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On the eigenvalues of truncations of random unitary matrices*

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Abstract

We consider the empirical eigenvalue distribution of an $m \times m$ principle submatrix of an $n \times n$ random unitary matrix distributed according to Haar measure. Earlier work of Petz and Réffy identified the limiting spectral measure if $\frac{m}{n} \to \alpha$, as $n \to \infty$; under suitable scaling, the family $\{\mu_\alpha\}_{\alpha \in (0,1)}$ of limiting measures interpolates between uniform measure on the unit disc (for small α) and uniform measure on the unit circle (as $\alpha \to 1$). In this note, we prove an explicit concentration inequality which shows that for fixed n and m, the bounded-Lipschitz distance between the empirical spectral measure and the corresponding μ_α is typically of order $\sqrt{\frac{\log(m)}{m}}$ or smaller. The approach is via the theory of two-dimensional Coulomb gases and makes use of a new "Coulomb transport inequality" due to Chafaï, Hardy, and Maïda.

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1 Introduction

Let U be an $n \times n$ Haar-distributed unitary matrix. By a truncation of such a matrix, we mean a reduction to the upper-left $m \times m$ block, for some $m \leq n$. In the case that $m = o(\sqrt{n})$, the truncated matrix is close to a matrix of independent, identically distributed Gaussian random variables (see Jiang [5]); the circular law for the Ginibre ensemble would lead one to expect that the eigenvalue distribution was approximately uniform in a disc, and this was indeed verified by Jiang in [5]. At the opposite extreme, namely $m = n$, we have the full original matrix U. The eigenvalues of U itself are also well-understood; it was first proved by Diaconis and Shahshahani [2] that for a sequence $\{U_n\}$ with U_n distributed according to Haar measure on $\mathbb{U}(n)$, the corresponding sequence of empirical spectral measures converges to the uniform measure on the circle, weakly in probability. In more recent work [6] of the first author and M. Meckes, it was shown that if μ_n denotes the spectral measure of U and ν is the uniform measure on the circle, then with probability one, for n large enough, $W_1(\mu_n, \nu) \leq \frac{C\sqrt{\log(n)}}{n}$ (here, $W_1(\cdot,\cdot)$ is the L^1-Wasserstein distance; the definition is given at the end of this section). This result

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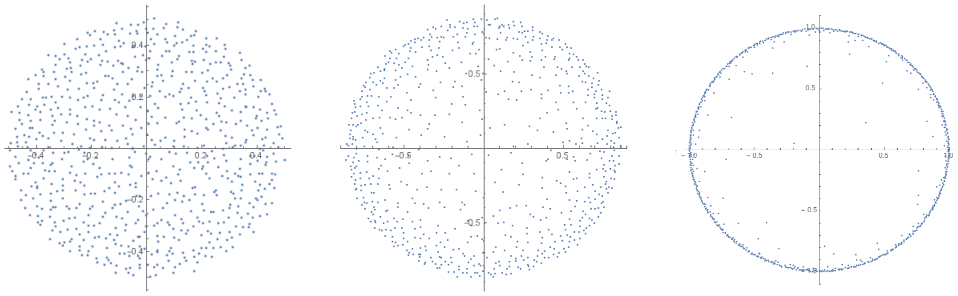

Figure 1: The eigenvalues of an $m \times m$ truncation of a $n \times n$ Haar-distributed unitary matrix, with $\frac{m}{n} = .25$, $\frac{m}{n} = .75$, and $\frac{m}{n} \approx .99$.

demonstrates a stronger uniformity of the eigenvalues of such a matrix than, for example, a collection of n i.i.d. uniform points on the circle (whose empirical measure typically has distance of the order $\frac{1}{\sqrt{n}}$ from the uniform measure).

It is thus natural to consider the evolution of the distribution of the eigenvalues of an $m \times m$ trucation of U, as $\alpha = \frac{m}{n}$ ranges from $o(1)$ to $1 - o(1)$. Figure 1 shows simulations of the eigenvalues of truncations for various values of α: in it, one can see the thinning out of the distribution in the center of the disc, as more of the original matrix is kept and the eigenvalues move from being uniform on a disc to uniform on the circle.

In fact, the exact eigenvalue distribution of an $m \times m$ submatrix with $m < n$ is known (see [8]; also [7]). For our purposes, it is most natural to consider the eigenvalues of an $m \times m$ truncation rescaled by $\sqrt{\frac{n}{m}}$; under this scaling, the joint distribution of the eigenvalues is supported on $\left\{|z| < \sqrt{\frac{n}{m}}\right\}^n$ and has density there given by

$$d\mathbb{P}_{n,m}(z_1, \dots, z_m) = \frac{1}{c_{n,m}} \prod_{1 \le j < k \le m} |z_j - z_k|^2 \prod_{j=1}^{m} \left(1 - \frac{m}{n}|z_j|^2\right)^{n-m-1} d\lambda(z_1) \dots d\lambda(z_m), \tag{1.1}$$

where

$$c_{n,m} = \pi^m m! \left(\frac{n}{m}\right)^{m(m+1)/2} \prod_{j=0}^{m-1} \binom{n-m+j-1}{j}^{-1} \frac{1}{n-m+j}$$

and λ denotes Lebesgue measure on $\mathbb{C}$.

Petz and Réffy [7] made use of the explicit eigenvalue density to identify the large-n limiting spectral measure, when $\frac{m}{n} \to \alpha \in (0,1)$; it has radial density with respect to Lebesgue measure on $\mathbb{C}$ (as it must, by rotation-invariance), given by

$$f_\alpha(z) = \begin{cases} \frac{(1-\alpha)}{\pi(1-\alpha|z|^2)^2}, & 0 < |z| < 1; \\ 0, & \text{otherwise.} \end{cases} \tag{1.2}$$

While the mathematical motivation in studying the eigenvalues of these truncations, and particularly the evolution of the ensemble as the ratio $\frac{m}{n}$ ranges from $o(1)$ to $1 - o(1)$, is clear, there are also many phyical systems in which large unitary matrices play a central role, and in which truncations of those matrices arise naturally. E.g., in chaotic scattering, the amplitudes of waves coming into the system are related to the amplitudes of outgoing waves by a large unitary matrix (called an S-matrix), and the so-called transmission matrix (related to long-lived resonances of the system) is a truncation of the S-matrix. See, e.g., [4], where the use of random unitary matrices in this context was explored.

