{l_n+1} := 1$. Then the sequence of processes $(\xi_n)_n$ converges weakly to ξ in $D[0,1]$. Moreover, the limiting stochastic process ξ has continuous paths almost surely.

We note that the statement above is also valid for nonincreasing processes $(\xi_n^\circ)_n$ and $(\xi_n^{\circ\circ})_n$ and is easily extended to processes on any interval $[a,b]$.

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