Electron. Commun. Probab. **24** (2019), no. 52, 1–8.
https://doi.org/10.1214/19-ECP264
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Local nondeterminism and the exact modulus of continuity for stochastic wave equation*

Cheuk Yin Lee† Yimin Xiao‡

Abstract

We consider the linear stochastic wave equation driven by a Gaussian noise which is white in time and colored in space. We show that the solution satisfies a certain form of strong local nondeterminism and we use this property to derive the exact uniform modulus of continuity for the solution.

Keywords: stochastic wave equation; strong local nondeterminism; uniform modulus of continuity; Gaussian random field.
AMS MSC 2010: 60G15; 60G17; 60H15.
Submitted to ECP on May 8, 2019, final version accepted on August 23, 2019.

1 Introduction

Let $k \geq 1$ and $\beta \in (0, k \wedge 2)$, or $k = 1 = \beta$. We consider the linear stochastic wave equation

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u(t,x) = \Delta u(t,x) + \dot{W}(t,x), \quad t \geq 0, x \in \mathbb{R}^k, \\ u(0,x) = \dfrac{\partial}{\partial t} u(0,x) = 0. \end{cases} \tag{1.1}$$

Here, $\dot{W}$ is the space-time Gaussian white noise if $k = 1 = \beta$; and is a Gaussian noise that is white in time and has a spatially homogeneous covariance given by the Riesz kernel with exponent β if $k \geq 1$ and $\beta \in (0, k \wedge 2)$, i.e.

$$\mathbb{E}(\dot{W}(t,x)\dot{W}(s,y)) = \delta(t-s)|x-y|^{-\beta}.$$

The existence of real-valued process solution to (1.1) was discussed in [13, 4]. Regarding the sample path properties of the solution, results on the Hölder regularity and hitting probability have been proved in [5]. In this present paper, we determine the exact uniform modulus of continuity of the solution $u(t,x)$ in the time and space variables (t,x). For this purpose, we show that the Gaussian random field $\{u(t,x), t \geq 0, x \in \mathbb{R}^k\}$ satisfies a form of strong local nondeterminism.

*The research of Yimin Xiao is partially supported by NSF grants DMS-1607089 and DMS-1855185.
†Dept. Statistics and Probability, Michigan State University, East Lansing, MI 48824, USA.
E-mail: leecheu1@msu.edu
‡Dept. Statistics and Probability, Michigan State University, East Lansing, MI 48824, USA.
E-mail: xiao@stt.msu.edu

The property of local nondeterminism is useful for investigating sample paths of Gaussian random fields. This notion was first introduced by Berman [3] for Gaussian processes and extended by Pitt [11] for Gaussian random fields to study their local times. Later, the property of strong local nondeterminism was developed to study exact regularity of local times, small ball probability and other sample paths properties for Gaussian random fields (see, e.g., [15, 16]).

It is well known that the Brownian sheet does not satisfy the property of (strong) local nondeterminism (in the sense of Pitt [11]) but it satisfies sectorial local nondeterminism [7, Proposition 4.2]. Recall from [13, Theorem 3.1] that when $k = 1 = \beta$ and $\dot{W}$ is the space-time white noise, the solution $u(t, x)$ of (1.1) has the representation

$$u(t,x) = \frac{1}{2}\,\hat{W}\left(\frac{t-x}{\sqrt{2}}, \frac{t+x}{\sqrt{2}}\right), \tag{1.2}$$

where $\hat{W}$ is a modified Brownian sheet (cf. [13, p.281]). In this case, many properties of the solution $u(t, x)$ can be derived from those of $\hat{W}(t, x)$. For $\beta \neq 1$ or $k \geq 2$, there are few precise results (such as the exact modulus of continuity, modulus of non-differentiability, multifractal analysis of exceptional oscillations) for the sample function $u(t, x)$. Investigation of these problems naturally leads to the study of local nondeterminism for the solution $u(t, x)$.

