Brazilian Journal of Probability and Statistics
2015, Vol. 29, No. 4, 778–789
DOI: 10.1214/14-BJPS246

Parameter estimations for the sub-fractional Brownian motion with drift at discrete observation

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Abstract. In this paper, we investigate the L^2-consistency and the strong consistency of the maximum likelihood estimators (MLE) of the mean and variance of the sub-fractional Brownian motion with drift at discrete observation. By combining the Stein's method with Malliavin calculus, we obtain the central limit theorem and the Berry–Esséen bounds for these estimators.

1 Introduction and main results

We consider the sub-fractional Brownian motion with drift determined by the following stochastic differential equation:

$$dX_t = \mu\,dt + \sigma\,dS_t^H, \qquad t \geq 0, X_0 = 0, \tag{1.1}$$

where S^H is a sub-fractional Brownian motion with Hurst index $H \in (\frac{1}{2}, 1)$, μ and σ are unknown parameters to be estimated from discrete observations of the process X. We assume that the process is observed at discrete time instants $(t_1, t_2, \dots, t_n)$. To simplify notation, we assume $t_k = kh$, $k = 1, 2, \dots, n$ for some fixed length $h > 0$. Thus, the observation vector is $\mathbf{X} = (X_{t_1}, X_{t_2}, \dots, X_{t_n})$. We will obtain the maximum likelihood estimators $\hat{\mu}$ and $\hat{\sigma}^2$ of μ and σ^2, respectively, and study their asymptotic behaviors (as $n \to \infty$ for a fixed h), in particular, the almost sure convergence, the central limit theorem, and the Berry–Esséen bounds. Shen and Yan (2014) discussed the problem of efficient estimation for the drift of sub-fractional Brownian motion using technique based on the Girsanov theorem and constructed a class of biased estimators of the James–Stein type.

The parameter estimation problem for long memory processes, in particular, the estimation for the Hurst parameter H, were extensively studied (see Beran (1994), Fox and Taqqu (1986), Hannan (1973), Palma (2007)). One of the most famous approaches for estimating H is the so-called R/S (rescaled analysis) method. Due to this fact, we will not discuss the estimation of H and will concentrate on μ and σ^2. Although most works require the process to be stationary, we may still

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Key words and phrases. Maximum likelihood estimator, sub-fractional Brownian motion, Stein's method, Malliavin calculus.
Received November 2013; accepted May 2014.

adapt their idea to analyze the above model (1.1). But we shall use the method of Hu and Nualart (2010) which seems to be the simplest one to us. This method is based on a result of Nualart and Ortiz-Latorre (2008) and uses the idea of Malliavin calculus (see also Privault and Réveillac (2008)). To obtain the Berry–Esséen bounds, we use the Stein's method, following the works of Nourdin and Peccati (2009) and Hu et al. (2011).

We introduce the notation

$$\mathbf{X} = \mu \mathbf{t} + \sigma \mathbf{S}_{\mathbf{t}}^H, \tag{1.2}$$

where, also for the rest of the paper, $\mathbf{t} = (h, 2h, \dots, nh)'$ and $\mathbf{S}_{\mathbf{t}}^H = (S_h^H, \dots, S_{nh}^H)'$. The joint probability density function of $\mathbf{X}$ is

$$f(\mathbf{X}) = \frac{1}{(\sqrt{2\pi\sigma^2})^n |\Sigma_H|^{1/2}} \exp\left(-\frac{1}{2\sigma^2}(\mathbf{X} - \mu\mathbf{t})'\Sigma_H^{-1}(\mathbf{X} - \mu\mathbf{t})\right),$$

where

$$\begin{aligned}\Sigma_H &= \left(\mathrm{Cov}\left(S_{ih}^H, S_{jh}^H\right)\right)_{i,j=1,2,\dots,n} \\ &= h^{2H}\left(i^{2H} + j^{2H} - \frac{1}{2}\left((i+j)^{2H} + |i-j|^{2H}\right)\right)_{i,j=1,2,\dots,n}.\end{aligned}$$

The maximum likelihood estimators of μ and σ^2 from the observation $\mathbf{X}$ are given by

$$\hat{\mu} = \frac{\mathbf{t}'\Sigma_H^{-1}\mathbf{X}}{\mathbf{t}'\Sigma_H^{-1}\mathbf{t}}, \tag{1.3}$$

$$\hat{\sigma}^2 = \frac{1}{n}\frac{(\mathbf{X}'\Sigma_H^{-1}\mathbf{X})(\mathbf{t}'\Sigma_H^{-1}\mathbf{t}) - (\mathbf{t}'\Sigma_H^{-1}\mathbf{X})^2}{\mathbf{t}'\Sigma_H^{-1}\mathbf{t}}. \tag{1.4}$$

Now we state our main results as follows.

