Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2016, Vol. 52, No. 3, 1248–1258
DOI: 10.1214/15-AIHP685

ANNALES
DE L'INSTITUT
HENRI
POINCARÉ
PROBABILITÉS
ET STATISTIQUES
www.imstat.org/aihp

Estimate for $P_t D$ for the stochastic Burgers equation

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Received 13 November 2014; revised 21 April 2015; accepted 21 April 2015

Abstract. We consider the Burgers equation on $H = L^2(0,1)$ perturbed by white noise and the corresponding transition semigroup P_t. We prove a new formula for $P_t D_x \varphi$ which depends on φ but not on its derivative. This formula allows us to provide a bound on $D_x \varphi$ in $L^2(H, \nu)$ where ν is the invariant measure of P_t. Some new consequences for the invariant measure ν of P_t are discussed as its Fomin differentiability and an integration by parts formula which generalises the classical one for Gaussian measures.

Résumé. Nous considèrons l'équation de Burgers stochastique sur $H = L^2(0,1)$ dirigée par un bruit blanc, de semi-groupe de transition P_t, et démontrons une nouvelle formule qui permet d'exprimer $P_t D_x \varphi$ en terme de φ mais pas de sa différentielle. Celle-ci nous permet d'obtenir des estimations sur $D_x \varphi$ dans $L^2(H, \nu)$, où ν est la mesure invariante de P_t, dont découlent quelques conséquences telles que l'existence de dérivées de Fomin pour ν ou encore une formule d'intégration par partie qui généralise celle bien connue pour les mesures gaussiennes.

MSC: 60H15; 35R15

Keywords: Stochastic Burgers equation; invariant measure; Fomin differentiability

1. Introduction

We consider the following stochastic Burgers equation in the interval $[0,1]$ with Dirichlet boundary conditions,

$$\begin{cases} dX(t,\xi) = (\partial_\xi^2 X(t,\xi) + \partial_\xi(X^2(t,\xi)))\,dt + dW(t,\xi), & t > 0, \xi \in (0,1), \\ X(t,0) = X(t,1) = 0, & t > 0, \\ X(0,\xi) = x(\xi), & \xi \in (0,1). \end{cases} \tag{1}$$

The unknown X is a real valued process depending on $\xi \in [0,1]$ and $t \geq 0$ and dW/dt is a space–time white noise on $[0,1] \times [0,\infty)$. This equation has been studied by several authors (see [4,7,9,11]) and it is known that there exists a unique solution with paths in $C([0,T]; L^p(0,1))$ if the initial data $x \in L^p(0,1)$, $p \geq 2$. In this article, we want to prove new properties on the transition semigroup associated to (1).

We rewrite (1) as an abstract differential equation in the Hilbert space $H = L^2(0,1)$,

$$\begin{cases} dX = (AX + b(X))\,dt + dW_t, \\ X(0) = x. \end{cases} \tag{2}$$

[1] Supported in part by GNAMPA from INDAM.
[2] Supported in part by the French government thanks to the ANR program Stosymap and also supported by the French government "Investissements d'Avenir" program ANR-11-LABX-0020-01.

As usual $A = \partial_{\xi\xi}$ with Dirichlet boundary conditions, on the domain $D(A) = H^2(0,1) \cap H_0^1(0,1)$, $b(x) = \partial_\xi(x^2)$. Here and below, for $s \geq 0$, $H^s(0,1)$ is the standard $L^2(0,1)$ based Sobolev space. Also, W is a cylindrical Wiener process on H. We denote by $X(t,x)$ the solution.

We denote by $(P_t)_{t\geq 0}$ the transition semigroup associated to equation (2) on $\mathcal{B}_b(H)$, the space of all real bounded and Borel functions on H endowed with the norm

$$\|\varphi\|_0 = \sup_{x\in H} |\varphi(x)|, \quad \forall \varphi \in \mathcal{B}_b(H).$$

We know that P_t possesses a unique invariant measure ν so that P_t is uniquely extendible to a strongly continuous semigroup of contractions on $L^2(H,\nu)$ (still denoted by P_t) whose infinitesimal generator we shall denote by $\mathcal{L}$. Let $\mathcal{E}_A(H)$ be the linear span of real parts of all φ of the form

$$\varphi_h(x) := e^{i\langle h,x\rangle}, \quad x \in D(A), x \in H,$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in H. We have proved in Section 4.1 of [6] that $\mathcal{E}_A(H)$ is a core for $\mathcal{L}$ and that

$$\mathcal{L}\varphi(x) = \frac{1}{2}\operatorname{Tr}\big[D_x^2\varphi(x)\big] + \big\langle Ax + b(x), D_x\varphi(x)\big\rangle, \quad \forall \varphi \in \mathcal{E}_A(H), x \in D(A). \tag{3}$$

Here and below, D_x denotes the differential with respect to $x \in H$. When φ is a real valued differentiable function, we often identify $D_x\varphi$ with its gradient. Similarly, D_x^2 is the second differential and for a real valued two times differentiable function $D_x^2\varphi$ can be identify with the Hessian.