In this paper, we investigate the property of local nondeterminism for the solution of (1.1) and use this property to study the uniform modulus of continuity of its sample functions. The main results of this paper are Proposition 2.1 and Theorem 3.1. Proposition 2.1 shows that for a general dimension k, the solution $u(t, x)$ satisfies an integral form of local nondeterminism. When $k = 1$ and $\beta = 1$, this property (see (2.4) below) can also be derived from the sectorial local nondeterminism for the Brownian sheet in [7, Proposition 4.2] after a change of coordinates. While for $k = 1$ and $\beta \in (0, 1)$, the property (2.4) is similar to the sectorial local nondeterminism in [14, Theorem 1] for a fractional Brownian sheet, which suggests that the sample function $u(t, x)$ may have some subtle properties that are different from those of Gaussian random fields with stationary increments (an important example of the latter is fractional Brownian motion). We believe that Proposition 2.1 is useful for studying precise regularity and other sample path properties of $u(t, x)$. In Theorem 3.1, we apply Proposition 2.1 to derive the exact uniform modulus of continuity of $u(t, x)$.

2 Local nondeterminism

Let G be the fundamental solution of the wave equation. Recall that if $k = 1$, $G(t, x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}}$; if $k \geq 2$ and k is even,

$$G(t,x) = c_k \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(k-2)/2} (t^2 - |x|^2)_+^{-1/2};$$

if $k \geq 3$ and k is odd,

$$G(t,x) = c_k \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(k-3)/2} \frac{\sigma_t^k(dx)}{t},$$

where σ_t^k is the uniform surface measure on the sphere $\{x \in \mathbb{R}^k : |x| = t\}$, see [6, Chapter 5]. Note that for $k \geq 3$, G is not a function but a distribution. Also recall that for any dimension $k \geq 1$, the Fourier transform of G in variable x is given by

$$\mathscr{F}(G(t,\cdot))(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \quad t \geq 0, \xi \in \mathbb{R}^k. \tag{2.1}$$

In [4], Dalang extended Walsh's stochastic integration and proved that the real-valued process solution of equation (1.1) is given by

$$u(t,x) = \int_0^t \int_{\mathbb{R}^k} G(t-s, x-y)\, W(ds\, dy),$$

where W is the martingale measure induced by the noise $\dot{W}$. The range of β has been chosen so that the stochastic integral exists. Recall from Theorem 2 of [4] that

$$\mathbb{E}\left[\left(\int_0^t \int_{\mathbb{R}^k} H(s,y) W(ds\, dy)\right)^2\right] = c_{k,\beta} \int_0^t ds \int_{\mathbb{R}^k} d\xi\, |\xi|^{\beta-k} |\mathscr{F}(H(s,\cdot))(\xi)|^2 \tag{2.2}$$

provided that $s \mapsto H(s,\cdot)$ is a deterministic function with values in the space of nonnegative distributions with rapid decrease and

$$\int_0^t ds \int_{\mathbb{R}^k} d\xi\, |\xi|^{\beta-k} |\mathscr{F}(H(s,\cdot)(\xi)|^2 < \infty.$$

The following result shows that the solution $u(t,x)$ satisfies a certain form of strong local nondeterminism.

Proposition 2.1. *Let $0 < a < a' < \infty$ and $0 < b < \infty$. There exist constants $C > 0$ and $\delta > 0$ depending on a, a' and b such that for all integers $n \geq 1$ and all $(t,x), (t^1,x^1), \ldots, (t^n,x^n)$ in $[a,a'] \times [-b,b]^k$ with $|t-t^j| + |x-x^j| \leq \delta$, we have*

$$\mathrm{Var}\left(u(t,x)|u(t^1,x^1), \ldots, u(t^n,x^n)\right) \geq C \int_{\mathbb{S}^{k-1}} \min_{1\leq j\leq n} |(t-t^j) + (x-x^j)\cdot w|^{2-\beta}\, dw, \tag{2.3}$$

where dw is the surface measure on the unit sphere $\mathbb{S}^{k-1}$.