Theorem 1.1. *The estimator $\hat{\mu}$ of μ is unbiased and converges in mean square to μ as $n \to \infty$.*

Theorem 1.2. *We have*

$$\mathbf{E}(\hat{\sigma}^2) = \frac{n-1}{n}\sigma^2 \quad \textit{and} \quad \mathrm{Var}(\hat{\sigma}^2) \xrightarrow{a.s.} 0 \qquad \textit{as } n \to \infty. \tag{1.5}$$

Theorem 1.3. *The estimators $\hat{\mu}$ and $\hat{\sigma}^2$ are strongly consistent, that is,*

$$\hat{\mu} \xrightarrow{a.s.} \mu \qquad \textit{as } n \to \infty, \tag{1.6}$$

$$\hat{\sigma}^2 \xrightarrow{a.s.} \sigma^2 \qquad \textit{as } n \to \infty. \tag{1.7}$$

Theorem 1.4. *We have*

$$\sqrt{\mathbf{t}'\Sigma_H^{-1}\mathbf{t}}(\hat{\mu}-\mu) \xrightarrow{law} N(0,\sigma^2) \qquad \text{as } n\to\infty, \tag{1.8}$$

$$\frac{1}{\sigma^2}\sqrt{\frac{n}{2}}(\hat{\sigma}^2-\sigma^2) \xrightarrow{law} N(0,1) \qquad \text{as } n\to\infty, \tag{1.9}$$

where $N(\mu,\sigma^2)$ is normal distribution.

Theorem 1.5. *Let*

$$G_n = \frac{1}{\sigma^2}\sqrt{\frac{n}{2}}(\hat{\sigma}^2-\sigma^2), \tag{1.10}$$

$$\bar{G}_n = G_n - \mathbf{E}(G_n). \tag{1.11}$$

Then we have

(1) $\sup_{z\in\mathbf{R}}|\mathbf{P}(\bar{G}_n\le z)-\Phi(z)| \le \frac{\sqrt{2n-1}}{n}$, *where* $\Phi(z)=\int_{-\infty}^{z}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx$.

(2) $\frac{n}{\sqrt{2n-1}}(\mathbf{P}(\bar{G}_n\le z)-\Phi(z)) \to -\frac{\Phi^{(3)}(z)}{3}$, *for every* $z\in\mathbf{R}$, *as* $n\to\infty$, *where* $\Phi^{(3)}(z)=(z^2-1)\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ *is the third-order derivative of* $\Phi(z)$.

(3) *There exists a constant* $C\in(0,1)$, *as well as an integer* $n_0\ge 1$, *such that*

$$C < \frac{n}{\sqrt{2n-1}}\sup_{z\in\mathbf{R}}\bigl|\mathbf{P}(\bar{G}_n\le z)-\Phi(z)\bigr| \le 1,$$

for every $n\ge n_0$.

The rest of our paper is organized as follows. Section 2 contains the preliminaries tools that we will need throughout the paper: sub-fractional Brownian motion, Malliavin derivative and Skorohod integral. Section 3 contains proofs of our results.

2 Preliminaries

In this section, we describe some basic facts on the stochastic calculus with respect to sub-fractional Brownian motion. Some surveys and complete literatures could be found in Alòs et al. (2001), Nualart (2006) and Tudor (2008). As an extension of Brownian motion, recently, Bojdecki et al. (2004) introduced and studied a rather special class of self-similar Gaussian processes which preserves many properties of fractional Brownian motion. This process arises from occupation time fluctuations of branching particle systems with poisson initial condition. This process is called the sub-fractional Brownian motion with index $H\in(0,1)$ is mean zero Gaussian process $\{S_t^H, t\ge 0\}$ with $S_0^H=0$ and the covariance

$$\begin{aligned} C_H(s,t) &= \mathbf{E}\bigl(S_s^H S_t^H\bigr) \\ &= s^{2H}+t^{2H}-\frac{1}{2}\bigl[(s+t)^{2H}+|s-t|^{2H}\bigr], \qquad t\ge 0, s\ge 0. \end{aligned} \tag{2.1}$$

For $H = \frac{1}{2}$, S^H coincides with the standard Brownian motion W. S^H is neither a semimartingale nor a Markov process unless $H = \frac{1}{2}$, so many of the powerful techniques from stochastic analysis are not available when dealing with S^H. The sub-fractional Brownian motion has properties analogous to those of fractional Brownian motion (self-similarity, long-range dependence, Hölder paths), and satisfies the following estimates:

$$\begin{aligned}\big[(2-2^{2H-1}) \wedge 1\big]|t-s|^{2H} &\le \mathbf{E}\big|S_t^H - S_s^H\big|^2 \\ &\le \big[(2-2^{2H-1}) \vee 1\big]|t-s|^{2H}.\end{aligned} \tag{2.2}$$