We write $C_b^1(H)$ for the space of continuously differentiable real valued functions with bounded differential.

In this paper, we use a formula for $P_t D_x\varphi$ (equation (4)) which depends on φ but not on its derivative. To our knowledge, this formula is new.

For a finite dimensional stochastic equation a formula for $P_t D_x$ can be obtained, under suitable assumptions, using the Malliavin calculus and it is the key tool for proving the existence of a density of the law of $X(t,x)$ with respect to the Lebesgue measure, see [12]. Concerning SPDEs, several results are available for densities of finite dimensional projections of the law of the solutions, see [15] and the references therein. For these results, Malliavin calculus is used on a finite dimensional random variable. Malliavin calculus is difficult to generalize to a true infinite dimensional setting and it does not seem useful to give estimate on $P_t D_x\varphi$ in terms of φ. The formula we use allows a completely different approach. It relates $P_t D_x$ to $D_x P_t$. In the recent years several formulae for $D_x P_t\varphi$ independent of $D_x\varphi$ have been proved thanks to suitable generalizations of the Bismut–Elworthy–Li formula (BEL). Thus, combining our formula to estimates obtained on $D_x P_t$ implies useful information on $P_t D_x$. As we shall show, these can be used to extend to the measure ν a basic integration by parts identity well known for Gaussian measures.

Let us explain the main ideas. They are based on the following identity proved in Section 1.2 below.

Proposition 1. *For any $\varphi \in C_b^1(H)$, $x \in H$ and any $h \in D(A)$ we have*

$$P_t\big(\big\langle D_x\varphi(x), h\big\rangle\big) = \big\langle D_x P_t\varphi(x), h\big\rangle - \int_0^t P_{t-s}\big(\big\langle Ah + b'(x)h, D_x P_s\varphi(x)\big\rangle\big)\,ds. \tag{4}$$

This formula could be used to prove the following pointwise estimate: For all $\varphi \in C_b^1(H)$, $\delta > 0$ and all $h \in H^{1+\delta}(0,1)$, we have

$$\big|P_t\big(\langle D_x\varphi, h\rangle\big)(x)\big| \leq ce^{ct}\big(1 + t^{-1/2}\big)\big(1 + |x|_{L^4}\big)^8 \|\varphi\|_0 |h|_{1+\delta}, \tag{5}$$

where $|\cdot|_{1+\delta}$ is the norm in $H^{1+\delta}(0,1)$. We do not give the proof here, it uses similar arguments as in Section 3.

Integrating (4) with respect to ν over H and taking into account the invariance of ν, yields

$$\begin{aligned}\int_H \big\langle D_x\varphi(x), h\big\rangle\nu(dx) &= \int_H \big\langle D_x P_t\varphi(x), h\big\rangle\nu(dx)\\ &\quad - \int_0^t \int_H \big(\big\langle Ah + b'(x)h, D_x P_s\varphi(x)\big\rangle\big)\nu(dx)\,ds.\end{aligned} \tag{6}$$

Using identity (6) we arrive at the main result of the paper, proved in Section 3.

Theorem 2. *For any $p>1$, $\delta>0$, there exists $C>0$ such that for all $\varphi\in C_b^1(H)$ and all $h\in H^{1+\delta}(0,1)$, we have*

$$\left|\int_H \langle D_x\varphi(x),h\rangle \nu(dx)\right| \le C\|\varphi\|_{L^p(H,\nu)}|h|_{1+\delta}, \tag{7}$$

for $t>0$, where $|\cdot|_{1+\delta}$ is the norm in $H^{1+\delta}(0,1)$.

For a Gaussian measure, it is easy to obtain such an estimate. In fact, if μ is the invariant measure of the stochastic heat equation on $(0,1)$, i.e., Equation (2) without the nonlinear term, then the same formula holds with $\delta=0$. Thus our result is not totally optimal and we except that it can be extended to $\delta=0$.

Also, identity (6) is general and we believe that it can be used in many other situations. For instance, we will investigate the generalization of our results to other SPDEs such as reaction–diffusion and 2D- Navier–Stokes equations. This will be the object of a future work.

Finally, we consider the case of a space time white noise in (2). This allows in particular to use previous results from [6]. It is easy to extend to the situation of a noise which has some correlation in space. For instance, if one takes $C^{1/2}\,dW$ in (2) for a bounded C such that $|C^{-1/2}x|\le c|x|_\beta$ for some $\beta\ge 0$ and a constant c, then Theorem 2 still holds provided $\beta<\delta<1$.

In Section 2, we show that Theorem 2 can be used to derive an integration by parts formula for the measure ν. Theorem 2 is proved in Section 3.

1.1. *Notations and preliminaries*

We shall denote by (e_k) an orthonormal basis in H and by (α_k) a sequence of positive numbers such that

$$Ae_k=-\alpha_k e_k,\quad k\in\mathbb{N}.$$

For any $k\in\mathbb{N}$, D_k will represent the directional derivative in the direction of e_k.