The purpose of this paper is to give non-asymptotic results; i.e., to describe the ensemble of eigenvalues of truncations of $U \in \mathbb{U}(n)$ for fixed (large) n. Our main result on approximation of the spectral measure is the following.

Theorem 1.1. *Let $n, m \in \mathbb{N}$ with $1 \leq m < n$. Let $U \in \mathbb{U}(n)$ be distributed according to Haar measure, and let $z_1, \ldots, z_m$ denote the eigenvalues of the top-left $m \times m$ block of $\sqrt{\frac{n}{m}}U$. The joint law of $z_1, \ldots, z_m$ is denoted $\mathbb{P}_{n,m}$.*

Let $\hat{\mu}_m$ be the empirical spectral measure given by

$$\hat{\mu}_m := \frac{1}{m}\sum_{i=1}^{m}\delta_{z_i}.$$

Let $\alpha = \frac{m}{n}$ and let μ_α be the probability measure on the unit disc with the density f_α defined in (1.2)*. For any $r > 0$,*

$$\mathbb{P}_{n,m}\left[d_{BL}\left(\hat{\mu}_m, \mu_\alpha\right) \geq r\right] \leq e^2 \exp\left\{-C_\alpha m^2 r^2 + 2m\log(m) + C'_\alpha m\right\} + \frac{e}{2\pi}\sqrt{\frac{m}{1-\alpha}}e^{-m},$$

where $C_\alpha = \frac{1}{128\pi(1+\sqrt{3+\log(\alpha^{-1})})^2}$ and $C'_\alpha = 6 + 3\log(\alpha^{-1})$, and $d_{BL}(\cdot,\cdot)$ denotes the bounded-Lipschitz distance (the definition is given at the beginning of Section 2).

The bounds in Theorem 1.1 are tight enough that we can in fact treat the evolution of the process of spectral measures of truncations of U, as the truncation ratio α ranges from $o(1)$ to $1 - o(1)$.

Theorem 1.2. *Let U be an $n \times n$ Haar-distributed matrix in $\mathbb{U}(n)$ and, for $1 \leq m < n$, let $\hat{\mu}_m$ be the empirical spectral measure of the top-left $m \times m$ block of $\sqrt{\frac{n}{m}}U$. Let $\alpha = \frac{m}{n}$, μ_α be the probability measure on the disc with density f_α as in* (1.2)*, and let $\{k_n\}_{n\geq 1} \subseteq \mathbb{N}$ be such that $k_n = o(n)$ and $\frac{k_n}{\log(n)^2} \to \infty$. Then with probability 1, for n large enough,*

$$d_{BL}\left(\hat{\mu}_m, \mu_\alpha\right) \leq \delta_m$$

for every $m \in \{k_n, \ldots, n-1\}$, where

$$\delta_m = \begin{cases} 48\sqrt{\frac{2\pi\log m}{m}}, & m \geq \frac{n}{e}; \\ \frac{165\sqrt{\log(\frac{n}{m})\log(n)}}{\sqrt{m}}, & m < \frac{n}{e}. \end{cases}$$

Note that if $m = o(n)$, then δ_m is of the order $\sqrt{\frac{\log(n)^2}{m}}$. The restriction on k_n in the statement of the theorem thus implies that with probability one,

$$\sup_{k_n \leq m \leq n-1} d_{BL}\left(\hat{\mu}_m, \mu_\alpha\right) \xrightarrow{n\to\infty} 0.$$

Observe that, although the support of the empirical spectral measure $\hat{\mu}_m$ is the disc of radius $\sqrt{\frac{n}{m}}$, the limiting spectral measure is supported on the unit disc; the following treats the question of how far into this intermediate regime the eigenvalues are likely to stray.

Theorem 1.3. *Let $z_1, \ldots, z_m$ be the eigenvalues of the top-left $m \times m$ block of $\sqrt{\frac{n}{m}}U$, with joint law $\mathbb{P}_{n,m}$, and let $\alpha = \frac{m}{n}$. Then for any $\epsilon \in \left(0, \frac{1}{\sqrt{\alpha}} - 1\right)$,*

$$\mathbb{P}_{n,m}\left[\max_{1\leq j\leq m}|z_j| > 1 + \epsilon\right] \leq \frac{e\left(1 - \alpha^m(1+\epsilon)^{2m}\right)}{2\pi\sqrt{n\alpha(1-\alpha)}\left(1 - \alpha(1+\epsilon)^2\right)}\left[\frac{\left(1 - \alpha(1+\epsilon)^2\right)^{(1-\alpha)}}{(1-\alpha)^{1-\alpha}\alpha^\alpha}\right]^n.$$

If $\epsilon \geq \frac{1}{\sqrt{\alpha}} - 1$, then

$$\mathbb{P}_{n,m}\left[\max_{1\le j\le m}|z_j|>1+\epsilon\right]=0.$$

This estimate requires some effort to parse. Note that in the non-trivial case that $(1+\epsilon)^2<\frac{1}{\alpha}$,

$$\frac{\left(1-\alpha^m(1+\epsilon)^{2m}\right)}{(1-\alpha(1+\epsilon)^2)}=1+\alpha(1+\epsilon)^2+\cdots+(\alpha(1+\epsilon)^2)^{m-1}\le m,$$

so that if we take ϵ to be such that

$$(1+\epsilon)^2>\frac{1}{\alpha}\left[1-(1-\alpha)\left(\frac{\alpha}{e}\right)^{\frac{\alpha}{1-\alpha}}\right], \tag{1.3}$$

then

$$\mathbb{P}_{n,m}\left[\max_{1\le j\le m}|z_j|>1+\epsilon\right]\le\frac{e\left(1-\alpha^m(1+\epsilon)^{2m}\right)}{2\pi\sqrt{n\alpha(1-\alpha)}\,(1-\alpha(1+\epsilon)^2)}e^{-\alpha n}\le\frac{e}{2\pi}\sqrt{\frac{m}{(1-\alpha)}}e^{-m}.$$

While the bound stated in Theorem 1.3 is formally stronger, we will use the simpler bound (1) in the following discussion, separated into three distinct regimes.

