The Annals of Statistics
2013, Vol. 41, No. 4, 1865–1891
DOI: 10.1214/13-AOS1135

VOLATILITY OCCUPATION TIMES

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We propose nonparametric estimators of the occupation measure and the occupation density of the diffusion coefficient (stochastic volatility) of a discretely observed Itô semimartingale on a fixed interval when the mesh of the observation grid shrinks to zero asymptotically. In a first step we estimate the volatility locally over blocks of shrinking length, and then in a second step we use these estimates to construct a sample analogue of the volatility occupation time and a kernel-based estimator of its density. We prove the consistency of our estimators and further derive bounds for their rates of convergence. We use these results to estimate nonparametrically the quantiles associated with the volatility occupation measure.

1. Introduction. Continuous-time Itô semimartingales are used widely to model stochastic processes in various areas such as finance. The general Itô semimartingale process is given by

$$X_t = X_0 + \int_0^t b_s\,ds + \int_0^t \sigma_s\,dW_s + J_t, \tag{1.1}$$

where b_t and σ_t are processes with càdlàg paths, W_t is a Brownian motion and J_t is a jump process; formal conditions are given in the next section. Inference for model (1.1) in the general case (either in a parametric or a nonparametric context) is quite complicated because of the many "layers of latency," for example, as typical in financial applications, σ_t and J_t can have randomness not captured by X_t.

When X is sampled at discrete times but with the mesh of the observation grid shrinking to zero, that is, high-frequency data of X are available, the distinct pathwise behavior of the components in (1.1) can be used to nonparametrically separate them. Indeed, various techniques have been already proposed to estimate nonparametrically the integrated variance $\int_0^T \sigma_s^2\,ds$ over a specific interval $[0, T]$ (see, e.g., [3] and [13]), and more generally integrated variance measures of the form $\int_0^T g(\sigma_s^2)\,ds$, where $g(\cdot)$ is a continuous function with polynomial growth

Received November 2012; revised May 2013.
[1]Supported in part by NSF Grant SES-1227448.
[2]Supported in part by NSF Grant SES-0957330.
MSC2010 subject classifications. Primary 62F12, 62M05; secondary 60H10, 60J75.
Key words and phrases. Occupation time, local approximation, stochastic volatility, spot variance, quantiles, nonparametric estimation, high-frequency data.

[and there are more smoothness requirements on $g(\cdot)$ for determining the rate of convergence]; see Theorems 3.4.1 and 9.4.1 in [10] and the recent work of [12].

This paper extends the existing literature on high-frequency nonparametric volatility estimation by developing a nonparametric jump-robust estimator of the occupation time of the latent volatility process $(V_t)_{t\geq 0} \equiv (\sigma_t^2)_{t\geq 0}$ where the volatility occupation time is defined by

$$F_t(x) = \int_0^t 1_{\{V_s \leq x\}}\, ds \qquad \forall x > 0,\ t \in [0, T]. \tag{1.2}$$

Evidently, the right-hand side of (1.2) is of the form $\int_0^t g(V_s)\, ds$ with $g(v) = 1_{\{v\leq x\}}$, which unlike earlier work is a discontinuous function.

If $F_t(\cdot)$ is absolutely continuous with respect to the Lebesgue measure, its derivative $f_t(\cdot)$, that is, the volatility occupation density, is well-defined. By the Lebesgue differentiation theorem, the occupation density can be equivalently defined as

$$f_t(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon}\bigl(F_t(x+\varepsilon) - F_t(x-\varepsilon)\bigr). \tag{1.3}$$

In addition to estimating $F_t(x)$, in this paper we also develop a consistent estimator for the volatility occupation density $f_t(x)$ using the high-frequency record of X.

The occupation measure of the volatility process "summarizes" in a convenient way the information regarding the volatility behavior over the given time interval. Indeed, for any bounded (or nonnegative) Borel function $g(\cdot)$ (see, e.g., Theorem 6.4 of [8]), we have

$$\int_0^t g(V_s)\, ds = \int_{\mathbb{R}_+} g(x) f_t(x)\, dx = \int_{\mathbb{R}_+} g(x)\, dF_t(x). \tag{1.4}$$

Thus, the occupation time and its density can be considered as the pathwise analogues of the cumulative distribution function and its density.

Our interest in occupation times stems from the fact that they are natural measures of risk, particularly in nonstationary settings where invariant distributions do not exist; see, for example, the discussion in [1]. Indeed, there has been a significant interest (both theoretically and in practice) in pricing options based on the occupation times of an underlying asset; see for example, [4] and [19] and references therein. Here, we show how to measure nonparametrically occupation times associated with the volatility risk of the price process. As a by-product, we also estimate the corresponding quantiles of the actual path of the volatility process over the fixed time interval. Since the pathwise volatility quantiles are preserved under monotone transformations, they provide a convenient way of studying the variability of the volatility and the relationship of the latter with the volatility process itself.

We summarize our estimation procedure as follows. We first split the fixed time interval into blocks of decreasing length and form local estimates of the unobserved stochastic variance over each of the blocks. The volatility estimates over the blocks are truncated variations (see, e.g., [13] and [10]), and we further allow for adaptive choice of the truncation level that makes use of some preliminary estimates of the stochastic variance. Then, our estimator of the volatility occupation time is simply the empirical cumulative distribution function of the local volatility estimates over the blocks. Analogously, we estimate the volatility occupation density from the local volatility estimates using kernel smoothing.

Our estimation problem can be compared with the recent work of [12]. Jacod and Rosenbaum [12] show that an estimator of $\int_0^T g(\sigma_s^2)\,ds$, for $g(\cdot)$ a C^3 function, formed by plugging in local variance estimates formed over blocks of decreasing length, can achieve the efficient $\Delta_n^{-1/2}$ rate of convergence (for Δ_n being the length of the high-frequency intervals). Similar to [12], our estimator here is formed by plugging local variance estimates in our function of interest.

The main difference between the current work and [12] is that in our case the function $g(\cdot)$ in (1.4) is discontinuous. As a result, the precision of estimating the volatility occupation time depends on the uniform rate of recovering the volatility process outside of the times of the "big" volatility jumps (with the size of the "big" jumps shrinking asymptotically to zero). Therefore, in the basic case when X and V are continuous, the rate of convergence of the volatility occupation time estimator is (almost) $\Delta_n^{-1/4}$ which, as we show in the paper, is the optimal uniform rate for recovering the volatility trajectory from high-frequency observations. By contrast, [12] derive a central limit theorem for the convergence of their estimator to $\int_0^T g(\sigma_s^2)\,ds$ by making use of the assumed smoothness of g and applying second-order Taylor expansion of the function g evaluated at the local volatility estimator in their bias-correction and asymptotic negligibility arguments.

Finally, our inference for the volatility occupation time and its density can be compared with the estimation of occupation time and density of a recurrent Markov diffusion process from discrete observations of the process; see, for example, [7] and [1]. The main difference is that here the state vector, and therefore the stochastic volatility, is not fully observed. Hence, we first need to recover nonparametrically the unobserved volatility trajectory, and the error associated with recovering the volatility trajectory determines the asymptotic behavior of our estimators.

The paper is organized as follows. In Section 2 we introduce the formal setup and state our assumptions. In Section 3 we develop our estimator of the volatility occupation measure and prove its consistency. In Section 4 we derive bounds for the rate of convergence of the volatility occupation time estimator. Section 5 derives a consistent estimator for the volatility occupation density. Section 6 reports results from a Monte Carlo study of our estimation technique. Section 7 concludes. Section 8 contains all proofs.

2. Setup and assumptions. We start with introducing the formal setup and stating our assumptions about X. The process X in (1.1) is defined on a filtered space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with b_t and σ_t being adapted to the filtration. Further, the jump component J_t is defined as

$$
\begin{aligned}
J_t = {} & \int_0^t \int_{\mathbb{R}} \delta(s,z) 1_{\{|\delta(s,z)|\leq 1\}} (\mu - \nu)(ds, dz) \\
& + \int_0^t \int_{\mathbb{R}} \delta(s,z) 1_{\{|\delta(s,z)|>1\}} \mu(ds, dz),
\end{aligned}
\tag{2.1}
$$

where μ is a Poisson measure on $\mathbb{R}_+ \times \mathbb{R}$ with compensator ν of the form $\nu(dt, dz) = dt \otimes \lambda(dz)$ for some σ-finite measure λ on $\mathbb{R}$ and $\delta : \Omega \times \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ is a predictable function. Regularity conditions on X_t are collected below.

ASSUMPTION A. Let $r \in [0, 2]$ be a constant. The process X is an Itô semimartingale given by (1.1) and (2.1), with b_t locally bounded and σ_t càdlàg. Moreover $|\delta(\omega, t, z)| \wedge 1 \leq \Gamma_m(z)$ for all (ω, t, z) with $t \leq \tau_m(\omega)$, where (τ_m) is a localizing sequence of stopping times, and each function Γ_m on $\mathbb{R}$ satisfies $\int_{\mathbb{R}} \Gamma_m(z)^r \lambda(dz) < \infty$.

Assumption A can be viewed as a regularity-type condition. The coefficient r in Assumption A controls the degree of activity of the jump component J and will play an important role in the rate of convergence of our estimator. We note that r provides an upper bound for the (generalized) Blumenthal–Getoor index of J_t; see, for example, Lemma 3.2.1 in [10]. We next state our assumption for the volatility occupation time.

ASSUMPTION B. Fix $x \geq 0$. We have $F_T(\cdot)$ a.s. differentiable with derivative $f_T(\cdot)$ in a neighborhood $\mathcal{N}_x$ containing x. Moreover, $\sup_{z\in\mathcal{N}_x} \mathbb{E}[f_T(z)] < \infty$.

Assumption B is mainly concerned with the pathwise smoothness of the occupation time. The differentiability condition amounts to the existence of occupation density $f_T(\cdot)$, which is not a strong requirement; see, for example, [8] and [14]. The condition $\sup_{z\in\mathcal{N}_x} \mathbb{E}[f_T(z)] < \infty$ only requires the temporal average (over $[0, T]$) of the probability density of V_t uniformly bounded in the neighborhood $\mathcal{N}_x$, which is satisfied by most stochastic volatility models. This condition is of course much weaker than requiring $\mathbb{E}[\sup_{z\in\mathcal{N}_x} f_T(z)] < \infty$, as the latter would demand more on the pathwise regularity of the occupation density.

EXAMPLE. Let V_t solve the following stochastic differential equation:

$$
dV_t = a_t^V \, dt + s(V_t) \, dW_t^V + dJ_t^V,
$$

where a_t^V is a locally bounded process, W_t^V is a Brownian motion, J_t^V is a finite-variational jump process and $s(\cdot)$ has twice continuously differentiable reciprocal. Assume further that V has an invariant distribution which is C^1 in a neighborhood of x. Then Assumption B holds. This follows from an application of Itô's formula and Theorem IV.75 in [14]. This example includes many parametric models of interest like the square-root diffusion model and the more general constant elasticity of variance model.

For some of the results we will need a stronger condition on the volatility occupation density, mainly its continuity which we state formally in the next assumption.

ASSUMPTION B′. $F_T(\cdot)$ is a.s. continuously differentiable on $\mathbb{R}$ with derivative $f_T(\cdot)$.

Assumption B′ is harder to verify than Assumption B. Necessary and sufficient conditions for the continuity of the occupation density (local time) of a Borel right Markov process are discussed in [6].

We finally state a slightly stronger condition on the volatility process that we will need for deriving the rate of convergence of our estimator.