The norm of $H=L^2(0,1)$ is denoted by $|\cdot|$. For $p\ge 1$, $|\cdot|_{L^p}$ is the norm of $L^p(0,1)$. The operator A is self–adjoint negative. For any $\alpha\in\mathbb{R}$, $(-A)^\alpha$ denotes the α power of the operator $-A$ and $|\cdot|_\alpha$ is the norm of $D((-A)^{\alpha/2})$ which is equivalent to the norm of the Sobolev space $H^\alpha(0,1)$. We have $|\cdot|_0=|\cdot|=|\cdot|_{L^2}$. We shall use the interpolatory estimate

$$|x|_\beta\le |x|_\alpha^{(\gamma-\beta)/(\gamma-\alpha)}|x|_\gamma^{(\beta-\alpha)/(\gamma-\alpha)},\quad \alpha<\beta<\gamma, \tag{8}$$

and Agmon's inequality

$$|x|_{L^\infty}\le |x|^{1/2}|x|_1^{1/2}. \tag{9}$$

1.2. *Galerkin approximations*

For $m\in\mathbb{N}$, we define the projector P_m onto the first m eigenvectors of A and set $b_m(x)=P_m b(P_m x)$, for $x\in H$. Then, we write the following approximations

$$\begin{cases} dX_m(t,x)=(AX_m(t,x)+b_m(X_m(t,x)))\,dt+P_m\,dW(t),\\ X_m(0,x)=P_m x=x_m,\end{cases} \tag{10}$$

and

$$\begin{cases} \frac{du_m}{dt}=\frac12\operatorname{Tr}[P_m D_x^2 u_m]+\langle Ax+b_m(x),D_x u_m\rangle=\mathcal{L}_m(u_m),\\ u_m(0)=\varphi.\end{cases} \tag{11}$$

We can extend the definition of $u_m(t,x)$ to any $x \in H$ by setting $u_m(t,x) = u_m(t, P_m x)$. Equation (11) has a unique solution given by

$$u_m(t,x) = P_t^m \varphi(x) = \mathbb{E}\big[\varphi\big(X_m(t,x)\big)\big]. \tag{12}$$

Let us sketch now the proof of Proposition 1.

Proof. Let $u(t,x) = P_t \varphi(x)$, where $\varphi \in C_b^1(H)$ and set $u_m(t,x) = P_t^m \varphi(x)$. Then $u_m(t,x)$ is smooth for $t > 0$ and is a solution of the Kolmogorov equation

$$\begin{cases} D_t u_m(t,x) = \frac{1}{2} \operatorname{Tr}[P_m D_x^2 u_m(t,x)] + \langle Ax + b_m(x), D_x u_m(t,x)\rangle, \\ u_m(0,x) = \varphi(P_m x). \end{cases} \tag{13}$$

Set $v_{h,m}(t,x) = \langle D_x u_m(t,x), h\rangle$, the differential of u with respect to x in the direction $h \in P_m(H)$. Then $v_{h,m}$ is a solution to the equation

$$\begin{cases} D_t v_{h,m}(t,x) = \mathcal{L}_m v_{h,m}(t,x) + \langle Ah + b_m'(x)h, D_x u_m(t,x)\rangle, \\ v_{h,m}(0,x) = \langle D_x \varphi(P_m x), h\rangle. \end{cases} \tag{14}$$

By (14) and variation of constants, it follows that

$$v_{h.m}(t,x) = P_t^m\big(\big\langle D_x \varphi(P_m x), h\big\rangle\big) + \int_0^t P_{t-s}^m\big(\big\langle Ah + b_m'(x)h, D_x u_m(s,x)\big\rangle\big)\, ds. \tag{15}$$

Letting $m \to \infty$ we find (4). $\square$

2. Integration by parts formula for ν

This section is devoted to some consequences of Theorem 2. Here we take $p = 2$ for simplicity. In this case (7) can be rewritten as

$$\left| \int_H \big\langle (-A)^{-\alpha} D_x \varphi(x), h\big\rangle \nu(dx) \right| \le C \|\varphi\|_{L^2(H,\nu)} |h|, \quad \forall h \in H, \tag{16}$$

where $\alpha = \frac{1+\delta}{2}$.

Proposition 3. *Let $\alpha > \frac{1}{2}$, then for any $h \in H$ the linear operator*

$$\varphi \in C_b^1(H) \mapsto \big\langle (-A)^{-\alpha} D_x \varphi(x), h\big\rangle \in C_b(H)$$

is closable in $L^2(H,\nu)$.

Proof. Let $(\varphi_n) \subset C_b^1(H)$ and $f \in L^2(H,\nu)$ such that

$$\varphi_n \to 0 \quad \text{in } L^2(H,\nu), \qquad \big\langle (-A)^{-\alpha} D_x \varphi_n(x), h\big\rangle \to f \quad \text{in } L^2(H,\nu).$$

Let $\psi \in C_b^1(H)$, then by (16) it follows that

$$\begin{aligned} &\left| \int_H \big[\psi(x)\big\langle (-A)^{-\alpha} D_x \varphi_n(x), h\big\rangle + \varphi_n(x)\big\langle (-A)^{-\alpha} D_x \psi(x), h\big\rangle\big] \nu(dx) \right| \\ &\quad \le \|\varphi_n \psi\|_{L^2(H,\nu)} |h| \le \|\psi\|_0 \|\varphi_n\|_{L^2(H,\nu)} |h|. \end{aligned}$$

Letting $n \to \infty$, yields

$$\int_H \psi(x) f(x) \nu(dx) = 0,$$

which yields $f = 0$ by the arbitrariness of ψ. $\square$

We can now define the Sobolev space $W^{1,2}_\alpha(H,\nu)$. First we improve Proposition 3.

