The Annals of Probability
2012, Vol. 40, No. 2, 648–694
DOI: 10.1214/10-AOP636
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RANDOM SUBSHIFTS OF FINITE TYPE

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Let X be an irreducible shift of finite type (SFT) of positive entropy, and let $B_n(X)$ be its set of words of length n. Define a random subset ω of $B_n(X)$ by independently choosing each word from $B_n(X)$ with some probability α. Let X_ω be the (random) SFT built from the set ω. For each $0 \le \alpha \le 1$ and n tending to infinity, we compute the limit of the likelihood that X_ω is empty, as well as the limiting distribution of entropy for X_ω. For α near 1 and n tending to infinity, we show that the likelihood that X_ω contains a unique irreducible component of positive entropy converges exponentially to 1. These results are obtained by studying certain sequences of random directed graphs. This version of "random SFT" differs significantly from a previous notion by the same name, which has appeared in the context of random dynamical systems and bundled dynamical systems.

1. Introduction. A shift of finite type (SFT) is a dynamical system defined by finitely many local transition rules. These systems have been studied for their own sake [36, 40], and they have also served as important tools for understanding other dynamical systems [9, 21, 30].

Each SFT can be described as the set of bi-infinite sequences on a finite alphabet that avoid a finite list of words over the alphabet. Thus, there are only countably many SFTs up to the naming of letters in an alphabet.

For the sake of simplicity, we state our results in terms of SFTs in the Introduction, even though we prove more general results in terms of sequences of directed graphs in the subsequent sections. Let X be a nonempty SFT (for definitions, see Section 2.1). Let $B_n(X)$ be the set of words of length n that appear in X. For α in $[0, 1]$, let $\mathbb{P}_\alpha$ be the probability measure on the power set of $B_n(X)$ given by choosing each word in $B_n(X)$ independently with probability α. The case $\alpha = 1/2$ puts uniform measure on the subsets of $B_n(X)$. For notation, let Ω_n be the power set of $B_n(X)$. To each subset ω of $B_n(X)$, we associate the SFT X_ω consisting of all points x in X such that each word of length n in x is contained in ω. With this association, we view $\mathbb{P}_\alpha$ as a probability measure on the SFTs X_ω that can be built out of the subsets of $B_n(X)$. Briefly, if X has entropy $\mathbf{h}(X) = \log\lambda > 0$ and n is large, then a typical random SFT X_ω is built from about $\alpha\lambda^n$ words, an α fraction of all the words in $B_n(X)$, but not all of these words will occur in any point in X_ω.

Received June 2010; revised October 2010.
MSC2010 subject classifications. Primary 37B10; secondary 37B40, 60C05.
Key words and phrases. Random subshifts of finite type, entropy.

Our main results can be stated as follows. Let $\zeta_X(t)$ denote the Artin–Mazur zeta function of X (see Definition 2.11). The first theorem deals with the likelihood that a randomly chosen SFT is empty.

THEOREM 1.1. *Let X be a nonempty SFT with entropy $\mathbf{h}(X) = \log\lambda$. Let $\mathcal{E}_n \subset \Omega_n$ be the event that X_ω is empty. Then for α in $[0, 1]$,*

$$\lim_{n\to\infty} \mathbb{P}_\alpha(\mathcal{E}_n) = \begin{cases} (\zeta_X(\alpha))^{-1}, & \text{if } \alpha \in [0, 1/\lambda), \\ 0, & \text{if } \alpha \in [1/\lambda, 1]. \end{cases}$$

Thus, when α is in $[0, 1/\lambda)$, there is an asymptotically positive probability of emptiness. The next theorem gives more information about what happens when α lies in $[0, 1/\lambda)$.

THEOREM 1.2. *Let X be a nonempty SFT with entropy $\mathbf{h}(X) = \log\lambda$. Let $\mathcal{Z}_n \subset \Omega_n$ be the event that X_ω has zero entropy, and let I_n be the random variable on Ω_n which is the number of irreducible components of X_ω. Then for $0 \le \alpha < 1/\lambda$,*

(1) $\lim_{n\to\infty} \mathbb{P}_\alpha(\mathcal{Z}_n) = 1$;

(2) *the sequence (I_n) converges in distribution to the random variable I_∞ such that $\mathbb{P}(I_\infty = 0) = (\zeta_X(\alpha))^{-1}$ and for $k \ge 1$,*

$$\mathbb{P}(I_\infty = k) = (\zeta_X(\alpha))^{-1} \sum_{\substack{S\subset\mathbb{N} \\ |S|=k}} \prod_{s\in S} \frac{\alpha^{|\gamma_s|}}{1-\alpha^{|\gamma_s|}},$$

where $\{\gamma_i\}_{i=1}^\infty$ is an enumeration of the periodic orbits in X;

(3) *the random variable I_∞ has exponentially decreasing tail and therefore finite moments of all orders.*

Our next result describes the entropy of the typical random SFT when α lies in $(1/\lambda, 1]$.

THEOREM 1.3. *Let X be an SFT with positive entropy $\mathbf{h}(X) = \log\lambda$. Then for $1/\lambda < \alpha \le 1$ and $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}_\alpha\big(|\mathbf{h}(X_\omega) - \log(\alpha\lambda)| \ge \varepsilon\big) = 0,$$

and the convergence to this limit is exponential in n.

Finally, we have a result concerning the likelihood that a random SFT will have a unique irreducible component of positive entropy when α is near 1.

THEOREM 1.4. *Let X be an irreducible SFT with positive entropy $\mathbf{h}(X) = \log\lambda$. Let $W_n \subset \Omega_n$ be the event that X_ω has a unique irreducible component C of positive entropy and C has the same period as X. Then there exists $c > 0$ such that for $1 - c < \alpha \leq 1$,*

$$\lim_{n\to\infty} \mathbb{P}_\alpha(W_n) = 1;$$

furthermore, the convergence to this limit is exponential in n.

There have been studies of other objects called random subshifts of finite type in the literature [7, 8, 25, 31–35], but the objects studied here are rather different in nature. The present work is more closely related to perturbations of SFTs, which have already appeared in works by Lind [38] in dimension 1 and by Pavlov [47] in higher dimensions. In those works, the main results establish good uniform bounds for the entropy of an SFT obtained by removing any single word of length n from a sufficiently mixing SFT as n tends to infinity. Random SFTs may also be interpreted as dynamical systems with holes [11–15, 17–20, 41, 42], in which case the words of length n in X that are forbidden in the random SFT X_ω are viewed as (random) holes in the original system X. The question of whether an SFT defined by a set of forbidden words is empty has been studied in formal language theory and automata theory, and in that context it amounts to asking whether the set of forbidden words is *unavoidable* [4, 10, 29]. Also, the random SFTs considered here can be viewed as specific instances of random matrices (see [3, 43]) or random graphs (see [2, 5, 22–24, 27, 28, 44]), and the concept of directed percolation on finite graphs has appeared in the physics literature in the context of directed networks [46, 49]. To the best of our knowledge, the specific considerations that arise for our random SFTs seem not to have appeared in any of this wider literature.

The paper is organized as follows. Section 2 contains the necessary background and notation, as well as some preliminary lemmas. The reader familiar with SFTs and directed graphs may prefer to skip Sections 2.1 and 2.2, referring back as necessary. In Section 3 we discuss the likelihood that a random SFT is empty, and, in particular, we prove Theorem 1.1. The remainder of the main results are split into two sections according to two cases: $\alpha \in [0, 1/\lambda)$ and $\alpha \in (1/\lambda, 1]$. The case $\alpha \in [0, 1/\lambda)$ is treated in Section 4, and the case $\alpha \in (1/\lambda, 1]$ is addressed in Section 5. Section 6 discusses some corollaries of the main results.

2. Preliminaries.

2.1. *Shifts of finite type and their presentations.* For a detailed treatment of SFTs and their presentations, see [40]. In this section we describe three ways to present an SFT: with a finite list of forbidden words over a finite alphabet, with a finite, directed graph, or with a square, nonnegative integer matrix.

Let $\mathcal{A}$ be a finite set, which we will call the *alphabet*. An element $b \in \mathcal{A}^n$ is called a *word* of length n. Let $\Sigma = \mathcal{A}^{\mathbb{Z}}$, endowed with the product topology induced by the discrete topology on $\mathcal{A}$. Then Σ is a compact metrizable space, which is called the *full shift* on $\mathcal{A}$. Let $\sigma : \Sigma \to \Sigma$ be the left shift, that is, for $x = (x_i)$ in Σ, let $(\sigma(x))_i = x_{i+1}$. With this definition σ is a homeomorphism of Σ.

A subset X of Σ is called shift-invariant if $\sigma(X) = X$. A closed, shift-invariant subset of Σ is called a *subshift*. For any subshift X, the *language* $\mathcal{B}(X)$ of X is the collection of all finite words (blocks) that appear in some sequence x in X. Note that $\mathcal{B}(X) = \bigcup B_n(X)$, where $B_n(X)$ is the set of all words of length n that appear in some sequence x in X. [By convention we set $B_0(X) = \{\varepsilon\}$, where ε denotes the empty word.] Given a set $\mathcal{F}$ of words on $\mathcal{A}$, we may define a subshift $X(\mathcal{F})$ as the set of sequences x in Σ such that no word in $\mathcal{F}$ appears in x. One may check that this procedure indeed defines a subshift. If X is a subshift and there exists a *finite* set of words $\mathcal{F} = \{F_1, \dots, F_k\}$ such that $X = X(\mathcal{F})$, then X is called a *subshift of finite type* (SFT).

The natural notion of isomorphism for SFTs is called conjugacy. Two SFTs X and Y are *conjugate*, written $X \cong Y$, if there exists a homeomorphism $\phi : X \to Y$ such that $\phi \circ \sigma = \sigma \circ \phi$. An SFT X is *irreducible* if for every two nonempty open sets U and V and every N in $\mathbb{N}$, there exists $n \geq N$ such that $\sigma^n(U) \cap V \neq \varnothing$. An SFT X is *mixing* if for every two nonempty open sets U and V in X, there exists n_0 in $\mathbb{N}$ such that for all $n \geq n_0$, we have $\sigma^n(U) \cap V \neq \varnothing$. Mixing and irreducibility are conjugacy-invariant. We now define the higher block presentations of an SFT.

DEFINITION 2.1. Let X be an SFT. The *n-block presentation* of X, denoted $X^{[n]}$, is defined as follows. The alphabet for $X^{[n]}$ is $B_n(X)$. We define the code $\phi_n : X \to B_n(X)^{\mathbb{Z}}$ by the equation

$$\phi_n(x)_i = x[i, i + n - 1] \tag{2.1}$$

for all x in X. Then $X^{[n]} = \phi_n(X)$. For all $n \geq 1$, we have that $X^{[n]} \cong X$, where the conjugacy is given by ϕ_n.

DEFINITION 2.2. The *entropy* of an SFT X is defined as $\mathbf{h}(X) = \lim_n \frac{1}{n} \times \log|B_n(X)|$.

Alternatively, one may define SFTs in terms of finite directed graphs. A directed graph $G = (V, E)$ consists of a set of vertices V and a set of edges E such that for each edge $e \in E$, there is a unique initial vertex, $i(e) \in V$, and a unique terminal vertex, $t(e) \in V$. We view the edge e as going from $i(e)$ to $t(e)$. We allow self-loops, but for the sake of convenience we assume (without loss of generality for our considerations) that there are no multiple edges. In this paper we make the standing convention that "graph" means directed graph. We will collect our standing assumptions in Standing Assumptions 2.21.

DEFINITION 2.3. Given a directed graph G, we define the *edge shift* X_G to be the set of all bi-infinite (oriented) walks on G, *that is*, $X_G = \{x \in E^{\mathbb{Z}} : t(x_j) = i(x_{j+1})$ for all $j \in \mathbb{Z}\}$.

Any edge shift is an SFT (trivially). Let us show that any SFT is conjugate to an edge shift. If $X = X(\mathcal{F})$ is an SFT and $\mathcal{F}$ is a finite set of forbidden words, then $X \cong X_G$, where $G = (V, E)$ is defined as follows. Let $n_0 = \max\{|F| : F \in \mathcal{F}\}$. Then let $V = B_{n_0-1}(X)$ and $E = B_{n_0}(X)$. Further, for any edge $e \in B_{n_0}(X)$, we let $i(e) = e[1, n_0 - 1]$ and $t(e) = e[2, n_0]$. The same construction works with n in place of n_0 for any $n \geq n_0$.

If G is a graph such that $X \cong X_G$, we say that X_G is an edge presentation of X, or sometimes just a presentation of X. The *adjacency matrix* A of a directed graph G may be defined as follows. Fix an enumeration of the vertices in G. Then let $A_{k\ell}$ be the number of distinct edges e in G such that $i(e) = v_k$ and $t(e) = v_\ell$. A square, nonnegative integral matrix A is *irreducible* if for each pair i, j and each N, there exists $n > N$ such that $(A^n)_{ij} > 0$. A matrix A is *nondegenerate* if it has no zero row and no zero column. If A is nondegenerate, then the edge shift X_G is irreducible if and only if A is irreducible. Also, if A is nondegenerate, then the edge shift X_G is mixing if and only if there exists n_0 such that for all $n \geq n_0$ and all pairs i, j, it holds that $(A^n)_{ij} > 0$. A matrix is *primitive* if it satisfies the latter property. A path in G is a finite sequence $\{e_j\}_{j=1}^n$ of edges such that $t(e_j) = i(e_{j+1})$ for $j = 1, \dots, n-1$. If $b = b_1 \cdots b_n$ is a path in G, we say that b goes from vertex $i(b_1)$ to vertex $t(b_n)$. We denote by $B_k(G)$ the set of paths of length k in G. By convention, we set $B_0(G) = V$.

DEFINITION 2.4. For a path b in G, let $V(b)$ and $E(b)$ be the set of vertices and the set of edges traversed by b, respectively.

DEFINITION 2.5. Let X be an SFT. An *irreducible component* Y of X is a nonempty, maximal SFT contained in X such that Y is irreducible. Let G be a graph. An *irreducible component* C of G is a nonempty, maximal subgraph of G such that the adjacency matrix of C is irreducible. The reader should be advised that in some papers the definition of irreducible component includes trivial components (a single vertex with no edges adjacent to it), but the definition given here does not include trivial components.

DEFINITION 2.6. Let G be a finite, directed graph. For $n \geq 1$, define $G^{[n]} = (V^{[n]}, E^{[n]})$, the *n-block graph* of G, as follows. Let $V^{[n]} = B_{n-1}(G)$ and $E^{[n]} = B_n(G)$, such that if $e \in E^{[n]}$, then $i(e) = e[1, n-1]$ and $t(e) = e[2, n]$. Note that $G^{[1]} = G$.

If $X = X_G$ for some graph G, then it follows immediately from the definitions that $X^{[n]} = X_{G^{[n]}}$.

DEFINITION 2.7. Let $G = (V, E)$ be a graph. For p in $\mathbb{N}$, we define the pth power graph, $G^p = (V^p, E^p)$, as follows. Let $V^p = V$ and $E^p = B_p(G)$. If $b = b_1 \cdots b_p$ is an edge in G^p, then we let $i(b) = i(b_1)$ and $t(b) = t(b_1)$.

DEFINITION 2.8. Let $G = (V, E)$ be a graph. Define the transpose graph, $G^T = (V^T, E^T)$, as follows. Let $V^T = V$ and $E^T = E$, where an edge e in G^T goes from $t(e)$ to $i(e)$. In other words, the transpose graph is just the graph formed by reversing the direction of all the edges in G.

Given a square, nonnegative, integral matrix A, one may also define an SFT X_A as follows. Let G be a directed graph whose adjacency matrix is exactly A (such a graph always exists). Then let X_A be the edge shift defined by G.

Recall the following basic facts (which may be found in [40]). For an SFT X, we have $\mathbf{h}(X) = \inf_n \frac{1}{n} \log|B_n(X)|$. If X is a nonempty SFT and $X = X_A$ for a square, nonnegative integral matrix A, then $\mathbf{h}(X) = \log \lambda$, where λ is the spectral radius of A. By the Perron–Frobenius theorem, if A is nonnegative and irreducible, then there exists a strictly positive (column) vector v such that $Av = \lambda v$, and there exists a strictly positive (row) vector w such that $wA = \lambda w$. Furthermore, v and w are each unique up to a positive scalar.

DEFINITION 2.9. For any nonnegative integer matrix A, let λ_A be the spectral radius of A, and let χ_A be the characteristic polynomial of A. Then let $\mathrm{Sp}_\times(A)$ be the *nonzero spectrum* of the matrix A, which is defined as the multiset of nonzero roots of χ_A listed according to their multiplicity. If A is the adjacency matrix of the graph G, we define $\lambda_G = \lambda_A$ and $\mathrm{Sp}_\times(G) = \mathrm{Sp}_\times(A)$.

If $X_A \cong X_B$ for two nonnegative integral matrices A and B, then $\mathrm{Sp}_\times(A) = \mathrm{Sp}_\times(B)$. Also, if A is primitive, then $\max\{|\beta| : \beta \in \mathrm{Sp}_\times(A) \setminus \{\lambda_A\}\} < \lambda_A$. Finally, if A is irreducible, then there exists a unique σ-invariant Borel probability measure μ on X_A of maximal entropy. Let us describe some basic properties of μ. We associate a word $b = b_1 \cdots b_k$ in X to the cylinder set $C_b = \{x \in X : x[1, k] = b\}$. In this way we interpret the measure of words in $\mathcal{B}(X)$ as the measure of the corresponding cylinder set. Let v be a positive right eigenvector of A and w a positive left eigenvector of A, and suppose they are normalized so that $w \cdot v = 1$. Our standing assumption that there are no multiple edges means that $A_{ij} \leq 1$ for all i, j. Then for a vertex u in V, we have $\mu(u) = w_u v_u$, and for $b \in B_n(X_A)$, we have that

$$\mu(b) = w_{i(b_1)} \lambda_A^{-n} v_{t(b_n)}. \tag{2.2}$$

Now we define two objects, the period and the zeta function, which contain combinatorial information about the cycles in a graph G (alternatively, one may refer to the periodic points in an SFT X).