ASSUMPTION C. The process σ_t is an Itô semimartingale with the form

$$\sigma_t = \sigma_0 + \int_0^t \tilde{b}_s\,ds + \int_0^t \tilde{\sigma}_s\,dW_s + \int_0^t \tilde{\sigma}'_s\,dW'_s + \int_0^t \int_{\mathbb{R}} \tilde{\delta}(s,z)(\mu-\nu)(ds,dz),$$

where the processes $\tilde{b}$, $\tilde{\sigma}$, $\tilde{\sigma}'$ are locally bounded and adapted, W' is a Brownian motion orthogonal to W, and $\tilde{\delta}(\cdot)$ is a predictable function. Moreover $|\tilde{\delta}(\omega,t,z)| \wedge 1 \le \tilde{\Gamma}_m(z)$ for all (ω,t,z) with $t \le \tau_m(\omega)$, where (τ_m) is a localizing sequence of stopping times, and for some $\tilde{r} \in (0,2]$, each function $\tilde{\Gamma}_m$ on $\mathbb{R}$ satisfies $\int_{\mathbb{R}} \tilde{\Gamma}_m(z)^{\tilde{r}} \lambda(dz) < \infty$.

Assumption C assumes that σ_t is an Itô semimartingale, an assumption that is satisfied by most stochastic volatility models. We impose no restriction on the activity of the volatility jumps as well as the dependence between σ_t and X_t, a generality that is important in practical applications (particularly in finance).

3. The estimator and its consistency. We next introduce our estimator of the volatility occupation time and derive its consistency. We suppose that the process X_t is observed at discrete times $i\Delta_n$, $i = 0, 1, \dots,$ on $[0,T]$ for a fixed $T > 0$ with the time lag $\Delta_n \to 0$ when $n \to \infty$. The assumption for equidistant observations is merely for simplicity, and the theoretical results that follow (except Theorem 3) will continue to hold in the case of irregular (but nonrandom) sampling with Δ_n replaced by the mesh of the irregular observation grid. In what follows the high-frequency increment of any process Y is denoted as $\Delta_i^n Y = Y_{i\Delta_n} - Y_{(i-1)\Delta_n}$.

Our strategy of estimating $F_T(\cdot)$ is to first form an approximation of the volatility trajectory and then use the latter to form a sample analogue of $F_T(\cdot)$. To recover the volatility trajectory we construct local approximations for the spot variance process V over blocks of shrinking length. To this end, let k_n be a sequence of integers with $k_n \to \infty$ and $k_n\Delta_n \to 0$. Henceforth we use the shorthand notation $u_n = k_n\Delta_n$. We also set a truncation process $\upsilon_{n,t}$ verifying the following assumption, which is maintained throughout the paper without further mention.

ASSUMPTION D. We have $\upsilon_{n,t} = \alpha_{n,t}\Delta_n^{\varpi}$, where $\varpi \in (0, 1/2)$ is constant, and $\alpha_{n,t}$ is a strictly positive real-valued process such that for some localizing sequence of stopping times (τ_m), $(\sup_{t\in[0,T]}(\alpha_{n,t\wedge\tau_m} \vee \alpha_{n,t\wedge\tau_m}^{-1}))_{n\geq 1}$ is tight for each $m \geq 1$.

With this notation, for each $i = 0, \dots, \lfloor T/\Delta_n\rfloor - k_n$, we set

$$\begin{cases} \widehat{V}^*_{i\Delta_n} = \dfrac{1}{u_n}\displaystyle\sum_{j=1}^{k_n}(\Delta_{i+j}^n X)^2, \\ \widehat{V}_{i\Delta_n} = \dfrac{1}{u_n}\displaystyle\sum_{j=1}^{k_n}(\Delta_{i+j}^n X)^2 1_{\{|\Delta_{i+j}^n X|\leq \upsilon_{n,i\Delta_n}\}}. \end{cases} \tag{3.1}$$

Here, $\widehat{V}^*_{i\Delta_n}$ is a local approximation of $V_{i\Delta_n}$ when X is continuous, while $\widehat{V}_{i\Delta_n}$ serves the same purpose but is robust to the presence of jumps in X. As a generalization to the standard truncation-based methods (see e.g., Chapter 9 of [10]), we allow explicitly the truncation parameter $\alpha_{n,t}$ to be time-varying and depend on $\{X_{i\Delta_n}\}_{i=1,\dots,\lfloor T/\Delta_n\rfloor}$. For example, one convenient and commonly used choice is to set $\alpha_{n,t} = c\overline{\sigma}_n$, where c is a constant [typically in the range $(3, 5)$], and $\overline{\sigma}_n$ is a preliminary estimate of the average volatility over $[0, T]$. Assumption D is verified as soon as $\overline{\sigma}_n$ and $\overline{\sigma}_n^{-1}$ are tight. Another possibility is to make $\alpha_{n,t}$ adaptive by setting $\alpha_{n,t} = c\hat{\sigma}_i$ for $t \in [(i-1)u_n, iu_n)$, where $\hat{\sigma}_i$ is a preliminary estimate for the volatility in the local window $[(i-1)u_n, iu_n)$. For example, one may take $\hat{\sigma}_i$ to be a localized version of the Bipower variation estimator of [2]: $\hat{\sigma}_i = ((\pi/2)u_n^{-1}\sum_{j=1}^{k_n}|\Delta_{i+j}^n X||\Delta_{i+j+1}^n X|)^{1/2}$. The tightness requirement in Assumption D can be easily fulfilled by replacing $\hat{\sigma}_i$ with $(\hat{\sigma}_i \vee (1/C)) \wedge C$ for some pre-specified regularization constant $C \geq 1$. Finally, in the above two examples for $\alpha_{n,t}$, we can further replace the constant c with a deterministic sequence c_n increasing at a logarithmic rate as $\Delta_n \to 0$.

We will use $\widehat{V}^*_{i\Delta_n}$ and $\widehat{V}_{i\Delta_n}$ to approximate for the volatility trajectory within the block. That is for $0 \leq i \leq \lfloor T/u_n\rfloor - 1$,

$$\begin{cases} \widehat{V}^*_t = \widehat{V}^*_{iu_n} \quad \text{and} \quad \widehat{V}_t = \widehat{V}_{iu_n}, & t \in [iu_n, (i+1)u_n), \\ \widehat{V}^*_t = \widehat{V}^*_{(\lfloor T/u_n\rfloor-1)u_n} \quad \text{and} \quad \widehat{V}_t = \widehat{V}_{(\lfloor T/u_n\rfloor-1)u_n}, & \lfloor T/u_n\rfloor u_n \leq t \leq T. \end{cases} \tag{3.2}$$

REMARK 3.1. We can alternatively define local estimators of volatility for each $i = 1, \dots, \lfloor T/\Delta_n\rfloor$ by averaging the k_n past squared increments below the threshold. All the results in the paper, except for Theorem 3 below, will hold for this alternative way of recovering the spot volatility.

Using $\widehat{V}_t^*$ and $\widehat{V}_t$, our proposed estimators of $F_T(\cdot)$ are defined as

$$\widehat{F}_{n,T}^*(x) = \int_0^T 1_{\{\widehat{V}_s^* \le x\}}\, ds, \qquad \widehat{F}_{n,T}(x) = \int_0^T 1_{\{\widehat{V}_s \le x\}}\, ds, \qquad x \in \mathbb{R}.$$

We first consider the pointwise consistency of $\widehat{F}_{n,T}^*(x)$ and $\widehat{F}_{n,T}(x)$. As a matter of fact, it is not much harder to prove a more general result as follows.

LEMMA 1. *Let $g : \mathbb{R}_+ \mapsto [0, 1]$ be a measurable function and D_g be the collection of discontinuity points of g. Suppose*:

(i) *Assumption* A *holds for $r = 2$;*
(ii) *for Lebesgue a.e. $t \in [0, T]$, $\mathbb{P}(V_t \in D_g) = 0$.*

Then we have

(a)

$$\int_0^T g(\widehat{V}_s)\, ds \xrightarrow{\mathbb{P}} \int_0^T g(V_s)\, ds; \tag{3.3}$$

(b) *if, in addition, X is continuous, then* (3.3) *also holds when replacing $\widehat{V}$ with $\widehat{V}^*$.*

Lemma 1 extends Theorem 9.4.1 in [10] by allowing for discontinuities in the test function $g(\cdot)$. For fixed $x \ge 0$, the pointwise consistency of $\widehat{F}_{n,T}(x)$, and $\widehat{F}_{n,T}^*(x)$ if X is continuous, follows immediately [with $g(\cdot) = 1_{\{\cdot \le x\}}$] provided that $\mathbb{P}(V_t = x) = 0$ for Lebesgue a.e. $t \in [0, T]$. The uniform consistency of $\widehat{F}_{n,T}^*(\cdot)$ and $\widehat{F}_{n,T}(\cdot)$ is available if the occupation time $F_T(\cdot)$ is a.s. continuous, as shown below.

THEOREM 1. *Suppose Assumption* A *holds for $r = 2$ and $F_T(\cdot)$ is a.s. continuous. We have*:

(a) $\sup_{x \in \mathbb{R}} |\widehat{F}_{n,T}(x) - F_T(x)| \xrightarrow{\mathbb{P}} 0$;
(b) *if, in addition, X is continuous, then* (a) *still holds when replacing $\widehat{F}_{n,T}$ with $\widehat{F}_{n,T}^*$.*

Analogous to the classical notion of quantile for cumulative distribution functions, the quantile of the occupation time $F_T(\cdot)$ is naturally defined as the left-continuous functional inverse of $F_T(\cdot)$: for $\alpha \in (0, T)$, we set $Q_T(\alpha) = \inf\{x \in \mathbb{R} : F_T(x) \ge \alpha\}$. A natural estimator for $Q_T(\alpha)$ is $\widehat{Q}_{n,T}(\alpha) = \inf\{x \in \mathbb{R} : \widehat{F}_{n,T}(x) \ge \alpha\}$, and $\widehat{Q}_{n,T}^*(\alpha)$ can be defined analogously for $\widehat{F}_{n,T}^*(\cdot)$. The consistency of the quantile estimators is given by the next corollary.

COROLLARY 1. *Suppose Assumption* A *holds for $r = 2$ and $F_T(\cdot)$ is a.s. continuous. Let $\mathcal{Q} \equiv \{\alpha \in (0, T) : Q_T(\cdot)$ is continuous at α a.s.$\}$. We have for each $\alpha \in \mathcal{Q}$:*

(a) $\widehat{Q}_{n,T}(\alpha) \xrightarrow{\mathbb{P}} Q_T(\alpha)$;

(b) *if, in addition, X is continuous, then* $\widehat{Q}^*_{n,T}(\alpha) \xrightarrow{\mathbb{P}} Q_T(\alpha)$.

4. Rate of convergence of $\widehat{F}_{n,T}$. We next study the rate of convergence of $\widehat{F}_{n,T}(x)$. We first consider in Section 4.1 the case when V is continuous and then the general case with discontinuous V is studied in Section 4.2, where the uniform rate of convergence of $\widehat{F}_{n,T}(\cdot)$ is also considered.

4.1. *The continuous volatility case and the uniform approximation of V.* In the continuous volatility case we can link the rate of convergence of our volatility occupation time estimators with the rate of convergence of $\widehat{V}^*_t$ and $\widehat{V}_t$ toward V_t on the space of càdlàg functions equipped with the uniform norm. We denote the latter as

$$\eta^*_n = \sup_{t\in[0,T]} \left|\widehat{V}^*_t - V_t\right|, \qquad \eta_n = \sup_{t\in[0,T]} |\widehat{V}_t - V_t|.$$

The rate of convergence of $\widehat{F}_{n,T}$ and $\widehat{Q}_{n,T}$ is then related with η_n through Lemma 2 below; an analogous result holds for $\widehat{F}^*_{n,T}$, $\widehat{Q}^*_{n,T}$ and η^*_n, but is omitted here for brevity.