Corollary 4. *Let $\alpha > \frac{1}{2}$, then the linear operator*

$$\varphi \in C^1_b(H) \mapsto (-A)^{-\alpha} D_x \varphi \in C_b(H; H)$$

is closable in $L^2(H,\nu)$.

Proof. By Proposition 3 taking $h = e_k$ we see that D_k is a closed operator on $L^2(H,\nu)$ for any $k \in \mathbb{N}$. Set, for $\varphi \in C^1_b(H)$,

$$(-A)^{-\alpha} D_x \varphi(x) = \sum_{k=1}^{\infty} \alpha_k^{-\alpha} D_k \varphi(x) e_k, \quad \forall x \in H,$$

the series being convergent in $L^2(H,\nu)$. Then

$$\left|(-A)^{-\alpha} D_x \varphi(x)\right|^2 = \sum_{k=1}^{\infty} \alpha_k^{-2\alpha} \left|D_k \varphi(x)\right|^2.$$

Let $(\varphi_n) \subset C^1_b(H)$ and $F \in L^2(H,\nu; H)$ such that

$$\varphi_n \to 0 \quad \text{in } L^2(H,\nu), \qquad (-A)^{-\alpha} D_x \varphi_n \to F \quad \text{in } L^2(H,\nu; H).$$

We have to show that $F = 0$.

Now for any $k \in \mathbb{N}$ we have $D_k \varphi_n(x) \to \alpha_k^{\alpha} \langle F(x), e_k \rangle$ in $L^2(H,\nu)$. So, $\langle F, e_k \rangle = 0$ and the conclusion follows. $\square$

Let us denote by $W^{1,2}_\alpha(H,\nu)$ the domain of the closure of $(-A)^{-\alpha} D_x$. Then if M^* denotes the adjoint of $(-A)^{-\alpha} D_x$ we have

$$\int_H \left\langle (-A)^{-\alpha} D_x \varphi(x), F(x) \right\rangle \nu(dx) = \int_H \varphi(x) M^*(F)(x) \nu(dx). \tag{17}$$

Set now $F_h(x) = h$ where $h \in H$. By Theorem 2 F_h belongs to the domain of M^*. Setting $M^*(F_h) = v_h$ we obtain the following integration by parts formula.

Proposition 5. *Let $\alpha > \frac{1}{2}$, then for any $h \in H$ there exists a function $v_h \in L^2(H,\nu)$ such that*

$$\int_H \left\langle (-A)^{-\alpha} D_x \varphi(x), h \right\rangle \nu(dx) = \int_H \varphi(x) v_h(x) \nu(dx), \tag{18}$$

for any $\varphi \in W^{1,2}_\alpha(H,\nu)$.

By (18) it follows that the measure ν possesses the Fomin derivative in all directions $(-A)^{-\alpha} h$ for $h \in H$, see, e.g., [14].

If, in (2), $b = 0$ then the Gaussian measure $\mu = N_Q$, where $Q = -\frac{1}{2}A^{-1}$, is the invariant measure and $v_h(x) = \sqrt{2}\langle Q^{-1/2}x, h\rangle$. Then (18) reduces to the usual integration by parts formula for the Gaussian measure μ. Note that it follows that, as already mentioned, Theorem 2 is true with $\delta = 0$ in this case.

We recall the importance of formula (18) for different topics as Malliavin calculus [12], definition of integral on infinite dimensional surfaces of H [3,5,10], definition of BV functions in abstract Wiener spaces [2], infinite dimensional generalization of DiPerna-Lions theory [1,8] and so on.

We think that Theorem 2 opens the possibility to study these topics in the more general situations of non Gaussian measures. Obviously this requires much work to be done. As a first application we prove the following result.

Proposition 6. *Assume that $g \in W^{1,2}_\alpha(H, \nu)$ where $\alpha > 0$ and that*

$$\left|(-A)^{-\alpha} D_x g(x)\right| > 0, \quad \nu\text{-a.e.}$$

Then $\nu \circ g^{-1} \ll \lambda$ where λ is the Lebesgue measure in $\mathbb{R}$.