1. $m=o(n)$:

   For $\frac{m}{n}=\alpha$ small, the lower bound on $(1+\epsilon)^2$ in (1.3) is

   $$\frac{1}{\alpha}\left[1-(1-\alpha)\left(\frac{\alpha}{e}\right)^{\frac{\alpha}{1-\alpha}}\right]=2-\log(\alpha)+o(1).$$

   If $m=o(n)$ and $m\ge 2\log(n)$, then the bound in (1) tends to zero at least as quickly as $n^{-\frac{3}{2}}$, and so it follows from the Borel–Cantelli lemma that if m_n is any sequence with $m_n=o(n)$ and $m_n\ge 2\log(n)$, then for any $\delta>0$, with probability one, for n large enough the support of the empirical spectral measure $\hat{\mu}_{m_n}$ lies within the disc of radius $\sqrt{2+\delta+\log(\frac{n}{m_n})}$, as opposed to the *a priori* support of the disc of radius $\sqrt{\frac{n}{m_n}}$.

2. There are $c>0$ and $C<1$ such that $cn\le m\le Cn$:

   Here the bound in (1) tends to zero exponentially with n, and in this case the lower bound on ϵ from (1.3) results in a fixed radius r_m (somewhat smaller than $\sqrt{\frac{n}{m}}$ but still bounded away from one, in terms of c and C), such that, if m_n is a sequence with $cn\le m_n\le Cn$ for all n, then with probability one for n large enough, $\hat{\mu}_{m_n}$ is supported in a disc of radius r_{m_n}.

3. $\frac{m}{n}\to 1$:

   The bound in (1) tends to zero exponentially with n, and for α tending to one,

   $$\frac{1}{\alpha}\left[1-(1-\alpha)\left(\frac{\alpha}{e}\right)^{\frac{\alpha}{1-\alpha}}\right]=2-\alpha+O((1-\alpha)^2).$$

   It thus follows from the Borel–Cantelli lemma that for any $\epsilon>0$, if m_n is a sequence with $m_n<n$ for each n and $\frac{m_n}{n}\to 1$, then with probability one, for n large enough, the empirical spectral measure $\hat{\mu}_{m_n}$ is contained within a disc of radius $1+\epsilon$.

2 Proofs of the main results

Throughout the paper, U will denote a Haar-distributed random unitary matrix in $\mathbb{U}(n)$ and, for $1 \leq m < n$, $z_1, \ldots, z_m$ will denote the eigenvalues of the top-left $m \times m$ block of $\sqrt{\frac{n}{m}}U$, with associated spectral measure $\hat{\mu}_m$.

Results below are formulated in terms of the L^1-Wasserstein distance and the bounded-Lipschitz distance, defined as follows. The L^1-Wasserstein distance between probability measures μ and ν is given by

$$W_1(\mu,\nu) = \sup_f \left| \int f d\mu - \int f d\nu \right|,$$

where the supremum is taken over Lipschitz functions with Lipschitz constant 1. The bounded-Lipschitz distance between μ and ν is given by

$$d_{BL}(\mu,\nu) = \sup_f \left| \int f d\mu - \int f d\nu \right|,$$

where the supremum is taken over functions which are bounded by 1 and have Lipschitz constant bounded by 1.

In some of the proofs, we will need to make use of the following uniform version of Stirling's approximation, which is an easy consequence of equation (9.15) in [3].

Lemma 2.1. *For each positive integer n,*

$$\sqrt{2\pi} n^{n+\frac{1}{2}} e^{-n} \leq n! \leq e n^{n+\frac{1}{2}} e^{-n}.$$

We now proceed with the proofs.

Proof of Theorem 1.3. The form of the eigenvalue density (1.1), specifically the presence of the Vandermonde determinant, gives that $z_1, \ldots, z_m$ form a determinantal point process on $\mathbb{C}$ with the kernel (with respect to Lebesgue measure)

$$\begin{aligned} K_{n,m}(z_1,z_2) = \frac{m}{n} \sum_{j=1}^m \frac{1}{N_j} \left(\frac{m}{n} z_1 \overline{z_2} \right)^{j-1} \left(1 - \frac{m}{n}|z_1|^2\right)^{\frac{n-m-1}{2}} \left(1 - \frac{m}{n}|z_2|^2\right)^{\frac{n-m-1}{2}} \\ \times \mathbb{1}_{(0,\infty)} \left(1 - \frac{m}{n}|z_1|^2\right) \mathbb{1}_{(0,\infty)} \left(1 - \frac{m}{n}|z_2|^2\right), \end{aligned}$$

where the normalization factor N_j is given by

$$N_j = \frac{\pi (j-1)!(n-m-1)!}{(n-m+j-1)!}.$$

Let B_r denote the ball of radius r, and let $\epsilon \in \left(0, \frac{1}{\sqrt{\alpha}} - 1\right)$. Then the expected number of z_i outside $B_{1+\epsilon}$ is given by

$$\begin{aligned} \mathbb{E}[\mathcal{N}_{B_{1+\epsilon}^c}] &= \int_{B_{1+\epsilon}^c} K_{n,m}(z,z) dz \\ &= 2\pi \int_{1+\epsilon}^{\frac{1}{\sqrt{\alpha}}} \sum_{j=1}^m \frac{1}{N_j} \alpha^j r^{2(j-1)} \left(1 - \alpha r^2\right)^{n-m-1} r dr \\ &\leq 2\pi \left(1 - \alpha(1+\epsilon)^2\right)^{n-m-1} \sum_{j=1}^m \frac{1}{N_j} \alpha^j \int_{1+\epsilon}^{\frac{1}{\sqrt{\alpha}}} r^{2(j-1)} r dr \\ &= 2\pi \left(1 - \alpha(1+\epsilon)^2\right)^{n-m-1} \sum_{j=1}^m \frac{1}{N_j} \left(\frac{1}{2j} \left(1 - \alpha^j (1+\epsilon)^{2j}\right) \right) \\ &\leq 2\pi \left(1 - \alpha(1+\epsilon)^2\right)^{n-m-1} \left(1 - \alpha^m (1+\epsilon)^{2m}\right) \sum_{j=1}^m \frac{1}{(2j)N_j}. \end{aligned}$$