Let $(S_t^H, t \in [0, T])$ be a sub-fractional Brownian motion with $\frac{1}{2} < H < 1$, defined on the complete probability space $(\Omega, \mathcal{F}, P)$. By Kolmogorov's continuity criterion and (2.2), we deduce that S^H has Hölder continuous paths of order $H - \epsilon$, for all $\epsilon \in (0, H)$. The main properties of a sub-fractional Brownian motion were studied by Bojdecki et al. (2004).

Fix a time interval $[0, T]$. We denote by $\mathcal{H}_{S^H}$ canonical Hilbert space associated to the sub-fractional Brownian motion S^H. That is, $\mathcal{H}_{S^H}$ is the closure of the linear span ε generated by the indicator function with respect to the scalar product

$$\langle \mathbf{I}_{[0,s]}, \mathbf{I}_{[0,t]} \rangle = C_H(s, t).$$

We know that the covariance of sub-fractional Brownian motion can be written as

$$C_H(s, t) = \mathbf{E}\big(S_s^H S_t^H\big) = \int_0^t \int_0^s \phi_H(u, v)\, du\, dv, \tag{2.3}$$

where $\phi_H(u, v) = H(2H-1)[|u-v|^{2H-2} - (u+v)^{2H-2}]$ and $\frac{1}{2} < H < 1$.

We can find a linear space of functions contained in $\mathcal{H}_{S^H}$ in the following way. Let $|\mathcal{H}_{S^H}|$ be the linear space of measurable functions on $[0, T]$ such that

$$\|\varphi\|^2_{|\mathcal{H}_{S^H}|} = c^2(H) \int_0^T \left(\int_s^T |\varphi_t| \frac{\partial n_H}{\partial t}(t, s)\, dt \right)^2 ds < \infty,$$

where

$$\begin{aligned} c^2(H) &= \frac{\Gamma(1+2H)\sin(\pi H)}{\pi}, \\ n_H(t, s) &= \frac{2^{1-H}\sqrt{\pi} s^{3/2-H}}{\Gamma(H-1/2)} \left(\int_s^t \big(x^2 - s^2\big)^{H-3/2} dx \right) \mathbf{I}_{(0,t)}(s). \end{aligned}$$

It is easy to check that (see Mendy (2013))

$$\|\varphi\|^2_{|\mathcal{H}_{S^H}|} = \int_0^T \int_0^T |\varphi_t| |\varphi_s| \phi_H(t, s)\, dt\, ds.$$

It is not difficult to show that $|\mathcal{H}_{S^H}|$ is a Banach space with the norm $\|\cdot\|_{|\mathcal{H}_{S^H}|}$ and ε is dense in $|\mathcal{H}_{S^H}|$. We have $L^2([0, T]) \subset L^{1/H}([0, T]) \subset |\mathcal{H}_{S^H}| \subset \mathcal{H}_{S^H}$.

We will introduce some elements of the Malliavin calculus associated with S^H. We refer to Nualart (2006) for detailed account these notions. Let $C_b^\infty(\mathbf{R}^n, \mathbf{R})$ be the class of infinitely differentiable functions $f : \mathbf{R}^n \to \mathbf{R}$ such that f and all its partial derivatives are bounded. We denote by S the class of smooth cylindrical random variables $F = f(S^H(\varphi_1), \dots, S^H(\varphi_n))$, for $\varphi_i \in \mathcal{H}_{S^H}, i = 1, \dots, n$ and $f \in C_b^\infty(\mathbf{R}^n, \mathbf{R})$. The Malliavin derivative operator D of a smooth cylindrical random variables $F = f(S^H(\varphi_1), \dots, S^H(\varphi_n))$ is defined as the $\mathcal{H}_{S^H}$-valued random variable

$$D_s F = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(S^H(\varphi_1), \dots, S^H(\varphi_n))\varphi_j(s), \qquad s \in [0, T].$$

In particular, $D_s S_t^H = \mathbf{I}_{[0,t]}(s)$. As usual, $\mathbf{D}^{1,2}$ denotes the closure of the set of smooth random variables with respect to the norm

$$\|F\|_{1,2}^2 = \mathbf{E}(F^2) + \mathbf{E}\big[\|DF\|_{\mathcal{H}_{S^H}}^2\big].$$

The Skorohod integral δ is the adjoint of the derivative operator D. If a random variable $u \in L^2(\Omega, \mathcal{H}_{S^H})$ belongs to the domain of the Skorohod integral (denoted by $\operatorname{dom}\delta$), that is, if it verifies

$$\big|\mathbf{E}\langle DF, u\rangle_{\mathcal{H}_{S^H}}\big| \le c_u \sqrt{\mathbf{E}(F^2)} \qquad \text{for any } F \in S,$$

then $\delta(u)$ is defined by the duality relationship

$$\mathbf{E}\big[\delta(u)F\big] = \mathbf{E}\big[\langle DF, u\rangle_{\mathcal{H}_{S^H}}\big],$$

for every $F \in \mathbf{D}^{1,2}$.