Proof. Let I be a Borel set of $\mathbb{R}$ such that $\lambda(I) = 0$. Choose a sequence $(\beta_n)_{n\in\mathbb{N}}$ of nonnegative functions in $C^1_b(\mathbb{R})$ such that

$$\beta_n(r) \to \beta(r) = \mathbb{1}_I(r), \quad \lambda\text{-a.e. and } \nu \circ g^{-1}\text{-a.e.}$$

Set

$$\psi_n(r) := \int_0^r \beta_n(s)\,ds$$

and

$$\psi(r) := \int_0^r \beta(s)\,ds = \lambda\big([0, r] \cap I\big) = 0.$$

Then

$$\psi_n(g) \to \psi(g) = 0 \quad \text{pointwise and in } L^2(H, \nu).$$

Moreover

$$(-A)^{-\alpha} D_x\big(\psi_n(g)\big) = \beta_n(g)(-A)^{-\alpha} D_x g \to \beta(g)(-A)^{-\alpha} D_x g \quad \text{in } L^2(H, \nu; H).$$

Therefore

$$\psi_n(g) \to 0 \quad \text{in } L^2(H, \nu),$$
$$(-A)^{-\alpha} D_x\big(\psi_n(g)\big) \to \beta(g)(-A)^{-\alpha} D_x g \quad \text{in } L^2(H, \nu; H).$$

Since $(-A)^{-\alpha} D_x$ is closable we have $\beta(g)(-A)^{-\alpha} D_x g = 0$ and since $|(-A)^{-\alpha} D_x g(x)| > 0\nu$-a.e. we have $\beta(g) = 0$ ν a.e. so that $(\nu \circ g^{-1})(I) = 0$ as required. □

We notice that when $\alpha = \frac{1}{2}$ and ν is Gaussian Proposition 6 is well known, see, e.g., [13], Theorem 2.1.3.

3. Proof of Theorem 2

For $h \in H$, $\eta^h(t, x)$ is the differential of $X(t, x)$ in the direction h and $(\eta^h(t, x))_{t\geq 0}$ satisfies the equation

$$\begin{cases} \frac{d\eta^h(t,x)}{dt} = A\eta^h(t, x) + b'(X(t, x))\eta^h(t, x), \\ \eta^h(0, x) = h. \end{cases} \tag{19}$$

Note that this equation as well as the computations below are done at a formal level. They could easily be justified rigorously by an approximation argument, such as Galerkin approximation, see the end of Section 1. The following result is proved in [6], see Proposition 3.1.

Lemma 7. *For any $\alpha \in [-1, 0]$, there exists $c = c(\alpha) > 0$ such that for all $t \geq 0$, $x, h \in H$*

$$e^{-c\int_0^t |X(s,x)|_{L^4}^{8/3}\,ds}\left|\eta^h(t,x)\right|_\alpha^2 + \int_0^t e^{-c\int_s^t |X(\tau,x)|_{L^4}^{8/3}\,d\tau}\left|\eta^h(s,x)\right|_{1+\alpha}^2 ds \leq |h|_\alpha^2. \tag{20}$$

We introduce the following Feynman–Kac semigroup

$$S_t\varphi(x) = \mathbb{E}\left[\varphi\big(X(t,x)\big)e^{-K\int_0^t |X(s,x)|_{L^4}^4\,ds}\right].$$

Next lemma is a slight generalization of Lemma 3.2 in [6].

Lemma 8. *For any $\varphi \in \mathcal{B}_b(H)$, $t \geq 0$, $x \in H$, $\alpha \in [0, 1]$ and $p > 1$, if K is chosen large enough then we have*

$$\left|D_x S_t\varphi(x)\right|_\alpha \leq ce^{ct}\big(1 + t^{-(1+\alpha)/2}\big)\big(1 + |x|_{L^6}^3\big)\left[\mathbb{E}\big(\varphi^p\big(X(t,x)\big)\big)\right]^{1/p}, \tag{21}$$

where c depends on p, K, α.

Proof. It is clearly sufficient to prove the result for $p \leq 2$. We proceed as in [6] and write

$$\left\langle D_x S_t\varphi(x), h\right\rangle = I_1 + I_2,$$

where

$$I_1 = \frac{1}{t}\mathbb{E}\left(e^{-K\int_0^t |X(s,x)|_{L^4}^4\,ds}\varphi\big(X(t,x)\big)\int_0^t \left\langle\eta^h(s,x), dW(s)\right\rangle\right)$$

and

$$I_2 = -4K\mathbb{E}\left(e^{-K\int_0^t |X(s,x)|_{L^4}^4\,ds}\varphi\big(X(t,x)\big)\int_0^t \left(1 - \frac{s}{t}\right)\left\langle X^3(s,x), \eta^h(s,x)\right\rangle ds\right).$$

For I_1 we have with $\frac{1}{p} + \frac{1}{q} = 1$:

$$\begin{aligned}I_1 &\leq \frac{1}{t}\left[\mathbb{E}\big(\varphi^p\big(X(t,x)\big)\big)\right]^{1/p}\\ &\quad\times\left[\mathbb{E}\left(e^{-Kq\int_0^t |X(s,x)|_{L^4}^4\,ds}\left|\int_0^t \left\langle\eta^h(s,x), dW(s)\right\rangle\right|^q\right)\right]^{1/q}.\end{aligned}$$