LEMMA 2. (a) *For any $x \geq 0$ and $\alpha \in (0, T)$, we have* $|\widehat{F}_{n,T}(x) - F_T(x)| \leq F_T(x + \eta_n) - F_T(x - \eta_n)$ *and* $|\widehat{Q}_{n,T}(\alpha) - Q_T(\alpha)| \leq \eta_n$.

(b) *Suppose Assumption B and $\eta_n = O_p(a_n)$ for some nonrandom sequence $a_n \to 0$. Then* $\widehat{F}_{n,T}(x) - F_T(x) = O_p(a_n)$.

In view of Lemma 2, bounding the rate of convergence of the occupation time estimators boils down to establishing the asymptotic order of magnitude of η_n and η^*_n. Our main result concerning the uniform approximation of the V process is given by the following theorem.

THEOREM 2. *Suppose Assumptions A and C with $\Delta V_s = 0$ for $s \in [0, T]$. Let $k_n \asymp \Delta_n^{-\gamma}$ for some $\gamma \in (r\varpi + (1 \vee r)(1 - 2\varpi), 1)$, $\varpi \in ((1 \vee r - 1)/(2(1 \vee r) - r), 1/2)$ and $\iota > 0$ be arbitrarily small but fixed. We have*:

(a) $\eta_n = O_p(a_n)$, *where*

$$a_n = \begin{cases} \Delta_n^{\gamma-1+(2-r)\varpi} \vee \Delta_n^{\gamma/2-\iota} \vee \Delta_n^{(1-\gamma)/2-\iota}, & \text{if } r \leq 1, \\ \Delta_n^{\gamma/r-(1-\varpi)-\iota} \vee \Delta_n^{\gamma/2-\iota} \vee \Delta_n^{(1-\gamma)/2-\iota}, & \text{if } r > 1, \\ \Delta_n^{\gamma/2-\iota} \vee \Delta_n^{(1-\gamma)/2-\iota}, & \text{if } X \text{ is continuous}; \end{cases} \tag{4.1}$$

(b) *if X is continuous, we also have* $\eta^*_n = O_p(a_n)$.

When X is continuous, the terms $\Delta_n^{\gamma/2-\iota}$ and $\Delta_n^{(1-\gamma)/2-\iota}$ capture, respectively, the sampling variability and the discretization bias in the approximation of the spot variance. When X is discontinuous, a_n contains an additional term arising from the elimination of jumps, which of course depends on the concentration of "small" jumps through r (recall Assumption A). The conditions on ϖ and γ imply $a_n \to 0$. In particular, when r is close to 2, ϖ and γ need to be chosen close to $1/2$ and 1, respectively, to ensure that $a_n \to 0$, rendering the rate of convergence arbitrarily slow. Nonetheless, if X is discontinuous, η_n still has the same rate of convergence as in the continuous case, that is, $\Delta_n^{1/4-\iota}$, provided $r \in (0, 1/2)$. This rate can be achieved by setting $\varpi \in (3/(8-4r), 1/2)$ and $\gamma = 1/2$.

Of course Theorem 2 provides only a bound for the rate of convergence of η_n and η_n^*. The following theorem, however, establishes the exact asymptotic distributions of η_n and η_n^* in a simple model with constant volatility.

THEOREM 3. *Suppose*:

(i) *Assumption* A *holds with V_t constant and $b_t = 0$ on* $[0, T]$;
(ii) $r < 1/2$ *and* $\varpi \in (3/(8-4r), 1/2)$.
Then

$$\sqrt{\log(\lfloor T/u_n \rfloor)}(\sqrt{k_n}\eta_n - \sqrt{2}Vm_n) \xrightarrow{\mathcal{L}} V \times \Lambda, \tag{4.2}$$

provided $k_n \asymp \Delta_n^{-1/2}$ *and where* Λ *is a random variable with c.d.f.* $\exp(-2\exp(-x))$, *and*

$$m_n = \sqrt{2\log(\lfloor T/u_n \rfloor)} - \frac{\log(\log(\lfloor T/u_n \rfloor)) + \log(4\pi)}{2\sqrt{2\log(\lfloor T/u_n \rfloor)}}. \tag{4.3}$$

If we further assume (iii) X_t *is continuous*, *then* (4.2) *still holds with η_n replaced by η_n^*.*

REMARK 4.1. Theorem 3 shows that the rates given in (4.1) are almost optimal when the jumps of X_t are not very active ($r < 1/2$). To be precise, we observe that (4.1) suggests $\eta_n = O_p(\Delta_n^{1/4-\iota})$ for $\iota > 0$ fixed but arbitrarily small, while the optimal rate in Theorem 3 provides a slightly sharper bound $\eta_n = O_p(\Delta_n^{1/4} \log(\lfloor T\Delta_n^{-1/2} \rfloor)^{1/2})$.

The rate of convergence of our volatility occupation time estimators and their quantiles is a direct corollary of Lemma 2 and Theorem 2; the proof is omitted for brevity.

COROLLARY 2. *Let $x \geq 0$ and $\alpha \in (0, T)$. Suppose Assumption* B *and the same setting as in Theorem* 2. *Then $\widehat{F}_{n,T}(x) - F_T(x)$ and $\widehat{Q}_{n,T}(\alpha) - Q_T(\alpha)$ are $O_p(a_n)$. If X is continuous and $k_n \asymp \Delta_n^{-1/2}$, then $\widehat{F}_{n,T}(x) - F_T(x)$, $\widehat{F}^*_{n,T}(x) -$*

$F_T(x)$, $\widehat{Q}_{n,T}(\alpha) - Q_T(\alpha)$ and $\widehat{Q}^*_{n,T}(\alpha) - Q_T(\alpha)$ are $O_p(\Delta_n^{1/4-\iota})$ for $\iota > 0$ arbitrarily small but fixed.

We should point out that in the trivial cases when $x < \inf_{t\in[0,T]} V_t$ or $x > \sup_{t\in[0,T]} V_t$ on a given path, the error in recovering the occupation time will become identically zero for n sufficiently high (up to taking a subsequence).

REMARK 4.2. More generally, we can use Theorem 2 to show in the setting of the theorem that if $\mathcal{L}:\mathcal{D}([0,T]) \to \mathbb{R}$, where $\mathcal{D}([0,T])$ is the space of càdlàg functions on the interval $[0,T]$ equipped with the uniform topology, is a continuous function, we have $\mathcal{L}(\widehat{V}) \xrightarrow{\mathbb{P}} \mathcal{L}(V)$. If further $|\mathcal{L}(f) - \mathcal{L}(g)| \le K \sup_{s\in[0,T]} |f_s - g_s|$ for any elements $f, g \in \mathcal{D}([0,T])$ and some positive constant K, then the rate of convergence of $\mathcal{L}(\widehat{V})$ to $\mathcal{L}(V)$ is bounded by the order of magnitude of η_n (under the conditions of Theorem 2). An example of such a function is $\mathcal{L}(f) = \sup_{s\in[0,T]} f_s$.

4.2. *The discontinuous volatility case.* We now turn to the general case when the volatility process contains jumps. When V is discontinuous, bounding the rate of convergence of the volatility occupation time estimators is much less straightforward. Lemma 2 is still valid, however, and the uniform approximation error η_n no longer vanishes asymptotically, due to the discontinuity in V. This is true even if we consider the ideal case where $\widehat{V}_{iu_n} = V_{iu_n}$, that is, perfect pointwise approximation is available. Indeed, the lack of uniform approximation for a discontinuous process with its discretized version is well known in the study of convergence of processes.

Our strategy of bounding the rate of convergence of $\widehat{F}_{n,T}(x)$ and $\widehat{F}^*_{n,T}(x)$ is to pick out the "big" jumps in the V process and then consider the uniform rate of approximation to the "remainder" process. The idea is best illustrated in the basic case where the jumps in V is finitely active. In this case, the volatility jumps only occur within finitely many time blocks with the form $[iu_n, (i+1)u_n)$ and hence their total effect on the estimation is $O_p(u_n)$. On time blocks not containing the jumps of V, V is continuous, so Theorem 2 can be used to provide uniform bound. The situation becomes considerably more complicated when V has infinitely active, or even infinite variational, jumps. In this case, one needs to compute the trade-off between picking out a smaller number of big jumps with less accurate uniform approximation to the remainder process, and picking out a larger number of big jumps with more accurate uniform approximation to the remainder process. The end result of this calculation is Theorem 4 below.

THEOREM 4. *Suppose Assumptions A, B and C. Let $k_n \asymp \Delta_n^{-\gamma}$ for some $\gamma \in (r\varpi + (1\vee r)(1-2\varpi), 1)$, $\varpi \in ((1\vee r - 1)/(2(1\vee r) - r), 1/2)$ and $\iota > 0$ be arbitrarily small but fixed. We have*:

(a) $\widehat{F}_{n,T}(x) - F_T(x) = O_p(d_n)$, *where* $d_n = a_n \vee \Delta_n^{(1-\gamma)/(1+\tilde{r})-\iota}$ *and* a_n *is given by* (4.1);

(b) *if* X *is continuous*, *we also have* $\widehat{F}^*_{n,T}(x) - F_T(x) = O_p(d_n)$.

Theorem 4 establishes an upper bound for the pointwise rate of convergence of the occupation time estimators. We remind the reader that the rate d_n depends on r through a_n; recall (4.1) and the discussion following Theorem 2. Whether the rate is optimal or not is an open question. The rate optimality for jump-robust estimation of the integrated variance, that is, $\int_0^T V_s\,ds = \int_0^\infty x F_T(dx)$, is studied by [11].

In order to establish a uniform bound, we invoke the stronger Assumption B′ which assumes continuity of the volatility occupation density.

THEOREM 5. *Consider the same setting as in Theorem* 4 *except with Assumption* B′ *replacing Assumption* B′. *Then* (a) *and* (b) *in Theorem* 4 *hold uniformly in* $x \in \mathbb{R}$. *Moreover, for* $\alpha \in (0, T)$ *with* $f_T(Q_T(\alpha)) > 0$ *a.s.*, *we have* $|\widehat{Q}_{n,T}(\alpha) - Q_T(\alpha)| = O_p(d_n)$ *and, if* X *is continuous*, $|\widehat{Q}^*_{n,T}(\alpha) - Q_T(\alpha)| = O_p(d_n)$.

Theorem 5 establishes a bound for the uniform rate of convergence of the occupation time estimators; the rate of convergence of the quantiles follows as a consequence.

5. Estimation of the volatility occupation density. We now turn to estimating the volatility occupation density $f_T(x)$. Clearly, the uniform convergence of the occupation time (Theorem 1) does not directly lead to valid estimation for the occupation density. While the focus of the current paper is on the occupation time, we consider the estimation of $f_T(\cdot)$ theoretically complementary and empirically relevant. Our occupation density estimator is based on the local volatility estimates $\widehat{V}_t$ and $\widehat{V}_t^*$ and kernel smoothing. In particular, we propose the following kernel estimator of Nadaraya–Watson type:

$$\widehat{f}_{n,T}(x) \equiv \int_0^T \frac{1}{h_n}\kappa\left(\frac{\widehat{V}_s - x}{h_n}\right) ds,$$

where $h_n \to 0$ is a bandwidth sequence and the kernel function $\kappa : \mathbb{R} \mapsto \mathbb{R}_+$ is bounded C^1 with bounded derivative and $\int_{\mathbb{R}} \kappa(x)\,dx = 1$. We can define $\widehat{f}^*_{n,T}(x)$ similarly but with $\widehat{V}_s^*$ replacing $\widehat{V}_s$.