For every $q \ge 1$, let $\mathcal{H}_q$ be the qth Wiener chaos of S^H, that is, the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{H_q(S^H(h)), h \in \mathcal{H}_{S^H}, \|h\|_{\mathcal{H}_{S^H}} = 1\}$, where H_q is the qth Hermite polynomial. The mapping $I_q(h^{\otimes q}) = H_q(S^H(h))$ provides a linear isometry between the symmetric tensor product $\mathcal{H}_{S^H}^{\odot q}$ (equipped with the modified norm $\|\cdot\|_{\mathcal{H}_{S^H}^{\odot q}} = \sqrt{q!}\|\cdot\|_{\mathcal{H}_{S^H}^{\otimes q}}$) and $\mathcal{H}_q$. Specifically, for all $f, g \in \mathcal{H}_{S^H}^{\odot q}$ and $q \ge 1$, one has

$$\mathbf{E}\big[I_q(f)I_q(g)\big] = q!\langle f, g\rangle_{\mathcal{H}_{S^H}^{\otimes q}}.$$

On the other hand, it is well known that any random variable Z belonging to $L^2(\Omega)$ admits the following chaotic expansion:

$$Z = \mathbf{E}[Z] + \sum_{q=1}^\infty I_q(f_q),$$

where the series converges in $L^2(\Omega)$ and the kernels f_q, belonging to $\mathcal{H}_{S^H}^{\odot q}$, are uniquely determined by Z.

The Ornstein–Uhlenbeck operator L is defined by $LF = -\delta DF$. If $F = I_q(f_q)$ is in the qth Wiener chaos of S^H, namely, $f_q \in \mathcal{H}_{S^H}^{\odot q}$, then $LF = -qF$.

3 Proofs of theorems

This section is devoted to the proofs of Theorems 1.1–1.5. In what follows, we use the same symbol c for all constants whose precise value is not important for our consideration.

Proof of Theorem 1.1. By (1.2) and (1.3), we have

$$\hat{\mu} = \mu + \sigma \frac{\mathbf{t}'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H}{\mathbf{t}'\Sigma_H^{-1}\mathbf{t}}. \tag{3.1}$$

Thus, $\mathbf{E}(\hat{\mu}) = \mu$, and hence $\hat{\mu}$ is unbiased. On the other hand, we have

$$\begin{aligned}\mathbf{E}\big[(\hat{\mu}-\mu)^2\big] &= \sigma^2 \mathbf{E}\Big[\frac{\mathbf{t}'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H(\mathbf{S}_\mathbf{t}^H)'\Sigma_H^{-1}\mathbf{t}}{(\mathbf{t}'\Sigma_H^{-1}\mathbf{t})^2}\Big] \\ &= \sigma^2 \frac{\mathbf{t}'\Sigma_H^{-1}\Sigma_H\Sigma_H^{-1}\mathbf{t}}{(\mathbf{t}'\Sigma_H^{-1}\mathbf{t})^2} \\ &= \frac{\sigma^2}{\mathbf{t}'\Sigma_H^{-1}\mathbf{t}}.\end{aligned}$$

Denote

$$M = (m_{ij})_{i,j=1,2,\dots,n}, \qquad \text{where } m_{ij} = i^{2H} + j^{2H} - \frac{1}{2}\big[(i+j)^{2H} + |i-j|^{2H}\big],$$

and denote by m_{ij}^{-1} the entry of the inverse matrix M^{-1} of M. Then we may write

$$\begin{aligned}\mathbf{E}\big[(\hat{\mu}-\mu)^2\big] &= h^{2H}\frac{\sigma^2}{\mathbf{t}'M^{-1}\mathbf{t}} \\ &= \frac{h^{2H-2}\sigma^2}{\sum_{i,j=1}^n ijm_{ij}^{-1}}.\end{aligned}$$