Remark 2.2. When $k = 1$, the surface measure dw in (2.3) is supported on $\{-1, 1\}$. It follows that $u(t,x)$ satisfies sectorial local nondeterminism:

$$\begin{aligned}&\mathrm{Var}(u(t,x)|u(t^1,x^1), \ldots, u(t^n,x^n))\\ &\geq C\left(\min_{1\leq j\leq n} |(t-t^j) + (x-x^j)|^{2-\beta} + \min_{1\leq j\leq n} |(t-t^j) - (x-x^j)|^{2-\beta}\right).\end{aligned} \tag{2.4}$$

When $\beta = 1$, this can be derived from (1.2) and Proposition 4.2 in [7] by a change of coordinates $(t,x) \mapsto (t+x, t-x)$. When $\beta \neq 1$, (2.4) is similar to Theorem 1 in [14] for a fractional Brownian sheet, after the change of coordinates.[1] We remark that (2.4) is different from the strong local nondeterminism for Gaussian random fields with stationary increments in [8]. This suggests that the solution process $u(t,x)$ may have some subtle properties that are different from those of Gaussian random fields with stationary increments such as a fractional Brownian motion.

Proof of Proposition 2.1. Take $\delta = a/2$. For each $w \in \mathbb{S}^{k-1}$, let

$$r(w) = \min_{1\leq j\leq n} |(t^j - t) - (x^j - x)\cdot w|.$$

Since u is a centered Gaussian random field, the conditional variance

$$\mathrm{Var}(u(t,x)|u(t^1,x^1), \ldots, u(t^n,x^n))$$

[1] Professor Ciprian Tudor showed us that the relation (1.2) still holds if $\hat{W}$ is replaced by an appropriate Gaussian random field related to a fractional Brownian sheet. This connection provides an explanation for the similarity between (2.4) and Theorem 1 in [14].

is the squared distance of $u(t,x)$ from the linear subspace spanned by $u(t^1,x^1),\dots,$ $u(t^n,x^n)$ in $L^2(\mathbb{P})$. Thus, it suffices to show that there exist constants $C>0$ and $\delta>0$ such that for all $(t,x),(t^1,x^1),\dots,(t^n,x^n)$ in $[a,a']\times[-b,b]^k$ with $|t-t^j|+|x-x^j|\le\delta$, we have

$$\mathbb{E}\bigg[\bigg(u(t,x)-\sum_{j=1}^n \alpha_j u(t^j,x^j)\bigg)^2\bigg]\ge C\int_{\mathbb{S}^{k-1}} r(w)^{2-\beta}\,dw \tag{2.5}$$

for any choice of real numbers $\alpha_1,\dots,\alpha_n$. Using (2.1), (2.2) and spherical coordinate $\xi=\rho w$, we have

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg(u(t,x)-\sum_{j=1}^n \alpha_j u(t^j,x^j)\bigg)^2\bigg]\\
&= c_{k,\beta}\int_0^\infty ds\int_{\mathbb{R}^k}\frac{d\xi}{|\xi|^{2+k-\beta}}\bigg|\sin((t-s)|\xi|)\mathbf{1}_{[0,t]}(s)-\sum_{j=1}^n\alpha_j e^{-i(x^j-x)\cdot\xi}\sin((t^j-s)|\xi|)\mathbf{1}_{[0,t^j]}(s)\bigg|^2\\
&\ge c_{k,\beta}\int_0^{a/2}ds\int_0^\infty\frac{d\rho}{\rho^{3-\beta}}\int_{\mathbb{S}^{k-1}}dw\bigg|\sin((t-s)\rho)-\sum_{j=1}^n\alpha_j e^{-i\rho(x^j-x)\cdot w}\sin((t^j-s)\rho)\bigg|^2\\
&=\frac{c_{k,\beta}}{8}\int_0^{a/2}ds\int_{-\infty}^\infty\frac{d\rho}{|\rho|^{3-\beta}}\int_{\mathbb{S}^{k-1}}dw\bigg|\Big(e^{i(t-s)\rho}-e^{-i(t-s)\rho}\Big)\\
&\qquad\qquad -\sum_{j=1}^n\alpha_j e^{-i\rho(x^j-x)\cdot w}\Big(e^{i(t^j-s)\rho}-e^{-i(t^j-s)\rho}\Big)\bigg|^2\\
&=:\frac{c_{k,\beta}}{8}\int_{\mathbb{S}^{k-1}}A(w)\,dw.
\end{aligned}$$