The sum on the right can be computed using the hockey stick identity:

$$\begin{aligned}\sum_{j=1}^{m}\frac{1}{(2j)N_j} &= \sum_{j=1}^{m}\frac{(n-m+j-1)!}{2\pi j!(n-m-1)!}\\ &= \frac{1}{2\pi}\sum_{j=1}^{m}\binom{n-m+j-1}{n-m-1}\\ &= \frac{1}{2\pi}\sum_{k=n-m}^{n-1}\binom{k}{n-m-1}\\ &= \frac{1}{2\pi}\left[\binom{n}{n-m}-1\right].\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}[\mathcal{N}_{B^c_{1+\epsilon}}] &\le \left(1-\alpha(1+\epsilon)^2\right)^{n-m-1}\left(1-\alpha^m(1+\epsilon)^{2m}\right)\binom{n}{m}\\ &= \frac{\left(1-\alpha^m(1+\epsilon)^{2m}\right)}{(1-\alpha(1+\epsilon)^2)}\left(1-\alpha(1+\epsilon)^2\right)^{(1-\alpha)n}\binom{n}{m}.\end{aligned}$$

The version of Stirling's formula in Lemma 2.1 gives that, for $m=\alpha n$,

$$\binom{n}{m}\le\frac{e}{2\pi\sqrt{n\alpha(1-\alpha)}}\left[\frac{1}{(1-\alpha)^{1-\alpha}\alpha^\alpha}\right]^n,$$

and so by Markov's inequality,

$$\begin{aligned}\mathbb{P}\left[\max_{1\le j\le m}|z_j|\ge 1+\epsilon\right] &= \mathbb{P}\left[\mathcal{N}_{B^c_{1+\epsilon}}\ge 1\right]\\ &\le \frac{e\left(1-\alpha^m(1+\epsilon)^{2m}\right)}{2\pi\sqrt{n\alpha(1-\alpha)}(1-\alpha(1+\epsilon)^2)}\left[\frac{\left(1-\alpha(1+\epsilon)^2\right)^{(1-\alpha)}}{(1-\alpha)^{1-\alpha}\alpha^\alpha}\right]^n.\end{aligned}$$

If $\epsilon\ge\frac{1}{\sqrt{\alpha}}-1$, then

$$\mathbb{P}\left[\max_{1\le j\le m}|z_j|\ge 1+\epsilon\right]\le\mathbb{P}\left[\max_{1\le j\le m}|z_j|\ge\frac{1}{\sqrt{\alpha}}\right]=0,$$

since the eigenvalues of a principal submatrix of U necessarily have modulus bounded by 1. □

We now proceed with Theorem 1.1. The proof is an adaptation of the approach in [1], using the framework of Coulomb gases. Specifically, the form of the eigenvalue density given in Equation (1.1) means that the z_i can be viewed as the (random) locations of m unit charges in a two-dimensional Coulomb gas with external potential, as follows. If the energy $H_{n,m}(z_1,\ldots,z_m)$ is defined by

$$H_{n,m}(z_1,\ldots,z_m)=\sum_{j\ne k}\log\left(\frac{1}{|z_j-z_k|}\right)+m\sum_{j=1}^{m}V_{n,m}(z_j),$$

with the potential $V_{n,m}(z)$ defined by

$$V_{n,m}(z)=\begin{cases}-\frac{n-m-1}{m}\log\left(1-\frac{m}{n}|z|^2\right), & |z|<\sqrt{\frac{n}{m}};\\ \infty, & \text{otherwise},\end{cases}$$

then the Gibbs measure on $\mathbb{C}^m$ (taking the inverse temperature β to be 2) is

$$d\mathbb{P}_{n,m}(z_1,\ldots,z_m) = \frac{1}{Z_{n,m}} \exp\left\{-H_{n,m}(z_1,\ldots,z_m)\right\} d\lambda(z_1)\ldots d\lambda(z_m),$$

where λ denotes Lebesgue measure on $\mathbb{C}$. That is, the Gibbs measure in this Coulomb gas model is exactly the same as the density of the eigenvalues of the top-left $m \times m$ block of $\sqrt{\frac{n}{m}}U$, and so the empirical measure of the charges $z_1,\ldots,z_m$ has the same distribution as the empirical spectral measure $\hat{\mu}_m$. This was the viewpoint taken by Petz and Réffy in [7] to identify the large-n limiting spectral measure; the limiting measure with density f_α as in (1.2) is exactly the equilibrium measure for the 2-dimensional Coulomb gas model with potential

$$V_\alpha = \begin{cases} -\left(\frac{1}{\alpha}-1\right)\log\left(1-\alpha|z|^2\right), & |z| < \frac{1}{\sqrt{\alpha}}; \\ \infty, & \text{otherwise}, \end{cases}.$$

It should be noted that the viewpoint here is slightly removed from the usual Coulomb gas model, where the potential would not depend on m or n; allowing such a dependence is possible because the approach taken here is non-asymptotic; i.e., n and m are fixed throughout.

In recent work, Chafaï, Hardy and Maïda [1] have developed an approach to studying the non-asymptotic behavior of Coulomb gases, using new inequalities they call Coulomb transport inequalities. Specifically, if $\mathcal{E}(\mu) = \iint g(x-y)d\mu(x)d\mu(y)$ is the Coulomb energy, with

$$g(x) = \begin{cases} \log\frac{1}{|x|}, & d=2; \\ \frac{1}{|x|^{d-2}}, & d>2 \end{cases}$$

the d-dimensional Coulomb kernel, they showed that if D is a compact subset of $\mathbb{R}^d$, then there is a constant $C_D > 0$ such that for any pair of probability measures μ and ν supported on D with $\mathcal{E}(\mu), \mathcal{E}(\nu) < \infty$,

$$W_1(\mu,\nu)^2 \leq C_D \mathcal{E}(\mu-\nu).$$

When comparing to the equilibrium measure μ_V of the Coulomb gas model with potential V, this leads to the estimate

$$d_{BL}(\mu,\mu_V)^2 \leq W_1(\mu,\mu_V)^2 \leq C_V\left[\mathcal{E}_V(\mu) - \mathcal{E}_V(\mu_V)\right], \tag{2.1}$$

where $\mathcal{E}_V$ is the modified energy functional

$$\mathcal{E}_V(\mu) = \mathcal{E}(\mu) + \int V d\mu. \tag{2.2}$$

The estimate (2.1) is the key ingredient in the proof of Theorem 1.1. The proof follows the analysis in [1] closely, although their analysis does not apply directly to our potential. In particular, certain technical lemmas in [1], e.g., Theorem 1.9, require modifications because boundedness assumptions made there are not satisfied by $V_{n,m}$.