We shall use the following inequality (with $x = \mathbf{n} = (1, 2, \dots, n)$):

$$x'M^{-1}x \geq \frac{\|x\|_2^2}{\lambda_{\max}},$$

where $\lambda_{\max}$ is the largest eigenvalue of the matrix M. Thus, we have

$$\mathbf{E}\big[(\hat{\mu}-\mu)^2\big] \leq h^{2H-2}\sigma^2 \frac{\lambda_{\max}}{\|\mathbf{n}\|_2^2}.$$

Since $\|\mathbf{n}\|_2^2 = \frac{n(n+1)(2n+1)}{6}$, we know that $\|\mathbf{n}\|_2^2 \approx \frac{1}{3}n^3$. On the other hand, by the Gerschgorin circle theorem (see Golub and van Loan (1996), Theorem 8.1.3), we have

$$\lambda_{\max} \le \max_{i=1,2,\dots,n} \sum_{j=1}^{n} |m_{ij}| \le cn^{2H+1}.$$

Consequently, we have

$$\mathbf{E}\big[(\hat{\mu}-\mu)^2\big] \le ch^{2H-2}\sigma^2 n^{2H+1} n^{-3} \le cn^{2H-2},$$

which converges to zero as $n \to \infty$. Therefore, the proof is complete. □

Remark. The essence of the proof of Theorem 1.1 is the same as that of Theorem 1 in Kuang and Xie (2013), however, we express $\hat{\mu}$ in (3.1) in the form of matrix, and it is simpler on aspect of calculation than (5) in Kuang and Xie (2013).

Proof of Theorem 1.2. By (1.2) and (1.4), we have

$$\hat{\sigma}^2 = \frac{\sigma^2}{n}\left((\mathbf{S}_{\mathbf{t}}^H)'\Sigma_H^{-1}\mathbf{S}_{\mathbf{t}}^H - \frac{(\mathbf{t}'\Sigma_H^{-1}\mathbf{S}_{\mathbf{t}}^H)^2}{\mathbf{t}'\Sigma_H^{-1}\mathbf{t}}\right).$$

Thus,

$$\begin{aligned}\mathbf{E}(\hat{\sigma}^2) &= \frac{\sigma^2}{n}\mathbf{E}\left((\mathbf{S}_{\mathbf{t}}^H)'\Sigma_H^{-1}\mathbf{S}_{\mathbf{t}}^H - \frac{(\mathbf{t}'\Sigma_H^{-1}\mathbf{S}_{\mathbf{t}}^H)^2}{\mathbf{t}'\Sigma_H^{-1}\mathbf{t}}\right)\\ &= \frac{\sigma^2}{n}\left(n - \frac{\mathbf{t}'\Sigma_H^{-1}\mathbf{E}[\mathbf{S}_{\mathbf{t}}^H(\mathbf{S}_{\mathbf{t}}^H)']\Sigma_H^{-1}\mathbf{t}}{\mathbf{t}'\Sigma_H^{-1}\mathbf{t}}\right)\\ &= \frac{n-1}{n}\sigma^2.\end{aligned}$$

To compute the variance of $\hat{\sigma}^2$, we introduce $\mathbf{Y} = \Sigma_H^{-1/2}\mathbf{S}_{\mathbf{t}}^H$. Then

$$\mathbf{E}(\mathbf{Y}\mathbf{Y}') = \mathbf{E}(\Sigma_H^{-1/2}\mathbf{S}_{\mathbf{t}}^H(\mathbf{S}_{\mathbf{t}}^H)'\Sigma_H^{-1/2}) = I.$$

Therefore, $\mathbf{Y}$ is a standard Gaussian vector of dimension n. For any λ small enough and $\epsilon \in \mathbf{R}$, let us compute the following expectation:

$$\begin{aligned}&\mathbf{E}\big(\exp\big(\lambda(\mathbf{S}_{\mathbf{t}}^H)'\Sigma_H^{-1}\mathbf{S}_{\mathbf{t}}^H + \epsilon\mathbf{t}'\Sigma_H^{-1}\mathbf{S}_{\mathbf{t}}^H\big)\big)\\ &\quad= \mathbf{E}\big(\exp\big(\lambda\|\mathbf{Y}\|^2 + \epsilon\mathbf{t}'\Sigma_H^{-1/2}\mathbf{Y}\big)\big)\\ &\quad= \frac{1}{(\sqrt{2\pi})^n}\int_{\mathbf{R}^n}\exp\left(-\frac{\|\mathbf{Y}\|^2}{2} + \lambda\|\mathbf{Y}\|^2 + \epsilon\mathbf{t}'\Sigma_H^{-1/2}\mathbf{Y}\right)d\mathbf{Y}.\end{aligned}$$