Let $\lambda=\min\{1,a/[2(a'+2\sqrt{k}b)]\}$ and consider the bump function $\varphi:\mathbb{R}\to\mathbb{R}$ defined by

$$\varphi(y)=\begin{cases}\exp\Big(1-\frac{1}{1-|\lambda^{-1}y|^2}\Big), & |y|<\lambda,\\ 0, & |y|\ge\lambda.\end{cases}$$

Let $\varphi_r(y)=r^{-1}\varphi(y/r)$. For each $w\in\mathbb{S}^{k-1}$ such that $r(w)>0$, consider the integral

$$\begin{aligned}
I(w):=\int_0^{a/2}ds\int_{-\infty}^\infty d\rho\bigg[&\Big(e^{i(t-s)\rho}-e^{-i(t-s)\rho}\Big)\\
&-\sum_{j=1}^n\alpha_j e^{-i\rho(x^j-x)\cdot w}\Big(e^{i(t^j-s)\rho}-e^{-i(t^j-s)\rho}\Big)\bigg]e^{-i(t-s)\rho}\widehat{\varphi}_{r(w)}(\rho).
\end{aligned}$$

By the inverse Fourier transform (or one can apply the Plancherel theorem), we have

$$\begin{aligned}
I(w)=2\pi\int_0^{a/2}ds\bigg[&\varphi_{r(w)}(0)-\varphi_{r(w)}\big(2(t-s)\big)\\
&-\sum_{j=1}^n\alpha_j\Big(\varphi_{r(w)}\big((x^j-x)\cdot w-(t^j-t)\big)-\varphi_{r(w)}\big((x^j-x)\cdot w-(t^j-t)+2(t^j-s)\big)\Big)\bigg].
\end{aligned}$$

Note that $r(w)\le|t^j-t|+|x^j-x|\le a'+2\sqrt{k}b$. For any $s\in[0,a/2]$, we have $2(t-s)/r(w)\ge a/[(a'+2\sqrt{k}b)]$ and $|(x^j-x)\cdot w-(t^j-t)|/r(w)\ge1$, thus

$$\varphi_{r(w)}\big(2(t-s)\big)=0\ \text{ and }\ \varphi_{r(w)}\big((x^j-x)\cdot w-(t^j-t)\big)=0\ \text{ for } j=1,\dots,n.$$

Also, $[(x^j-x)\cdot w-(t^j-t)+2(t^j-s)]/r(w)\ge(-\delta+a)/[(a'+2\sqrt{k}b)]\ge\lambda$, thus

$$\varphi_{r(w)}\big((x^j-x)\cdot w-(t^j-t)+2(t^j-s)\big)=0.$$

It follows that

$$I(w) = a\pi\, r(w)^{-1}.$$

On the other hand, by the Cauchy–Schwarz inequality and scaling, we obtain

$$\begin{aligned}(a\pi)^2 r(w)^{-2} = |I(w)|^2 &\leq A(w) \times \int_0^{a/2} ds \int_{-\infty}^{\infty} d\rho\, |\widehat{\varphi}(r(w)\rho)|^2 |\rho|^{3-\beta} \\ &= (a/2)A(w)r(w)^{\beta-4} \int_{-\infty}^{\infty} d\rho\, |\widehat{\varphi}(\rho)|^2 |\rho|^{3-\beta} \\ &= CA(w)r(w)^{\beta-4}\end{aligned}$$

for some finite constant C. Hence we have

$$A(w) \geq C' r(w)^{2-\beta} \tag{2.6}$$

and this remains true if $r(w) = 0$. Integrating both sides of (2.6) over $\mathbb{S}^{k-1}$ yields (2.5). □

3 Exact uniform modulus of continuity

It is known that sectorial local nondeterminism is useful for proving the exact uniform modulus of continuity for Gaussian random fields [10]. In this section we show that the form of local nondeterminism in Proposition 2.1 can serve the same purpose for deriving the exact uniform modulus of continuity of $u(t,x)$.