DEFINITION 2.10. For an SFT X, let $\mathrm{per}(X)$ be the greatest common divisor of the sizes of all periodic orbits in X. For a graph G, let $\mathrm{per}(G)$ be the greatest common divisor of the lengths of all cycles in G.

DEFINITION 2.11. Let X be an SFT and $N_p = |\{x \in X : \sigma^p(x) = x\}|$. Then the Artin–Mazur zeta function of X (see [40]) is, by definition,

$$\zeta_X(t) = \exp\left(\sum_{p=1}^{\infty} \frac{N_p}{p} t^p\right).$$

For a graph G, let $\zeta_G = \zeta_{X_G}$.

For a graph G, note that $|\{x \in X_G : \sigma^p(x) = x\}|$ is the number of cycles of (not necessarily least) period p in G, and

$$\zeta_G(t) = \frac{1}{\det(I - tA)} = \prod_{\lambda \in \mathrm{Sp}_\times(G)} \frac{1}{1 - \lambda t}.$$

Also, ζ_G has radius of convergence $1/\lambda_G$ and $\lim_{t \to 1/\lambda_G^-} \zeta_G(t) = +\infty$.

2.2. *Sequences of graphs under consideration.* In this work we consider sequences of graphs (G_n) that grow in some way. A particular example of such a sequence is the sequence of n-block graphs of an SFT X. Indeed, by taking (G_n) to be such a sequence in Theorems 3.1, 4.2, 5.13 and 5.15, we obtain the theorems stated in the Introduction. Generalizing to the graph setting also allows one to consider sequences of graphs presenting SFTs which are conjugate to a fixed SFT X, where the sequences need not be the n-block sequence for X. To indicate the generality of the arguments further, though, we formulate and prove the results for sequences of graphs that do not necessarily present conjugate SFTs. Before we move on to these results, we need to define several notions regarding the manner of growth of the sequence (G_n).

Let G be a finite, directed graph with adjacency matrix A. We will have use for the following notation.

DEFINITION 2.12. Let

$$\mathrm{Per}_p(G) = \{b \in B_p(G) : i(b_1) = t(b_p)\} \quad \text{and} \quad \mathrm{Per}(G) = \bigcup_{p=1}^{\infty} \mathrm{Per}_p(G).$$

For b in $\mathrm{Per}_p(G)$, let $\theta(b)$ be the set of all paths c in $\mathrm{Per}_p(G)$ such that there exists a natural number ℓ such that $c = b_{\tau^\ell(1)} \cdots b_{\tau^\ell(p)}$, where τ is the permutation of $\{1, \ldots, k\}$ defined in cycle notation by $(1 \cdots k)$.

DEFINITION 2.13. For each vertex u in G, let $d_{\mathrm{out}}(u) = |\{e \in E : i(e) = u\}|$ and $d_{\mathrm{in}}(u) = |\{e \in E : t(e) = u\}|$. Then let

$$d_{\max}(G) = \max\{\max(d_{\mathrm{out}}(u), d_{\mathrm{in}}(u)) : u \in V\}.$$

In order to measure the separation of periodic orbits in G, we make the following definition.

DEFINITION 2.14. Let

$$z(G) = \max\left\{n \geq 0 : \forall b, c \in \bigcup_{p=1}^{n} \mathrm{Per}_p(G) \text{ with } c \notin \theta(b), V(b) \cap V(c) = \varnothing\right\},$$

where $V(b)$ is the set of vertices traversed by the path b.

As a measure of the size of G, we consider the following quantity.

DEFINITION 2.15. If A has spectral radius $\lambda > 1$, then let

$$m(G) = \lceil \log_\lambda |V| \rceil.$$

To measure a range for uniqueness of paths in G, we make the following definitions.

DEFINITION 2.16. Let

$$\begin{aligned} U_1(G) &= \sup\{n : \forall i, j \text{ it holds that } (A^n)_{ij} \leq 1\}, \\ U_2(G) &= \sup\bigl\{n : \forall u \in V \text{ and } 1 \leq s < t \leq n, \\ &\qquad\qquad |\{b \in B_t(X) : i(b_1) = u, b_s = b_t\}| \leq 1\bigr\}, \\ U(G) &= \min(U_1(G), U_2(G)). \end{aligned}$$

We use the transition length as a type of diameter of G.

DEFINITION 2.17. Let

$$R(G) = \inf\{n : \forall i, j, \exists k \leq n, (A^k)_{ij} > 0\}.$$

Here we briefly recall the notion of the weighted Cheeger constant of an irreducible, directed graph G. The weighted Cheeger constant was defined and studied in [16]. Let μ be the measure of maximal entropy of X_G, and let $F : E \to [0, 1]$ be given by $F(e) = \mu(e)$. For any vertex v in V, let $F(v) = \sum_{i(e)=v} F(e) = \sum_{t(e)=v} F(e)$. Then for any subset of vertices $S \subseteq V$, let $F(S) = \sum_{v \in S} F(v)$, and for any two subsets $S, T \subseteq V$, let

$$F(S, T) = \sum_{\substack{i(e) \in S \\ t(e) \in T}} F(e).$$

In general, $F(S,T)$ is not symmetric in S and T since G is directed. Let $E(S,T)$ be the set of edges e in G such that $i(e) \in S$ and $t(e) \in T$. Let $\overline{S} = V \setminus S$.

DEFINITION 2.18. The weighted Cheeger constant of G is defined as

$$c_w(G) = \inf_{\varnothing \subsetneq S \subsetneq V} \frac{F(S,\overline{S})}{\min(F(S), F(\overline{S}))},$$

and the unweighted Cheeger constant of G is defined as

$$c(G) = \inf_{0<|S|\leq |V|/2} \frac{|E(S,\overline{S})|}{|S|}.$$

DEFINITION 2.19. We say that G is a directed b-expander graph if $c(G) \geq b$. Also, a sequence of directed graphs (G_n) is a *uniform expander sequence*, if there exists a $b > 0$ such that G_n is a directed b-expander for each n.

We will also have use for the following quantity related to the spectral gap of G.

DEFINITION 2.20. Let $g(G) = \min\{1 - \frac{|\lambda_i|}{\lambda} : \lambda_i \in \mathrm{Sp}_\times(G) \setminus \{\lambda\}\}$.

We make the following standing assumptions, even though some of the statements we make may hold when these restrictions are relaxed. In particular, Theorems 3.1 and 4.2 do not require that A_n is irreducible, nor do they require that $\lambda > 1$ (see Remark 6.1).

STANDING ASSUMPTIONS 2.21. Recall that "graph" means directed graph. Let (G_n) be a sequence of graphs with an associated sequence of adjacency matrices (A_n). Unless otherwise stated, we will make the following assumptions:

- for each n, each entry of A_n is contained in $\{0, 1\}$;
- each A_n is irreducible;
- for each n, $\mathrm{Sp}_\times(A_n) = \mathrm{Sp}_\times(A_1)$;
- $\lambda := \lambda_{A_1} > 1$;
- $\lim_n m(G_n) = \infty$.

REMARK 2.22. Note that $|\mathrm{Per}_p(G_n)| = \mathrm{tr}(A_n^p)$, which depends only on $\mathrm{Sp}_\times(A_n)$ and p. Therefore, the standing assumptions imply that $|\mathrm{Per}_p(G_n)|$ does not depend on n, and, therefore, $\mathrm{per}(G_n)$ and ζ_{G_n} do not depend on n.

Additional conditions that we place on sequences of graphs will come from the following list. [Different theorems will require different assumptions, but the sequence of n-block graphs of an irreducible graph with spectral radius greater than 1 will satisfy conditions (C1)–(C8) below by Proposition 2.29.]

DEFINITION 2.23. We define the following conditions on a sequence of graphs (G_n) with a sequence of adjacency matrices (A_n):

(C1) there exists $\Delta > 0$ such that $d_{\max}(G_n) \leq \Delta$ for all n (bounded degree);

(C2) $z(G_n)$ tends to infinity as n tends to infinity (separation of periodic points);

(C3) there exists $C > 0$ such that $z(G_n) \geq Cm(G_n)$ for all n (fast separation of periodic points);

(C4) there exists $C > 0$ such that $U(G_n) \geq m(G_n) - C$ for all n (local uniqueness of paths);

(C5) there exists $C > 0$ such that $R(G_n) \leq m(G_n) + C$ for all n (small diameter);

(C6) there exists $K > 0$ such that $\max_{u \in V_n} \mu(u) \leq K \min_{u \in V_n} \mu(u)$ for all n (bounded distortion of vertices) and $\max_{e \in E_n} \mu(e) \leq K \min_{e \in E_n} \mu(e)$ for all n (bounded distortion of edges);

(C7) there exists $K > 0$ such that $\max_i w_i^n \leq K \min_i w_i^n$ and $\max_i v_i^n \leq K \min_i v_i^n$ for all n, where w^n is a positive left eigenvector of A_n and v^n is a positive right eigenvector of A_n (bounded distortion of weights);

(C8) (G_n) is a uniform expander sequence, and (G_n^T) is a uniform expander sequence (forward/backward expansion).

Now we establish some lemmas, which will be used in the subsequent sections.

LEMMA 2.24. *Let (G_n) be a sequence of graphs satisfying the Standing Assumptions 2.21. Then* (C7) *implies* (C1) *and* (C6) *for both (G_n) and (G_n^T).*

PROOF. First note that if (C7) holds for (G_n), then it also holds for (G_n^T) since a positive left eigenvector for A_n^T is given by $(v^n)^T$ and a positive right eigenvector for A_n^T is given by $(w^n)^T$. Therefore, we only need to show that (C7) for (G_n) implies (C1) and (C6) for (G_n) [since the same argument will apply to (G_n^T)].

Let w^n and v^n be positive left and right eigenvectors for A_n, respectively, and assume that $w^n \cdot v^n = 1$. Recall with this normalization, if u is a vertex in V_n, then $\mu(u) = w_u^n v_u^n$. Then condition (C7) implies that there exists $K > 0$ such that for all n,

$$\begin{aligned}\max_u \mu(u) &\leq \max_u w_u^n \max_u v_u^n \leq K^2 \min_u w_u^n \min_u v_u^n \\ &\leq K^2 \min_u w_u^n v_u^n = K^2 \min_u \mu(u).\end{aligned}$$

Similarly, (C7) implies that there exists $K' > 0$ such that for all n, we have that $\max_{e \in E_n} \mu(e) \leq K' \min_{e \in E_n} \mu(e)$ [recall that $\mu(e) = w_{i(e)}^n \lambda^{-1} v_{t(e)}^n$]. Thus, (C7) implies (C6).

Note that for e in E_n, we have that

$$\mu(e|i(e)) = \frac{w^n_{i(e)}\lambda^{-1}v^n_{t(e)}}{w^n_{i(e)}v^n_{i(e)}} = \frac{v^n_{t(e)}}{\lambda v^n_{i(e)}}.$$

Then condition (C7) implies that there exists a uniform constant $K > 0$ such that $\mu(e|i(e)) \geq K^{-1}$ for all n and all e in E_n. We also have that

$$\mu(u) = \sum_{e:i(e)=u} \mu(e) \geq \sum_{e:i(e)=u} K^{-1}\mu(u) = |\{e : i(e) = u\}| K^{-1}\mu(u).$$

Since G_n is irreducible (by Standing Assumptions 2.21), we know that $\mu(u) > 0$, and, therefore, we have that for any n, and any u in V_n,

$$|\{e \in E_n : i(e) = u\}| \leq K,$$

which implies that $\max_u d_{\mathrm{out}}(u)$ is uniformly bounded in n. A similar argument shows that $\max_u d_{\mathrm{in}}(u)$ is uniformly bounded in n, which shows that $d_{\max}(G_n)$ is uniformly bounded in n and gives (C1). □

Recall that for a graph G, the quantities $g(G)$ and $c_w(G)$ were defined in Definitions 2.20 and 2.18, respectively.

LEMMA 2.25. *Let G be a graph with primitive adjacency matrix A. Then it holds that $c_w(G) \geq \frac{1}{2}g$.*

PROOF. This lemma is a consequence of [16], Theorems 4.3 and 5.1, as we now explain. Since A is primitive, there exists a strictly positive vector v and $\lambda \geq 1$ such that $Av = \lambda v$. Let P be the stochastic matrix defined by $P_{ij} = \frac{A_{ij}v_j}{\lambda v_i}$. Then P is the transition probability matrix corresponding to the random walk defined by the measure of maximal entropy μ on X_G. We have that $\mathrm{Sp}_\times(P) = \frac{1}{\lambda}\,\mathrm{Sp}\times(A)$. Given such a transition probability matrix, Chung defines a Laplacian L and proves ([16], Theorem 4.3), that the smallest nonzero eigenvalue of L, denoted λ_1, satisfies the following inequality:

$$\min\bigl\{1 - |\rho| : \rho \in \mathrm{Sp}_\times(P) \setminus \{1\}\bigr\} \leq \lambda_1. \tag{2.3}$$

We remark that the left-hand side of the inequality in [16], Theorem 4.3, is equal to the left-hand side of (2.3) since A is primitive (not just irreducible). Note that the left-hand side of (2.3) equals $g(G)$, as defined in Definition 2.20. After defining the weighted Cheeger constant (as in Definition 2.18), Chung proves ([16], Theorem 5.1), that

$$c_w(G) \geq \tfrac{1}{2}\lambda_1. \tag{2.4}$$

Combining the inequalities in (2.3) and (2.4), we obtain the desired inequality. □

Recall that the pth power graph was defined in Definition 2.7.

LEMMA 2.26. *Let G be a graph with an irreducible adjacency matrix. Let $p = \operatorname{per}(G)$. Let $G^{p,0}$ be an irreducible component of G^p, the pth power graph of G. Let $g = g(G^{p,0})$ (which does not depend on the choice of irreducible component in G^p). Then there exists $b > 0$, depending only on g and p, such that $c_w(G) \geq b$.*

PROOF. Let G, p and g be as in the statement of the lemma. If $p = 1$, then Lemma 2.25 immediately gives the result. Now we assume $p \geq 2$. The fact that G is irreducible and $\operatorname{per}(G) = p$ implies that there is a partition of the vertices into p nonempty subsets, $V = \bigcup_{j=0}^{p-1} V^j$, such that for each edge e with $i(e) \in V^j$, it holds that $t(e) \in V^{j+1}$, where the superscripts are taken modulo p. Let $X = X_G$ (Definition 2.3), and for each $j = 0, \dots, p-1$, let $X_j = \{x \in X : i(x_0) \in V^j\}$. For any set $S \subset V$ with $0 < |S| < |V|$ and $j = 0, \dots, p-1$, define

$$C_S = \{x \in X : i(x_0) \in S\}, \qquad \overline{C}_S = X_G \setminus C_S,$$

$$C_S^j = X_j \cap C_S \quad \text{and} \quad \overline{C}_S^j = X_j \cap \overline{C}_S.$$

Recall that we denote by μ the measure of maximal entropy on X, and we may write $c_w(G)$ as follows:

$$\begin{aligned} c_w(G) &= \inf_{\varnothing \subsetneq S \subsetneq V} \frac{\mu(C_S \cap \sigma^{-1}\overline{C}_S)}{\min(\mu(C_S), \mu(\overline{C}_S))} \\ &= \inf_{\varnothing \subsetneq S \subsetneq V} \max\left(\frac{\mu(C_S \cap \sigma^{-1}\overline{C}_S)}{\mu(C_S)}, \frac{\mu(C_S \cap \sigma^{-1}\overline{C}_S)}{\mu(\overline{C}_S)}\right). \end{aligned}$$

We also use the following notation:

$$r_i = \frac{\mu(C_S^i)}{\mu(C_S)} \quad \text{and} \quad \overline{r}_i = \frac{\mu(\overline{C}_S^i)}{\mu(\overline{C}_S)}. \tag{2.5}$$

Let us establish a useful inequality. For $i = 0, \dots, p-1$ and $1 \leq \ell \leq p$, note that each point x in $C_S^i \cap \sigma^{-\ell}\overline{C}_S^{i+\ell}$ also lies in $C_S^j \cap \sigma^{-1}\overline{C}_S^{j+1}$ for $j = \min\{k > 0 : \sigma^k x \notin C_S\}$. Thus,

$$\mu(C_S^i \cap \sigma^{-\ell}\overline{C}_S^{i+\ell}) \leq \sum_{j=0}^{p-1} \mu(C_S^j \cap \sigma^{-1}\overline{C}_S^{j+1}) = \mu(C_S \cap \sigma^{-1}\overline{C}_S). \tag{2.6}$$

To complete the proof, we will find $b > 0$ in terms of g and p so that for $S \subset V$ with $0 < |S| < |V|$, we have that

$$b \leq \max\left(\frac{\mu(C_S \cap \sigma^{-1}\overline{C}_S)}{\mu(C_S)}, \frac{\mu(C_S \cap \sigma^{-1}\overline{C}_S)}{\mu(\overline{C}_S)}\right). \tag{2.7}$$

The bound b will be the minimum of four bounds, each coming from a particular type of set $S \subset V$.

Consider the following conditions on the set S, which we will use to break our proof into cases:

(I) there exists $i \in \{0, \dots, p-1\}$ such that $\mu(C_S^i) \in \{0, 1\}$;
(II) $\mu(C_S^i) \leq 1/2p$ for each i, or $\mu(C_S^i) \geq 1/2p$ for each i;
(III) $1/4p \leq \mu(C_S^i) \leq 3/4p$ for each i.

Now we consider cases.