Below, we consider a weight function $w : \mathbb{R} \mapsto \mathbb{R}_+$ with $\int_{\mathbb{R}} w(x)\,dx < \infty$. For generic real-valued functions g_1 and g_2 on $\mathbb{R}$, we denote

$$\|g_1 - g_2\|_w \equiv \int_{\mathbb{R}} \big|g_1(x) - g_2(x)\big| w(x)\,dx.$$

THEOREM 6. *Suppose*:

(i) *Assumptions* A, B′ *and* C;
(ii) $r \in (0,2)$ *and* $\varpi \in ((1 \vee r - 1)/(2(1 \vee r) - r), 1/2)$;
(iii) $k_n \asymp \Delta_n^{-\gamma}$ *for some* $\gamma \in (0,1)$;
(iv) V_t^{-1} *is locally bounded*;
(v) *for some* $\beta \in (0,1]$ *and any compact* $\mathcal{K} \subset (0,\infty)$, *there exists a constant* $C_{\mathcal{K}} > 0$, *such that for all* $x, y \in \mathcal{K}$, $\mathbb{E}|f_T(x) - f_T(y)| \le C_{\mathcal{K}}|x-y|^\beta$;
(vi) $\int_{\mathbb{R}} \kappa(z)|z|^\beta\,dz < \infty$.

We set

$$\bar{a}_n^* \equiv \Delta_n^{\gamma/2} \vee \Delta_n^{(1-\gamma)/2}, \qquad \bar{a}_n \equiv \bar{a}_n^* \vee \Delta_n^{(1-r\varpi-\theta)/(1\vee r)-(1-2\varpi)},$$

where $\theta = 0$ *when* $r \le 1$ *and* $\theta > 0$ *is arbitrarily fixed when* $r > 1$. *Then for each* $x \ge 0$, *we have*:

(a) $\widehat{f}_{n,T}(x) - f_T(x)$ *and* $\|\widehat{f}_{n,T} - f_T\|_w$ *are* $O_p(h_n^{-2}\bar{a}_n \vee h_n^\beta)$;
(b) *if* X *is continuous*, $\widehat{f}_{n,T}(x) - f_{n,T}(x)$ *and* $\|\widehat{f}_{n,T} - f_T\|_w$ *are* $O_p(h_n^{-2}\bar{a}_n^* \vee h_n^\beta)$ *and moreover, the results still hold with* $\widehat{f}_{n,T}(\cdot)$ *replaced by* $\widehat{f}^*_{n,T}(\cdot)$.

REMARK 5.1. Condition (v) in Theorem 6 requires the occupation density of V_t to be Hölder continuous on compacta with exponent β under the L_1-norm. We preclude the analysis for cases in which $f_T(\cdot)$ is differentiable, or in a Hölder class of higher order, because occupation densities of semimartingales in general do not enjoy such higher-order smoothness; recall from Assumption C that V_t is a semimartingale. For example, the occupation density of a one-dimensional Brownian motion is Hölder continuous in L_1 with exponent $\beta = 1/2$; see Exercise VI.1.32 in [15]. That being said, occupation densities of other processes, such as certain Gaussian processes (see, e.g., Table 2 in [8]), may enjoy higher-order smoothness. Such models have rarely been studied in the analysis of high-frequency financial data and are not directly compatible with Assumption C, so we do not pursue further results here. Notice that the rate $\bar{a}_n^*$ is optimized by setting $\gamma = 1/2$, resulting in $\bar{a}_n^* = \Delta_n^{1/4}$. Furthermore, when X is continuous, the estimation error of the occupation density is $O_p(\Delta_n^{\beta/4(2+\beta)})$, which is achieved by setting $h_n \asymp \Delta_n^{1/4(2+\beta)}$. Not surprisingly, the smoother the occupation density (larger β), the faster the rate of convergence.

REMARK 5.2. Theorem 6(a) implies $\widehat{f}_{n,T}(x) - f_T(x) \xrightarrow{\mathbb{P}} 0$ and $\|\widehat{f}_{n,T} - f_T\|_w \xrightarrow{\mathbb{P}} 0$, provided that $h_n \to 0$ and $h_n^{-2}\bar{a}_n \to 0$. These results can be shown directly without conditions (iv)–(vi) using a very similar proof; the details are omitted for brevity. A similar comment applies to Theorem 6(b).

6. Monte Carlo. We test the performance of our nonparametric procedures on two popular stochastic volatility models. The first is the square-root diffusion volatility model, given by

$$(6.1) \qquad dX_t = \sqrt{V_t}\,dW_t, \qquad dV_t = 0.03(1.0 - V_t)\,dt + 0.2\sqrt{V_t}\,dB_t,$$

W_t and B_t are two independent Brownian motions. Our second model is a jump-diffusion volatility model in which the log-volatility is a Lévy-driven Ornstein–Uhlenbeck (OU) process, that is,

$$(6.2) \qquad dX_t = e^{V_t - 1}\,dW_t, \qquad dV_t = -0.03V_t\,dt + dL_t,$$

where L_t is a Lévy martingale uniquely defined by the marginal law of V_t which in turn has a selfdecomposable distribution (see Theorem 17.4 of [16]) with characteristic triplet (Definition 8.2 of [16]) of $(0, 1, \nu)$ for $\nu(dx) = \frac{2.33e^{-2.0|x|}}{|x|^{1+0.5}} 1_{\{x>0\}}\,dx$ with respect to the identity truncation function. The mean and persistence of both volatility specifications are calibrated realistically to observed financial data, and the two models differ in the presence of volatility jumps as well as in the modeling of the volatility of volatility: for model (6.1), the transformation $\sqrt{V_t}$ is with constant diffusion coefficient while for (6.2) this is the case for the transformation $\log V_t$.

In the Monte Carlo we fix the time span to $T = 22$ days (our unit of time is a day), equivalent to one calendar month, and we consider $n = 80$ and $n = 400$, which correspond to 5-minute and 1-minute, respectively, of intraday observations of X in a 6.5-hour trading day. We set $k_n = 20$ for $n = 80$ and we increase it to $k_n = 40$ when $n = 400$, which, respectively, correspond to 4 and 10 blocks per unit of time. We finally set the truncation process at $\upsilon_{n,t} = 3\sqrt{BV_j}\Delta_n^{0.49}$ for $t \in [j-1, j)$ and where $BV_j = \frac{\pi}{2}\sum_{i=\lfloor (j-1)/\Delta_n \rfloor + 2}^{\lfloor j/\Delta_n \rfloor} |\Delta_{i-1}^n X||\Delta_i^n X|$ is the Bipower Variation on the unit interval $[j-1, j)$. For each realization we compute the 25th, 50th and 75th volatility qunatiles over the interval $[0, T]$. The results from the Monte Carlo are summarized in Table 1. Overall, the performance of our volatility quantile estimator is satisfactory. The highest bias arises for the square-root diffusion volatility model when volatility was started from a high value (the 75th quantile of its invariant distribution). Intuitively, in this case volatility drifts toward its unconditional mean, and this results in its larger variation over $[0, T]$, which in turn is more difficult to accurately disentangle from the Gaussian noise in the price process, that is, the Brownian motion W_t in X_t. Consistently with our asymptotic results, the biases and the mean absolute deviations of all volatility quantiles shrink as we increase the sampling frequency from $n = 80$ to $n = 400$ in all considered scenarios.

7. Conclusion. In this paper we propose nonparametric estimators of the volatility occupation time and its density from discrete observations of the process over a fixed time interval with asymptotically shrinking mesh of the observation

TABLE 1
Monte Carlo results

	$\widehat{Q}_{T,n}(0.25)$			$\widehat{Q}_{T,n}(0.50)$			$\widehat{Q}_{T,n}(0.75)$		
Start value	**True**	**Bias**	**MAD**	**True**	**Bias**	**MAD**	**True**	**Bias**	**MAD**
Panel A: Square-root volatility model, $n = 80$									
$V_0 = Q^V(0.25)$	0.3798	−0.0536	0.0547	0.5394	−0.0478	0.0514	0.7324	−0.0190	0.0473
$V_0 = Q^V(0.50)$	0.6223	−0.0916	0.0929	0.8170	−0.0651	0.0703	1.0513	−0.0081	0.0626
$V_0 = Q^V(0.75)$	0.9865	−0.1516	0.1525	1.2359	−0.0949	0.1027	1.5310	0.0110	0.0911
Panel B: Square-root volatility model, $n = 400$									
$V_0 = Q^V(0.25)$	0.3798	−0.0305	0.0315	0.5394	−0.0304	0.0327	0.7324	−0.0178	0.0293
$V_0 = Q^V(0.50)$	0.6223	−0.0519	0.0529	0.8170	−0.0412	0.0453	1.0513	−0.0146	0.0375
$V_0 = Q^V(0.75)$	0.9865	−0.0868	0.0882	1.2359	−0.0596	0.0654	1.5310	−0.0043	0.0554
Panel C: Log-volatility model, $n = 80$									
$V_0 = Q^V(0.25)$	0.1737	−0.0231	0.0249	0.2860	−0.0269	0.0302	0.4519	−0.0171	0.0358
$V_0 = Q^V(0.50)$	0.3293	−0.0428	0.0455	0.5243	−0.0460	0.0524	0.8069	−0.0245	0.0610
$V_0 = Q^V(0.75)$	0.6337	−0.0809	0.0866	0.9945	−0.0807	0.0968	1.5162	−0.0434	0.1117
Panel D: Log-volatility model, $n = 400$									
$V_0 = Q^V(0.25)$	0.1737	−0.0131	0.0142	0.2860	−0.0158	0.0180	0.4519	−0.0116	0.0224
$V_0 = Q^V(0.50)$	0.3293	−0.0248	0.0268	0.5243	−0.0276	0.0318	0.8069	−0.0169	0.0358
$V_0 = Q^V(0.75)$	0.6337	−0.0452	0.0490	0.9945	−0.0480	0.0575	1.5162	−0.0305	0.0682

Notes: In all simulated scenarios $T = 22$, and we set $k_n = 20$ for $n = 80$ and $k_n = 40$ for $n = 400$. In each of the cases, the volatility is started from a fixed point being the 25th, 50th and 75th quantile of the invariant distribution of the volatility process, denoted correspondingly as $Q^V(0.25)$, $Q^V(0.50)$ and $Q^V(0.75)$. The columns "True" report the average value (across the Monte Carlo simulations) of the true variance quantile that is estimated; MAD stands for mean absolute deviation around true value. The Monte Carlo replica is 1000.

grid. We derive the asymptotic properties of our volatility occupation time estimator and further invert it to estimate the corresponding quantiles of the volatility path over the fixed time interval. Monte Carlo shows satisfactory performance of the proposed estimation techniques.

8. Proofs. This section contains all proofs. Throughout the proof, we denote by K a generic constant that may change from line to line. We sometimes emphasize its dependence on some parameter p by writing K_p. As is typical in this kind of problem, by a standard localization procedure, Assumptions A, C and D can be strengthened into the following stronger versions without loss of generality.

ASSUMPTION SA. We have Assumption A. The processes b_t and σ_t are bounded, and for some bounded nonnegative function Γ on $\mathbb{R}$, $|\delta(\omega, t, z)| \leq \Gamma(z)$ and $\int_{\mathbb{R}} \Gamma(z)^r \lambda(dz) < \infty$.