Let us denote

$$\sigma\big[(t,x),(t',x')\big] = \mathbb{E}[(u(t,x) - u(t',x'))^2]^{1/2}.$$

Recall from [5, Proposition 4.1] that for any $0 < a < a' < \infty$ and $0 < b < \infty$, there are positive constants C_1 and C_2 such that

$$C_1\left(|t-t'| + \sum_{j=1}^k |x_j - x_j'|\right)^{2-\beta} \leq \sigma[(t,x),(t',x')]^2 \leq C_2\left(|t-t'| + \sum_{j=1}^k |x_j - x_j'|\right)^{2-\beta} \tag{3.1}$$

for all $(t,x),(t',x') \in [a,a'] \times [-b,b]^k$.

The following result establishes the exact uniform modulus of continuity of $u(t,x)$ in the time and space variables (t,x).

Theorem 3.1. *Let $I = [a,a'] \times [-b,b]^k$, where $0 < a < a' < \infty$ and $0 < b < \infty$. Let*

$$\gamma\big[(t,x),(t',x')\big] = \sigma\big[(t,x),(t',x')\big]\sqrt{\log\big(1 + \sigma\big[(t,x),(t',x')\big]^{-1}\big)}.$$

Then there is a positive finite constant K such that

$$\lim_{\varepsilon\to 0+} \sup_{\substack{(t,x),(t',x')\in I,\\ \sigma[(t,x),(t',x')]\leq\varepsilon}} \frac{|u(t,x) - u(t',x')|}{\gamma\big[(t,x),(t',x')\big]} = K, \quad \text{a.s.} \tag{3.2}$$

Proof. For any $\varepsilon > 0$, let

$$J(\varepsilon) = \sup_{\substack{(t,x),(t',x')\in I,\\ \sigma[(t,x),(t',x')]\leq\varepsilon}} \frac{|u(t,x) - u(t',x')|}{\gamma\big[(t,x),(t',x')\big]}.$$

Since $\varepsilon \mapsto J(\varepsilon)$ is non-decreasing, we see that the limit $\lim_{\varepsilon\to 0+} J(\varepsilon)$ exists a.s. In order to prove (3.2), we prove the following statements: there exist positive and finite constants K^* and K_* such that

$$\lim_{\varepsilon\to 0+} J(\varepsilon) \leq K^*, \quad \text{a.s.} \tag{3.3}$$

and

$$\lim_{\varepsilon\to 0+} J(\varepsilon) \ge K_*, \quad \text{a.s.} \tag{3.4}$$

Then the conclusion of Theorem 3.1 follows from Lemma 7.1.1 of [9] where τ is chosen to be the Euclidean metric and d is the canonical metric $\sigma[(t,x),(t',x')]$. [It is a 0-1 law for the modulus of continuity which is obtained by applying Kolmogorov's 0-1 law to the Karhunen–Loève expansion of $u(t,x)$.]

The proof of the upper bound (3.3) is standard. For any $\varepsilon > 0$, denote by $N(I,\varepsilon,\sigma)$ the smallest number of balls of radius ε in the canonical metric $\sigma\big[(t,x),(t',x')\big]$ that are needed to cover the compact interval I. By the upper bound in (3.1), we have $N(I,\varepsilon,\sigma) \le C\varepsilon^{-(1+k)/(2-\beta)}$ and thus

$$\int_0^{\varepsilon} \sqrt{\log N(I,\tilde\varepsilon,\sigma)}\, d\tilde\varepsilon \le C\varepsilon\sqrt{\log(1+\varepsilon^{-1})}.$$

By Theorem 1.3.5 of [1], there is a positive finite constant K^* such that

$$\limsup_{\varepsilon\to 0+} \sup_{\substack{(t,x),(t',x')\in I,\\ \sigma[(t,x),(t',x')]\le\varepsilon}} \frac{|u(t,x)-u(t',x')|}{\varepsilon\sqrt{\log(1+\varepsilon^{-1})}} \le K^* \quad \text{a.s.}$$