Case: (I) holds, *that is*, there exists $i \in \{0, \dots, p-1\}$ such that $\mu(C_S^i) \in \{0, 1\}$. Assume first that $\mu(C_S^i) = 0$, which implies that $\mu(\overline{C}_S^i) = \mu(X_i)$. Choose j such that $\mu(C_S^j) = \max_k \mu(C_S^k)$, and finally choose $1 \leq \ell \leq p$ such that $j + \ell = i \pmod p$. Then by inequality (2.6) and the shift-invariance of μ, we have that

$$\frac{\mu(C_S \cap \sigma^{-1}\overline{C}_S)}{\mu(C_S)} \geq \frac{\mu(C_S^j \cap \sigma^{-\ell}\overline{C}_S^{j+\ell})}{\mu(C_S)} \geq \frac{\mu(C_S^j \cap \sigma^{-\ell} X_{j+\ell})}{p \max_k \mu(C_S^k)} = \frac{\mu(C_S^j)}{p\mu(C_S^j)} = \frac{1}{p}.$$

Now assume $\mu(C_S^i) = 1$. Choose j such that $\mu(\overline{C}_S^j) = \max_k \mu(\overline{C}_S^k)$, and finally choose $1 \leq \ell \leq p$ such that $i + \ell = j \pmod p$. Then by (2.6) and the shift-invariance of μ,

$$\frac{\mu(C_S \cap \sigma^{-1}\overline{C}_S)}{\mu(\overline{C}_S)} \geq \frac{\mu(C_S^i \cap \sigma^{-\ell}\overline{C}_S^{i+\ell})}{\mu(\overline{C}_S)} \geq \frac{\mu(X_i \cap \sigma^{-\ell}\overline{C}_S^j)}{p \max_k \mu(\overline{C}_S^k)} = \frac{\mu(\overline{C}_S^j)}{p\mu(\overline{C}_S^j)} = \frac{1}{p}.$$

Let $b_1 = 1/p$, and note that if condition (I) holds, then the inequality in (2.7) holds with b_1 in place of b.

Case: (I) does not hold, but (II) holds, *that is*, $0 < \mu(C_S^i) \leq 1/2p$ for all i, or $1 > \mu(C_S^i) \geq 1/2p$ for all i. Assume first that $0 < \mu(C_S^i) \leq 1/2p$ for all i. Since $\sum_i r_i = 1$ and $r_i \geq 0$ for all i, there exists j such that $r_j \geq 1/p$. Then by (2.6) and the definition of r_i in (2.5),

$$\begin{aligned}\frac{\mu(C_S \cap \sigma^{-1}\overline{C}_S)}{\mu(C_S)} &\geq \frac{\mu(C_S^j \cap \sigma^{-p}\overline{C}_S^j)}{\mu(C_S)} = r_j \frac{\mu(C_S^j \cap \sigma^{-p}\overline{C}_S^j)}{\mu(C_S^j)} \\ &\geq \frac{1}{p} \cdot \frac{\mu(C_S^j \cap \sigma^{-p}\overline{C}_S^j)}{\mu(C_S^j)}.\end{aligned}$$

Let $G^{p,j}$ be the irreducible component of G^p with vertex set V^j. Then $G^{p,j}$ has primitive adjacency matrix, and $g = g(G^{p,j}) > 0$. Lemma 2.25 gives that $c_w(G^{p,j}) \geq \frac{1}{2}g$. Since $\mu(C_S^j) \leq 1/2p$ and $\mu(C_S^j) + \mu(\overline{C}_S^j) = \mu(X_j) = 1/p$, we have that $\min(\mu(C_S^j), \mu(\overline{C}_S^j)) = \mu(C_S^j)$, and, thus,

$$\frac{\mu(C_S^j \cap \sigma^{-p}\overline{C}_S^j)}{\mu(C_S^j)} \geq c_w(G^{p,j}) \geq \frac{1}{2}g.$$

Let $b_2 = g/2p$. We have shown that for S such that $\mu(C_S^i) \leq 1/2p$ for each i, the inequality in (2.7) holds with b_2 in place of b. For S such that $1 > \mu(C_S^i) \geq 1/2p$

for each i, choose j such that $\overline{r}_j \geq 1/p$. Then an analogous argument gives that the inequality in (2.7) holds with b_2 in place of b.

Case: (III) holds, *that is*, $1/4p \leq \mu(C_S^i) \leq 3/4p$ for all i. A simple calculation yields that $r_i \geq 1/3p$ and $\overline{r}_i \geq 1/3p$ for each i. Using (2.6), we see that for each j,

$$\begin{aligned}\frac{\mu(C_S \cap \sigma^{-1}\overline{C}_S)}{\mu(C_S)} &\geq \frac{\mu(C_S^j \cap \sigma^{-p}\overline{C}_S^j)}{\mu(C_S)} = r_j \frac{\mu(C_S^j \cap \sigma^{-p}\overline{C}_S^j)}{\mu(C_S^j)} \\ &\geq \frac{1}{3p} \cdot \frac{\mu(C_S^j \cap \sigma^{-p}\overline{C}_S^j)}{\mu(C_S^j)}\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\frac{\mu(C_S \cap \sigma^{-1}\overline{C}_S)}{\mu(\overline{C}_S)} &\geq \frac{\mu(C_S^j \cap \sigma^{-p}\overline{C}_S^j)}{\mu(\overline{C}_S)} = \overline{r}_j \frac{\mu(C_S^j \cap \sigma^{-p}\overline{C}_S^j)}{\mu(\overline{C}_S^j)} \\ &\geq \frac{1}{3p} \cdot \frac{\mu(C_S^j \cap \sigma^{-p}\overline{C}_S^j)}{\mu(\overline{C}_S^j)}.\end{aligned} \tag{2.9}$$

Then since $G^{p,j}$ has a primitive adjacency matrix, Lemma 2.25 and inequalities (2.8) and (2.9) give that the inequality in (2.7) holds with $b_3 := g/6p$ in place of b.

Case: each of (I), (II) and (III) does not hold, *that is*, we assume that S is such that $0 < \mu(C_S^i) < 1$ for each i, there exists i_1 and i_2 such that $\mu(C_S^{i_1}) > 1/2p$ and $\mu(C_S^{i_2}) < 1/2p$, and there exists i_3 such that either $\mu(C_S^{i_3}) < 1/4p$ or $\mu(C_S^{i_3}) > 3/4p$. Suppose first that $\mu(C_S^{i_3}) < 1/4p$. Choose j such that $\mu(C_S^j) = \max_k \mu(C_S^k)$, and choose $1 \leq \ell \leq p$ such that $j + \ell = i_3 \pmod p$. Calculation gives that $\mu(C_S^{i_3}) < \frac{1}{2}\mu(C_S^j)$. Then by (2.6) and the shift-invariance of μ,

$$\begin{aligned}\frac{\mu(C_S \cap \sigma^{-1}\overline{C}_S)}{\mu(C_S)} &\geq \frac{\mu(C_S^j \cap \sigma^{-\ell}\overline{C}_S^{j+\ell})}{p\mu(C_S^j)} \geq \frac{\mu(C_S^j) - \mu(C_S^{i_3})}{p\mu(C_S^j)} \\ &\geq \frac{\mu(C_S^j) - (1/2)\mu(C_S^j)}{p\mu(C_S^j)} = \frac{1}{2p}.\end{aligned}$$

Now assume $\mu(C_S^{i_3}) > 3/4p$. Choose j such that $\mu(C_S^j) = \max_k \mu(C_S^k)$ and choose $1 \leq \ell \leq p$ such that $j + \ell = i_2 \pmod p$. Calculation reveals that $\mu(C_S^{i_2}) < \frac{2}{3}\mu(C_S^j)$. Then by (2.6) and the shift-invariance of μ,

$$\begin{aligned}\frac{\mu(C_S \cap \sigma^{-1}\overline{C}_S)}{\mu(C_S)} &\geq \frac{\mu(C_S^j \cap \sigma^{-\ell}\overline{C}_S^{j+\ell})}{p\mu(C_S^j)} \geq \frac{\mu(C_S^j) - \mu(C_S^{i_2})}{p\mu(C_S^j)} \\ &\geq \frac{\mu(C_S^j) - (2/3)\mu(C_S^j)}{p\mu(C_S^j)} = \frac{1}{3p}.\end{aligned}$$

Let $b_4 = 1/3p$. We have shown that for S in this case, the inequality in (2.7) holds with b_4 in place of b.

Now let $b = \min(b_1, b_2, b_3, b_4) = \min(1/p, g/2p, g/6p, 1/3p) = g/6p$, which depends only on g and p. We have shown that $c_w(G) \geq b$. $\square$

Recall that the transpose graph G^T of a graph G was defined in Definition 2.8.

LEMMA 2.27. *Let (G_n) be a sequence of graphs satisfying the Standing Assumptions 2.21 and such that both (G_n) and (G_n^T) have bounded degrees and bounded distortion of edges and vertices* [*conditions* (C1) *and* (C6) *in Definition* 2.23]. *Then (G_n) and (G_n^T) are both uniform expander sequences* [*condition* (C8) *in Definition* 2.23].

PROOF. We check that conditions (C1) and (C6) for (G_n) together imply that (G_n) is a uniform expander sequence, and then the same argument will apply to (G_n^T) since (C1) and (C6) also hold for (G_n^T).

Recall the following notation. Let $F : E_n \to [0, 1]$ be given by $F(e) = \mu(e)$, where μ is the measure of maximal entropy on X_{G_n}. Also, $c_w(G_n)$ denotes the weighted Cheeger constant of G_n (Definition 2.18). By the Standing Assumptions 2.21, $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$ for each n. Therefore, $\mathrm{per}(G_n)$ does not depend on n, and we let $p = \mathrm{per}(G_1)$. Let $G_n^{p,0}$ be an irreducible component of the pth power graph of G_n, and let $g_n = g(G_n^{p,0})$. Since g_n only depends on the nonzero spectrum of G_n, which is constant in n by the Standing Assumptions 2.21, we have the g_n is constant in n. Let $g = g_1$. By Lemma 2.26, there exists $b_n > 0$, depending only g_n and $\mathrm{per}(G_n)$, such that $c_w(G_n) \geq b_n$. Since we have that $g_n = g$ and $\mathrm{per}(G_n) = p$ for all n, we may choose $b := b_1$, and we obtain that $c_w(G_n) \geq b > 0$ for all n.

Now we relate $c_w(G_n)$ to $c(G_n)$ (Definition 2.18) using properties (C1) and (C6). For notation, let $m = m(G_n)$. Since (G_n) satisfies conditions (C1) and (C6), there exists $K_1, K_2 > 0$ such that for every n and every subset $S \subset V_n$,

$$K_1|S|\lambda^{-m} \leq F(S) \leq K_2|S|\lambda^{-m}$$

and

$$K_1|E_n(S, \overline{S})|\lambda^{-m} \leq F(S, \overline{S}) \leq K_2|E_n(S, \overline{S})|\lambda^{-m}.$$

We already have that $c_w(G_n) \geq b$, which implies that for every S such that $\varnothing \subsetneq S \subsetneq V_n$,

$$b \leq \frac{F(S, \overline{S})}{\min(F(S), F(\overline{S}))} \leq \frac{K_2|E_n(S, \overline{S})|\lambda^{-m}}{\min(F(S), F(\overline{S}))}.$$

Now assume $0 < |S| \leq |V_n|/2$. If $\min(F(S), F(\overline{S})) = F(S)$, then $\min(F(S), F(\overline{S})) = F(S) \geq K_1|S|\lambda^{-m}$. If $\min(F(S), F(\overline{S})) = F(\overline{S})$, then we have $\min(F(S),$

$F(\overline{S})) = F(\overline{S}) \geq K_1|\overline{S}|\lambda^{-m} \geq K_1|S|\lambda^{-m}$. Combining these estimates gives that for all S such that $0 < |S| \leq |V_n|/2$, we obtain that

$$|E_n(S, \overline{S})| \geq b\frac{K_1}{K_2}|S|,$$

which shows that (G_n) is a uniform $(b\frac{K_1}{K_2})$-expander sequence. □

LEMMA 2.28. *Let* (G_n) *be a sequence of graphs satisfying the Standing Assumptions* 2.21 *and bounded distortion of weights* [*condition* (C7) *in Definition* 2.23]. *Then*:

(1) *there exists* $K > 0$ *such that for all* n, k *and* $S \subset B_k(G_n)$,

$$K^{-1}|S| \leq \lambda^{m(G_n)+k}\mu(S) \leq |S|K;$$

(2) *there exists a* $K > 0$ *such that for all* $n, k, e \in E_n$, *and* $S \subset B_k(G_n)$,

$$K^{-1}|S \cap C_e^{n,k}| \leq \lambda^k \mu(S|C_e^{n,k}) \leq K|S \cap C_e^{n,k}|,$$

where $C_e^{n,k} = \{b \in B_k(G_n) : b_1 = e\}$;

(3) *there exists* $K > 0$ *such that for all* n, k, *and* $1 \leq s < t \leq k$, *it holds that* $\mu(A_{s,t}) \leq K\lambda^{-m(G_n)}$, *where* $A_{s,t} = \{b \in B_k(G_n) : b_s = b_t\}$;

(4) *there exists* $K > 0$ *such that for all* n, $k > U(G_n)$, *and* $u \in V_n$, *it holds that* $\mu(\mathrm{Per}_k(G_n)|C_u^{n,k}) \leq K\lambda^{-U(G_n)}$, *where* $C_u^{n,k} = \{b \in B_k(G_n) : i(b_1) = u\}$ *and* $U(G_n)$ *was defined in Definition* 2.16.

PROOF. For notation, let $m = m(G_n)$ and $U = U(G_n)$.

Proof of (1). We have that

$$1 = \sum_{u \in V_n} \mu(u) = \sum_{u \in V_n} w_u^n v_u^n.$$

Then condition (C7) implies that there exists $K_1 > 0$ such that for each n and u in V_n,

$$K_1^{-1}|V_n|^{-1} \leq w_u^n v_u^n \leq K_1|V_n|^{-1}.$$

By the definition of m, there exists $K_2 > 0$ such that $K_2^{-1}|V_n|^{-1} \leq \lambda^{-m} \leq K_2|V_n|^{-1}$. It follows that there exists $K_3 > 0$ such that for each n and u in V_n,

$$K_3^{-1}\lambda^{-m} \leq w_u^n v_u^n \leq K_3\lambda^{-m}.$$

Then (C7) implies that there exists $K_4 > 0$ such that for any n and any three vertices u, u_1 and u_2 in V_n,

$$K_4^{-1} w_{u_1}^n v_{u_2}^n \leq w_u^n v_u^n \leq K_4 w_{u_1}^n v_{u_2}^n.$$

Finally, we conclude that there exists $K_5 > 0$ such that for each n, k and b in $B_k(G_n)$, we have that

$$K_5^{-1}\lambda^{-(m+k)} \leq \mu(b) = w_{i(b)}^n \lambda^{-k} v_{t(b)}^n \leq K_5\lambda^{-(m+k)}.$$

The statement in (1) follows.

Proof of (2). The statement in (2) follows from the statement in (1) and the fact that $\mu(C_e^{n,k}) = \mu(e)$.

Proof of (3). Note that from (1) we have that there exists $K > 0$ such that

$$\mu(A_{s,t}) = \sum_{\gamma \in \mathrm{Per}_{t-s}(G_n)} \mu(\gamma) \leq K\lambda^{-(m+t-s)}|\mathrm{Per}_{t-s}(G_n)|.$$

Since $\mathrm{Sp}_\times(A_n)$ does not depend on n by our Standing Assumptions 2.21, we have that $|\mathrm{Per}_{t-s}(G_n)|$ does not depend on n. Clearly, $|\mathrm{Per}_{t-s}(G_n)|\lambda^{-(t-s)}$ is bounded as $t - s$ tends to infinity. Therefore, there exists K' such that

$$\mu(A_{s,t}) \leq K'\lambda^{-m}$$

as desired.

Proof of (4). By (2), we have that there exists $K_1 > 0$ such that for all $n, k > U$, and u in V_n,

$$\mu(\mathrm{Per}_k(G_n)|C_u^{n,k}) \leq K_1\lambda^{-k}|\mathrm{Per}_k(G_n) \cap C_u^{n,k}|.$$

By (2), there exists $K_2 > 0$ such that for all $n, k > U$, and u in V_n,

$$|B_{k-U}(G_n) \cap C_u^{n,k-U}| \leq K_2\lambda^{k-U}.$$

By definition of the uniqueness parameter U, each path in $B_{k-U}(G_n) \cap C_u^{n,k-U}$ can be continued in at most one way to form a path in $\mathrm{Per}_k(G_n) \cap C_u^{n,k}$. Therefore, with $K_3 = K_1K_2 > 0$, we have that for all $n, k > U$, and u in V_n,

$$\mu(\mathrm{Per}_k(G_n)|C_u^{n,k}) \leq K_1K_2\lambda^{-k}\lambda^{k-U} = K_3\lambda^{-U}. \qquad \square$$

PROPOSITION 2.29. *Let G_1 be a graph with irreducible adjacency matrix A_1 having entries in $\{0, 1\}$ and spectral radius $\lambda > 1$. Let $G_n = G_1^{[n]}$ for $n \geq 2$. Then the sequence (G_n) satisfies the Standing Assumptions* 2.21 *and conditions* (C1)–(C8). *Moreover,*

(i) *$d_{\max}(G_n) = d_{\max}(G_1)$ for all n;*
(ii) *there exists $C > 0$ such that $|m(G_n) - n| \leq C$ for all n;*
(iii) *$z(G_n) \geq \frac{1}{2}(n - 1)$ for all n;*
(iv) *$U(G_n) \geq n - 1$ for all n;*
(v) *$R(G_n) \leq n + R(G_1)$ for all n.*

PROOF. One may easily check from the definitions that each A_n has entries in $\{0, 1\}$, each A_n is irreducible, and $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$. We show below that $m(G_n)$ tends to infinity as n tends to infinity, which gives that (G_n) satisfies the Standing Assumptions 2.21.

The set of in-degrees that appear in G_n is constant in n, and so is the set of out-degrees that appear in G_n. Therefore, $d_{\max}(G_n) = d_{\max}(G_1)$, which implies condition (C1).

By definition, $m(G_n) = \lceil \log_\lambda |V_n| \rceil$. Since $G_n = G_1^{[n]}$, we have that $|V_n| = |B_{n-1}(G_1)|$. By the standard Perron–Frobenius theory, there exist constants K_1 and K_2 such that $K_1\lambda^n \le |B_n(G_1)| \le K_2\lambda^n$. It follows that there exists a constant $C > 0$ such that $|m(G_n) - n| \le C$, and, in particular, $m(G_n)$ tends to infinity.