Using Itô's formula for $|z(t)|^q = e^{-Kq\int_0^t |X(s,x)|_{L^4}^4\,ds}\,|\int_0^t \langle\eta^h(s,x), dW(s)\rangle|^q$, we get:

$$\begin{aligned}\left|z(t)\right|^q &= -4Kq\int_0^t \left|X(s,x)\right|_{L^4}^4\left|z(s)\right|^q ds\\ &\quad + q\int_0^t e^{-K\int_0^s |X(s,x)|_{L^4}^4\,ds}\left|z(s)\right|^{q-2}z(s)\left\langle\eta^h(s,x), dW(s)\right\rangle\\ &\quad + \frac{1}{2}q(q-1)\int_0^t e^{-2K\int_0^t |X(s,x)|_{L^4}^4\,ds}\left|z(s)\right|^{q-2}\left|\eta^h(s,x)\right|^2 ds.\end{aligned}$$

We deduce:

$$\begin{aligned}\mathbb{E}\Big(\sup_{r\in[0,t]}\big|z(r)\big|^q\Big) &\le q\mathbb{E}\Big(\sup_{r\in[0,t]}\Big|\int_0^r e^{-K\int_0^s |X(s,x)|_{L^4}^4\,ds}\big|z(s)\big|^{q-2}z(s)\big\langle\eta^h(s,x),dW(s)\big\rangle\Big|\Big)\\ &\quad+\frac{1}{2}q(q-1)\mathbb{E}\Big(\int_0^t e^{-2K\int_0^t |X(s,x)|_{L^4}^4\,ds}\big|z(s)\big|^{q-2}\big|\eta^h(s,x)\big|^2\,ds\Big)\\ &= A_1+A_2.\end{aligned}$$

By a standard martingale inequality, (8) and Lemma 7, we have

$$\begin{aligned}A_1 &\le 3q\mathbb{E}\Big(\Big|\int_0^t e^{-2K\int_0^s |X(s,x)|_{L^4}^4\,ds}\big|z(s)\big|^{2(q-1)}\big|\eta^h(s,x)\big|^2\,ds\Big|^{1/2}\Big)\\ &\le 3q\mathbb{E}\Big(\sup_{r\in[0,t]}\big|z(r)\big|^{q-1}\Big(\int_0^t e^{-2K\int_0^s |X(s,x)|_{L^4}^4\,ds}\big|\eta^h(s,x)\big|^2\,ds\Big)^{1/2}\Big)\\ &\le 3q\mathbb{E}\Big(\sup_{r\in[0,t]}\big|z(r)\big|^{q-1}\Big(\int_0^t e^{-2K\int_0^s |X(s,x)|_{L^4}^4\,ds}\big|\eta^h(s,x)\big|_{-\alpha}^{2(1-\alpha)}\big|\eta^h(s,x)\big|_{1-\alpha}^{2\alpha}\,ds\Big)^{1/2}\Big)\\ &\le 3qt^{(1-\alpha)/2}|h|_{-\alpha}\mathbb{E}\Big(\sup_{r\in[0,t]}\big|z(r)\big|^{q-1}\Big)\\ &\le 3qt^{(1-\alpha)/2}|h|_{-\alpha}\Big[\mathbb{E}\Big(\sup_{r\in[0,t]}\big|z(r)\big|^{q}\Big)\Big]^{(q-1)/q}\\ &\le \frac{1}{4}\mathbb{E}\Big(\sup_{r\in[0,t]}\big|z(r)\big|^{q}\Big)+ct^{(q(1-\alpha))/2}|h|_{-\alpha}^q.\end{aligned}$$

Similarly:

$$\begin{aligned}A_2 &\le \frac{1}{2}q(q-1)\mathbb{E}\Big(\sup_{r\in[0,t]}\big|z(r)\big|^{q-2}\int_0^t e^{-2K\int_0^s |X(s,x)|_{L^4}^4\,ds}\big|\eta^h(s,x)\big|^2\,ds\Big)\\ &\le \frac{1}{2}q(q-1)\mathbb{E}\Big(\sup_{r\in[0,t]}\big|z(r)\big|^{q-2}\int_0^t e^{-2K\int_0^t |X(s,x)|_{L^4}^4\,ds}\big|\eta^h(s,x)\big|_{-\alpha}^{2(1-\alpha)}\big|\eta^h(s,x)\big|_{1-\alpha}^{2\alpha}\,ds\Big)\\ &\le \frac{1}{2}q(q-1)t^{1-\alpha}|h|_{-\alpha}^2\mathbb{E}\Big(\sup_{r\in[0,t]}\big|z(r)\big|^{q-2}\Big)\\ &\le \frac{1}{2}q(q-1)t^{1-\alpha}|h|_{-\alpha}^2\Big[\mathbb{E}\Big(\sup_{r\in[0,t]}\big|z(r)\big|^{q}\Big)\Big]^{(q-2)/q}\\ &\le \frac{1}{4}\mathbb{E}\Big(\sup_{r\in[0,t]}\big|z(r)\big|^{q}\Big)+ct^{q(1-\alpha)/2}|h|_{-\alpha}^q.\end{aligned}$$

We deduce:

$$I_1\le ct^{-(1+\alpha)/2}|h|_{-\alpha}\big[\mathbb{E}\big(\varphi^p\big(X(t,x)\big)\big)\big]^{1/p}.$$