From this we can deduce (3.3) by considering $\varepsilon_{n+1} \le \sigma[(t,x),(t',x')] \le \varepsilon_n$ where $\varepsilon_n = 1/n$, and using the fact that the function $\varepsilon \mapsto \varepsilon\sqrt{\log(1+\varepsilon^{-1})}$ is increasing for ε small, and

$$\lim_{n\to\infty} \frac{\varepsilon_n\sqrt{\log(1+\varepsilon_n^{-1})}}{\varepsilon_{n+1}\sqrt{\log(1+\varepsilon_{n+1}^{-1})}} = 1.$$

Next we prove the lower bound (3.4). This is accomplished by applying Proposition 2.1, a conditioning argument and the Borel–Cantelli lemma. We first choose δ according to Proposition 2.1 and let $\delta' = \min\{\delta/(1+\sqrt{k}), a'-a, 2b\}$. Note that δ' depends only on a, a' and b. For each $n \ge 1$, let

$$\varepsilon_n = [C_2((1+k)\delta')^{2-\beta} 2^{-(2-\beta)n}]^{1/2}.$$

For $i = 0,1,\dots,2^n$, let $t^{n,i} = a + i\delta' 2^{-n}$ and $x_j^{n,i} = -b + i\delta' 2^{-n}$. Then

$$\begin{aligned}
\lim_{\varepsilon\to 0+} J(\varepsilon) &= \lim_{n\to\infty} \sup_{\substack{(t,x),(t',x')\in I,\\ \sigma[(t,x),(t',x')]\le\varepsilon_n}} \frac{|u(t,x)-u(t',x')|}{\gamma[(t,x),(t',x')]} \\
&\ge \liminf_{n\to\infty} \max_{1\le i\le 2^n} \frac{|u(t^{n,i},x^{n,i})-u(t^{n,i-1},x^{n,i-1})|}{\varepsilon_n\sqrt{\log(1+\varepsilon_n^{-1})}} \\
&=: \liminf_{n\to\infty} J_n.
\end{aligned}$$

To obtain the inequality, we have used the fact that $\sigma[(t^{n,i},x^{n,i}),(t^{n,i-1},x^{n,i-1})] \le \varepsilon_n$ and that the function $\varepsilon \mapsto \varepsilon\sqrt{\log(1+\varepsilon^{-1})}$ is increasing for ε small.

Let $K_* > 0$ be a constant whose value will be determined later. Fix n and write $t^{n,i} = t^i$, $x^{n,i} = x^i$ to simplify notations. By conditioning, we can write

$$\begin{aligned}
&\mathbb{P}\left(J_n \le K_*\right) \\
&= \mathbb{P}\left(\max_{1\le i\le 2^n} \frac{|u(t^i,x^i)-u(t^{i-1},x^{i-1})|}{\varepsilon_n\sqrt{\log(1+\varepsilon_n^{-1})}} \le K_* \right) \\
&= \mathbb{E}\left[\mathbf{1}_A \mathbb{P}\left(\frac{|u(t^{2^n},x^{2^n})-u(t^{2^n-1},x^{2^n-1})|}{\varepsilon_n\sqrt{\log(1+\varepsilon_n^{-1})}} \le K_* \middle| u(t^i,x^i) : 0\le i\le 2^n-1 \right)\right],
\end{aligned} \tag{3.5}$$

where A is the event defined by

$$A = \left\{ \max_{1 \le i \le 2^n - 1} \frac{|u(t^i, x^i) - u(t^{i-1}, x^{i-1})|}{\varepsilon_n \sqrt{\log(1 + \varepsilon_n^{-1})}} \le K_* \right\}.$$