Recall the higher-block coding map $\phi_n : X_{G_1} \to X_{G_n}$ (see Definition 2.1). If x is a point in X_{G_1}, then let $V_n(x)$ be the set of vertices in G_n traversed by $\phi_n(x)$. Let us show that $z(G_n) \ge (n-1)/2$. Recall Fine and Wilf's theorem [26], which can be stated as follows. Let x be a periodic sequence with period p, and y be a periodic sequence with period q. If $x[i+1, i+n] = y[i+1, i+n]$ for $n \ge p + q - \gcd(p, q)$ and i in $\mathbb{Z}$, then $x = y$. It follows from this theorem that if x and y lie in distinct periodic orbits of X_{G_1} and have periods less than or equal to $(n-1)/2$, then $V_n(x) \cap V_n(y) = \varnothing$. Thus, $z(G_n) \ge (n-1)/2$, and, in particular, (G_n) satisfies conditions (C2) and (C3).

Note that the map ϕ_n gives a bijection between $B_k(G_n)$ and $B_{k+n-1}(G_1)$ for all $k \ge 0$. Using this map, we check that $U(G_n) \ge n-1$ as follows. For any two paths $b, c \in B_{n-1}(G_1)$, there is at most one path of length $2n-2$ in G_1 of the form bc (since every edge in such a path is specified by either b or c). This fact implies that $U_1(G_n) \ge n-1$. Now if b is in $B_{n-1}(G_1)$ and $1 \le s < t \le n-1$ are given, then there is at most one path c in $B_{t+n-2}(G_1)$ such that $c[1, n-1] = b$ and $c[s, s+n-2] = c[t, t+n-2]$; indeed, if c is such a path, then $c[1, n-1]$ is determined by b, and $c[n, t+n-1]$ is determined by the periodicity condition $c[s, s+n-2] = c[t, t+n-2]$. This fact implies that $U_2(G_n) \ge n-1$, and, thus, we have that $U(G_n) \ge n-1$, which, in particular, gives condition (C4).

Let us check that $R(G_n) \le n + R(G_1)$, which will imply that (G_n) satisfies condition (C5). The statement that $R(G_n) \le n + R(G_1)$ is equivalent to the statement that for any two paths $b, c \in B_{n-1}(G_1)$, there exists a path d in G_1 of length less than or equal to $R(G_1)$ such that bdc is a path in G_1. In this formulation, the statement is clearly true, since, by the definition of $R(G_1)$, there is a path d from $t(b)$ to $i(c)$ of length less than or equal to $R(G_1)$, and then the concatenation bdc gives a path in G_1.

Let w^1 be a positive left (row) eigenvector for A_1 (corresponding to the eigenvalue λ), and let v^1 be a positive right (column) eigenvector for A_1 (corresponding to the eigenvalue λ). Let $b \in B_{n-1}(G_1) = V_n$. Then let $w^n_b = w^1_{i(b)}$ and $v^n_b = v^1_{t(b)}\lambda^{-(n-1)}$. Then w^n is a positive left eigenvector for A_n and v^n is a positive right eigenvector for A_n. It follows that (G_n) satisfies conditions (C6) and (C7). In fact, to satisfy (C7), we may choose $K = \max(K_1, K_2)$, where $K_1 = (\max_i w^1_i)(\min_i w^1_i)^{-1}$ and $K_2 = (\max_i v^1_i)(\min_i v^1_i)^{-1}$.

Condition (C8) follows from the fact that (G_n) satisfies condition (C7) (by applying Lemmas 2.24 and 2.27 in succession). $\square$

2.3. *Probabilistic framework.* Let Ω be the probability space consisting of the set $\{0, 1\}^n$ and the probability measure $\mathbb{P}_\alpha$, where $\mathbb{P}_\alpha$ is the product of the Bernoulli measures on each coordinate with parameter $\alpha \in [0, 1]$. There is a natural partial order on Ω, given by the relation $\omega \le \tau$ if and only if $\omega_i \le \tau_i$ for $i = 1, \dots, n$. We say that a random variable χ on Ω is *monotone increasing* if $\chi(\omega) \le \chi(\tau)$ whenever $\omega \le \tau$. An event A is monotone increasing if its characteristic function is monotone increasing. Monotone decreasing is defined analogously. Monotone random variables and events have been studied extensively [27]; however, we require only a small portion of that theory. In particular, we will make use of the following proposition, a proof of which may be found in [27].

PROPOSITION 2.30 (FKG Inequality). *If X and Y are monotone increasing random variables on $\{0, 1\}^n$, then $\mathbb{E}_\alpha(XY) \ge \mathbb{E}_\alpha(X)\mathbb{E}_\alpha(Y)$.*

It follows easily from the FKG Inequality that if $\bigcap F_j$ is a finite intersection of monotone decreasing events, then $\mathbb{P}_\alpha(\bigcap F_j) \ge \prod \mathbb{P}_\alpha(F_j)$ (use induction and note that if χ_F is the characteristic function of the monotone decreasing event F, then $-\chi_F$ is monotone increasing). In fact, we only use this corollary, but we nonetheless refer to it as the FKG Inequality.

For a finite, directed graph G, we consider the discrete probability space on the set $\Omega_G = \{0, 1\}^E$, where $\mathbb{P}_\alpha$ is the product of the Bernoulli(α) measures on each coordinate. The set Ω_G corresponds to the power set of E in the usual way: ω in Ω_G corresponds to the set F in 2^E such that e is in F if and only if $\omega(e) = 1$. Furthermore, Ω_G corresponds to the space of subgraphs of G: for ω in Ω_G, define the subgraph $G(\omega)$ to have vertex set V and edge set F_ω, where an edge e in E is included in $F_\omega \subset E$ if and only if $\omega(e) = 1$. In the percolation literature, the edges e such that $\omega(e) = 1$ are often called "open," and the remaining edges are called "closed." Since we are interested in studying edge shifts defined by graphs, we will refer to an edge e as "allowed" when $\omega(e) = 1$ and "forbidden" when $\omega(e) = 0$. Finally, each ω in Ω_G can be associated to the SFT X_ω defined as the set of all bi-infinite, directed walks on G that traverse only *allowed* edges (with respect to ω). The probability measure $\mathbb{P}_\alpha$ corresponds to allowing each edge of G with probability α, independently of all other edges. For the sake of notation, we suppress the dependence of $\mathbb{P}_\alpha$ on the graph G.

DEFINITION 2.31. In this work we consider the following conjugacy invariants of SFTs. Let $\mathcal{E}$ be the property containing only the empty shift. Let $\mathcal{Z}$ be the property containing all SFTs with zero entropy. By convention, we let $\mathcal{E} \subset \mathcal{Z}$. For any SFT X, let $\mathbf{h}(X)$ be the topological entropy, and let $I(X)$ be the number of irreducible components of X. If X is nonempty, let $\beta(X)$ be defined by the equation $\mathbf{h}(X) = \log(\beta(X))$. If X is empty, let $\beta(X) = 0$. If $\mathcal{S}$ is a property of SFTs and G is a finite directed graph, then let $\mathcal{S}_G \subset \Omega_G$ be the set of ω in Ω_G such that X_ω has property $\mathcal{S}$. If f is a function from SFTs to the real numbers and G is a finite directed graph, then let $f_G : \Omega_G \to \mathbb{R}$ be the function $f_G(\omega) = f(X_\omega)$.

3. Emptiness. Recall that $\mathrm{Sp}_\times(G)$, ζ_G and $z(G)$ were defined in Definitions 2.9, 2.11 and 2.14, respectively.

THEOREM 3.1. *Let (G_n) be a sequence of graphs such that $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$ for all n and either* (i) $\lambda = \lambda_{G_1} = 1$ *or* (ii) $\lambda = \lambda_{G_1} > 1$ *and $z(G_n)$ tends to infinity as n tends to infinity. Let $\zeta = \zeta_{G_1}$. Then*

$$\lim_{n\to\infty} \mathbb{P}_\alpha(\mathcal{E}_{G_n}) = \begin{cases} (\zeta(\alpha))^{-1}, & \text{if } \alpha \in [0, 1/\lambda), \\ 0, & \text{if } \alpha \in [1/\lambda, 1]. \end{cases}$$

REMARK 3.2. Theorem 1.1 can be obtained as a corollary of Theorem 3.1 by taking (G_n) to be the sequence of n-block graphs of X. Indeed, if the SFT X in Theorem 1.1 has zero entropy, then $\lambda = 1$, and the conclusion of Theorem 1.1 follows from case (i) in Theorem 3.1. If the SFT X in Theorem 1.1 has positive entropy, then $\lambda > 1$ and $z(G_n)$ tends to infinity by the exact same argument in the proof of Proposition 2.29(iii), and, therefore, the conclusion of Theorem 1.1 follows from case (ii) in Theorem 3.1.

In this section we provide a proof of Theorem 3.1. Before proceeding with the proof, we state a fact that will be useful in the investigations that follow. Recall that for a path b, we denote by $V(b)$ the set of vertices traversed by b.

LEMMA 3.3. *Suppose G is a directed graph. Suppose b is in* $\mathrm{Per}(G)$ *such that* $|V(b)| < \mathrm{per}(b)$. *Then there exists a path c in* $\mathrm{Per}(G)$ *such that* $\mathrm{per}(c) < \mathrm{per}(b)$ *and* $V(c) \subset V(b)$.

PROOF. Let v be in $V(b)$. Then there exists a return path to v following b, and we may choose a shortest return path c to v using only vertices in $V(b)$. Then c is in $\mathrm{Per}(G)$ and $\mathrm{per}(c) < \mathrm{per}(b)$, as desired. □

PROOF OF THEOREM 3.1. Recall that an SFT is nonempty if and only if it contains a periodic point (see [40]).

First, assume that case (i) holds, which means that $\lambda = 1$. In this case, each X_{G_n} contains finitely many orbits. Further, the number of periodic orbits of each period in X_{G_n} is constant, and the probability of each periodic orbit being allowed in X_ω is constant. Therefore, the conclusion follows immediately, since the sequence $\mathbb{P}_\alpha(\mathcal{E}_{G_n})$ is constant.

Now assume that case (ii) holds. For the moment, consider a fixed natural number n. Let $\{\gamma_j\}_{j\in\mathbb{N}}$ be an enumeration of the periodic orbits of X_{G_n} such that if $i \le j$, then $\mathrm{per}(\gamma_i) \le \mathrm{per}(\gamma_j)$. Let $p_i = \mathrm{per}(\gamma_i) = |\gamma_i|$. Let $V_n(\gamma_j)$ be the vertices in G_n traversed in the orbit γ_j and let $E_n(\gamma_j)$ be the edges in G_n traversed in the orbit γ_j.

Now for each j, let A_j be the event that γ_j is allowed, which is the event that all of the edges in $E_n(\gamma_j)$ are allowed. Let F_j be the event that γ_j is forbidden, which is A_j^c, the complement of A_j. Notice that A_j is a monotone increasing event (if ω is in A_j and $\omega \le \omega'$, then ω' is in A_j), and F_j is a monotone decreasing event. The fact that an SFT is nonempty if and only if it contains a periodic point implies that $\mathcal{E}_{G_n} = \bigcap F_j$.

Combining the definition of $z(G_n)$ and Lemma 3.3, we obtain that if $\mathrm{per}(\gamma_i) \le z(G_n)$, then $|E_n(\gamma_i)| = p_i$. It follows that $\mathbb{P}_\alpha(F_i) = 1 - \alpha^{p_i}$ for each i such that $p_i \le z(G_n)$. Furthermore, the definition of $z(G_n)$ implies that the events F_i such that $p_i \le z(G_n)$ are all jointly independent. These observations give that

$$\mathbb{P}_\alpha(\mathcal{E}_{G_n}) = \mathbb{P}_\alpha\biggl(\bigcap_{j\in\mathbb{N}} F_j\biggr) \le \mathbb{P}_\alpha\biggl(\bigcap_{p_i \le z(G_n)} F_i\biggr) \tag{3.1}$$

$$= \prod_{p_i \le z(G_n)} \mathbb{P}_\alpha(F_i) = \prod_{p_i \le z(G_n)} (1 - \alpha^{p_i}). \tag{3.2}$$

Using Lemma 3.3, we see that there is great redundancy in the intersection $\bigcap F_j$. Eliminating some of this redundancy, we obtain the following:

$$\bigcap_{j\in\mathbb{N}} F_j = \bigcap_{j:|E_n(\gamma_j)|=p_j} F_j. \tag{3.3}$$

Then using Lemma 3.3 again and the fact that $|E_n(\gamma_j)| \le |E_n|$, we see that the intersection on the right in (3.3) is actually a finite intersection. Applying the FKG Inequality, we obtain that

$$\mathbb{P}_\alpha(\mathcal{E}_{G_n}) = \mathbb{P}_\alpha\biggl(\bigcap_{j\in\mathbb{N}} F_j\biggr) = \mathbb{P}_\alpha\biggl(\bigcap_{j:|E_n(\gamma_j)|=p_j} F_i\biggr) \ge \prod_{j:|E_n(\gamma_j)|=p_j} \mathbb{P}_\alpha(F_i) \tag{3.4}$$

$$= \prod_{j:|E_n(\gamma_j)|=p_j} (1 - \alpha^{p_j}) \ge \prod_{j:p_j \le |E_n|} (1 - \alpha^{p_j}). \tag{3.5}$$

Combining the inequalities in (3.1), (3.2), (3.4) and (3.5) gives that for each n,

$$\prod_{p_j \le |E_n|} (1 - \alpha^{p_j}) \le \mathbb{P}_\alpha(\mathcal{E}_{G_n}) \le \prod_{p_i \le z(G_n)} (1 - \alpha^{p_i}). \tag{3.6}$$

By the standing assumptions that $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$, we have that $|\mathrm{Per}_p(G_n)|$ is independent of n. Since $z(G_n)$ and $|E_n|$ tend to infinity as n tends to infinity, equation (3.6) gives that

$$\lim_{n\to\infty} \mathbb{P}_\alpha(\mathcal{E}_{G_n}) = \prod_{j=1}^{\infty} (1 - \alpha^{p_j}).$$

Then Theorem 3.1 follows from the well-known product formula for ζ (see [40]), which may be stated as

$$(\zeta(t))^{-1} = \prod_{j=1}^{\infty} (1 - t^{p_j}),$$

along with the fact that $\zeta(t)$ converges for $t < 1/\lambda$ and diverges to $+\infty$ for $t \geq 1/\lambda$. $\square$

4. Subcritical phase. In this section we study random SFTs in the subcritical phase: $0 \leq \alpha < 1/\lambda$. The main result of this section is Theorem 4.2. Let us fix some notation for this section. We consider a sequence of graphs (G_n) such that $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$ and $z(G_n)$ tends to infinity as n tends to infinity, with $\lambda = \lambda_{G_1} \geq 1$ and $\zeta = \zeta_{G_1}$. Since $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$, there exist shift-commuting bijections $\phi_n : \mathrm{Per}(X_{G_1}) \to \mathrm{Per}(X_{G_n})$. In other words, there exist bijections ϕ_n from the set of cyclic paths in G_1 to the set of cyclic paths in G_n such that if b is in $\mathrm{Per}_p(G_1)$, then $\phi_n(b)$ is $\mathrm{Per}_p(G_n)$. If b is in $\mathrm{Per}(G)$, then we refer to $\theta(b)$ (recall Definition 2.12) as a cycle. Using the fixed bijections ϕ_n, we may refer to a cycle γ as being in G_n for any n. We fix an enumeration of the cycles in G_1, $\{\gamma_i\}_{i\in\mathbb{N}}$, and then since the bijections ϕ_n are fixed, this choice simultaneously gives enumerations of all the cycles in each G_n. For any s in $\mathbb{N}$, let $p_s = \mathrm{per}(\gamma_s)$. Let us begin with a lemma.

LEMMA 4.1. *Let (G_n) be a sequence of graphs such that $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$ and $z(G_n)$ tends to infinity as n tends to infinity, with $\lambda = \lambda_{G_1} \geq 1$ and $\zeta = \zeta_{G_1}$. Given a nonempty, finite set S in $\mathbb{N}$, let $D_{G_n}(S)$ be the event that the set of allowed cycles is $\{\gamma_s : s \in S\}$. Then*

$$\lim_{n\to\infty} \mathbb{P}_\alpha(D_{G_n}(S)) = \begin{cases} (\zeta(\alpha))^{-1} \prod_{j\in S} \frac{\alpha^{p_j}}{1-\alpha^{p_j}}, & \text{if } \alpha \in [0, 1/\lambda), \\ 0, & \text{if } \alpha \in [1/\lambda, 1]. \end{cases}$$

The proof of Lemma 4.1 is an easy adaptation of the proof of Theorem 3.1, and we omit it for the sake of brevity.

Recall that $I(X)$ denotes the number of irreducible components in the SFT X, and for any graph G, the random variable $I_G : \Omega_G \to \mathbb{Z}_{\geq 0}$ is defined by the equation $I_G(\omega) = I(X_\omega)$.

THEOREM 4.2. *Let (G_n) be a sequence of graphs such that $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$ and either* (i) $\lambda = \lambda_{G_1} = 1$ *or* (ii) $\lambda = \lambda_{G_1} > 1$ *and $z(G_n)$ tends to infinity as n tends to infinity. Let $\zeta = \zeta_{G_1}$. Then for* $0 \leq \alpha < 1/\lambda$,

(1) $\lim_{n\to\infty} \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) = 1$;

(2) *the sequence (I_{G_n}) converges in distribution to the random variable I_∞ such that $\mathbb{P}(I_\infty = 0) = (\zeta(\alpha))^{-1}$ and for $k \geq 1$,*

$$\mathbb{P}(I_\infty = k) = (\zeta(\alpha))^{-1} \sum_{\substack{S\subset\mathbb{N} \\ |S|=k}} \prod_{s\in S} \frac{\alpha^{p_s}}{1-\alpha^{p_j}},$$

where $\{\gamma_i\}_{i=1}^\infty$ is an enumeration of the cycles in G_1;

(3) *the random variable I_∞ has exponentially decreasing tail and therefore finite moments of all orders*.