ASSUMPTION SC. We have Assumption C. The processes $\tilde{b}_t$, $\tilde{\sigma}_t$ and $\tilde{\sigma}'_t$ are bounded, and for some bounded nonnegative function Γ_σ on $\mathbb{R}$, $|\tilde{\delta}(\omega, t, z)| \leq \Gamma_\sigma(z)$ and $\int_{\mathbb{R}} \Gamma_\sigma(z)^{\tilde{r}} \lambda(dz) < \infty$.

ASSUMPTION SD. We have Assumption D. Moreover, $\alpha_{n,t}$ and $\alpha_{n,t}^{-1}$ are uniformly bounded for all n, t.

8.1. *Proofs in Section* 3.

PROOF OF LEMMA 1. (a) We set $\widehat{V}_t^+ = \widehat{V}_{iu_n}$ for $t \in [(i-1)u_n, iu_n)$. Denote the left-hand side of (3.3) by S_n and $T_n = \lfloor T/u_n \rfloor u_n$. We have

$$S_n = \int_0^{(\lfloor T/u_n \rfloor - 1)u_n} g(\widehat{V}_s^+)\, ds + \int_0^{u_n} g(\widehat{V}_s)\, ds + \int_{T_n}^T g(\widehat{V}_s)\, ds.$$

Since g is bounded,

$$\mathbb{E}\left| S_n - \int_0^T g(V_s)\, ds \right| \leq K u_n + \int_0^{(\lfloor T/u_n \rfloor - 1)u_n} \mathbb{E}\left| g(\widehat{V}_s^+) - g(V_s) \right| ds. \tag{8.1}$$

Observe that for each $s \in [0, (\lfloor T/u_n \rfloor - 1)u_n)$, $\widehat{V}_s^+ \xrightarrow{\mathbb{P}} V_s$. To see this, we recall from Assumption SD that $\alpha_{n,t} \in [\underline{\alpha}, \overline{\alpha}]$ for some constant $\overline{\alpha} \geq \underline{\alpha} > 0$. Let $\widehat{V}_s^+(\overline{\alpha})$ and $\widehat{V}_s^+(\underline{\alpha})$ be defined as $\widehat{V}_s^+$ except with $\alpha_{n,t}$ replaced, respectively, by $\overline{\alpha}$ and $\underline{\alpha}$. By Theorem 9.3.2 in [10], the right continuity of V and $u_n \to 0$, $\widehat{V}_s^+(\overline{\alpha})$ and $\widehat{V}_s^+(\underline{\alpha})$ converge in probability to V_s. The claim then follows $\widehat{V}_s^+(\underline{\alpha}) \leq \widehat{V}_s^+ \leq \widehat{V}_s^+(\overline{\alpha})$.

Hence, by condition (ii) and bounded convergence, for Lebesgue a.e. $s \in [0, T]$, $\mathbb{E}|g(\widehat{V}_s^+) - g(V_s)| = \mathbb{E}|(g(\widehat{V}_s^+) - g(V_s))1_{\{V_s \notin D_g\}}| \to 0$. Applying bounded convergence on (8.1), we readily obtain (3.3). Part (b) can be shown similarly. $\square$

PROOF OF THEOREM 1. (a) For each $x \geq 0$, $F_T(x) = F_T(x-)$ a.s. by the continuity of $F_T(\cdot)$. Hence,

$$\int_0^T \mathbb{P}(V_s = x)\, ds = \mathbb{E}\big[F_T(x) - F_T(x-)\big] = 0.$$

Therefore, $\mathbb{P}(V_s = x) = 0$ for Lebesgue a.e. $s \in [0, T]$. By Lemma 1 with $g(\cdot) = 1_{\{\cdot \leq x\}}$, $\widehat{F}_{n,T}(x) \xrightarrow{\mathbb{P}} F_T(x)$. Since $\widehat{F}_{n,T}(\cdot)$ and $F_T(\cdot)$ are increasing, and $F_T(\cdot)$ is continuous, this convergence also holds locally uniformly. Since V is càdlàg, $\bar{V} \equiv \sup_{t \in [0,T]} V_t = O_p(1)$. For any $\eta > 0$, there exists some $M > 0$, such that $\mathbb{P}(\bar{V} > M) < \eta$, yielding $\mathbb{P}(T \neq F_T(M)) < \eta$. Hence, for any $\varepsilon > 0$,

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{P}\Big(\sup_{x\in\mathbb{R}} \big|\widehat{F}_{n,T}(x) - F_T(x)\big| > \varepsilon\Big) \\
&\quad \leq \limsup_{n\to\infty} \mathbb{P}\Big(\sup_{0\leq x\leq M} \big|\widehat{F}_{n,T}(x) - F_T(x)\big| > \varepsilon\Big) \\
&\qquad + \limsup_{n\to\infty} \mathbb{P}\Big(\sup_{x\geq M} \big|\widehat{F}_{n,T}(x) - F_T(x)\big| > \varepsilon\Big) \\
&\quad \leq \limsup_{n\to\infty} \mathbb{P}\big(\big|\widehat{F}_{n,T}(M) - F_T(M)\big| > \varepsilon/2\big) + \mathbb{P}\big(T - F_T(M) > \varepsilon/2\big) \\
&\quad < \eta.
\end{aligned}$$

Sending $\eta \to 0$, we readily derive the assertion in part (a). Part (b) can be proved similarly. □

PROOF OF COROLLARY 1. By Theorem 1, $\widehat{F}_{n,T}(\cdot) \xrightarrow{\mathbb{P}} F_T(\cdot)$ uniformly. By a subsequence argument, we can then assume $\widehat{F}_{n,T}(\cdot) \xrightarrow{\text{a.s.}} F_T(\cdot)$ uniformly without loss. The assertion in part (a) then follows Lemma 21.2 of [17]. The proof of part (b) is similar. □

8.2. *Proofs in Section 4.1.*

PROOF OF LEMMA 2. (a) Observe

$$\begin{aligned}
F_T(x - \eta_n) &= \int_0^T 1_{\{V_s \leq x - \eta_n\}}\, ds \\
&\leq \widehat{F}_{n,T}(x) \leq \int_0^T 1_{\{V_s \leq x + \eta_n\}}\, ds = F_T(x + \eta_n).
\end{aligned} \tag{8.2}$$

Since $F_T(x - \eta_n) \leq F_T(x) \leq F_T(x + \eta_n)$, the first assertion in part (a) readily follows. Now consider the quantiles. By definition, $\widehat{F}_{n,T}(\widehat{Q}_{n,T}(\alpha)) \geq \alpha$. By (8.2), $F_T(\widehat{Q}_{n,T}(\alpha) + \eta_n) \geq \alpha$. Therefore, $Q_T(\alpha) \leq \widehat{Q}_{n,T}(\alpha) + \eta_n$. For any $\varepsilon > 0$, by (8.2), we have $F_T(\widehat{Q}_{n,T}(\alpha) - \eta_n - \varepsilon) \leq \widehat{F}_{n,T}(\widehat{Q}_{n,T}(\alpha) - \varepsilon) < \alpha$. Hence,

$\widehat{Q}_{n,T}(\alpha) - \eta_n - \varepsilon < Q_T(\alpha)$. Since $\varepsilon > 0$ is arbitrary, $\widehat{Q}_{n,T}(\alpha) - \eta_n \le Q_T(\alpha)$. The second assertion of part (a) is then obvious.

(b) Fix some $\varepsilon > 0$. There exists $M > 0$ such that $\mathbb{P}(\eta_n \ge M a_n) < \varepsilon/2$ for n sufficiently large. Since $a_n \to 0$, $[x-\eta_n, x+\eta_n]$ is contained in $\mathcal{N}_x$ with probability approaching one (w.p.a.1) and by part (a),

$$\left|\widehat{F}_{n,T}(x) - F_T(x)\right| \le \int_{x-\eta_n}^{x+\eta_n} f_T(z)\,dz.$$

Let $M' = 4M \sup_{z\in\mathcal{N}_x} \mathbb{E}[f_T(z)]/\varepsilon$. We have for n sufficiently large,

$$\begin{aligned}
&\mathbb{P}\left(\int_{x-\eta_n}^{x+\eta_n} f_T(z)\,dz > M' a_n\right)\\
&\quad \le \mathbb{P}\left(\int_{x-Ma_n}^{x+Ma_n} f_T(z)\,dz > M' a_n\right) + \mathbb{P}(\eta_n \ge M a_n)\\
&\quad < \frac{2M \sup_{z\in\mathcal{N}_x} \mathbb{E}[f_T(z)]}{M'} + \varepsilon/2\\
&\quad \le \varepsilon.
\end{aligned}$$

Hence, $\int_{x-\eta_n}^{x+\eta_n} f_T(z)\,dz = O_p(a_n)$. The assertion in part (b) then readily follows. $\square$

We now prove Theorem 2, starting with two lemmas. Below, $\|\cdot\|_p$ denotes the L_p norm.

LEMMA 3. *Let $p \ge 1$ be a constant and $k_n \asymp \Delta_n^{-\gamma}$ for some $\gamma \in (0,1)$. Suppose Assumption SA holds with X continuous and Assumption SD. Then for each $0 \le i \le \lfloor T/u_n \rfloor - 1$,*

$$|\widehat{V}^*_{iu_n} - V_{iu_n}| \vee |\widehat{V}_{iu_n} - V_{iu_n}| \le \xi_{n,i} + \sup_{s\in[iu_n,(i+1)u_n)} |V_s - V_{iu_n}|, \tag{8.3}$$

where the variable $\xi_{n,i}$ satisfies $\|\xi_{n,i}\|_p \le K_p k_n^{-1/2}$. If we further have Assumption SC with $\Delta V_s = 0$ for $s \in [0,T]$, then the majorant side of the above can be bounded by $K_p(k_n^{-1/2} + u_n^{1/2})$ in L_p.

PROOF. By Itô's formula, $\widehat{V}^*_{iu_n} - V_{iu_n} = \zeta'_{n,i} + \zeta''_{n,i}$, where

$$\begin{aligned}
\zeta'_{n,i} &= \frac{2}{u_n}\int_{iu_n}^{(i+1)u_n} (X_s - X_{n,s})\,dX_s,\\
\zeta''_{n,i} &= \frac{1}{u_n}\int_{iu_n}^{(i+1)u_n} (V_s - V_{iu_n})\,ds,
\end{aligned}$$

and $X_{n,s}$ is the discretized process given by $X_{n,s} = X_{iu_n+(j-1)\Delta_n}$ when $s \in [iu_n + (j-1)\Delta_n, iu_n + j\Delta_n)$. By classical estimates (note that X is continuous),

$$\mathbb{E}\left|\frac{2}{u_n}\int_{iu_n}^{(i+1)u_n}(X_s - X_{n,s})b_s\,ds\right|^p \le K_p\Delta_n^{p/2},$$
$$\mathbb{E}\left|\frac{2}{u_n}\int_{iu_n}^{(i+1)u_n}(X_s - X_{n,s})\sigma_s\,dW_s\right|^p \le K_p k_n^{-p/2}.$$

Since $k_n = o(\Delta_n^{-1})$, we have $\|\zeta'_{n,i}\|_p \le K_p k_n^{-1/2}$ by Minkowski's inequality. We also observe $|\zeta''_{n,i}| \le \sup_{s\in[iu_n,(i+1)u_n)}|V_s - V_{iu_n}|$.