Since $|t^{2^n} - t^i| + |x^{2^n} - x^i| \le \delta$, by Proposition 2.1 we have

$$\begin{aligned}
&\mathrm{Var}\left(u(t^{2^n}, x^{2^n}) | u(t^i, x^i) : 0 \le i \le 2^n - 1\right) \\
&\ge C \int_{\mathbb{S}^{k-1}} \min_{0 \le i \le 2^n - 1} |(t^{2^n} - t^i) + (x^{2^n} - x^i) \cdot w|^{2-\beta} \, dw \\
&\ge C \int_{\{w \in \mathbb{S}^{k-1} :\, (1,\ldots,1)\cdot w \ge 0\}} \min_{0 \le i \le 2^n - 1} |\delta'(2^n - i) 2^{-n} + \delta'(2^n - i) 2^{-n} (1, \ldots, 1) \cdot w|^{2-\beta} \, dw \\
&\ge C (\delta')^{2-\beta} \, 2^{-(2-\beta)n} \int_{\{w \in \mathbb{S}^{k-1} :\, (1,\ldots,1)\cdot w \ge 0\}} dw \\
&= C_0 \, \varepsilon_n^2
\end{aligned} \tag{3.6}$$

for some constant $C_0 > 0$ depending on a, a' and b.

Since the conditional distribution of $u(t^{2^n}, x^{2^n})$, given $u(t^i, x^i)$, $(0 \le i \le 2^n - 1)$, is Gaussian with conditional variance $\mathrm{Var}\left(u(t^{2^n}, x^{2^n}) | u(t^i, x^i) : 0 \le i \le 2^n - 1\right)$, it follows from Anderson's inequality [2] and (3.6) that

$$\begin{aligned}
&\mathbb{P}\left(\frac{|u(t^{2^n}, x^{2^n}) - u(t^{2^n - 1}, x^{2^n - 1})|}{\varepsilon_n \sqrt{\log(1 + \varepsilon_n^{-1})}} \le K_* \middle| u(t^i, x^i) : 0 \le i \le 2^n - 1 \right) \\
&\le \mathbb{P}\left(\frac{|u(t^{2^n}, x^{2^n})|}{\varepsilon_n \sqrt{\log(1 + \varepsilon_n^{-1})}} \le K_* \middle| u(t^i, x^i) : 0 \le i \le 2^n - 1 \right) \\
&\le \mathbb{P}\left(|Z| \le K_* \sqrt{C_0^{-1} \log\left(1 + \varepsilon_n^{-1}\right)} \right)
\end{aligned}$$

where Z is a standard normal random variable. Using $\mathbb{P}(|Z| > x) \ge (\sqrt{2\pi} x)^{-1} \exp(-x^2/2)$ for $x \ge 1$ and $1 + \varepsilon^{-1} < 2/\varepsilon$ for ε small, we deduce that when n is large the above probability is bounded from above by

$$1 - \frac{C(\varepsilon_n/2)^{K_*^2/(2C_0)}}{K_* \sqrt{\log\left(2/\varepsilon_n\right)}} \le \exp\left(-\frac{C(\varepsilon_n/2)^{K_*^2/(2C_0)}}{K_* \sqrt{\log\left(2/\varepsilon_n\right)}} \right) \le \exp\left(- \frac{C_{K_*} 2^{-\frac{(2-\beta)K_*^2}{4C_0} n}}{\sqrt{n}} \right)$$

where $C_{K_*} > 0$ is a constant depending on K_*. Then by (3.5) and induction, we have

$$\mathbb{P}\big(J_n \le K_*\big) \le \exp\left(- 2^n \frac{C_{K_*} 2^{-\frac{(2-\beta)K_*^2}{4C_0} n}}{\sqrt{n}} \right).$$

We can now choose $K_* > 0$ to be a sufficiently small constant such that $1 - \frac{(2-\beta)K_*^2}{4C_0} > 0$. Then $\sum_{n=1}^\infty \mathbb{P}\big(J_n \le K_*\big) < \infty$. Hence, by the Borel–Cantelli lemma, $\liminf_n J_n \ge K_*$ a.s. and the proof is complete. □

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Acknowledgments. The authors thank Professor Raluca Balan and Ciprian Tudor for stimulating discussions and for their generosity in encouraging the authors to publish this paper.