For I_2 we write

$$\begin{aligned}I_2 &= 4K\mathbb{E}\Big(e^{-K\int_0^t |X(s,x)|_{L^4}^4\,ds}\varphi\big(X(t,x)\big)\int_0^t\Big(1-\frac{s}{t}\Big)\big\langle X^3(s,x),\eta^h(s,x)\big\rangle\,ds\Big)\\ &\le 4K\big[\mathbb{E}\big(\varphi^p\big(X(t,x)\big)\big)\big]^{1/p}\Big[\mathbb{E}\Big(e^{-Kq\int_0^t |X(s,x)|_{L^4}^4\,ds}\Big(\int_0^t\big|X(s,x)\big|_{L^6}^3\big|\eta^h(s,x)\big|\,ds\Big)^q\Big)\Big]^{1/q}.\end{aligned}$$

By Lemma 7 and Proposition 2.2 in [6]

$$I_2 \le c_q\big(1+|x|_{L^6}^3\big)\big[\mathbb{E}\big(\varphi^p\big(X(t,x)\big)\big)\big]^{1/p}|h|_{-1}.$$

Gathering the estimates on I_1 and I_2 gives the result. □

Lemma 9. *For any $\alpha \in [0,1)$, $p>1$, $q>1$ satisfying $\frac{1}{p}+\frac{1}{q}<1$, if K is chosen large enough there exists a constants c depending on α, p, q such that for any φ Borel bounded and $h : H \to D((-A)^{-\alpha/2})$ Borel such that $\int_H |h(x)|_{-\alpha}^q \nu(dx) < \infty$ we have*

$$\left|\int_H \langle D_x P_t\varphi(x), h(x)\rangle \nu(dx)\right| \le ce^{ct}\big(1+t^{-(1+\alpha)/2}\big)\|\varphi\|_{L^p(H,\nu)}\left(\int_H \big|h(x)\big|_{-\alpha}^q \nu(dx)\right)^{1/q}. \tag{22}$$

Proof. We first prove a similar estimate for S_t. Using Lemma 8 we have by Hölder's inequality

$$\begin{aligned}
&\left|\int_H \langle D_x S_t\varphi(x), h(x)\rangle \nu(dx)\right| \\
&\quad\le ce^{ct}\big(1+t^{-(1+\alpha)/2}\big)\int_H \big(1+|x|_{L^6}^3\big)\big[\mathbb{E}\big(\varphi^p\big(X(t,x)\big)\big)\big]^{1/p}\big|h(x)\big|_{-\alpha}\nu(dx) \\
&\quad\le ce^{ct}\big(1+t^{-(1+\alpha)/2}\big)\left[\int_H \big(1+|x|_{L^6}^3\big)^r \nu(dx)\right]^{1/r} \\
&\qquad\times\left[\int_H \mathbb{E}\big(\varphi^p\big(X(t,x)\big)\big)\nu(dx)\right]^{1/p}\left[\int_H \big|h(x)\big|_{-\alpha}^q \nu(dx)\right]^{1/q},
\end{aligned}$$

with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$. Thus by Proposition 2.3 in [6] and the invariance of ν:

$$\left|\int_H \langle D_x S_t\varphi(x), h(x)\rangle \nu(dx)\right| \le ce^{ct}\big(1+t^{-(1+\alpha)/2}\big)\|\varphi\|_{L^p(H,\nu)}\left(\int_H \big|h(x)\big|_{-\alpha}^q \nu(dx)\right)^{1/q}.$$

We then proceed as in [6] to get a similar estimate on P_t. We write

$$P_t\varphi(x) = S_t\varphi(x) + K\int_0^t S_{t-s}\big(|x|_{L^4}^4 P_s\varphi\big)\,ds.$$

It follows that, using the estimate above with $p>\tilde{p}>1$ such that $\frac{1}{\tilde{p}}+\frac{1}{q}<1$:

$$\begin{aligned}
\left|\int_H \langle D_x P_t\varphi(x), h(x)\rangle \nu(dx)\right| &\le ce^{ct}\big(1+t^{-(1+\alpha)/2}\big)\|\varphi\|_{L^p(H,\nu)}\left(\int_H \big|h(x)\big|_{-\alpha}^q \nu(dx)\right)^{1/q} \\
&\quad + K\int_0^t ce^{c(t-s)}\big(1+(t-s)^{-(1+\alpha)/2}\big)\left(\int_H |x|_{L^4}^4\big|P_s\varphi(x)\big|^{\tilde{p}}\,d\nu(dx)\right)^{1/\tilde{p}} \\
&\quad \times\left(\int_H \big|h(x)\big|_{-\alpha}^q \nu(dx)\right)^{1/q} ds.
\end{aligned}$$

The result follows by Hölder's inequality and the invariance of ν. □

Theorem 2 follows directly from the following result thanks to the invariance of ν and taking for instance $t=1$.