Now note that $\widehat{V}^*_{iu_n} - \widehat{V}_{iu_n} = k_n^{-1}\sum_{j=1}^{k_n}(\Delta^n_{ik_n+j}X/\Delta_n^{1/2})^2 1_{\{|\Delta^n_{ik_n+j}X|>v_{n,iu_n}\}}$. Under Assumption SD, $\alpha_{n,t} \ge \underline{\alpha}$ for some constant $\underline{\alpha} > 0$. Hence, $v_{n,t} \ge \underline{v}_n \equiv \underline{\alpha}\Delta_n^{\varpi}$. Since X is continuous, for any $q \ge 0$,

$$\begin{aligned}
&\|(\Delta^n_{ik_n+j}X/\Delta_n^{1/2})^2 1_{\{|\Delta^n_{ik_n+j}X|>v_{n,iu_n}\}}\|_p \\
&\quad \le \left(\frac{\mathbb{E}|\Delta^n_{ik_n+j}X/\Delta_n^{1/2}|^{2p+q}}{(\underline{v}_n/\Delta_n^{1/2})^q}\right)^{1/p} \\
&\quad \le K_{p,q}\Delta_n^{q(1/2-\varpi)/p}.
\end{aligned}$$

Since $\varpi \in (0, 1/2)$, when q is taken sufficiently large, terms in the above display can be further bounded by $K_p k_n^{-1/2}$. Hence, $\|\widehat{V}^*_{iu_n} - \widehat{V}_{iu_n}\|_p \le K_p k_n^{-1/2}$. The first assertion then readily follows by setting $\xi_{n,i} = |\zeta'_{n,i}| + |\widehat{V}^*_{iu_n} - \widehat{V}_{iu_n}|$.

Now, suppose Assumption SC together with V being continuous. By standard estimates, for each $p \ge 1$, the second term on the right-hand side of (8.3) can be bounded by $K_p u_n^{1/2}$ in L_p. The second assertion of the lemma is then obvious. $\square$

Under Assumption SA, we set

$$X'_t = X_t - X''_t,$$
$$X''_t = \begin{cases} \displaystyle\int_0^t\int_{\mathbb{R}}\delta(s,z)\mu(ds,dz), & \text{if } r \le 1,\\ \displaystyle\int_0^t\int_{\mathbb{R}}\delta(s,z)(\mu-\nu)(ds,dz), & \text{if } r > 1.\end{cases}$$

We define $\widehat{V}'$ as $\widehat{V}^*$ in (3.1) but with X' in place of X; in particular, $\widehat{V}'_{iu_n} \equiv u_n^{-1}\sum_{j=1}^{k_n}(\Delta^n_{ik_n+j}X')^2$.

LEMMA 4. *Suppose that Assumption SA holds with some $r \in (0, 2)$ and Assumption SD. Let $p \ge r \vee 1$ and $\varpi \in (\frac{p-1}{2p-r}, \frac{1}{2})$. Let $\theta \in (0, \infty)$ be arbitrarily fixed* *if $r > 1$ and $\theta = 0$ if $r \le 1$. We have for each i,*

$$\|\widehat{V}_{iu_n} - \widehat{V}'_{iu_n}\|_p \le K_p \Delta_n^{(1-r\varpi-\theta)/p-(1-2\varpi)}.$$

PROOF. Under Assumption SD, $\alpha_{n,t} \in [\underline{\alpha}, \overline{\alpha}]$ for constants $\overline{\alpha} \ge \underline{\alpha} > 0$. We set $\bar{v}_n = \overline{\alpha}\Delta_n^{\varpi}$ and $\underline{v}_n = \underline{\alpha}\Delta_n^{\varpi}$. By applying Lemma 13.2.6 in [10] [with $s = 1$, $s' = 2$, $m = p$, $p' = 1$, $k = 1$, $F(x) = x^2$], we have

$$\begin{aligned}
&\mathbb{E}\big|(\Delta_{ik_n+j}^n X/\Delta_n^{1/2})^2 1_{\{|\Delta_{ik_n+j}^n X| \le \bar{v}_n\}} - (\Delta_{ik_n+j}^n X'/\Delta_n^{1/2})^2 1_{\{|\Delta_{ik_n+j}^n X'| \le \bar{v}_n\}}\big|^p \\
&\quad \le K_p \Delta_n^{(2-r)/2-\theta} + K_p \Delta_n^{1-r\varpi-p(1-2\varpi)-\theta} \\
&\quad \le K_p \Delta_n^{1-r\varpi-p(1-2\varpi)-\theta}.
\end{aligned}$$

By a similar argument as in the proof of Lemma 3,

$$\big\|(\Delta_{ik_n+j}^n X'/\Delta_n^{1/2})^2 1_{\{|\Delta_{ik_n+j}^n X'| > \bar{v}_n\}}\big\|_p \le K_{p,q} \Delta_n^{q(1/2-\varpi)/p}$$

for any $q \ge 0$. Taking q sufficiently large, we then derive

$$\begin{aligned}
&\big\|(\Delta_{ik_n+j}^n X/\Delta_n^{1/2})^2 1_{\{|\Delta_{ik_n+j}^n X| \le \bar{v}_n\}} - (\Delta_{ik_n+j}^n X'/\Delta_n^{1/2})^2\big\|_p \\
&\qquad \le K_p \Delta_n^{(1-r\varpi)/p-(1-2\varpi)-\theta/p}.
\end{aligned} \tag{8.4}$$

By a similar argument, we can derive (8.4) when $\bar{v}_n$ is replaced with $\underline{v}_n$. Since $\underline{v}_n \le v_{n,iu_n} \le \bar{v}_n$, (8.4) also holds when $\bar{v}_n$ is replaced with v_{n,iu_n}. The assertion of the lemma then follows from Minkowski's inequality. □

PROOF OF THEOREM 2. *Step* 1. We first suppose X is continuous. Observe that

$$\eta_n \le \sup_{0 \le i \le \lfloor T/u_n \rfloor - 1} |\widehat{V}_{iu_n} - V_{iu_n}| + 2 \sup_{0 \le i \le \lfloor T/u_n \rfloor} \sup_{s \in [iu_n,(i+1)u_n)} |V_s - V_{iu_n}|.$$

Since V is continuous, $\|\sup_{s\in[iu_n,(i+1)u_n)} |V_s - V_{iu_n}|\|_p \le K_p u_n^{1/2}$ for any $p \ge 1$ by standard estimates. By Lemma 3 and the maximal inequality (e.g., Lemma 2.2.2 in [18]), for any $p \ge 1$, $\|\eta_n\|_p \le K_p u_n^{-1/p}(k_n^{-1/2} + u_n^{1/2})$. Since $k_n \asymp \Delta_n^{-\gamma}$ by assumption, we derive $\|\eta_n\|_p \le K\Delta_n^{(\gamma\wedge(1-\gamma))/2-\iota}$ by taking p sufficiently large. The same argument yields $\|\eta_n^*\|_p \le K\Delta_n^{(\gamma\wedge(1-\gamma))/2-\iota}$. This finishes the proof of part (a) with X continuous, as well as part (b).

Step 2. We now consider part (a) allowing X to be discontinuous. Let $\widehat{V}'$ be defined as in Lemma 4. Observe that

$$\eta_n \le \sup_{0 \le i \le \lfloor T/u_n \rfloor - 1} |\widehat{V}_{iu_n} - \widehat{V}'_{iu_n}| + \eta'_n, \qquad \text{where}$$

$$\eta'_n = \sup_{0 \le i \le \lfloor T/u_n \rfloor - 1} |\widehat{V}'_{iu_n} - V_{iu_n}| + 2 \sup_{0 \le i \le \lfloor T/u_n \rfloor} \sup_{s \in [iu_n,(i+1)u_n)} |V_s - V_{iu_n}|.$$

A similar argument as in part (a) yields $\eta_n' = O_p(\Delta_n^{(\gamma\wedge(1-\gamma))/2-\iota})$. By the maximal inequality and Lemma 4 for $p = 1 \vee r$,

$$\begin{aligned}\Big\| \sup_{0\le i\le \lfloor T/u_n\rfloor -1} \big|\widehat{V}_{iu_n} - \widehat{V}'_{iu_n}\big| \Big\|_p &\le K u_n^{-1/p} \Delta_n^{(1-r\varpi-\theta)/p-(1-2\varpi)} \\ &\le K \Delta_n^{(\gamma-r\varpi-\theta)/p-(1-2\varpi)} \\ &\le \begin{cases} K\Delta_n^{\gamma-r\varpi-(1-2\varpi)}, & \text{if } r\le 1, \\ K\Delta_n^{(\gamma-\theta)/r-(1-\varpi)}, & \text{if } r>1.\end{cases}\end{aligned}$$

Taking θ sufficiently small in the $r > 1$ case, we readily derive the assertion in part (a). □

PROOF OF THEOREM 3. *Step* 1. We first prove the assertion on η_n^*, so condition (iii) is in force. In the constant volatility setting of the theorem, we have $\sqrt{k_n}\eta_n^* = \sqrt{2}V \times M_n$, where we denote

$$\begin{aligned} M_n &= \sup_{i=0,\dots,\lfloor T/u_n\rfloor-1} \big|Z_i^n\big|, \\ Z_i^n &= \frac{\sqrt{k_n}}{\sqrt{2}V}\big(\widehat{V}^*_{iu_n} - V\big), \qquad i = 0,\dots,\lfloor T/u_n\rfloor - 1.\end{aligned} \tag{8.5}$$

Under our constant volatility assumption $\{Z_i^n\}_i$ are independent and identically distributed with distribution which is approximately standard normal. Therefore, we can use Edgeworth expansion of the c.d.f. together with extreme value theory to pin down the limit distribution of M_n. To this end, we set

$$\begin{cases} c_n = \big(2\log(b_n)\big)^{-1/2}, \qquad m_n = \sqrt{2\log(b_n)} - \dfrac{\log(\log(b_n)) + \log(4\pi)}{2\sqrt{2\log(b_n)}}, \\ \tau_n(x) = c_n x + m_n, \qquad b_n = \lfloor T/u_n\rfloor, \qquad x\in\mathbb{R}_+. \end{cases} \tag{8.6}$$

Note that $\tau_n(x) \asymp \sqrt{2\log(b_n)}$ and hence increases to infinity as the number of blocks increases to infinity for every fixed x.

Using second-order Edgeworth expansion and denoting with $\Phi(\cdot)$, the c.d.f. of standard normal random variable (see Theorem 2.2 and Lemma 5.4 of [9]), we have

$$\mathbb{P}\big(\big|Z_i^n\big| \le \tau_n(x)\big) = \Phi\big(\tau_n(x)\big) - \Phi\big(-\tau_n(x)\big) + \frac{(\log(b_n))^4}{b_n\sqrt{k_n}} K(x) + o\left(\frac{1}{k_n}\right) \tag{8.7}$$

for any x where $K(x)$ is a polynomial of x. Then we have

$$\lim_{n\to\infty}\left[\frac{\mathbb{P}(|Z_i^n| \le \tau_n(x))}{\Phi(\tau_n(x)) - \Phi(-\tau_n(x))}\right]^{b_n} = 1, \tag{8.8}$$

provided $k_n \asymp \Delta_n^{-1/2}$. This assumption on the rate of growth of k_n guarantees that the distribution of Z_i^n is "sufficiently close" to standard normal. Now we can use (8.8) to get

$$\mathbb{P}(c_n^{-1}(M_n - m_n) \le x) = [\mathbb{P}(|Z_i^n| \le \tau_n(x))]^{b_n} \sim [\Phi(\tau_n(x)) - \Phi(-\tau_n(x))]^{b_n} \tag{8.9}$$

as $n \to \infty$. From here, using the results for the maximum domain of attraction of the Gumbel distribution (see e.g., Example 1.1.7 of [5]), we have

$$[\Phi(\tau_n(x)) - \Phi(-\tau_n(x))]^{b_n} = [2\Phi(\tau_n(x)) - 1]^{b_n} \longrightarrow \exp(-2\exp(-x)) \qquad \forall x, \tag{8.10}$$

and hence

$$c_n^{-1}(M_n - m_n) \xrightarrow{\mathcal{L}} \Lambda \tag{8.11}$$

for Λ being a random variable with c.d.f. $\exp(-2\exp(-x))$. From here the result in (4.2) (with η_n replaced by η_n^*) follows.