The central idea of the proof of Theorem 1.1 is the following simple application of the bound (2.1). Let $\mathbf{z} = (z_1,\ldots,z_m)$, and let $\hat{\mu}_{\mathbf{z}} := \frac{1}{m}\sum_{j=1}^m \delta_{z_j}$. Let μ_α have density f_α, for $\alpha = \frac{m}{n}$. Given $r > 0$,

$$\begin{aligned} &\mathbb{P}\left[d_{BL}(\hat{\mu}_{\mathbf{z}},\mu_\alpha) > r\right] \\ &\quad = \frac{1}{Z_{n,m}} \int_{d_{BL}(\hat{\mu}_{\mathbf{z}},\mu_\alpha)>r} \exp\left\{-\sum_{j\neq k}\log\left(\frac{1}{|z_j-z_k|}\right) - m\sum_{j=1}^m V_{n,m}(z_j)\right\} d\lambda^n(z) \\ &\quad \approx \frac{1}{Z_{n,m}} \int_{d_{BL}(\hat{\mu}_{\mathbf{z}},\mu_\alpha)>r} \exp\left\{-m^2\mathcal{E}_{V_{n,m}}(\hat{\mu}_{\mathbf{z}})\right\} d\lambda^n(z), \end{aligned}$$

and on $\{z : d_{BL}(\hat{\mu}_{\mathbf{z}}, \mu_\alpha) > r\}$, (2.1) gives that

$$\exp\left\{-m^2\mathcal{E}_{V_{n,m}}(\hat{\mu}_{\mathbf{z}})\right\} \leq \exp\left\{-cm^2r^2 + m^2\mathcal{E}_{V_{n,m}}(\mu_\alpha)\right\}.$$

Of course, since the measures $\hat{\mu}_{\mathbf{z}}$ are singular, the approximate inequality above is invalid, and so part of the argument is to mollify the empirical measures under consideration. Since our potential $V_{n,m}$ is only finite on $\left\{|z| < \sqrt{\frac{n}{m}}\right\}$, this requires in particular that the probability of any eigenvalues lying too close to the boundary of this disc is small, which follows from Theorem 1.3. In fact, some further truncation is useful in order to obtain improved control on the constants. Beyond that, all that is really needed is to give estimates for the normalizing constant and the modified Coulomb energy at the equilibrium measure.

The following lemma relates the energy $H_{n,m}(x_1, \ldots, x_m)$ to the modified Coulomb energy of the mollified spectral measure.

Lemma 2.2. *For $\mathbf{z} = (z_1, \ldots, z_m) \in \mathbb{C}^m$, let $\hat{\mu}_{\mathbf{z}} = \frac{1}{m}\sum_{j=1}^m \delta_{z_j}$. That is, $\hat{\mu}_{\mathbf{z}}$ is the probability measure putting equal mass at each of the z_j. For any $\epsilon > 0$, define*

$$\hat{\mu}_{\mathbf{z}}^{(\epsilon)} := \hat{\mu}_{\mathbf{z}} * \lambda_\epsilon,$$

where λ_ϵ is the uniform probability measure on the ball B_ϵ. Then for $z_1, \ldots, z_m \in B_{\frac{1}{\sqrt{\alpha}} - \sqrt{\epsilon}}$, with $\epsilon < \left(\frac{\alpha}{4+2\sqrt{\alpha}}\right)^2$,

$$H_{n,m}\left(z_1, \ldots, z_m\right) \geq m^2\mathcal{E}_{V_{n,m}}\left(\hat{\mu}_{\mathbf{z}}^{(\epsilon)}\right) - m\mathcal{E}\left(\lambda_\epsilon\right) - \frac{m^2(1-\alpha)\epsilon}{2\alpha}.$$

Proof. Lemma 4.2 from [1] gives that

$$H_{n,m}\left(z_1, \ldots, z_m\right) \geq m^2\mathcal{E}_{V_{n,m}}\left(\hat{\mu}_{\mathbf{z}}^{(\epsilon)}\right) - m\mathcal{E}\left(\lambda_\epsilon\right) - m\sum_{i=1}^m \left(V_{n,m} * \lambda_\epsilon - V_{n,m}\right)(z_i),$$

so that the only task is to give an upper bound for $\left(V_{n,m} * \lambda_\epsilon - V_{n,m}\right)(z_i)$.

Let $0 < \epsilon < \left(\frac{\alpha}{4+2\sqrt{\alpha}}\right)^2$, and suppose that $z < \frac{1}{\sqrt{\alpha}} - \sqrt{\epsilon}$. Then in particular, $V_{n,m}(y) < \infty$ for $|y - z| < \epsilon$, so that

$$\left(V_{n,m} * \lambda_\epsilon - V_{n,m}\right)(z) = \int \left(V_{n,m}(z - \epsilon u) - V_{n,m}(z)\right) d\lambda_1(u).$$

Note that by symmetry, $\int \langle \nabla V_{n,m}(z), u\rangle \, d\lambda_1(u) = 0$ for fixed z. Moreover, $V_{n,m}$ is convex, so that $\operatorname{Hess} V_{n,m}$ is positive semi-definite; it thus follows from Taylor's theorem that

$$\begin{aligned}\left(V_{n,m} * \lambda_\epsilon - V_{n,m}\right)(z) &\leq \frac{\epsilon^2}{2} \sup_{\substack{y \in \mathbb{R}^2 \\ |y-z| \leq \epsilon}} \int \langle \operatorname{Hess}(V_{n,m})_y u, u\rangle \, d\lambda_1(u) \\ &\leq \frac{\epsilon^2}{2} \sup_{|y| < \frac{1}{\sqrt{\alpha}} - \sqrt{\epsilon} + \epsilon} \frac{1}{4}\Delta V_{n,m}(y).\end{aligned}$$