REMARK 4.3. One obtains Theorem 1.2 as a consequence of Theorem 4.2 by taking (G_n) to be the sequence of n-block graphs of a nonempty SFT X. Indeed, if the SFT X in Theorem 1.2 has zero entropy, then $\lambda = 1$, and the conclusions of Theorem 1.2 follow from the case (i) in Theorem 4.2. If the SFT X in Theorem 1.2 has positive entropy, then $\lambda > 1$ and $z(G_n)$ tends to infinity by the exact same argument in the proof of Proposition 2.29(iii), and, therefore, the conclusions of Theorem 1.2 follow from case (ii) in Theorem 4.2.

PROOF OF THEOREM 4.2. Let (G_n) be as above. Let $0 \le \alpha < 1/\lambda$.

First, assume that case (i) holds, which means that $\lambda = 1$. Conclusion (1) follows immediately, since for each n, we have that $\mathbb{P}_\alpha(\mathcal{Z}_{G_n}) = 1$ [the random SFT X_ω satisfies $0 = \mathbf{h}(X_\omega) \le \mathbf{h}(X_{G_n}) = \log \lambda = 0$]. Also, the fact that $\lambda = 1$ is equivalent to the fact that G_1 (and therefore G_n) contains only finitely many cycles. Then conclusions (2) and (3) also follow immediately, since the sequence I_{G_n} is constant.

Now assume that case (ii) holds. Recall that we have an enumeration $\{\gamma_i\}_{i\in\mathbb{N}}$ of the cycles in G_1, which we refer to as an enumeration of the cycles in G_n, for any n, using the bijections ϕ_n. Also recall that for any nonempty, finite set $S \subset \mathbb{N}$, we denote by $D_{G_n}(S)$ the event in Ω_{G_n} consisting of all ω such that the set of cycles in $G_n(\omega)$ is exactly $\{\gamma_s : s \in S\}$.

Proof of (1). Recall that an SFT has zero entropy if and only if it has at most finitely many periodic points [40]. Then we have that

$$\mathcal{Z}_{G_n} = \mathcal{E}_{G_n} \cup \left(\bigcup_{\substack{S\subset\mathbb{N} \\ 0<|S|<\infty}} D_{G_n}(S) \right). \tag{4.1}$$

Also note that by the definition of $D_{G_n}(S)$, the union in (4.1) is a disjoint union. Thus, we have that

$$\mathbb{P}_\alpha(\mathcal{Z}_{G_n}) = \mathbb{P}_\alpha(\mathcal{E}_{G_n}) + \sum_{\substack{S\subset\mathbb{N} \\ 0<|S|<\infty}} \mathbb{P}_\alpha(D_{G_n}(S)).$$

Now let $S_1, \dots, S_J$ be distinct, nonempty, finite subsets of $\mathbb{N}$. Then by Theorem 3.1 and Lemma 4.1, we have that

$$\begin{aligned}
\liminf_{n\to\infty} \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) &\ge \lim_{n\to\infty} \mathbb{P}_\alpha(\mathcal{E}_{G_n}) + \sum_{j=1}^{J} \lim_{n\to\infty} \mathbb{P}_\alpha(D_{G_n}(S_j)) \\
&= (\zeta(\alpha))^{-1} \left(1 + \sum_{j=1}^{J} \prod_{s\in S_j} \frac{\alpha^{p_s}}{1-\alpha^{p_s}} \right).
\end{aligned}$$

Since J and $S_1, \dots, S_J$ were arbitrary, we conclude that

$$\liminf_{n\to\infty} \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) \geq (\zeta(\alpha))^{-1}\left(1 + \sum_{\substack{S\subset\mathbb{N}\\ 0<|S|<\infty}} \prod_{s\in S} \frac{\alpha^{p_s}}{1-\alpha^{p_s}}\right).$$

Using the facts that $\alpha^{p_s}/(1-\alpha^{p_s}) = \sum_{k=1}^{\infty}(\alpha^{p_s})^k$ and $\alpha < 1/\lambda$ (which implies that the relevant infinite products and series converge uniformly), one may easily check that

$$\left(1 + \sum_{\substack{S\subset\mathbb{N}\\ 0<|S|<\infty}} \prod_{s\in S} \frac{\alpha^{p_s}}{1-\alpha^{p_s}}\right) = \zeta(\alpha).$$

Thus, we have shown that $\liminf_n \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) \geq 1$. Since $\limsup_n \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) \leq 1$, we conclude that $\lim_n \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) = 1$.

Proof of (2). Since I_{G_n} takes values in $\mathbb{Z}_{\geq 0}$, the sequence (I_{G_n}) converges in distribution to I_∞ if and only if $\mathbb{P}_\alpha(I_{G_n} = k)$ converges to $\mathbb{P}_\alpha(I_\infty = k)$ for each k in $\mathbb{Z}_{\geq 0}$.

Note that $I_{G_n}(\omega) = 0$ if and only if ω is in $\mathcal{E}_{G_n}$, which implies that $\mathbb{P}_\alpha(I_{G_n} = 0) = \mathbb{P}_\alpha(\mathcal{E}_{G_n})$. Thus, for $\alpha < 1/\lambda$, Theorem 3.1 implies that $\mathbb{P}_\alpha(I_{G_n} = 0)$ converges to $(\zeta(\alpha))^{-1}$ as n tends to infinity.

Now let k be in $\mathbb{N}$. Recall that $\{\gamma_i\}_{i=1}^{\infty}$ is an enumeration of the cycles in G_1, and we have fixed bijections between these cycles and the cycles in each G_n. By Theorem 4.2(1), we have that $\lim_n \mathbb{P}_\alpha(\mathcal{Z}_{G_n}) = 1$, and, therefore, $\mathbb{P}_\alpha(I_{G_n} = k) = \mathbb{P}_\alpha(\{I_{G_n} = k\} \cap \mathcal{Z}_{G_n}) + \varepsilon_n$, where ε_n tends to 0 as n tends to infinity. Thus, we need only focus on events of the form $\{I_{G_n} = k\} \cap \mathcal{Z}_{G_n}$ for some k. Now if ω is in $\mathcal{Z}_{G_n}$, then $I_{G_n}(\omega)$ is the number of periodic orbits in X_ω. Thus,

$$\mathbb{P}_\alpha(\{I_{G_n} = k\} \cap \mathcal{Z}_{G_n}) = \sum_{\substack{S\subset\mathbb{N}\\ |S|=k}} \mathbb{P}_\alpha(D_{G_n}(S)).$$

For any n in $\mathbb{N}$, let $T_n^0 = \mathbb{P}_\alpha(\mathcal{E}_{G_n})$. For k in $\mathbb{N}$ and n in $\mathbb{N}$, let

$$T_n^k = \sum_{\substack{S\subset\mathbb{N}\\ |S|=k}} \mathbb{P}_\alpha(D_{G_n}(S)).$$

We have that $\sum_{k=0}^{\infty} T_n^k = \mathbb{P}_\alpha(\mathcal{Z}_{G_n})$, and, therefore, $\lim_n \sum_{k=0}^{\infty} T_n^k = 1$ by Theorem 4.2(1). Also, using Lemma 4.1, we have that $\liminf_n T_n^k \geq T^k$, where $T^0 = (\zeta(\alpha))^{-1}$ and for k in $\mathbb{N}$,

$$T^k = (\zeta(\alpha))^{-1} \sum_{\substack{S\subset\mathbb{N}\\ |S|=k}} \prod_{s\in S} \frac{\alpha^{p_s}}{1-\alpha^{p_s}}.$$

Further, we have that $\sum_{k=0}^{\infty} T^k = 1$. It follows from these facts that $\lim_n T_n^k = T^k$. Thus, we have shown that for k in $\mathbb{N}$,

$$\lim_n \mathbb{P}_\alpha(I_{G_n} = k) = \lim_n \mathbb{P}_\alpha(\{I_{G_n} = k\} \cap \mathcal{Z}_{G_n}) = (\zeta(\alpha))^{-1} \sum_{\substack{S \subset \mathbb{N} \\ |S|=k}} \prod_{s \in S} \frac{\alpha^{p_s}}{1-\alpha^{p_s}}$$

as desired.

Proof of (3). For k in $\mathbb{N}$, let

$$T^k = \mathbb{P}_\alpha(I_\infty = k) = (\zeta(\alpha))^{-1} \sum_{\substack{S \subset \mathbb{N} \\ |S|=k}} \prod_{s \in S} \frac{\alpha^{p_s}}{1-\alpha^{p_s}}.$$

We show that there for any real number $\delta > 0$, there exists k_0 such that $T^{k+1} \leq \delta T^k$ for all $k \geq k_0$. Let $\delta > 0$. Since $\alpha < 1/\lambda$, we have that

$$\sum_{i \in \mathbb{N}} \frac{\alpha^{p_i}}{1-\alpha^{p_i}} < \infty.$$

Now choose k_0 such that

$$\sum_{i \geq k_0} \frac{\alpha^{p_i}}{1-\alpha^{p_i}} < \delta.$$

In the following sums, we will use that any set $S \subset \mathbb{N}$ with $|S| = j$ can be written as $S = \{s_1, \dots, s_j\}$, where $s_1 < \cdots < s_j$. Note that in this case $s_j \geq j$. Then for $k \geq k_0$ we have

$$\begin{aligned}
(\zeta(\alpha))T^{k+1} &= \sum_{\substack{S \subset \mathbb{N} \\ |S|=k+1}} \prod_{i=1}^{k+1} \frac{\alpha^{p_{s_i}}}{1-\alpha^{p_{s_i}}} = \sum_{\substack{S \subset \mathbb{N} \\ |S|=k}} \prod_{i=1}^{k} \frac{\alpha^{p_{s_i}}}{1-\alpha^{p_{s_i}}} \sum_{j > s_k} \frac{\alpha^{p_j}}{1-\alpha^{p_j}} \\
&\leq \sum_{\substack{S \subset \mathbb{N} \\ |S|=k}} \prod_{i=1}^{k} \frac{\alpha^{p_{s_i}}}{1-\alpha^{p_{s_i}}} \sum_{j > k_0} \frac{\alpha^{p_j}}{1-\alpha^{p_j}} \leq \left(\sum_{\substack{S \subset \mathbb{N} \\ |S|=k}} \prod_{i=1}^{k} \frac{\alpha^{p_{s_i}}}{1-\alpha^{p_{s_i}}} \right) \delta \\
&= (\zeta(\alpha))T^k \delta.
\end{aligned}$$

Since $\alpha < 1/\lambda$, we have that $0 < \zeta(\alpha) < \infty$, and we conclude that $T^{k+1} \leq \delta T^k$ for all $k \geq k_0$. $\square$

We recognize the distribution of I_∞ as the sum of countably many independent Bernoulli trials, where the probability of success of trial $i \in \mathbb{N}$ is given by α^{p_i} for some enumeration $\{\gamma_i\}_{i \in \mathbb{N}}$ of the cycles in G_1 (or any G_n). We record some facts about this distribution in the following corollary.

COROLLARY 4.4. *With the same hypotheses as in Theorem 4.2, the characteristic function of I_∞ is given by*

$$\varphi_{I_\infty}(t) = (\zeta(\alpha))^{-1} \prod_s \left(1 + e^{it} \frac{\alpha^{p_s}}{1-\alpha^{p_s}}\right),$$

where the product is over all periodic orbits in X. It follows that the moment generating function of I_∞ is given by

$$M_{I_\infty}(t) = (\zeta(\alpha))^{-1} \prod_s \left(1 + e^{t} \frac{\alpha^{p_s}}{1-\alpha^{p_s}}\right).$$

REMARK 4.5. In Theorems 3.1 and 4.2, we assert the existence of various limits to certain values. Beyond the bounds given in our proofs, we do not know at which rates these sequences converge to their limits.

5. Supercritical phase. In this section we study random SFTs in the supercritical phase. The main results are Theorems 5.13 and 5.15. On a first reading, the reader may prefer to skip Section 5.1 and refer back to it as necessary. Our proof of Theorem 5.13 relies, in part, on showing that with large probability the number of allowed words of length k in a random SFT is close to $(\alpha\lambda)^k$, for a particular choice of k. In our proof, we choose k to be polynomial in $m = m(G_n)$ for two reasons. First, we need k to dominate m, so that the kth root of the number of words of length k gives a good upper bound on the Perron eigenvalue of the random SFT. Second, k should be subexponential in m, essentially because most paths in G_n with length subexponential in m are self-avoiding, and we need good bounds on the probability of paths of length k that exhibit "too-soon-recurrence." For context, we recall a result of Ornstein and Weiss [45]. In fact, their result is quite general, but we only recall it in a very specific case. Let X be an irreducible SFT with measure of maximal entropy μ. For x in X, let $R_n(x)$ be the first return time (greater than 0) of x to the cylinder set $x[1,n]$ under σ. Then the result of Ornstein and Weiss implies that for μ-a.e. x in X, $\lim_n n^{-1} \log R_n(x) = h(X)$. It follows from this result that for k polynomial in n, the μ-measure of the set of words of length k with a repeated n-word tends to 0. In the following lemmas, we give some quantitative bounds on the μ-measure of the set of paths of length k in G_n with $k-j$ repeated edges, where the important point for our purposes is that the bounds improve exponentially as j decreases. To get these bounds, we employ some of the language and tools of information theory. After getting a handle on the μ-measure of paths in G_n with certain self-intersection properties, our assumption that (G_n) satisfies condition (C7) in Definition 2.23 implies that the μ-measure on paths is the same as the counting measure up to uniform constants.

5.1. *Information theory and lemmas.* In keeping with the convention of information theory, $\log(x)$ denotes the base 2 logarithm of x.

DEFINITION 5.1. A binary *n-code* on an alphabet $\mathcal{A}$ is a mapping $C : \mathcal{A}^n \to \{0, 1\}^*$, where $\{0, 1\}^*$ is the set of all finite words on the alphabet $\{0, 1\}$. We may refer to such mappings simply as codes. A code is *faithful* if it is injective. The function that assigns to each w in $\mathcal{A}^n$ the length of the word $C(w)$ is called the *length function* of the code, and it will be denoted by $\mathcal{L}$ when the code is understood. A code is a *prefix code* if $w = w'$ whenever $C(w)$ is a prefix of $C(w')$. A *Shannon code* with respect to a measure ν on $\mathcal{A}^n$ is a code such that $\mathcal{L}(w) = \lceil -\log \nu(w) \rceil$.

We note that for a measure ν on $\mathcal{A}^n$, there is a prefix Shannon code on $\mathcal{A}^n$ with respect to ν [50]. We will also require the following two lemmas from information theory.

LEMMA 5.2 [50]. *Let $\mathcal{A}$ be an alphabet. Let C_n be a prefix-code on $\mathcal{A}^n$, and let μ be a shift-invariant Borel probability measure on $\mathcal{A}^{\mathbb{Z}}$. Then*

$$\mu(\{w \in \mathcal{A}^n : \mathcal{L}(w) + \log \mu(w) \le -a\}) \le 2^{-a}.$$

PROOF. Let $B = \{w \in \mathcal{A}^n : \mathcal{L}(w) + \log \mu(w) \le -a\}$. Then for any w in B, we have that $\mu(w) \le 2^{-\mathcal{L}(w)} 2^{-a}$. The Kraft inequality for prefix codes ([50], page 73) states that since $\mathcal{L}$ is a prefix code, $\sum_{w \in \mathcal{A}^n} 2^{-\mathcal{L}(w)} \le 1$. Hence,

$$\mu(B) = \sum_{w \in B} \mu(w) \le 2^{-a} \sum_{w \in B} 2^{-\mathcal{L}(w)} \le 2^{-a}.$$

□

LEMMA 5.3 [50]. *There is a prefix code $C : \mathbb{N} \to \{0, 1\}^*$ such that $\ell(C(n)) = \log(n) + o(\log(n))$, where $\ell(C(n))$ is the length of $C(n)$.*

DEFINITION 5.4. A prefix code satisfying the conclusion of Lemma 5.3 is called an *Elias code*.

Recall that if b is a path in the graph $G = (V, E)$, then we denote by $E(b)$ the set of edges traversed by b. Let (G_n) be a sequence of graphs satisfying our Standing Assumptions 2.21.

DEFINITION 5.5. For each n, k, and $1 \le j \le k - 1$, let

$$N_{n,k}^j = \{b \in B_k(G_n) : |E_n(b)| \le j\}.$$

DEFINITION 5.6. For each n, k, and $1 \le j \le 2k - 1$, let

$$D_{n,k}^j = \{(b, c) \in B_k(G_n) \times B_k(G_n) : E_n(b) \cap E_n(c) \ne \varnothing, |E_n(b) \cup E_n(c)| \le j\}.$$

DEFINITION 5.7. For each n, k, and $1 \le j \le k - 1$, let

$$Q_{n,k}^j = \{b \in \mathrm{Per}_k(G_n) : |E_n(b)| \le j\}.$$

DEFINITION 5.8. For each n, k, and $1 \le j \le 2k-1$, let

$$S_{n,k}^{j} = \{(b, c) \in \mathrm{Per}_k(G_n) \times \mathrm{Per}_k(G_n) : E_n(b) \cap E_n(c) \neq \varnothing, |E_n(b) \cup E_n(c)| \le j\}.$$

For any of the sets defined in Definitions 5.5–5.8, we use a "hat" to denote the set with "$\le$" replaced by "$=$" in the definition. For example,

$$\hat{N}_{n,k}^{j} = \{b \in B_k(G_n) : |E_n(b)| = j\}.$$

The "hat" notation will only appear in the proof of Theorem 5.13. The following four lemmas find bounds on $|N_{n,k}^{j}|$, $|D_{n,k}^{j}|$, $|S_{n,k}^{2k-1}|$ and $|S_{n,k}^{j}|$.

The following lemma bounds the μ-measure (and therefore the cardinality) of the set of paths of length k in G_n that traverse at most $j < k$ edges. The proof relies on a general principle in information theory (made precise by Lemma 5.2): a set of words that can be encoded "too efficiently" must have small measure. In order to use this principle, we find an efficient encoding of the paths of length k in G_n that traverse at most j edges. The basic observation behind the coding is trivial: a path of length k that only traverses $j < k$ edges must have $k - j$ repeated edges. Therefore, instead of encoding each of the $k - j$ repeated edges explicitly, we simply encode some combinatorial data that specifies when "repeats" happen and when the corresponding edges are first traversed.