A standard technique of completing the squares yields

$$\begin{aligned}&\mathbf{E}\big(\exp\big(\lambda(\mathbf{S}_{\mathbf{t}}^H)'\Sigma_H^{-1}\mathbf{S}_{\mathbf{t}}^H + \epsilon\mathbf{t}'\Sigma_H^{-1}\mathbf{S}_{\mathbf{t}}^H\big)\big)\\ &\qquad= (1-2\lambda)^{-n/2}\exp\left(\frac{\epsilon^2\mathbf{t}'\Sigma_H^{-1}\mathbf{t}}{2(1-2\lambda)}\right) =: f(\lambda,\epsilon).\end{aligned}$$

We are only interested in the coefficients of λ^2, $\lambda\epsilon^2$ and ϵ^4 in the above expression. We have

$$\begin{aligned}
f(\lambda,\epsilon) &= \left(1+n\lambda+\frac{n(n+2)}{2}\lambda^2+\cdots\right)\left(1+\frac{\epsilon^2\mathbf{t}'\Sigma_H^{-1}\mathbf{t}}{2}(1+2\lambda+\cdots)\right. \\
&\qquad\left.+\frac{\epsilon^4(\mathbf{t}'\Sigma_H^{-1}\mathbf{t})^2}{8}(1+2\lambda+\cdots)^2+\cdots\right) \\
&= 1+n\lambda+\frac{n(n+2)}{2}\lambda^2+\cdots+\frac{1}{2}(n+2)\lambda\epsilon^2\mathbf{t}'\Sigma_H^{-1}\mathbf{t} \\
&\quad+\cdots+\frac{\epsilon^4(\mathbf{t}'\Sigma_H^{-1}\mathbf{t})^2}{8}+\cdots.
\end{aligned}$$

Comparing the coefficients of λ^2, $\lambda\epsilon^2$ and ϵ^4, we have

$$\begin{aligned}
\mathbf{E}\big(\big((\mathbf{S}_\mathbf{t}^H)'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H\big)^2\big) &= n(n+2), \\
\mathbf{E}\big((\mathbf{S}_\mathbf{t}^H)'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H\big(\mathbf{t}'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H\big)^2\big) &= (n+2)\big(\mathbf{t}'\Sigma_H^{-1}\mathbf{t}\big), \\
\mathbf{E}\big(\big(\mathbf{t}'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H\big)^4\big) &= 3\big(\mathbf{t}'\Sigma_H^{-1}\mathbf{t}\big)^2.
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}
\mathrm{Var}(\hat\sigma^2) &= \mathbf{E}\big[(\hat\sigma^2)^2\big]-\big[\mathbf{E}(\hat\sigma^2)\big]^2 \\
&= \frac{\sigma^4}{n^2}\Big\{\mathbf{E}\big(\big((\mathbf{S}_\mathbf{t}^H)'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H\big)^2\big)-\frac{2}{\mathbf{t}'\Sigma_H^{-1}\mathbf{t}}\mathbf{E}\big((\mathbf{S}_\mathbf{t}^H)'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H\big(\mathbf{t}'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H\big)^2\big) \\
&\qquad+\frac{1}{(\mathbf{t}'\Sigma_H^{-1}\mathbf{t})^2}\mathbf{E}\big(\big(\mathbf{t}'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H\big)^4\big)-(n-1)^2\Big\} \\
&= \frac{\sigma^4}{n^2}\big(n(n+2)-2(n+2)+3-(n-1)^2\big) \\
&= \frac{2(n-1)}{n^2}\sigma^4,
\end{aligned}$$

which is convergent to 0. Thus, we complete the proof of (1.5). □

Proof of Theorem 1.3. Let us prove the convergence of $\hat\mu$ first. We will use the Borel–Cantelli lemma. To this end, we will show that

$$\sum_{n=1}^{\infty}\mathbf{P}\left(|\hat\mu-\mu|>\frac{1}{n^\gamma}\right)<\infty, \tag{3.2}$$

for some $\gamma>0$.

Take $0 < \gamma < 1 - H$. Then from the Chebyshev's inequality and the Nelson's hypercontractivity inequality (see Hu (2000)), we have

$$\begin{aligned}
\mathbf{P}\left(|\hat{\mu} - \mu| > \frac{1}{n^{\gamma}}\right) &\leq n^{q\gamma}\mathbf{E}\left(|\hat{\mu} - \mu|^{q}\right) \\
&\leq cn^{q\gamma}\left(\mathbf{E}\left(|\hat{\mu} - \mu|^{2}\right)\right)^{q/2} \\
&\leq c\sigma^{q}h^{(H-1)q}n^{q\gamma+(H-1)q}.
\end{aligned}$$

For sufficiently large q, we have $q\gamma + (H-1)q < -1$. Thus (3.2) is proved, which implies (1.6) by the Borel–Cantelli lemma.