If $|y| < \frac{1}{\sqrt{\alpha}} - \sqrt{\epsilon} + \epsilon$, then

$$\Delta V_{n,m}(y) = \frac{4\left(1 - \alpha - \frac{1}{n}\right)}{(1 - \alpha|y|^2)^2} \leq \frac{4(1-\alpha)}{(2\sqrt{\alpha}(\sqrt{\epsilon} - \epsilon) - \alpha(\sqrt{\epsilon} - \epsilon)^2)^2} \leq \frac{4(1-\alpha)}{\alpha\epsilon},$$

for ϵ in the range specified above. □

In the proof of Theorem 1.1, we will use the following version of the Coulomb transport inequality from [1], which is an immediate consequence of Lemma 3.1 together with Theorem 1.1 of that paper. The lemma refers to an *admissible* external potential V; we refer the reader to [1] for the definition, which is satisfied for our potentials $V_{n,m}$. A key fact is that such a potential is associated with an equilibrium measure μ_V, which is the unique minimizer of the modified energy $\mathcal{E}_V$ as defined in (2.2). For our potential $V_{n,m}$, the equilibrium measure is μ_α for $\alpha = \frac{m}{n}$.

Lemma 2.3 (Coulomb Transport Inequality [1])**.** *Let V be an admissible external potentialon $\mathbb{R}^d$ with equilibrium measure μ_V. If $D \subset \mathbb{R}^d$ is compact then for any $\mu \in \mathcal{P}(\mathbb{R}^d)$ supported in D,*

$$d_{BL}(\mu, \mu_V)^2 \leq W_1(\mu, \mu_V)^2 \leq C_{D \cup supp(\mu_V)} \left(\mathcal{E}_V(\mu) - \mathcal{E}_V(\mu_V)\right),$$

where if $R > 0$ is such that $D \cup supp(\mu_V) \subset B_R$, then $C_{D \cup supp(\mu_V)}$ can be taken to be $vol(B_{4R})$.

Proof of Theorem 1.1. Fix $r > 0$. By Theorem 1.3 and the discussion following it, it is possible to choose $\eta_\alpha > 0$ such that $1 + \eta_\alpha < \frac{1}{\sqrt{\alpha}}$ and so that

$$\mathbb{P}_{n,m}\left[\max_{1 \leq j \leq m} |z_j| > 1 + \eta_\alpha\right] \leq \frac{e}{2\pi}\sqrt{\frac{m}{1-\alpha}} e^{-m}.$$

In particular, we may take $1 + \eta_\alpha = \sqrt{3 + \log(\alpha^{-1})}$, (although when $\alpha \to 1$, η_α may in fact be taken to be any fixed positive number). Take $\epsilon \in \left(0, \left(\frac{\alpha}{4+2\sqrt{\alpha}}\right)^2\right)$ as in Lemma 2.2, such that also $1 + \eta_\alpha < \frac{1}{\sqrt{\alpha}} - \sqrt{\epsilon}$. Let

$$A_{\alpha,r} := \{(z_1, \ldots, z_m) \in \mathbb{C}^m : |z_j| < 1 + \eta_\alpha, j = 1, \ldots, n, \text{ and } d_{BL}\left(\hat{\mu}_{\mathbf{z}}, \mu_\alpha\right) \geq r\}.$$

The probability that the eigenvalues of the top-left $m \times m$ block of $\sqrt{\frac{n}{m}}U$ lie in the set $A_{\alpha,r}$ is given by

$$\begin{aligned}
\mathbb{P}_{n,m}(A_{\alpha,r}) &= \frac{1}{Z_{n,m}} \int_{A_{\alpha,r}} e^{-H_{n,m}(z_1,\ldots,z_m)} \prod_{i=1}^m d\lambda(z_i) \\
&\leq \frac{1}{Z_{n,m}} \int_{A_{\alpha,r}} e^{-\left[m^2 \mathcal{E}_{V_{n,m}}\left(\hat{\mu}_{\mathbf{z}}^{(\epsilon)}\right) - m\mathcal{E}(\lambda_\epsilon) - \frac{m^2(1-\alpha)\epsilon}{2\alpha}\right]} \prod_{i=1}^m d\lambda(z_i) \qquad (2.3) \\
&\leq \frac{1}{Z_{n,m}} e^{-\left[m^2 \inf_{A_{\alpha,r}} \mathcal{E}_{V_{n,m}}\left(\hat{\mu}_{\mathbf{z}}^{(\epsilon)}\right) - m\mathcal{E}(\lambda_\epsilon) - \frac{m^2(1-\alpha)\epsilon}{2\alpha}\right]} \left(\frac{\pi}{\alpha}\right)^m,
\end{aligned}$$

by Lemma 2.2.

The normalizing constant $Z_{n,m}$ can be bounded in terms of μ_α as follows:

$$\begin{aligned}
&\log(Z_{n,m}) \\
&\quad = \log \int \cdots \int e^{-\sum_{j \neq k} \log\left(\frac{1}{|z_j - z_k|}\right) + (n-m-1)\sum_{j=1}^m \log\left(1 - \alpha|z_j|^2\right)} d\lambda(z_1) \cdots d\lambda(z_m) \\
&\quad = \log \int \cdots \int e^{-\sum_{j \neq k} \log\left(\frac{1}{|z_j - z_k|}\right) + (n-m+1)\sum_{j=1}^m \log\left(1 - \alpha|z_j|^2\right)} \\
&\qquad \times \left(\frac{1}{2(1-\alpha)}\right)^m d\mu_\alpha(z_1) \cdots d\mu_\alpha(z_m) \\
&\quad \geq -m\log(2(1-\alpha)) \\
&\qquad + \int \left[-\sum_{j \neq k} \log\left(\frac{1}{|z_j - z_k|}\right) + (n-m+1)\sum_{j=1}^m \log\left(1 - \alpha|z_j|^2\right)\right] d\mu_\alpha(z_1) \cdots d\mu_\alpha(z_m) \\
&\quad = -m\log(2(1-\alpha)) - m(m-1)\mathcal{E}(\mu_\alpha) + \left(\frac{n-m+1}{n-m-1}\right) m^2 \int V_{n,m} d\mu_\alpha,
\end{aligned}$$

where the inequality is by Jensen's inequality.