LEMMA 5.9. *Let (G_n) be a sequence of graphs satisfying the Standing Assumptions 2.21 and such that (G_n) has local uniqueness of paths and bounded distortion of weights* [*conditions* (C4) *and* (C7) *in Definition* 2.23]. *Then there exists a polynomial $p_0(x)$ and n_0 such that for each $n \ge n_0$, $k > U(G_n)$ and $1 \le j \le k-1$,*

$$\mu(N_{n,k}^{j}) \le p_0(k)^{\min(k-j, k/U(G_n))} \lambda^{-(m(G_n)+k-j)}$$

and

$$|N_{n,k}^{j}| \le p_0(k)^{\min(k-j, k/U(G_n))} \lambda^{j}.$$

PROOF. Consider (G_n), n, k and j as in the hypotheses. Let $m = m(G_n)$ and $U = U(G_n)$. A path b in $N_{n,k}^{j}$ from vertex s to vertex t contributes $w_s^n v_t^n \lambda^{-k}$ to $\mu(N_{n,k}^{j})$. The condition (C7) gives a uniform constant K such that $w_s^n v_t^n$ is bounded below by $(K^2|V_n|)^{-1} = (K^2\lambda^m)^{-1}$. Therefore, the bound on $|N_{n,k}^{j}|$ follows from the bound on $\mu(N_{n,k}^{j})$, since $|N_{n,k}^{j}| \le K^2\lambda^{m+k}\mu(N_{n,k}^{j})$ [as in Lemma 2.28(1)]. We now proceed to show the bound on $\mu(N_{n,k}^{j})$.

Let $r = k - j$. Consider b in $N_{n,k}^{j}$. Then there exists $1 < t_1 < \cdots < t_r \le k$ such that $b_{t_i} = b_{s_i}$ for some $1 \le s_i < t_i$, for each $i = 1, \dots, r$, where $s_i = \min\{s \ge 1 : b_s = b_{t_i}\}$. Now we define a set $\mathcal{I} \subset \{1, \dots, r\}$ by induction. Let $i_1 = 1$ and $\mathcal{I}_1 = \{i_1\}$. Assuming by induction that i_j and $\mathcal{I}_j$ have been defined and that $i_j < r$, we define i_{j+1} and $\mathcal{I}_{j+1}$ as follows:

- if $t_{i_j+1} - t_{i_j} > U$, let $i_{j+1} = i_j + 1$;
- otherwise, if $t_{i_j+1} - t_{i_j} \le U$, then let

$$i_{j+1} = \max\{i_j < i \le r : t_i - t_{i_j} \le U\}.$$

Let $\mathcal{I}_{j+1} = \mathcal{I}_j \cup \{i_{j+1}\}$. This induction procedure terminates when $i_j = r$ for some $j \le r$, and we denote this terminal j by j_*. Let $\mathcal{I} = \mathcal{I}_{j_*}$. Note that for each $0 \le s \le k - U$, we have that

$$|\{i \in \mathcal{I} : s + 1 \le t_i \le s + U\}| \le 2.$$

It follows that $|\mathcal{I}| \le \min(r, 2k/U + 2)$.

Having defined the set $\mathcal{I}$, we now decompose the integer interval $\{1, \dots, k\}$ into subintervals. First, let

$$J = \bigcup_{j=1}^{j_*} \{t_{i_j}\} \cup \{1 \le s \le k : \exists i_j, i_{j+1} \in \mathcal{I}, t_{i_{j+1}} - t_{i_j} \le U \text{ and } t_{i_j} \le s \le t_{i_{j+1}}\}.$$

Let $J_1, \dots, J_N$ be the maximal disjoint subintervals (with singletons allowed) of $\{1, \dots, k\}$ such that $J = J_1 \cup \dots \cup J_N$ and $J_\ell < J_{\ell+1}$. Note that $\sum_{\ell=1}^N |J_\ell| = |J| \ge r$ and $N \le |\mathcal{I}|$. Then let $I_1, \dots, I_{N+1}$ be the maximal disjoint subintervals of $\{1, \dots, k\}$ such that:

- $I_\ell \subset \{1, \dots, k\} \setminus J$ for each $\ell = 1, \dots, N + 1$;
- $\bigcup_{\ell=1}^{N+1} I_\ell = \{1, \dots, k\} \setminus J$;
- and for each $\ell = 1, \dots, N$, we have that I_ℓ is nonempty and $I_\ell < I_{\ell+1}$.

In summary, we have that $\{1, \dots, k\} = I_1 \cup J_1 \cup \dots \cup I_N \cup J_N \cup I_{N+1}$, and only I_{N+1} may be empty.

For any $1 \le s < t \le k$, let $A_{s,t} = \{b \in B_k(G_n) : b_s = b_t\}$. By Lemma 2.28(3), there exists a uniform constant K_1 such that

$$\mu(A_{s,t}) \le K_1 \lambda^{-m}. \tag{5.1}$$

For notation, if I is a subset of $\{1, \dots, k\}$, then b_I is b restricted to I. Since μ is a 1-step Markov on X_{G_n}, we have that

$$\begin{aligned} \mu(b|A_{s_1,t_1}) &= \mu(b_{I_1}|A_{s_1,t_1}) \prod_{\ell=1}^N \mu(b_{J_\ell}|A_{s_1,t_1} \cap b_{I_1 \cdots I_\ell}) \\ &\quad \times \prod_{\ell=2}^{N+1} \mu(b_{I_\ell}|A_{s_1,t_1} \cap b_{I_1 \cdots J_{\ell-1}}) \end{aligned} \tag{5.2}$$

$$= \mu(b_{I_1}|A_{s_1,t_1}) \prod_{\ell=1}^N \mu(b_{J_\ell}|b_{I_1 \cdots I_\ell}) \prod_{\ell=2}^{N+1} \mu(b_{I_\ell}|b_{I_1 \cdots J_{\ell-1}}). \tag{5.3}$$

Given b, we may form s_i, t_i, I_ℓ and J_ℓ as above, and then we encode b as follows:

(1) encode s_1 and t_1 using an Elias code;

(2) encode b_{I_1} using a prefix Shannon code with respect to $\mu(\cdot|A_{s_1,t_1})$;

(3) assuming $b_{I_1\cdots I_\ell}$ has been encoded, we encode b_{J_ℓ} by encoding s_i and t_i for each i in $\mathcal{I}$ such that $t_i \in J_\ell$, using an Elias code (and note that this information completely determines b_{J_ℓ} by definition of U and construction of J);

(4) assuming $b_{I_1\cdots J_{\ell-1}}$ has been encoded, we encode b_{I_ℓ} using a prefix Shannon code with respect to $\mu(\cdot|b_{I_1\cdots J_{\ell-1}})$.

Now we analyze the performance of the code. Since the code is a concatenation of prefix codes, it is a prefix code. Since U tends to infinity as n tends to infinity [by (C4)] and $k > U$, there exists n_0 such that for $n \geq n_0$ and $1 \leq s \leq k$, the length of the codeword in the Elias encoding of s is less than or equal to $2\log k$. Then we have, neglecting bits needed to round up,

$$\mathcal{L}(b) \leq -\log\mu(b_{I_1}|A_{s_1,t_1}) + |\mathcal{I}|(4\log k) + \sum_{\ell=2}^{N+1} -\log\mu(b_{I_\ell}|b_{I_1\cdots J_{\ell-1}}). \tag{5.4}$$

Combining (5.2), (5.3) and (5.4), we have that

$$\mathcal{L}(b) + \log\mu(b) \leq |\mathcal{I}|(4\log k) + \log\mu(A_{s_1,t_1}) + \sum_{\ell=1}^{N}\log\mu(b_{J_\ell}|b_{I_1\cdots I_\ell}). \tag{5.5}$$

Now by Lemma 2.28(2) and (3), there exist uniform constants K_2 and K_3 such that

$$\mathcal{L}(b) + \log\mu(b) \leq |\mathcal{I}|(4\log k) + K_2 - m\log\lambda + NK_3 - |J|\log\lambda \tag{5.6}$$

$$= |\mathcal{I}|(4\log k) + K_2 + NK_3 - (m + |J|)\log\lambda. \tag{5.7}$$

By construction, $|\mathcal{I}| \leq \min(k-j, 2k/U + 2)$, $N \leq |\mathcal{I}|$ and $|J| \geq r = k - j$. Then by Lemma 5.2, there exists a uniform constant $K_4 > 0$ such that

$$\mu(N_{n,k}^j) \leq (K_4 k^4)^{\min(k-j,2k/U+2)}\lambda^{-(m+k-j)}. \tag{5.8}$$

Letting $p_0(x) = K_5 x^{12}$, for some uniform constant $K_5 > 0$, we obtain that

$$\mu(N_{n,k}^j) \leq p_0(k)^{\min(k-j,k/U)}\lambda^{-(m+k-j)},$$

which completes the proof. □

The following lemma bounds the $\mu \times \mu$-measure (and therefore the cardinality) of the set of pairs paths of length k in G_n that share at least one edge and together traverse at most $j < 2k$ edges. The general strategy of encoding pairs of paths using combinatorial data and appealing to information theory is similar to that of Lemma 5.9. Lemma 5.10 involves the additional hypothesis that there exists a uniform bound R such that for any pair of paths (u, w) in G_n, there exists a path uvw in G_n with $|v| \leq R$. Using this hypothesis, one observes that pairs of paths can essentially be concatenated in G_n and then treated as single paths as in Lemma 5.9.

LEMMA 5.10. *Let (G_n) be a sequence of graphs satisfying the Standing Assumptions 2.21 and such that (G_n) has local uniqueness of paths, small diameter and bounded distortion of weights* [*conditions* (C4), (C5) *and* (C7) *in Definition* 2.23]. *Then there exists a polynomial $p_1(x)$ and n_1 such that for $n \geq n_1$, $k > R(G_n)$ and $1 \leq j \leq 2k-1$,*

$$\mu \times \mu(D^j_{n,k}) \leq p_1(k)^{\min(2k-j,k/U(G_n))} \lambda^{-(m(G_n)+2k-j)}$$

and

$$|D^j_{n,k}| \leq p_1(k)^{\min(2k-j,k/U(G_n))} \lambda^{j+m(G_n)}.$$

PROOF. Consider (G_n), n, k and j as in the hypotheses. Let $m = m(G_n)$, $U = U(G_n)$ and $R = R(G_n)$. Note that the bound on $|D^j_{n,k}|$ follows from the bound on $\mu \times \mu(D^j_{n,k})$, since condition (C7) implies that there exists a uniform constant K such that $|D^j_{n,k}| \leq K\lambda^{2m+2k} \mu \times \mu(D^j_{n,k})$ [as in Lemma 2.28(1)]. We now proceed to show the bound on $\mu \times \mu(D^j_{n,k})$.

By the definition of R, for every pair $(b,c) \in B_k(G_n) \times B_k(G_n)$, there exists a path d_1 in G_n such that $|b| \leq R$ and bdc is in $B_{2k+|d_1|}(G_n)$. We choose a single such d_1 for each pair (b,c), and we choose a (possibly empty) path d_2 such that bd_1cd_1 is in $B_{2k+R(G_n)}$ (whose existence is guaranteed by the fact that G_n is irreducible). If $(b,c) \in D^j_{n,k}$, then bd_1cd_2 is in $N^{j+R}_{n,2k+R}$. Using condition (C5), we have that $R \leq m + C$ for a uniform constant C. Then we have that there exist uniform constants K_1, K_2 and K_3 such that for each n, each k and each pair (b,c) in $B_k(G_n) \times B_k(G_n)$,

$$\mu \times \mu((b,c)) \leq K_1\lambda^{-(2m+2k)} \leq K_2\lambda^{-(m+R+2k)} \leq K_3\mu(bd_1cd_2).$$

Thus, Lemma 5.9 implies that there exists a polynomial $p_0(x)$ and n_0 such that for $n \geq n_0$,

$$\mu \times \mu(D^j_{n,k}) \leq K_3\mu(N^{j+R}_{n,2k+R}) \leq K_3 p_0(2k+R)^{\min(2k-j,(2k+R)/U)} \lambda^{-(m+2k-j)}.$$

With $n_1 = n_0$ and $p_1(x) = K_4 p_0(3x)^3$ for a uniform constant K_4, we have

$$\mu \times \mu(D^j_{n,k}) \leq p_1(k)^{\min(2k-j,k/U)} \lambda^{-(m+2k-j)},$$

which completes the proof. $\square$

The following two lemmas (Lemmas 5.11 and 5.12) give bounds on the $\mu \times \mu$ measure (and therefore the cardinality) of the set of pairs of periodic paths in G_n with certain overlap properties. The general ideas are similar to those in Lemmas 5.9 and 5.10, but in order to get precise bounds on the relevant sets, we exploit the fact that these sets consist of pairs of periodic paths. In other words, when we encode paths using their pattern of "repeats," we also take into account their assumed periodicity.

LEMMA 5.11. *Let* (G_n) *be a sequence of graphs satisfying the Standing Assumptions* 2.21 *and bounded distortion of weights* [*condition* (C7) *in Definition* 2.23]. *Then there exists a polynomial* $p_2(x)$ *and* n_2 *such that for each* $n \geq n_2$ *and* $k > U(G_n)$,

$$\mu \times \mu(S_{n,k}^{2k-1}) \leq p_2(k)\lambda^{-(2m(G_n)+U(G_n))}$$

and

$$|S_{n,k}^{2k-1}| \leq p_2(k)\lambda^{2k-U(G_n)}.$$

PROOF. Consider (G_n), n and k as in the hypotheses. Let $m = m(G_n)$ and $U = U(G_n)$. Note that the bound on $|S_{n,k}^{2k-1}|$ follows from the bound on $\mu \times \mu(S_{n,k}^{2k-1})$, since condition (C7) implies that there exists a uniform constant K such that $|S_{n,k}^{2k-1}| \leq K\lambda^{2m+2k}\mu \times \mu(S_{n,k}^{2k-1})$ [as in Lemma 2.28(1)]. We now proceed to show the bound on $\mu \times \mu(S_{n,k}^{2k-1})$.

Let b be in $\mathrm{Per}_k(G_n)$. Let e be in $E_n(b)$. For $i = 1, \dots, k$, let $C_i \subset B_k(G_n)$ be the set of paths c of length k in G_n such that $c_i = e$. Then Lemma 2.28 [parts (1) and (4)] implies that there exist uniform constants K_1 and K_2 such that

$$\mu\big(\mathrm{Per}_k(G_n) \cap C_1\big) = \mu(C_1)\mu(\mathrm{Per}_k(G_n)|C_1) \leq K_1\lambda^{-m}\mu(\mathrm{Per}_k(G_n)|C_1) \tag{5.9}$$

$$\leq K_2\lambda^{-(m+U)}. \tag{5.10}$$

Let C be the set of paths c of length k in G_n such that $e \in E_n(c)$. Then $C = \bigcup_{i=1}^k C_i$, and by shift-invariance of μ,

$$\mu\big(\mathrm{Per}_k(G_n) \cap C\big) \leq \sum_{i=1}^{k} \mu\big(\mathrm{Per}_k(G_n) \cap C_i\big) \leq K_2 k\lambda^{-(m+U)}. \tag{5.11}$$

Since $e \in E_n(b)$ was arbitrary, it follows from inequality (5.11) that

$$\begin{aligned}
&\mu\big(\{c \in \mathrm{Per}_k(G_n) : E_n(c) \cap E_n(b) \neq \varnothing\}\big) \\
&\quad \leq \sum_{e \in E_n(b)} \mu\big(\{c \in \mathrm{Per}_k(G_n) : e \in E_n(c)\}\big) \\
&\quad \leq K_2 \sum_{e \in E_n(b)} k\lambda^{-(m+U)} \leq K_2 k^2 \lambda^{-(m+U)}.
\end{aligned}$$

Since $b \in \mathrm{Per}_k(G_n)$ was arbitrary, we conclude that there exists a uniform constant K_3 such that

$$\mu \times \mu(S_{n,k}^{2k-1}) \leq K_2\mu(\mathrm{Per}_k(G_n))k^2\lambda^{-(m+U)} \leq K_3 k^2 \lambda^{-(2m+U)},$$

where the last inequality follows from Lemma 2.28(4). This inequality completes the proof. □

LEMMA 5.12. *Let (G_n) be a sequence of graphs satisfying the Standing Assumptions 2.21 and such that (G_n) has local uniqueness of paths, small diameter and bounded distortion of weights* [*conditions* (C4), (C5) *and* (C7) *in Definition* 2.23]. *Then there exists a polynomial $p_3(x)$ and n_3 such that for $n \geq n_3$, $k > U(G_n)$ and $1 \leq j \leq 2k-1$,*

$$\mu \times \mu(S^j_{n,k}) \leq p_3(k)^{k/U(G_n)} \lambda^{-(m(G_n)+U(G_n)+2k-j)}$$

and

$$|S^j_{n,k}| \leq p_3(k)^{k/U(G_n)} \lambda^{j+m(G_n)-U(G_n)}.$$

PROOF. Consider (G_n), n, k and j as in the hypotheses. Let $m = m(G_n)$, $U = U(G_n)$ and $R = R(G_n)$. Note that the bound on $|S^j_{n,k}|$ follows from the bound on $\mu \times \mu(S^j_{n,k})$, since condition (C7) implies that there exists a uniform constant K such that $|S^j_{n,k}| \leq K\lambda^{2m+2k} \mu \times \mu(S^j_{n,k})$ [as in Lemma 2.28(1)]. We now proceed to show the bound on $\mu \times \mu(S^j_{n,k})$.