Now we prove the convergence of $\hat{\sigma}^2$. Take $0 < \gamma < \frac{1}{2}$, then

$$\begin{aligned}
\mathbf{P}\left(\left|\hat{\sigma}^2 - \sigma^2\right| > \frac{1}{n^{\gamma}}\right) &\leq n^{q\gamma}\mathbf{E}\left(\left|\hat{\sigma}^2 - \sigma^2\right|^{q}\right) \\
&\leq cn^{q\gamma}\left(\mathbf{E}\left(\left|\hat{\sigma}^2 - \sigma^2\right|^{2}\right)\right)^{q/2}.
\end{aligned}$$

Since

$$\mathbf{E}\left(\left|\hat{\sigma}^2 - \sigma^2\right|^{2}\right) = \frac{2n-1}{n^2}\sigma^4,$$

we have

$$\begin{aligned}
\mathbf{P}\left(\left|\hat{\sigma}^2 - \sigma^2\right| > \frac{1}{n^{\gamma}}\right) &\leq cn^{q\gamma}\left(\frac{2n-1}{n^2}\right)^{q/2}\sigma^{2q} \\
&\leq cn^{q\gamma}\left(\frac{2}{n}\right)^{q/2}\sigma^{2q} \\
&\leq cn^{q(\gamma-1/2)}.
\end{aligned}$$

For sufficiently large q, we have $q(\gamma - \frac{1}{2}) < -1$. Thus, (1.7) is proved by Borel–Cantelli lemma. □

In order to prove Theorem 1.4, we need the following lemma.

Lemma 3.1. *Recall the definition of G_n defined by* (1.10). *We have*

$$\|DG_n\|^2_{\mathcal{H}_{SH}} = \frac{2\hat{\sigma}^2}{\sigma^2}.$$

Proof. Note that

$$G_n = \frac{1}{\sqrt{2n}}\left(\left(\mathbf{S}_{\mathbf{t}}^{H}\right)'\Sigma_H^{-1}\mathbf{S}_{\mathbf{t}}^{H} - \frac{\left(\mathbf{t}'\Sigma_H^{-1}\mathbf{S}_{\mathbf{t}}^{H}\right)^2}{\mathbf{t}'\Sigma_H^{-1}\mathbf{t}}\right) - \sqrt{\frac{n}{2}}.$$

Denote by $(D\mathbf{S}_\mathbf{t}^H)'$ the vector $(I_{[0,h]}, I_{[0,2h]}, \dots, I_{[0,nh]})'$. Then

$$DG_n = \frac{2}{\sqrt{2n}}\left((D\mathbf{S}_\mathbf{t}^H)'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H - \frac{(\mathbf{t}'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H)(\mathbf{t}'\Sigma_H^{-1}D\mathbf{S}_\mathbf{t}^H)}{\mathbf{t}'\Sigma_H^{-1}\mathbf{t}}\right).$$

As a consequence,

$$\begin{aligned}\|DG_n\|^2_{\mathcal{H}_{SH}} = \frac{2}{n}\bigg\{&\|(D\mathbf{S}_\mathbf{t}^H)'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H\|^2_{\mathcal{H}_{SH}} + \frac{(\mathbf{t}'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H)^2\|\mathbf{t}'\Sigma_H^{-1}D\mathbf{S}_\mathbf{t}^H\|^2_{\mathcal{H}_{SH}}}{(\mathbf{t}'\Sigma_H^{-1}\mathbf{t})^2}\\ &- \frac{2(\mathbf{t}'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H)\langle(D\mathbf{S}_\mathbf{t}^H)'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H, \mathbf{t}'\Sigma_H^{-1}D\mathbf{S}_\mathbf{t}^H\rangle_{\mathcal{H}_{SH}}}{\mathbf{t}'\Sigma_H^{-1}\mathbf{t}}\bigg\}.\end{aligned}$$