Step 2. We now prove (4.2) with condition (iii) relaxed. Let $\eta_n'^*$ be defined as η_n^* but with $\widehat{V}_t^*$ replaced by $\widehat{V}_t'$. By step 1, (4.2) holds with η_n replaced by $\eta_n'^*$. It remains to show that

$$\log(\lfloor T/u_n \rfloor)^{1/2} \Delta_n^{-1/4}(\eta_n - \eta_n'^*) = o_p(1). \tag{8.12}$$

Note that $|\eta_n - \eta_n'^*| \le \sup_{0 \le i \le \lfloor T/u_n \rfloor - 1} |\widehat{V}_{iu_n} - \widehat{V}'_{iu_n}| = O_p(\Delta_n^{(2-r)\varpi - 1/2})$, where the stochastic order is shown in step 2 of the proof of Theorem 2. (8.12) then follows condition (ii). This completes the proof. □

8.3. *Proofs in Section* 4.2. Under Assumption C, by Itô's formula, we can represent V as

$$V_t = V_0 + \int_0^t \tilde{b}_{V,s}\,ds + \int_0^t \tilde{\sigma}_{V,s}\,dW_s + \int_0^t \tilde{\sigma}'_{V,s}\,dW'_s + \int_0^t \int_{\mathbb{R}} \tilde{\delta}_V(s,z)(\mu - \nu)(ds,dz),$$

where, by localization, we can assume without loss that the coefficients $\tilde{b}_V$, $\tilde{\sigma}_V$, $\tilde{\sigma}'_V$ are bounded, and $|\tilde{\delta}_V(\omega,s,z)| \le \tilde{\Gamma}(z)$ for any (ω,s,z), where $\tilde{\Gamma}(\cdot)$ is bounded and deterministic, and satisfies $\int_{\mathbb{R}} \tilde{\Gamma}(z)^{\tilde{r}} \lambda(dz) < \infty$.

We consider the following decomposition: for $q > 0$,

$$V_t = V_t'(q) + V_t''(q), \qquad \text{where}$$

$$V_t^c = V_0 + \int_0^t \tilde{b}_{V,s}\,ds + \int_0^t \tilde{\sigma}_{V,s}\,dW_s + \int_0^t \tilde{\sigma}'_{V,s}\,dW'_s,$$

$$V_t'(q) = V_t^c + \int_0^t \int_{\{z : \tilde{\Gamma}(z) \le q\}} \tilde{\delta}_V(s,z)(\mu - \nu)(ds,dz)$$

$$- \int_0^t \int_{\{z : \tilde{\Gamma}(z) > q\}} \tilde{\delta}_V(s,z)\nu(ds,dz),$$

$$V_t''(q) = \int_0^t \int_{\{z : \tilde{\Gamma}(z) > q\}} \tilde{\delta}_V(s,z)\mu(ds,dz).$$

Denote $I(n,i) = [iu_n, (i+1)u_n)$. We also set $\mathcal{I}_n(q) = \{0 \le i \le \lfloor T/u_n \rfloor - 1 : \mu(I(n,i) \times \{z : \tilde{\Gamma}(z) > q\}) = 0\}$ and $\mathcal{T}_n(q) = \bigcup_{i \in \mathcal{I}_n(q)} I(n,i)$. Here, $\mathcal{I}_n(q)$ collects indices of intervals not containing "big" jumps. We can decompose $\widehat{F}_{n,T}(x) = \widehat{F}_{n,T}(x;q) + \widehat{R}_{n,T}(x;q)$ where

$$\widehat{F}_{n,T}(x;q) = u_n \sum_{i \in \mathcal{I}_n(q)} 1_{\{\widehat{V}_{iu_n} \le x\}}, \qquad \widehat{R}_{n,T}(x;q) = \int_{[0,T]\setminus\mathcal{T}_n(q)} 1_{\{\widehat{V}_s \le x\}}\,ds.$$

Analogously, we have $F_T(x) = F_{n,T}(x;q) + R_{n,T}(x;q)$, where

$$F_{n,T}(x;q) = \int_{\mathcal{T}_n(q)} 1_{\{V_s \le x\}}\,ds, \qquad R_{n,T}(x;q) = \int_{[0,T]\setminus\mathcal{T}_n(q)} 1_{\{V_s \le x\}}\,ds.$$

Finally, we set

$$\hat{\eta}_n(q) = \sup_{i \in \mathcal{I}_n(q)} |\widehat{V}_{iu_n} - V_{iu_n}|,$$

$$\eta_n'(q) = \sup_{0 \le i \le \lfloor T/u_n \rfloor} \sup_{s \in [iu_n,(i+1)u_n)} |V_s'(q) - V_{iu_n}'(q)|,$$

$$\eta_n(q) = \hat{\eta}_n(q) + \eta_n'(q).$$

We now generalize Lemma 2 as follows.

LEMMA 5. *Suppose $\eta_n(q_n) = O_p(w_n)$ for some nonrandom sequences $q_n \to 0$ and $w_n \to 0$ and Assumption SA with $r = 2$. Then* (a) *under Assumption B, $\widehat{F}_{n,T}(x) - F_T(x) = O_p(w_n) + O_p(u_n q_n^{-\tilde{r}})$*; (b) *under Assumption B′, the assertion in* (a) *holds uniformly in $x \in \mathbb{R}$ and moreover* (c) *$|\widehat{Q}_{n,T}(\alpha) - Q_T(\alpha)| \le \xi_n \sup_{x\in\mathbb{R}} |\widehat{F}_{n,T}(x) - F_T(x)|$ for some tight sequence of variables ξ_n, provided $f_T(Q_T(\alpha)) > 0$ a.s.*

PROOF. (a) Observe that $\mathbb{E}[\int_{[0,T]\setminus\mathcal{T}_n(q_n)} ds] \le K u_n q_n^{-\tilde{r}}$, yielding

$$\sup_{x\in\mathbb{R}} R_{n,T}(x;q_n) = O_p(u_n q_n^{-\tilde{r}}), \qquad \sup_{x\in\mathbb{R}} \widehat{R}_{n,T}(x;q_n) = O_p(u_n q_n^{-\tilde{r}}). \tag{8.13}$$

By definition, $\widehat{F}_{n,T}(x;q_n) = \int_{\mathcal{T}_n(q_n)} 1_{\{\widehat{V}_s \le x\}}\,ds$. Note that over $\mathcal{T}_n(q_n)$, the process $V_t''(q_n)$ is identically zero. Hence, $\sup_{t\in\mathcal{T}_n(q_n)} |\widehat{V}_t - V_t| \le \eta_n(q_n)$. By a similar argument as in (8.2), we deduce

$$\begin{aligned} &|\widehat{F}_{n,T}(x;q_n) - F_{n,T}(x;q_n)| \\ &\quad \le F_{n,T}(x + \eta_n(q_n); q_n) - F_{n,T}(x - \eta_n(q_n); q_n) \\ &\quad \le F_T(x + \eta_n(q_n)) - F_T(x - \eta_n(q_n)). \end{aligned} \tag{8.14}$$

By an argument similar to part (b) of Lemma 2, we derive $F_T(x+\eta_n(q_n)) - F_T(x-\eta_n(q_n)) = O_p(w_n)$. The assertion of part (a) then follows (8.13) and (8.14).

(b) By localization, we can suppose that V is bounded and thus $f_T(\cdot)$ is compactly supported. Since $f_T(\cdot)$ is continuous, $\sup_{x\in\mathbb{R}} f_T(x) = O_p(1)$. By (8.14),

$$\sup_{x\in\mathbb{R}}\bigl|\widehat{F}_{n,T}(x;q_n) - F_{n,T}(x;q_n)\bigr| \le 2\eta_n(q_n)\sup_{z\in\mathbb{R}} f_T(z). \tag{8.15}$$

The assertion then readily follows (8.13) and (8.15).

(c) Observe that $\widehat{F}_{n,T}(\widehat{Q}_{n,T}(\alpha)) \ge \alpha = F_T(Q_T(\alpha))$, where the inequality follows the definition of quantiles and the equality is due to the continuity of $F_T(\cdot)$. Hence,

$$\begin{aligned} F_T\bigl(Q_T(\alpha)\bigr) - F_T\bigl(\widehat{Q}_{n,T}(\alpha)\bigr) &\le \widehat{F}_{n,T}\bigl(\widehat{Q}_{n,T}(\alpha)\bigr) - F_T\bigl(\widehat{Q}_{n,T}(\alpha)\bigr) \\ &\le \sup_{x\in\mathbb{R}}\bigl|\widehat{F}_{n,T}(x) - F_T(x)\bigr|. \end{aligned} \tag{8.16}$$

For any $\varepsilon > 0$, $\widehat{F}_{n,T}(\widehat{Q}_{n,T}(\alpha)-\varepsilon) < \alpha = F_T(Q_T(\alpha))$, yielding

$$\begin{aligned} F_T\bigl(\widehat{Q}_{n,T}(\alpha)-\varepsilon\bigr) - F_T\bigl(Q_T(\alpha)\bigr) &< F_T\bigl(\widehat{Q}_{n,T}(\alpha)-\varepsilon\bigr) - \widehat{F}_{n,T}\bigl(\widehat{Q}_{n,T}(\alpha)-\varepsilon\bigr) \\ &\le \sup_{x\in\mathbb{R}}\bigl|\widehat{F}_{n,T}(x) - F_T(x)\bigr|. \end{aligned}$$

Since $F_T(\cdot)$ is continuous, by sending $\varepsilon \downarrow 0$ we deduce

$$F_T\bigl(\widehat{Q}_{n,T}(\alpha)\bigr) - F_T\bigl(Q_T(\alpha)\bigr) \le \sup_{x\in\mathbb{R}}\bigl|\widehat{F}_{n,T}(x) - F_T(x)\bigr|. \tag{8.17}$$

Let $\mathcal{K}_{n,T}(\alpha)$ be the closed interval with endpoints $Q_T(\alpha)$ and $\widehat{Q}_{n,T}(\alpha)$. Set $\xi_n \equiv \sup_{x\in\mathcal{K}_{n,T}(\alpha)} f_T^{-1}(x)$. By a mean-value expansion,

$$\bigl|F_T\bigl(\widehat{Q}_{n,T}(\alpha)\bigr) - F_T\bigl(Q_T(\alpha)\bigr)\bigr| \ge \inf_{x\in\mathcal{K}_{n,T}(\alpha)} f_T(x)\bigl|\widehat{Q}_{n,T}(\alpha) - Q_T(\alpha)\bigr|. \tag{8.18}$$

Since $f_T(Q_T(\alpha)) > 0$ a.s., $\widehat{Q}_{n,T}(\alpha) \xrightarrow{\mathbb{P}} Q_T(\alpha)$ by Corollary 1. Since $f_T(\cdot)$ is continuous, $\inf_{x\in\mathcal{K}_{n,T}(\alpha)} f_T(x) \xrightarrow{\mathbb{P}} f_T(Q_T(\alpha)) > 0$; hence ξ_n is tight. The assertion then follows (8.16), (8.17) and (8.18). □