Let e be in E_n and let C_1 be the set of paths b of length k in G_n such that $b_1 = e$. Then it follows from Lemma 2.28(4) that there exists a uniform constant K_1 such that

$$\mu(\mathrm{Per}_k(G_n)|C_1) \leq K_1 \lambda^{-U}. \tag{5.12}$$

To each pair (b,c) in $S^j_{n,k}$, let us associate a particular path of length $2k+R$ in G_n, which we construct as follows. Let (b,c) be in $S^j_{n,k}$. By definition of $S^j_{n,k}$, there is at least one edge e in $E_n(b) \cap E_n(c)$. Let τ be the cyclic permutation of $\{1,\dots,k\}$ of order k given by $(12\cdots k)$. Let τ act on periodic paths of length k in G_n by permuting the indices: $\tau(b_1 \cdots b_k) = b_{\tau(1)} \cdots b_{\tau(k)}$. Then let b' be in $\{\tau^\ell(b) : \ell \in \{1,\dots,k\}, \tau^\ell(b)_k = e\}$. Similarly, let c' be in $\{\tau^\ell(c) : \ell \in \{1,\dots,k\}, \tau^\ell(c)_1 = e\}$. Now choose a path d_1 in G_n such that $|b'd_1c'| \leq R$ and $b'd_1c'$ is a path in G_n (the existence of such a path d_1 is guaranteed by the definition of R). By irreducibility of G_n, we also choose a (possibly empty) path d_2 in G_n such that $b'd_1c'd_2$ is in B_{2k+R}. We associate the path $b'd_1c'd_2$ to the pair (b,c), and note that there exist uniform constants K_2, K_3 and K_4 [by Lemma 2.28(1) and condition (C5)] such that

$$\mu \times \mu((b,c)) \leq K_2 \lambda^{-(2m+2k)} \leq K_3 \lambda^{-(m+R+2k)} \leq K_4 \mu(b'd_1c'd_2). \tag{5.13}$$

Now we use the same construction as in the proof of Lemma 5.9 with only slight modification. We encode the words $b'd_1c'd_2$ as follows:

(1) Construct $\mathcal{I}$, J and the partition of $\{1,\dots,2k+R\}$ as in the proof of Lemma 5.9, with the additional condition that $J \cap \{k+1,\dots,k+R\} = \varnothing$. (In other words, we ignore any "repeats" introduced by d.)

(2) Encode b' as in the proof of Lemma 5.9.

(3) To encode the path d_1, we first encode the fact that $b'_k = c'_1$ (by encoding k and $k + |d_1|$ using an Elias code), and then encode d_1 using a prefix Shannon code with respect to $\mu(\cdot|A_{k,k+|d_1|} \cap b')$.

(4) Encode c' as in the proof of Lemma 5.9.

(5) Encode d_2 using a prefix Shannon code with respect to $\mu(\cdot|b'd_1c')$.

For large n, encoding the fact that $b'_k = c'_1$ adds less than $4\log(2k + R)$ to $\mathcal{L}(b'd_1c'd_2)$. On the other hand, we have that there is a uniform constant $K_5 > 0$ such that $\mu(A_{k,k+|d_1|}|b') \le K_5\lambda^{-U}$, by Lemma 2.28(4). Thus, there exists n_3 and a uniform constant K_6 such that for $n \ge n_3$, we have

$$\begin{aligned}&\mathcal{L}(b'd_1c'd_2) + \log\mu(b'd_1c'd_2)\\&\quad\le (|\mathcal{I}| + 1)\bigl(4\log(2k + R)\bigr) + NK_6 - (m + U + |J|)\log\lambda\end{aligned} \tag{5.14}$$

with $|\mathcal{I}| \le 2k/U + 2$, $N \le |\mathcal{I}|$ and $|J| \ge 2k - j - 1$. Then by Lemma 5.2, there is a polynomial $p_4(x)$ such that for $n \ge n_3$,

$$\mu\bigl(\{b'd_1c'd_2 : (b, c) \in S^j_{n,k}\}\bigr) \le p_4(k)^{k/U}\lambda^{-(m+U+2k-j)}. \tag{5.15}$$

Note that the number of pairs (b, c) associated to the path $b'd_1c'd_2$ is at most k^2, and, hence,

$$\mu \times \mu(S^j_{n,k}) \le k^2 p_4(k)^{k/U}\lambda^{-(m+U+2k-j)}. \tag{5.16}$$

Now let $p_3(x) = x^2 p_4(x)$, and the proof is complete. □

5.2. *Entropy.* Recall that if G is a graph, then β_G is the random variable such that $\beta_G(\omega)$ is the spectral radius of the adjacency matrix of $G(\omega)$.

THEOREM 5.13. *Let (G_n) be a sequence of graphs that satisfies the Standing Assumptions 2.21 and such that (G_n) has local uniqueness of paths, small diameter and bounded distortion of weights [conditions (C4), (C5) and (C7) in Definition 2.23]. Then for $1/\lambda < \alpha \le 1$ and $\varepsilon > 0$,*

$$\lim_{n\to\infty} \mathbb{P}_\alpha(|\beta_{G_n} - \alpha\lambda| \ge \varepsilon) = 0,$$

and the convergence to the limit is exponential in $m(G_n)$.

REMARK 5.14. If we assume that X is irreducible in the statement of Theorem 1.3, then Theorem 1.3 is a direct corollary of Theorem 5.13, obtained by choosing (G_n) to be the sequence of n-block graphs of an irreducible SFT with positive entropy (and using the fact that such a sequence satisfies the hypotheses of Theorem 5.13 by Proposition 2.29). In the case when X is reducible, X has a finite number of irreducible components of positive entropy, $X_1, \dots, X_r$, and there exists i such that $\mathbf{h}(X_i) = \mathbf{h}(X)$. For all large n, we have that $B_n(X_i) \cap B_n(X_j) = \varnothing$ for $i \ne j$, which means that the entropies of the random subshifts appearing inside each of these components are mutually independent. Applying Theorem 5.13 to each of these components, we obtain Theorem 1.3 for reducible X.

PROOF OF THEOREM 5.13. Let α be in $(1/\lambda, 1]$. Let $m = m(G_n)$ and $U = U(G_n)$. Let b be a path in $G_n = (V_n, E_n)$. Let $\xi_b : \Omega_n \to \mathbb{R}$ be the random variable defined by

$$\xi_b(\omega) = \begin{cases} 1, & \text{if } b \text{ is allowed in } G_n(\omega), \\ 0, & \text{else.} \end{cases}$$

Now let

$$\phi_{n,k} = \sum_{b \in B_k(G_n)} \xi_b \quad \text{and} \quad \psi_{n,k} = \frac{1}{|V_n|} \sum_{b \in \mathrm{Per}_k(G_n)} \xi_b.$$

For each n and k, we have that $\psi_{n,k} \leq \beta_n^k \leq \phi_{n,k}$. Indeed, $\psi_{n,k}$ is the average number of loops of length k based at a vertex in G_n. Thus, there is at least one vertex v with at least $\psi_{n,k}$ loops of length k based at v, and it follows that $k^{-1} \log \psi_{n,k} \leq \log \beta_n$ since these loops may be concatenated freely. Also, it follows from subadditivity that $\log \beta_n = \lim_k k^{-1} \log \phi_{n,k} = \inf_k k^{-1} \log \phi_{n,k}$, which implies that $\beta_n^k \leq \phi_{n,k}$ for all n and k.

Fix $0 < \nu < 1$, and let $k = \lceil m^{1+\nu} \rceil + i$, where i is chosen such that $0 \leq i \leq \mathrm{per}(G_1) - 1$ and $\mathrm{per}(G_1)$ divides k. Recall that if (G_n) is the sequence of n-block graphs of a fixed graph G, then by Proposition 2.29 we have that m and n differ by at most a uniform constant, and, thus, $k \sim n^{1+\nu}$. We will show below that as n tends to infinity,

(I) $(\mathbb{E}_\alpha \phi_{n,k})^{1/k}$ tends to $\alpha\lambda$;
(II) $(\mathbb{E}_\alpha \psi_{n,k})^{1/k}$ tends to $\alpha\lambda$;
(III) there exists $K_1 > 0$ and $\rho_1 > 0$ such that $\frac{\mathrm{Var}(\phi_{n,k})}{(\mathbb{E}_\alpha \phi_{n,k})^2} \leq K_1 e^{-\rho_1 m}$;
(IV) there exists $K_2 > 0$ and $\rho_2 > 0$ such that $\frac{\mathrm{Var}(\psi_{n,k})}{(\mathbb{E}_\alpha \psi_{n,k})^2} \leq K_2 e^{-\rho_2 m}$.

Recall Definitions 5.5–5.8, as well as the modification of these definitions using "hats." Notice that

$$\mathbb{E}_\alpha \phi_{n,k} = \sum_{b \in B_k(G_n)} \mathbb{E}_\alpha \xi_b = \sum_{b \in B_k(G_n)} \alpha^{|E_n(b)|} = \sum_{j=1}^{k} \alpha^j |\hat{N}_{n,k}^j|.$$

Also,

$$|V_n| \mathbb{E}_\alpha \psi_{n,k} = \sum_{b \in \mathrm{Per}_k(G_n)} \mathbb{E}_\alpha \xi_b = \sum_{b \in \mathrm{Per}_k(G_n)} \alpha^{|E_n(b)|} = \sum_{j=1}^{k} \alpha^j |\hat{Q}_{n,k}^j|.$$

Regarding variances, we have

$$\mathrm{Var}(\phi_{n,k}) = \sum_{(b,c) \in B_k(G_n)^2} \alpha^{|E_n(b) \cup E_n(c)|} \big(1 - \alpha^{|E_n(b) \cap E_n(c)|}\big) \leq \sum_{j=1}^{2k-1} \alpha^j |\hat{D}_{n,k}^j|$$

and

$$|V_n|^2 \operatorname{Var}(\psi_{n,k}) = \sum_{(b,c)\in \operatorname{Per}_k(G_n)^2} \alpha^{|E_n(b)\cup E_n(c)|}\big(1-\alpha^{|E_n(b)\cap E_n(c)|}\big) \le \sum_{j=1}^{2k-1} \alpha^j |\hat{S}^j_{n,k}|.$$

For the remainder of this proof, we use the following notation: if (x_n) and (y_n) are two sequences, then $x_n \sim y_n$ means that the limit of the ratio of x_n and y_n tends to 1 as n tends to infinity.

Proof of (I). By Lemma 2.28(1), there exists a uniform constant $K_1 > 0$ such that

$$\mathbb{E}_\alpha \phi_{n,k} = \sum_{j=1}^{k} \alpha^j |\hat{N}^j_{n,k}| \ge \alpha^k \sum_{j=1}^{k} |\hat{N}^j_{n,k}| = \alpha^k |B_k(G_n)| \ge K_1 \alpha^k \lambda^{m+k}. \tag{5.17}$$

Taking kth roots, letting n tend to infinity, and using that $m/k \sim m^{-\nu}$ tends to 0, we obtain that $\liminf_n (\mathbb{E}_\alpha \phi_{n,k})^{1/k} \ge \alpha\lambda$.

By Lemmas 2.28(1) and 5.9, we have that there exists n_0, a polynomial $p_0(x)$, and a uniform constant $K_2 > 0$ such that for $n \ge n_0$,

$$\begin{aligned}
\mathbb{E}_\alpha \phi_{n,k} &= \sum_{j=1}^{k} \alpha^j |\hat{N}^j_{n,k}| \\
&\le \sum_{j=1}^{k-1} \alpha^j |N^j_{n,k}| + \alpha^k |B_k(G_n)| \\
&\le (p_0(k))^{k/U} \Bigg(\sum_{j=1}^{k-1} (\alpha\lambda)^j\Bigg) + K_2 \alpha^k \lambda^{k+m} \\
&\le (\alpha\lambda)^k \lambda^m \bigg(\frac{1}{\alpha\lambda - 1} p_0(k)^{k/U} \lambda^{-m} + K_2\bigg).
\end{aligned}$$

By condition (C4) and the fact that $k \sim m^{1+\nu}$, we have that:

- m tends to infinity as n tends to infinity by the Standing Assumptions 2.21;
- $m/k \sim m^{-\nu}$, which tends to zero as n tends to infinity;
- $U \ge m - C$, which tends to infinity as n tends to infinity.

Thus, taking kth roots and letting n tend to infinity, we have that

$$\limsup_n (\mathbb{E}_\alpha \phi_{n,k})^{1/k} \le \alpha\lambda,$$

which concludes the proof of (I).

Proof of (II). Let $p = \operatorname{per}(G_1) = \operatorname{per}(G_n)$. Note that since p divides k, there exists a uniform constant $K_3 > 0$ such that $|\operatorname{Per}_k(G_n)| \ge K_3 \lambda^k$ for large enough k.

We choose n large enough so that this inequality is satisfied. Then we have that

$$\begin{aligned}\mathbb{E}_\alpha \psi_{n,k} &= |V_n|^{-1} \sum_{j=1}^{k} \alpha^j |\hat{Q}_{n,k}^j| \\ &\geq |V_n|^{-1} \alpha^k \sum_{j=1}^{k} |\hat{Q}_{n,k}^j| \\ &= |V_n|^{-1} \alpha^k |\mathrm{Per}_k(G_n)| \\ &\geq K_3 \lambda^{-m} \alpha^k \lambda^k.\end{aligned}$$

Taking kth roots, letting n tend to infinity, and using that $m/k \sim m^{-\nu}$ tends to 0, we get that $\liminf_n (\mathbb{E}_\alpha \psi_{n,k})^{1/k} \geq \alpha\lambda$. Recall that $0 \leq \psi_{n,k} \leq \phi_{n,k}$. Therefore, it follows from (I) that $\limsup_n (\mathbb{E}_\alpha \psi_{n,k})^{1/k} \leq \alpha\lambda$. Thus, we have shown (II).

Proof of (III). For $j \leq 2k-1$, Lemma 5.10 implies that there is n_1 and a polynomial p_1 such that $|D_{n,k}^j| \leq p_1(k)^{k/U} \lambda^{j+m}$ and $|D_{n,k}^{2k-1}| \leq p_1(k)\lambda^{2k+m}$ for $n \geq n_1$. Now using that $\mathbb{E}_\alpha \phi_{n,k} \geq K_1 \alpha^k \lambda^{m+k}$ [see (5.17)], we obtain that there exists a uniform constant $K_5 > 0$ such that

$$\begin{aligned}\frac{\operatorname{Var} \phi_{n,k}}{(\mathbb{E}_\alpha \phi_{n,k})^2} &\leq \frac{\sum_{j=1}^{2k-1} \alpha^j |\hat{D}_{n,k}^j|}{K_1^2 \alpha^{2k} \lambda^{2m+2k}} \\ &= \frac{\sum_{j=1}^{2k-m-1} \alpha^j |\hat{D}_{n,k}^j| + \sum_{j=2k-m}^{2k-1} \alpha^j |\hat{D}_{n,k}^j|}{K_1^2 \alpha^{2k} \lambda^{2m+2k}} \\ &\leq \frac{\sum_{j=1}^{2k-m-1} \alpha^j |D_{n,k}^j| + \alpha^{2k-m} \sum_{j=2k-m}^{2k-1} |\hat{D}_{n,k}^j|}{K_1^2 \alpha^{2k} \lambda^{2m+2k}} \\ &\leq \frac{p_1(k)^{k/U} \lambda^m \sum_{j=1}^{2k-m-1} (\alpha\lambda)^j + \alpha^{2k-m} |D_{n,k}^{2k-1}|}{K_1^2 \alpha^{2k} \lambda^{2m+2k}} \\ &\leq \frac{K_5 p_1(k)^{k/U} \lambda^m (\alpha\lambda)^{2k-m} + \alpha^{2k-m} p_1(k) \lambda^{2k+m}}{K_1^2 \alpha^{2k} \lambda^{2m+2k}} \\ &\leq \frac{K_5}{K_1^2} \frac{p_1(k)^{k/U}}{(\alpha\lambda)^m \lambda^m} + \frac{p_1(k)}{K_1^2 (\alpha\lambda)^m} \\ &\leq \frac{K_5}{K_1^2} \left(\frac{p_1(k)^{k/Um}}{(\alpha\lambda)} \right)^m + \frac{p_1(k)}{K_1^2 (\alpha\lambda)^m}.\end{aligned}$$

Using the facts that $U \geq m - C$ and $k \sim m^{1+\nu}$, we have that k/Um is asymptotically bounded above by $2m^{\nu-1}$. Since $\nu - 1 < 0$, it holds that $p_1(k)^{k/Um}$ tends to 1. Thus, we obtain that for any $0 < \rho_1 < \ln \alpha\lambda$, there exists $K_6 > 0$ and n_2 such that for $n \geq n_2$, it holds that $\operatorname{Var} \phi_{n,k} (\mathbb{E}_\alpha \phi_{n,k})^{-2} \leq K_6 e^{-\rho_1 m}$, which proves (III).

Proof of (IV). For $j \leq 2k-1$, Lemma 5.12 together with (C4) implies that there is n_3 and a polynomial p_3 such that $|S^j_{n,k}| \leq p_3(k)^{k/U}\lambda^j$ for $n \geq n_3$. Also, Lemma 5.11 implies that there is n_4 and a polynomial p_2 such that $|S^{2k-1}_{n,k}| \leq p_2(k)\lambda^{2k-U}$ for $n \geq n_4$. Now using that $|V_n|\mathbb{E}_\alpha\psi_{n,k} \geq K_3\alpha^k\lambda^k$, we obtain that there exists $K_7 > 0$ such that, with $K := K_3$,

$$\begin{aligned}
\frac{\operatorname{Var}\psi_{n,k}}{(\mathbb{E}_\alpha\psi_{n,k})^2} &\leq \frac{\sum_{j=1}^{2k-1}\alpha^j|\hat{S}^j_{n,k}|}{K^2\alpha^{2k}\lambda^{2k}} \\
&= \frac{\sum_{j=1}^{2k-U-1}\alpha^j|\hat{S}^j_{n,k}| + \sum_{j=2k-U}^{2k-1}\alpha^j|\hat{S}^j_{n,k}|}{K^2\alpha^{2k}\lambda^{2k}} \\
&\leq \frac{\sum_{j=1}^{2k-U-1}\alpha^j|S^j_{n,k}| + \alpha^{2k-U}\sum_{j=2k-U}^{2k-1}|\hat{S}^j_{n,k}|}{K^2\alpha^{2k}\lambda^{2k}} \\
&\leq \frac{p_3(k)^{k/U}\sum_{j=1}^{2k-U-1}(\alpha\lambda)^j + \alpha^{2k-U}|S^{2k-1}_{n,k}|}{K^2\alpha^{2k}\lambda^{2k}} \\
&\leq \frac{K_7 p_3(k)^{k/U}(\alpha\lambda)^{2k-U} + \alpha^{2k-U}p_2(k)\lambda^{2k-U}}{K^2\alpha^{2k}\lambda^{2k}} \\
&\leq \frac{K_7}{K^2}\frac{p_3(k)^{k/U}}{(\alpha\lambda)^U} + \frac{p_2(k)}{K^2(\alpha\lambda)^U} \\
&\leq \frac{K_7}{K^2}\left(\frac{p_3(k)^{k/U^2}}{(\alpha\lambda)}\right)^U + \frac{p_2(k)}{K^2(\alpha\lambda)^U}.
\end{aligned}$$

Using the facts that $U \geq m - C$ and $k \sim m^{1+\nu}$, we have that k/U^2 is asymptotically bounded above by $2m^{\nu-1}$. Since $\nu - 1 < 0$, it holds that $p_3(k)^{k/U^2}$ tends to 1. Thus, we obtain that for any $0 < \rho_2 < \log\alpha\lambda$, there exists $K_8 > 0$ and n_5 such that for $n \geq n_5$,

$$\frac{\operatorname{Var}\phi_{n,k}}{(\mathbb{E}_\alpha\phi_{n,k})^2} \leq K_8 e^{-\rho_2 m},$$

which proves (IV).