Proposition 10. *For all $p > 1$, $\delta > 0$, there exists a constant $c > 0$ such that for $\varphi \in C_b^1(H)$, and all $h \in H^{1+\delta}(0,1)$, we have*

$$\left|\int_H P_t\big(\langle D_x\varphi, h\rangle\big)(x)\nu(dx)\right| \le ce^{ct}\big(1+t^{-1/2}\big)\|\varphi\|_{L^p(H,\nu)}|h|_{1+\delta}. \tag{23}$$

Proof. By the Poincaré inequality, it is no loss of generality to assume $\delta < \min\{2(1-\frac{1}{p}), \frac{1}{2}\}$. Integrating (4) on H, yields

$$\int_H P_t\big(\langle D_x\varphi, h\rangle\big)(x)\nu(dx) = \int_H \big\langle D_x P_t\varphi(x), h\big\rangle\nu(dx) - \int_0^t\int_H P_{t-s}\big[\big\langle Ah + b'(x)h, D_x P_s\varphi(x)\big\rangle\big]\,ds\,\nu(dx). \tag{24}$$

Then by Lemma 9 we deduce

$$\left|\int_H P_t\big(\langle D_x\varphi, h\rangle\big)(x)\nu(dx)\right| \le ce^{ct}\big(1+t^{-1/2}\big)\|\varphi\|_{L^p(H,\nu)}|h| + \left|\int_H\int_0^t P_{t-s}\big[\big\langle Ah + b'(x)h, D_x P_s\varphi\big\rangle\big]\,ds\,\nu(dx)\right|.$$

By the invariance of ν:

$$\int_H\int_0^t P_{t-s}\big[\big\langle Ah + b'(\cdot)h, D_x P_s\varphi\big\rangle\big]\,ds\,\nu(dx) = \int_H\int_0^t \big\langle Ah + b'(x)h, D_x P_s\varphi\big\rangle\,ds\,\nu(dx).$$

Therefore, by Lemma 9 with $\alpha = 1-\delta$ and $q = \frac{2}{\delta}$:

$$\left|\int_H\int_0^t P_{t-s}\big[\big\langle Ah + b'(x)h, D_x P_s\varphi\big\rangle\big]\,ds\,\nu(dx)\right| \le \int_0^t ce^{c(t-s)}\big(1+s^{-1+\delta/2}\big)\|\varphi\|_{L^p(H,\nu)}\left(\int_H \big|Ah + b'(\cdot)h\big|_{-1+\delta}^{\delta/2}\nu(dx)\right)^{\delta/2}.$$

Note that

$$\big|b'(x)h\big|_{-1+\delta} = \big|\partial_\xi(xh)\big|_{-1+\delta} \le c|xh|_\delta.$$

Then, we have:

$$|xh| \le c|x||h|_1$$

by the embedding $H^1 \subset L^\infty$ and

$$|xh|_1 \le c|x|_1|h|_1,$$

since H^1 is an algebra. We deduce by interpolation

$$|xh|_\delta \le c|x|_\delta|h|_1.$$

It follows

$$\left|\int_H\int_0^t P_{t-s}\big[\big\langle Ah + b'(x)h, D_x P_s\varphi\big\rangle\big]\,ds\,\nu(dx)\right| \le c_\delta e^{ct}\|\varphi\|_{L^p(H,\nu)}\left(1 + \int_H |x|_\delta^{\delta/2}\nu(dx)\right)^{2/\delta}|h|_{1+\delta}.$$

We need to estimate $\int_H |x|_\delta^{\delta/2}\nu(dx)$. We use the notation of [6], Proposition 2.2,

$$\begin{aligned}\big|X(s,x)\big|_\delta &\le \big|Y(s,x)\big|_\delta + \big|z_\alpha(s)\big|_\delta \\ &\le \big|Y(s,x)\big|^{1-\delta}\big|Y(s,x)\big|_1^\delta + \big|z_\alpha(s)\big|_\delta.\end{aligned}$$

Using computation in [6], Proposition 2.2, we obtain

$$\sup_{t\in[0,1]}\big|Y(t,x)\big|^2 + \int_0^1 \big|Y(s,x)\big|_1^2\,ds \le c\big(|x|^2+\kappa\big),$$

where κ is a random variable with all moments finite. It follows by (8):

$$\mathbb{E}\bigg(\int_0^1 \big|Y(s,x)\big|_\delta^{2/\delta}\,ds\bigg) \le \mathbb{E}\bigg(\int_0^1 \big|Y(s,x)\big|^{2(1-\delta)/\delta}\big|Y(s,x)\big|_1^2\,ds\bigg) \le c\big(|x|^2+1\big)^{1/\delta}.$$

Generalizing slightly Proposition 2.1 in [6], we have:

$$\mathbb{E}\big(\big|z_\alpha(t)\big|_\delta^p\big) \le c_{\delta,p}$$

for $t\in[0,1]$, $\delta<1/2$, $\alpha\ge 1$, $p\ge 1$. We deduce:

$$\mathbb{E}\bigg(\int_0^1 \big|X(s,x)\big|_\delta^{2/\delta}\,ds\bigg) \le c\big(|x|^2+1\big)^{1/\delta}.$$

Integrating with respect to ν and using Proposition 2.3 in [6] we deduce:

$$\int_H |x|_\delta^{\delta/2}\nu(dx) \le c_\delta$$

Then (23) follows. □

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