Notice that

$$\begin{aligned}\|(D\mathbf{S}_\mathbf{t}^H)'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H\|^2_{\mathcal{H}_{SH}} &= \|(\mathbf{S}_\mathbf{t}^H)'\Sigma_H^{-1}D\mathbf{S}_\mathbf{t}^H\|^2_{\mathcal{H}_{SH}}\\ &= \langle(D\mathbf{S}_\mathbf{t}^H)'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H, (\mathbf{S}_\mathbf{t}^H)'\Sigma_H^{-1}D\mathbf{S}_\mathbf{t}^H\rangle_{\mathcal{H}_{SH}}\\ &= (\mathbf{S}_\mathbf{t}^H)'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H,\\ \|\mathbf{t}'\Sigma_H^{-1}D\mathbf{S}_\mathbf{t}^H\|^2_{\mathcal{H}_{SH}} &= \mathbf{t}'\Sigma_H^{-1}\mathbf{t},\\ \langle(D\mathbf{S}_\mathbf{t}^H)'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H, \mathbf{t}'\Sigma_H^{-1}D\mathbf{S}_\mathbf{t}^H\rangle_{\mathcal{H}_{SH}} &= \mathbf{t}'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H.\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}\|DG_n\|^2_{\mathcal{H}_{SH}} &= \frac{2}{n}\left((\mathbf{S}_\mathbf{t}^H)'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H - \frac{(\mathbf{t}'\Sigma_H^{-1}\mathbf{S}_\mathbf{t}^H)^2}{\mathbf{t}'\Sigma_H^{-1}\mathbf{t}}\right)\\ &= \frac{2\hat{\sigma}^2}{\sigma^2}.\end{aligned}$$

□

Proof of Theorem 1.4. First, from (3.1), it is easy to see that (1.8) holds. Second, by Theorem 1.2, $\|DG_n\|^2_{\mathcal{H}_{SH}}$ converges in L^2 to the constant 2, and we can use the theorem in Nualart and Ortiz-Latorre to conclude the proof of (1.9). □

Proof of Theorem 1.5. We shall use Theorem 3.1 in Nourdin and Peccati (2009). When the theorem is applied to our case, it suffices to verify the following:

(i) $\varphi(n) := \sqrt{\mathbf{E}[(1 - \langle DG_n, -DL^{-1}G_n\rangle_{\mathcal{H}_{SH}})^2]}$ converges to 0 as $n \to \infty$.

(ii) The two-dimensional random vector $(\bar{G}_n, \frac{1-\langle D\bar{G}_n, -DL^{-1}\bar{G}_n\rangle_{\mathcal{H}_{SH}}}{\varphi(n)})$ converges to the centered two-dimensional standard normal (N_1, N_2) with covariance ρ.

First, we have

$$\begin{aligned}\mathbf{E}\big[\big(1-\langle D\bar{G}_n,-DL^{-1}\bar{G}_n\rangle_{\mathcal{H}_{SH}}\big)^2\big]&=\mathbf{E}\bigg[\bigg(1-\Big\langle D\bar{G}_n,\frac{1}{2}D\bar{G}_n\Big\rangle_{\mathcal{H}_{SH}}\bigg)^2\bigg]\\&=\frac{1}{4}\mathbf{E}\big(\|DG_n\|^2_{\mathcal{H}_{SH}}-2\big)^2\\&=\frac{1}{4}\big(\mathbf{E}\|DG_n\|^4_{\mathcal{H}_{SH}}-4\mathbf{E}\|DG_n\|^2_{\mathcal{H}_{SH}}+4\big).\end{aligned}$$

Thus from Lemma 3.1, we have

$$\mathbf{E}\big[\big(1-\langle DG_n,-DL^{-1}G_n\rangle_{\mathcal{H}_{SH}}\big)^2\big]=\frac{2n-1}{n^2},$$

and hence

$$\varphi(n)=\sqrt{\mathbf{E}\big[\big(1-\langle DG_n,-DL^{-1}G_n\rangle_{\mathcal{H}_{SH}}\big)^2\big]}=\frac{\sqrt{2n-1}}{n}.$$

Again from Lemma 3.1, the item (ii) can also be checked easily. Now the computation of ρ is shown by

$$\begin{aligned}\rho&=\frac{1}{2}\lim_{n\to\infty}\mathbf{E}\bigg(\bar{G}_n\cdot\frac{2-\|DG_n\|^2_{\mathcal{H}_{SH}}}{\varphi(n)}\bigg)\\&=\lim_{n\to\infty}\frac{n}{\sqrt{2n-1}\sigma^2}\mathbf{E}\big(\bar{G}_n\cdot(\sigma^2-\hat{\sigma}^2)\big)\\&=-\lim_{n\to\infty}\frac{n\sqrt{n}}{\sqrt{2(2n-1)}\sigma^4}\operatorname{Var}\big(\hat{\sigma}^2\big)\\&=-1.\end{aligned}$$

From Theorem 3.1 in Nourdin and Peccati (2009), we obtain the desired result. □

Acknowledgments

The authors would like to express our gratitude to the editor and the referee, whose very kind comments helped improve this paper. The authors are very grateful to Professor F. Q. Gao for the useful discussions. Research supported by the National Natural Science Foundation of China under Grant 11101137 and by the Education Department Foundation of Hunan Province under Grant 14C0456.

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