Observe that

$$\begin{aligned} m^2 \inf_{A_{\alpha,r}} \mathcal{E}_{V_{n,m}}\left(\hat{\mu}_{\mathbf{z}}^{(\epsilon)}\right) &= m^2 \inf_{A_{\alpha,r}} \left(\mathcal{E}_{V_{n,m}}\left(\hat{\mu}_{\mathbf{z}}^{(\epsilon)}\right) - \mathcal{E}_{V_{n,m}}\left(\mu_\alpha\right)\right) + m^2 \mathcal{E}_{V_{n,m}}\left(\mu_\alpha\right) \\ &= m^2 \inf_{A_{\alpha,r}} \left(\mathcal{E}_{V_{n,m}}\left(\hat{\mu}_{\mathbf{z}}^{(\epsilon)}\right) - \mathcal{E}_{V_{n,m}}\left(\mu_\alpha\right)\right) + m^2 \left[\mathcal{E}(\mu_\alpha) + \int V_{n,m} d\mu_\alpha\right], \end{aligned}$$

and by Lemma 2.3,

$$\inf_{A_{\alpha,r}} \left(\mathcal{E}_{V_{n,m}}\left(\hat{\mu}_{\mathbf{z}}^{(\epsilon)}\right) - \mathcal{E}_{V_{n,m}}\left(\mu_\alpha\right)\right) \geq \frac{1}{16\pi(1+\eta_\alpha+\epsilon)^2} \inf_{A_{\alpha,r}} d_{BL}\left(\hat{\mu}_{\mathbf{z}}^{(\epsilon)}, \mu_\alpha\right)^2.$$

Since $d_{BL}(\hat{\mu}_{\mathbf{z}}^{(\epsilon)}, \hat{\mu}_{\mathbf{z}}) \leq \epsilon$,

$$\begin{aligned} \frac{1}{2} d_{BL}\left(\hat{\mu}_{\mathbf{z}}, \mu_\alpha\right)^2 &\leq \frac{1}{2}\left(d_{BL}\left(\hat{\mu}_{\mathbf{z}}^{(\epsilon)}, \mu_\alpha\right) + d_{BL}\left(\hat{\mu}_{\mathbf{z}}^{(\epsilon)}, \hat{\mu}_{\mathbf{z}}\right)\right)^2 \\ &\leq d_{BL}\left(\hat{\mu}_{\mathbf{z}}^{(\epsilon)}, \mu_\alpha\right)^2 + d_{BL}\left(\hat{\mu}_{\mathbf{z}}^{(\epsilon)}, \hat{\mu}_{\mathbf{z}}\right)^2 \leq d_{BL}\left(\hat{\mu}_{\mathbf{z}}^{(\epsilon)}, \mu_\alpha\right)^2 + \epsilon^2. \end{aligned}$$

It follows that

$$\inf_{A_{\alpha,r}} \left(\mathcal{E}_{V_{n,m}}\left(\hat{\mu}_{\mathbf{z}}^{(\epsilon)}\right) - \mathcal{E}_{V_{n,m}}\left(\mu_\alpha\right)\right) \geq \frac{1}{16\pi(1+\eta+\epsilon)^2}\left(\frac{r^2}{2} - \epsilon^2\right),$$

and so combining the estimate in (2.3) with the analysis above gives that

$$\begin{aligned} \mathbb{P}_{n,m}(A_{\alpha,r}) &\leq \exp\left\{-\frac{m^2}{16\pi(1+\eta_\alpha+\epsilon)^2}\left(\frac{r^2}{2}-\epsilon^2\right) - m\mathcal{E}(\mu_\alpha) - \frac{2(n-m)m^2}{n-m-1}\int V_{n,m} d\mu_\alpha \right. \\ &\qquad \left. + m\left(\log\left(\frac{2\pi(1-\alpha)}{\alpha}\right)\right) + \frac{m^2(1-\alpha)\epsilon}{2\alpha} + m\mathcal{E}(\lambda_\epsilon)\right\} \\ &\leq \exp\left\{-\frac{m^2}{16\pi(1+\eta_\alpha+\epsilon)^2}\left(\frac{r^2}{2}-\epsilon^2\right) + m\left(\log\left(\frac{2\pi(1-\alpha)}{\alpha}\right)\right) \right. \\ &\qquad \left. + \frac{m^2(1-\alpha)\epsilon}{2\alpha} + m\mathcal{E}(\lambda_\epsilon)\right\}. \end{aligned}$$

Now, the Coulomb energy of λ_ϵ is

$$\mathcal{E}(\lambda_\epsilon) = \int\int \log\left(\frac{1}{|\epsilon x - \epsilon y|}\right) d\lambda_1(x) d\lambda_1(y) = -\log(\epsilon) + \mathcal{E}(\lambda_1) = -\log(\epsilon) + \frac{1}{4} = \log\left(\frac{e^{1/4}}{\epsilon}\right),$$

and so

$$\begin{aligned} &\mathbb{P}_{n,m}(A_{\alpha,r}) \\ &\quad \leq \exp\left\{-\frac{m^2}{16\pi(1+\eta_\alpha+\epsilon)^2}\left(\frac{r^2}{2}-\epsilon^2\right) + m\left(\log\left(\frac{2\pi e^{1/4}(1-\alpha)}{\alpha\epsilon}\right)\right) + \frac{m^2(1-\alpha)\epsilon}{2\alpha}\right\}. \end{aligned} \tag{2.4}$$

Now take

$$\epsilon = \min\left\{\left(\frac{\alpha}{2(2+\sqrt{\alpha})}\right)^2, \left(\frac{1}{\sqrt{\alpha}} - (1+\eta_\alpha)\right)^2, \frac{8\sqrt{\pi}(1+\eta_\alpha)}{m}, 1+\eta_\alpha, \frac{2\alpha}{(1-\alpha)m^2}\right\}.$$

The analysis above required that $\epsilon \leq \left(\frac{\alpha}{4+2\sqrt{\alpha}}\right)^2$ and that $1+\eta_\alpha \leq \frac{1}{\sqrt{\alpha}} - \sqrt{\epsilon}$, which is guaranteed by the first two terms in the minimum.