PROOF OF THEOREM 4. *Step* 1. We first consider $\eta_n'(q_n)$. For each i,

$$\sup_{s\in[iu_n,(i+1)u_n)}\bigl|V_s'(q_n) - V_{iu_n}'(q_n)\bigr| \le \zeta_{n,i} + \zeta_{n,i}' + \zeta_{n,i}'', \qquad \text{where}$$

$$\begin{aligned} \zeta_{n,i} &= \sup_{s\in[iu_n,(i+1)u_n)}\left|\int_{iu_n}^{s}\int_{\{z:\,\tilde{\Gamma}(z)>q_n\}} \tilde{\delta}_V(s,z)\nu(ds,dz)\right|, \\ \zeta_{n,i}' &= \sup_{s\in[iu_n,(i+1)u_n)}\left|\int_{iu_n}^{s}\int_{\{z:\,\tilde{\Gamma}(z)\le q_n\}} \tilde{\delta}_V(s,z)(\mu-\nu)(ds,dz)\right|, \\ \zeta_{n,i}'' &= \sup_{s\in[iu_n,(i+1)u_n)}\bigl|V_s^c - V_{iu_n}^c\bigr|. \end{aligned}$$

For $\zeta_{n,i}$, observe that

$$\left|\sup_{0\le i\le\lfloor T/u_n\rfloor}\zeta_{n,i}\right|\le\sup_{0\le i\le\lfloor T/u_n\rfloor}\int_{iu_n}^{(i+1)u_n}\int_{\{z:\tilde{\Gamma}(z)>q_n\}}\tilde{\Gamma}(z)\nu(ds,dz)$$
$$\le Ku_nq_n^{(1-\tilde{r})\wedge 0}. \tag{8.19}$$

Now turn to $\zeta'_{n,i}$. Let $p\ge 2$. For each $i\ge 0$, by Lemma 2.1.5 in [10],

$$\begin{aligned}\mathbb{E}|\zeta'_{n,i}|^p &\le K_pu_n\int_{\{z:\tilde{\Gamma}(z)\le q_n\}}\tilde{\Gamma}(z)^p\lambda(dz)\\ &\quad+K_pu_n^{p/2}\left(\frac{1}{u_n}\int_{I(n,i)}ds\int_{\{z:\tilde{\Gamma}(z)\le q_n\}}\tilde{\Gamma}(z)^2\lambda(dz)\right)^{p/2}\\ &\le K_pu_nq_n^{p-\tilde{r}}+K_pu_n^{p/2}q_n^{(2-\tilde{r})p/2}.\end{aligned}$$

Hence, $\|\zeta'_{n,i}\|_p\le K_pu_n^{1/p}q_n^{1-\tilde{r}/p}+K_pu_n^{1/2}q_n^{1-\tilde{r}/2}$. By the maximal inequality (Lemma 2.2.2 in [18]),

$$\left\|\sup_{0\le i\le\lfloor T/u_n\rfloor}\zeta'_{n,i}\right\|_p\le K_pq_n^{1-\tilde{r}/p}+K_pu_n^{1/2-1/p}q_n^{1-\tilde{r}/2}. \tag{8.20}$$

Since V^c is continuous, by a standard estimate for $\zeta''_{n,i}$ and the maximal inequality,

$$\left\|\sup_{0\le i\le\lfloor T/u_n\rfloor}\zeta''_{n,i}\right\|_p\le K_pu_n^{1/2-1/p}. \tag{8.21}$$

Combining (8.19), (8.20) and (8.21), we derive for $p\ge 2$, $\|\eta'_n(q_n)\|_p\le K_pa'_{n,p}$, where

$$a'_{n,p}\equiv u_nq_n^{(1-\tilde{r})\wedge 0}\vee q_n^{1-\tilde{r}/p}\vee u_n^{1/2-1/p}.$$

Step 2. Observe that

$$\hat{\eta}_n(q_n)\le\sup_{i\in\mathcal{I}_n(q_n)}\left|\widehat{V}_{iu_n}-\widehat{V}'_{iu_n}\right|+\sup_{i\in\mathcal{I}_n(q_n)}\left|\widehat{V}'_{iu_n}-V_{iu_n}\right|. \tag{8.22}$$

By Lemma 3, for $i\in\mathcal{I}_n(q_n)$, for some $\xi_{n,i}$ with $\|\xi_{n,i}\|_p\le K_pk_n^{-1/2}$,

$$\left|\widehat{V}'_{iu_n}-V_{iu_n}\right|\le\xi_{n,i}+\sup_{s\in[iu_n,(i+1)u_n)}\left|V'_s(q_n)-V'_{iu_n}(q_n)\right|.$$

Therefore, by a similar argument as in step 1,

$$\left\|\sup_{i\in\mathcal{I}_n(q_n)}\left|\widehat{V}'_{iu_n}-V_{iu_n}\right|\right\|_p\le K_pu_n^{-1/p}k_n^{-1/2}+K_pa'_{n,p}. \tag{8.23}$$

Similarly as in step 2 of the proof of Theorem 2 [recall that $\theta=0$ if $r\le 1$ and $\theta\in(0,\infty)$ can be arbitrarily fixed when $r>1$]

$$\sup_{i\in\mathcal{I}_n(q_n)}\left|\widehat{V}_{iu_n}-\widehat{V}'_{iu_n}\right|=O_p\left(\Delta_n^{(\gamma-r\varpi-\theta)/(1\vee r)-(1-2\varpi)}\right). \tag{8.24}$$

Combining (8.22)–(8.24), we derive $\hat{\eta}_n(q_n) = O_p(w_{n,p})$ for $p \geq 2$, where

$$w_{n,p} \equiv \Delta_n^{(\gamma - r\varpi - \theta)/(1\vee r)-(1-2\varpi)} \vee u_n^{-1/p} k_n^{-1/2} \vee a'_{n,p}.$$

By step 1, we further derive $\eta_n(q_n) = O_p(w_{n,p})$.

By Lemma 5(a), we have

$$\begin{aligned}
&\widehat{F}_{n,T}(x) - F_T(x) \\
&\quad = O_p(w_{n,p}) + O_p\bigl(u_n q_n^{-\tilde{r}}\bigr) \\
&\quad = O_p\bigl(\Delta_n^{(\gamma - r\varpi - \theta)/(1\vee r)-(1-2\varpi)} \vee u_n^{-1/p} k_n^{-1/2} \vee q_n^{1-\tilde{r}/p} \vee u_n^{1/2-1/p} \\
&\qquad\qquad \vee u_n q_n^{-\tilde{r}}\bigr).
\end{aligned}$$

Taking $q_n = u_n^{1/(1+\tilde{r}-\tilde{r}/p)}$ and recalling $k_n \asymp \Delta_n^{-\gamma}$, we have

$$\begin{aligned}
&\widehat{F}_{n,T}(x) - F_T(x) \\
&\quad = O_p\bigl(\Delta_n^{(\gamma - r\varpi - \theta)/(1\vee r)-(1-2\varpi)} \vee \Delta_n^{\gamma/2-(1-\gamma)/p} \vee \Delta_n^{(1-\gamma)(1/2-1/p)} \\
&\qquad\qquad \vee \Delta_n^{((1-\gamma)(1-\tilde{r}/p))/(1+\tilde{r}-\tilde{r}/p)}\bigr)
\end{aligned}$$

and by taking p sufficiently large,

$$\begin{aligned}
&\widehat{F}_{n,T}(x) - F_T(x) \\
&\quad = O_p\bigl(\Delta_n^{(\gamma - r\varpi - \theta)/(1\vee r)-(1-2\varpi)} \vee \Delta_n^{\gamma/2-\iota} \vee \Delta_n^{(1-\gamma)/2-\iota} \vee \Delta_n^{(1-\gamma)/(1+\tilde{r})-\iota}\bigr).
\end{aligned}$$

The discontinuous case in part (a) then readily follows the definitions of θ, a_n [see (4.1)] and d_n. The continuous case in part (a), as well as part (b), can be proved in a similar (but simpler) way. $\square$

PROOF OF THEOREM 5. We first show that $\sup_{x\in\mathbb{R}} |\widehat{F}_{n,T}(x) - F_T(x)| = O_p(d_n)$. The proof is similar as that of Theorem 4, except in step 2 of the proof, we use Lemma 5(b) instead of Lemma 5(a). The result for $\widehat{F}^*_{n,T}$ can be proved similarly. The assertion concerning $\widehat{Q}_{n,T}(\alpha)$ then follows from Lemma 5(c). The assertion on $\widehat{Q}^*_{n,T}(\alpha)$ can be proved in a similar (but simpler) way; the details are omitted for brevity. $\square$

8.4. *Proofs in Section 5.*

PROOF OF THEOREM 6. (a) By localization and condition (iv), we can assume that V_t takes value in some compact $\mathcal{K} \subset (0, \infty)$, and thus $f_T(\cdot)$ is supported on $\mathcal{K}$. We set $f_{n,T}(x) = \int_0^T h_n^{-1} \kappa(h_n^{-1}(V_s - x))\, ds$. For each $x \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{E}\bigl|\widehat{f}_{n,T}(x) - f_{n,T}(x)\bigr| &\leq \mathbb{E}\biggl[h_n^{-1} \int_0^T \biggl|\kappa\biggl(\frac{\widehat{V}_s - x}{h_n}\biggr) - \kappa\biggl(\frac{V_s - x}{h_n}\biggr)\biggr| ds\biggr] \\
&\leq K h_n^{-2} \mathbb{E}\biggl[\int_0^T |\widehat{V}_s - V_s| ds\biggr].
\end{aligned}$$

By Lemmas 3 and 4, $\mathbb{E}|\widehat{V}_s - V_s| \le K\bar{a}_n$. Hence,

$$\mathbb{E}\big|\widehat{f}_{n,T}(x) - f_{n,T}(x)\big| \le Kh_n^{-2}\bar{a}_n, \qquad \mathbb{E}\big[\|\widehat{f}_{n,T} - f_{n,T}\|_w\big] \le Kh_n^{-2}\bar{a}_n, \tag{8.25}$$

where K does not depend on x. Now observe that $f_{n,T}(x) = \int_{\mathbb{R}} h_n^{-1}\kappa(h_n^{-1}(y - x)) f_T(y)\,dy$. By a change of variable, $f_{n,T}(x) = \int_{\mathbb{R}} \kappa(z) f_T(x + h_n z)\,dz$. Hence,

$$\begin{aligned}\mathbb{E}\big|f_{n,T}(x) - f_T(x)\big| &\le \int_{\mathbb{R}} \kappa(z)\mathbb{E}\big|f_T(x + h_n z) - f_T(x)\big|\,dz \\ &\le Kh_n^{\beta}\int_{\mathbb{R}} \kappa(z)|z|^{\beta}\,dz \le Kh_n^{\beta},\end{aligned}$$

which further implies $\mathbb{E}[\|f_{n,T} - f_T\|_w] \le Kh_n^{\beta}$. Combining these estimates with (8.25) completes the proof of part (a). Part (b) can be proved in a similar way. □

Acknowledgments. We would like to thank Tim Bollerslev, Nathalie Eisenbaum, Jean Jacod, Andrew Patton and Philip Protter for helpful discussions as well as an Associate Editor and two referees for very helpful suggestions. We are particularly grateful to Markus Reiss for suggesting the direct estimation approach adopted in the paper and the link with the uniform error in estimating the volatility path given in Lemma 2 of the paper.

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