For the first term in the estimate (2.4), first estimating the ϵ in the denominator by $1+\eta_\alpha$ and then the factor of ϵ^2 in the numerator by $\frac{8\sqrt{\pi}(1+\eta_\alpha)}{m}$ gives that

$$-\frac{m^2}{16\pi(1+\eta_\alpha+\epsilon)^2}\left(\frac{r^2}{2}-\epsilon^2\right) \le -\frac{m^2}{64\pi(1+\eta_\alpha)^2}\left(\frac{r^2}{2}-\epsilon^2\right) \le -\frac{m^2r^2}{128\pi(1+\eta_\alpha)^2}+1.$$

For the second term of (2.4), one verifies each of the 5 possible choices of ϵ above: in all cases,

$$m\left(\log\left(\frac{2\pi e^{1/4}(1-\alpha)}{\alpha\epsilon}\right)\right) \le 2m\log(m)+mC'_\alpha,$$

where

$$\begin{aligned}C'_\alpha &= \max\left\{2\log(6)+3\log(\alpha^{-1}), -2\log(1-\sqrt{\alpha}(1+\eta_\alpha)), -\log(\alpha(1+\eta_\alpha))\right\}+\log(2\pi e^{1/4})\\ &\le 2\log(6)+3\log(\alpha^{-1})+\log(2\pi e^{1/4})\\ &\le 6+3\log(\alpha^{-1}),\end{aligned}$$

since $\eta_\alpha > 0$ and $\sqrt{\alpha}\eta_\alpha \le 1$. Since we take $\epsilon \le \frac{2\alpha}{(1-\alpha)m^2}$, the final term inside the exponential in the estimate (2.4) is bounded by 1. All together, then

$$\mathbb{P}_{n,m}(A_{\alpha,r}) \le \exp\left\{-C_\alpha m^2r^2+2m\log(m)+mC'_\alpha+2\right\},$$

where $C_\alpha = \frac{1}{128\pi(1+\eta_\alpha)^2} \ge \frac{1}{128\pi(1+\sqrt{3+\log(\alpha^{-1})})^2}$ and $C'_\alpha = 6+3\log(\alpha^{-1})$. □

The proof of Theorem 1.2 then follows from Theorem 1.1 and an application of the Borel-Cantelli Lemma.

Proof of Theorem 1.2. If $m \ge \frac{n}{e}$, then in Theorem 1.1, C_α can be taken to be $\frac{1}{1152\pi}$ and C'_α can be taken to be 9. Choosing

$$\delta_m = \sqrt{\frac{4\log m}{C_\alpha m}} = 48\sqrt{\frac{2\pi\log(m)}{m}},$$

Theorem 1.1 gives that

$$\begin{aligned}\mathbb{P}\left[d_{BL}\left(\hat{\mu}_m,\mu_\alpha\right)\ge\delta_m\right] &\le e^{\left\{2-C_\alpha m^2\delta_m^2+2m\log m+C'_\alpha m\right\}}+\frac{e}{2\pi}\sqrt{\frac{m}{1-\alpha}}e^{-m}\\ &\le e^{2-2m\log(m)+C'_\alpha m}+ne^{-\frac{n}{e}},\end{aligned}$$

which is summable.

If instead $m < \frac{n}{e}$, then in Theorem 1.1 we may take $C_\alpha = \frac{1}{128\pi(1+\sqrt{3\log(\alpha^-1)})^2}$ and $C'_\alpha = 9\log(\alpha^{-1})$. Choosing

$$\delta_m = \sqrt{\frac{9\log n}{C_\alpha m}} \le \frac{24(1+\sqrt{3\log(\frac{n}{m})})\sqrt{2\pi\log(n)}}{\sqrt{m}} \le \frac{165\sqrt{\log(\frac{n}{m})\log(n)}}{\sqrt{m}},$$

Theorem 1.1 gives that

$$\begin{aligned}\mathbb{P}\left[d_{BL}\left(\hat{\mu}_m,\mu_\alpha\right)\ge\delta_m\right] &\le e^{\left\{2-C_\alpha m^2\delta_m^2+2m\log m+mC'_\alpha\right\}}+\frac{e}{2\pi}\sqrt{\frac{m}{1-\alpha}}e^{-m}\\ &\le e^{2-7m\log(m)}+\frac{e}{2\pi}\sqrt{\frac{m}{1-\alpha}}e^{-m},\end{aligned}$$

which is summable since $m \ge k_n$ and $\frac{k_n}{\log(n)^2}\to\infty$. The claimed result thus follows from the Borel-Cantelli lemma. □

References

[1] Djalil Chafaï, Adrien Hardy, and Mylène Maïda. Concentration for Coulomb gases and Coulomb transport inequalities. *arXiv:1610.00980*, 2016. MR-3820329

[2] Persi Diaconis and Mehrdad Shahshahani. On the eigenvalues of random matrices. *J. Appl. Probab.*, 31A:49–62, 1994. Studies in applied probability. MR-1274717

[3] William Feller. *An introduction to probability theory and its applications. Vol. I.* Third edition. John Wiley & Sons, Inc., New York-London-Sydney, 1968. MR-0228020

[4] Yan V. Fyodorov and Hans-Jürgen Sommers. Statistics of resonance poles, phase shifts and time delays in quantum chaotic scattering: random matrix approach for systems with broken time-reversal invariance. *J. Math. Phys.*, 38(4):1918–1981, 1997. Quantum problems in condensed matter physics. MR-1450906

[5] Tiefeng Jiang. Approximation of Haar distributed matrices and limiting distributions of eigenvalues of Jacobi ensembles. *Probab. Theory Related Fields*, 144(1-2):221–246, 2009. MR-2480790

[6] Elizabeth S. Meckes and Mark W. Meckes. Spectral measures of powers of random matrices. *Electron. Commun. Probab.*, 18:no. 78, 13 pp., 2013. MR-3109633

[7] Dénes Petz and Júlia Réffy. Large deviation for the empirical eigenvalue density of truncated Haar unitary matrices. *Probab. Theory Related Fields*, 133(2):175–189, 2005. MR-2198697

[8] Karol Życzkowski and Hans-Jürgen Sommers. Truncations of random unitary matrices. *J. Phys. A*, 33(10):2045–2057, 2000. MR-1748745