Proof of Theorem 5.13 using (I)–(IV). Recall that $\psi_{n,k} \leq \beta^k_n \leq \phi_{n,k}$. Let $\varepsilon > 0$. Since $\alpha\lambda > 1$, we may assume without loss of generality that $\alpha\lambda - \varepsilon > 1$. Then

$$\begin{aligned}
&\mathbb{P}_\alpha(|\beta_n - \alpha\lambda| \geq \varepsilon) \\
&\quad = \mathbb{P}_\alpha(\beta_n \geq \alpha\lambda + \varepsilon) + \mathbb{P}_\alpha(\beta_n \leq \alpha\lambda - \varepsilon) \qquad (5.18)\\
&\quad = \mathbb{P}_\alpha(\beta^k_n \geq (\alpha\lambda + \varepsilon)^k) + \mathbb{P}_\alpha(\beta^k_n \leq (\alpha\lambda - \varepsilon)^k) \qquad (5.19)\\
&\quad \leq \mathbb{P}_\alpha(\phi_{n,k} \geq (\alpha\lambda + \varepsilon)^k) + \mathbb{P}_\alpha(\psi_{n,k} \leq (\alpha\lambda - \varepsilon)^k). \qquad (5.20)
\end{aligned}$$

We will bound each of the two terms in (5.20). Notice that

$$\mathbb{P}_\alpha\big(\phi_{n,k} \geq (\alpha\lambda+\varepsilon)^k\big) = \mathbb{P}_\alpha\big(\phi_{n,k} - \mathbb{E}_\alpha\phi_{n,k} \geq (\alpha\lambda+\varepsilon)^k - \mathbb{E}_\alpha\phi_{n,k}\big)$$

$$= \mathbb{P}_\alpha\biggl(\phi_{n,k} - \mathbb{E}_\alpha\phi_{n,k} \geq \mathbb{E}_\alpha\phi_{n,k}\biggl(\biggl(\frac{\alpha\lambda+\varepsilon}{(\mathbb{E}_\alpha\phi_{n,k})^{1/k}}\biggr)^k - 1\biggr)\biggr).$$

Let $d^1_{n,k} = (\mathrm{Var}(\phi_{n,k}))^{1/2}/\mathbb{E}_\alpha\phi_{n,k}$. Then by Chebyshev's Inequality,

$$\mathbb{P}_\alpha\big(\phi_{n,k} \geq (\alpha\lambda+\varepsilon)^k\big) \tag{5.21}$$

$$= \mathbb{P}_\alpha\biggl(\phi_{n,k} - \mathbb{E}_\alpha\phi_{n,k} \geq (\mathrm{Var}(\phi_{n,k}))^{1/2}\frac{1}{d^1_{n,k}}\biggl(\biggl(\frac{\alpha\lambda+\varepsilon}{(\mathbb{E}_\alpha\phi_{n,k})^{1/k}}\biggr)^k - 1\biggr)\biggr) \tag{5.22}$$

$$\leq \biggl(\frac{d^1_{n,k}}{((\alpha\lambda+\varepsilon)/(\mathbb{E}_\alpha\phi_{n,k})^{1/k})^k - 1}\biggr)^2. \tag{5.23}$$

The denominator in the right-hand side of (5.23) might be 0 for finitely many n, but by properties (I) and (III), there exists $K_9 > 0$ such that for large enough n,

$$\mathbb{P}_\alpha\big(\phi_{n,k} \geq (\alpha\lambda+\varepsilon)^k\big) \leq \biggl(\frac{d^1_{n,k}}{((\alpha\lambda+\varepsilon)/(\mathbb{E}_\alpha\phi_{n,k})^{1/k})^k - 1}\biggr)^2 \leq K_9 e^{-\rho_1 m}.$$

Similarly, we let $d^2_{n,k} = (\mathrm{Var}(\psi_{n,k}))^{1/2}/\mathbb{E}_\alpha\psi_{n,k}$, and then Chebyshev's Inequality gives that

$$\mathbb{P}_\alpha\big(\psi_{n,k} \leq (\alpha\lambda-\varepsilon)^k\big) \tag{5.24}$$

$$= \mathbb{P}_\alpha\biggl(\psi_{n,k} - \mathbb{E}_\alpha\psi_{n,k} \leq (\mathrm{Var}(\psi_{n,k}))^{1/2}\frac{1}{d^2_{n,k}}\biggl(\biggl(\frac{\alpha\lambda-\varepsilon}{(\mathbb{E}_\alpha\psi_{n,k})^{1/k}}\biggr)^k - 1\biggr)\biggr) \tag{5.25}$$

$$\leq \biggl(\frac{d^2_{n,k}}{((\alpha\lambda-\varepsilon)/(\mathbb{E}_\alpha\psi_{n,k})^{1/k})^k - 1}\biggr)^2. \tag{5.26}$$

Again, the denominator in the right-hand side might be 0 for finitely many n, but by properties (II) and (IV), there exists $K_{10} > 0$ such that for large enough n,

$$\mathbb{P}_\alpha\big(\psi_{n,k} \leq (\alpha\lambda-\varepsilon)^k\big) \leq \biggl(\frac{d^2_{n,k}}{((\alpha\lambda-\varepsilon)/(\mathbb{E}_\alpha\psi_{n,k})^{1/k})^k - 1}\biggr)^2 \leq K_{10} e^{-\rho_2 m}.$$

In conclusion, we obtain that there exists $K_{11} > 0$ such that for large enough n,

$$\mathbb{P}_\alpha(|\beta_n - \alpha\lambda| \geq \varepsilon) \leq K_{11} e^{-\min(\rho_1,\rho_2)m}. \qquad \square$$

5.3. *Irreducible components of positive entropy.*

THEOREM 5.15. *Let (G_n) be a sequence of graphs that satisfies the Standing Assumptions 2.21, with $p = \mathrm{per}(G_1) = \mathrm{per}(G_n)$, and such that*:

- (G_n) *has bounded degrees* [*condition* (C1) *in Definition* 2.23],
- (G_n) *has fast separation of periodic points* [*condition* (C3) *in Definition* 2.23],
- *and* (G_n) *has uniform forward and backward expansion* [*condition* (C8) *in Definition* 2.23].

Let $\mathcal{U}_{G_n}$ *be the event in* Ω_{G_n} *that* $G_n(\omega)$ *contains a unique irreducible component* C *of positive entropy. Also, let* $\mathcal{W}_{G_n}$ *be the event* (*contained in* $\mathcal{U}_{G_n}$) *that the induced edge shift on* C *has period* p. *Then there exists* $c > 0$ *such that for* $1 - c < \alpha \leq 1$,

$$\lim_{n\to\infty} \mathbb{P}_\alpha(\mathcal{U}_{G_n}) = 1 \quad \textit{and} \quad \lim_{n\to\infty} \mathbb{P}_\alpha(\mathcal{W}_{G_n}) = 1,$$

and the convergence to these limits is exponential in $m(G_n)$.

REMARK 5.16. Theorem 1.4 is a corollary of Theorem 5.15: if X is an irreducible SFT of positive entropy, then the sequence of n-block graphs for X satisfies the hypotheses of Theorem 5.15 by Proposition 2.29. In fact, if X is a reducible SFT, we may apply Theorem 1.4 to each irreducible component independently, which allows us to conclude the following. Let X be a reducible SFT with irreducible components $X_1, \dots, X_r$ such that $p_i = \mathrm{per}(X_i)$ for each i. Let $\mathcal{W}_n$ be the event in Ω_n that X_ω has exactly r irreducible components with periods $p_1, \dots, p_r$. Then there exists $c > 0$ such that for $\alpha \in (1 - c, 1]$, we have that $\lim_n \mathbb{P}_\alpha(\mathcal{W}_n) = 1$, with exponential (in n) convergence to the limit.

DEFINITION 5.17. Let G be a directed graph. For each vertex v in G, and for each ω in Ω_G, let $\Gamma^+_\omega(v)$ be the union of $\{v\}$ and the set of vertices u in G such that there is an allowed path from v to u in $G(\omega)$. Similarly, for each vertex v in G and each ω in Ω_G, let $\Gamma^-_\omega(v)$ be the union of $\{v\}$ and the set of vertices u in G such that there is an allowed path from u to v in $G(\omega)$. Also, let $I_\omega(v) = \Gamma^+_\omega(v) \cap \Gamma^-_\omega(v)$, which is the vertex set of the irreducible component containing v in $G(\omega)$.

The proof of the following proposition is an adaptation of the proof of Lemma 2.2 in [2].

PROPOSITION 5.18. *Let* (G_n) *be a sequence of graphs satisfying the Standing Assumptions* 2.21 *and such that* (G_n) *has bounded degrees and uniform forward and backward expansion* [*conditions* (C1) *and* (C8) *in Definition* 2.23]. *Let* r_n *be a sequence of integers such that* $r_n \geq am(G_n)$, *for some* $a > 0$, *for all large* n. *Let* $C^+_{G_n}$ *be the event in* Ω_{G_n} *consisting of all* ω *such that there exists a vertex* v *in* G_n *with* $r_n \leq \Gamma^+_\omega(v) \leq |V_n|/2$. *Then there exists* $c > 0$ *such that for* $\alpha > 1 - c$,

$$\lim_{n\to\infty} \mathbb{P}_\alpha(C^+_{G_n}) = 0, \tag{5.27}$$

and the convergence of this limit is exponential in $m(G_n)$. *Furthermore, the same statement holds with* "+" *replaced by* "−."

PROOF. Let $m = m(G_n)$. Let $b > 0$ be such that both (G_n) and (G_n^T) are b-expander sequences [where the existence of such a b is guaranteed by condition (C8)]. We use the notation in Definition 5.17. For any v in V_n and any ω in Ω_{G_n}, the set $\Gamma_\omega^+(v)$ has the property that all edges in $E_n(\Gamma_\omega^+(v), \overline{\Gamma_\omega^+(v)})$ are forbidden (by ω). Then the fact that G_n is a b-expander implies that for a particular subset S of V_n, the probability that $S = \Gamma_\omega^+(v)$ for some v is bounded above by $(1-\alpha)^{b|S|}$. The number of subsets S of V_n with $|S| = r$ that could appear as $\Gamma_\omega^+(v)$ for some v is bounded above by $(\Delta e)^r$, where e is the base of the natural logarithm ([2], Lemma 2.2) (see also [1], Lemma 2.1, or [37], page 396, Exercise 11). Then for α such that $\Delta e(1-\alpha) < 1$, we have that for any $0 \le r_n \le |V_n|/2$,

$$\mathbb{P}_\alpha(C_{G_n}^+) = \mathbb{P}_\alpha\left(\exists v \text{ such that } r_n \le |\Gamma_\omega^+(v)| \le \frac{|V_n|}{2}\right) \tag{5.28}$$

$$\le \sum_{r=r_n}^{|V_n|/2} |V_n|(\Delta e)^r(1-\alpha)^{br} \tag{5.29}$$

$$\le |V_n|\big(\Delta e(1-\alpha)^b\big)^{r_n} \frac{1}{1-\Delta e(1-\alpha)} \tag{5.30}$$

$$\le \big(\lambda^{1/a}\Delta e(1-\alpha)^b\big)^{am} \frac{1}{1-\Delta e(1-\alpha)}. \tag{5.31}$$

Thus, there is a $c > 0$ (depending only on a, b, λ and Δ) such that if $\alpha > 1-c$, then the right-hand side of the inequality in (5.31) tends to zero exponentially in $m(G_n)$ as n tends to infinity. In particular, we may take

$$c = \left(\frac{1}{\lambda}\right)^{1/ab}\left(\frac{1}{\Delta e}\right)^{1/b}.$$

Since (G_n^T) is also a uniform b-expander, the same estimates hold with $C_{G_n}^-$ in place of $C_{G_n}^+$. $\square$

PROOF OF THEOREM 5.15. Let (G_n) be as in the statement of Theorem 5.15. Let $m = m(G_n)$, $z = z(G_n)$, and $p = \mathrm{per}(G_1) = \mathrm{per}(G_n)$. We use the notation in Definition 5.17. Consider the following events:

$$F_n^+ = \{\omega \in \Omega_n : \forall v \in V_n, \Gamma_\omega^+(v) \le z(G_n) - 2p \text{ or } \Gamma_\omega^+(v) > |V_n|/2\},$$
$$F_n^- = \{\omega \in \Omega_n : \forall v \in V_n, \Gamma_\omega^-(v) \le z(G_n) - 2p \text{ or } \Gamma_\omega^-(v) > |V_n|/2\},$$
$$F_n = F_n^+ \cap F_n^-.$$

Recall that condition (C3) gives $a > 0$ such that $z \ge am$. Note that Proposition 5.18 gives $c > 0$ such that for $1-c < \alpha \le 1$, there exists $K_1, K_2 > 0$ and $\rho_1, \rho_2 > 0$ such that for large n,

$$\mathbb{P}_\alpha(\Omega_n \setminus F_n^+) \le K_1 e^{-\rho_1 m} \quad \text{and} \quad \mathbb{P}_\alpha(\Omega_n \setminus F_n^-) \le K_2 e^{-\rho_2 m}.$$

Fix such an α, and note that for all large enough n, we have the following estimate: $\mathbb{P}_\alpha(\Omega_n \setminus F_n) \le 2\max(K_1, K_2)e^{-\min(\rho_1,\rho_2)m}$.

Consider ω in F_n. Suppose that there exists v_1 and v_2 in V_n such that $|I_\omega(v_1)| > z - 2p$ and $|I_\omega(v_2)| > z - 2p$. Then by definition of F_n, we must have that $\Gamma_\omega^+(v_1) \cap \Gamma_\omega^-(v_2) \neq \varnothing$ and $\Gamma_\omega^-(v_1) \cap \Gamma_\omega^+(v_2) \neq \varnothing$. It follows that there is a path from v_1 to v_2 in $G_n(\omega)$, and there is a path from v_2 to v_1 in $G_n(\omega)$. Thus, $I_\omega(v_1) = I_\omega(v_2)$. We have shown that for ω in F_n, there is at most one irreducible component of cardinality greater than $z - 2p$. Note that this argument implies that for ω in F_n, all allowed periodic orbits γ such that $|V_n(\gamma)| > z - 2p$ must lie in the same irreducible component.

By definition of z, if I_ω is an irreducible component of $G_n(\omega)$ with positive entropy, then $|I_\omega| > z$ (since it must contain at least two periodic orbits with overlapping vertex sets). We deduce that for ω in F_n, there is at most one irreducible component of $G_n(\omega)$ with positive entropy.

We now show that there exists an irreducible component of positive entropy with probability tending exponentially to 1. Let $z_1 = z - i$, where i is chosen (for each n) such that $0 \le i \le p - 1$ and p divides z_1. Then let $z_2 = z_1 - p$. Consider the following sequences of random variables:

$$f_n = \sum_{b \in \mathrm{Per}_{z_1}(G_n)} \xi_b \quad \text{and} \quad g_n = \sum_{b \in \mathrm{Per}_{z_2}(G_n)} \xi_b. \tag{5.32}$$

Note that by the definition of z and Lemma 3.3, we have that $|E_n(b)| = |b|$ for any periodic path b with period less than or equal to z. Furthermore, any two such paths are disjoint. Therefore, the random variables $\{\xi_b\}_{b \in \mathrm{Per}_{z_1}(G_n)}$ are jointly independent, and the random variables $\{\xi_b\}_{b \in \mathrm{Per}_{z_2}(G_n)}$ are also jointly independent. Thus,

$$\mathbb{E}_\alpha f_n = \sum_{b \in \mathrm{Per}_z(G_n)} \alpha^{z_1} = \alpha^{z_1} |\mathrm{Per}_{z_1}(G_n)|,$$

$$\mathbb{E}_\alpha g_n = \sum_{b \in \mathrm{Per}_{z_2}(G_n)} \alpha^{z_2} = \alpha^{z_2} |\mathrm{Per}_{z_2}(G_n)|,$$

$$\mathrm{Var}(f_n) = \sum_{b \in \mathrm{Per}_{z_1}(G_n)} \alpha^{z_1}(1 - \alpha^{z_1}) = \alpha^{z_1}(1 - \alpha^{z_1}) |\mathrm{Per}_{z_1}(G_n)|,$$

$$\mathrm{Var}(g_n) = \sum_{b \in \mathrm{Per}_{z_2}(G_n)} \alpha^{z_2}(1 - \alpha^{z_2}) = \alpha^{z_2}(1 - \alpha^{z_2}) |\mathrm{Per}_{z_2}(G_n)|.$$

As n tends to infinity, z tends to infinity since $z \ge am$ and m tends to infinity. Then by the Standing Assumptions 2.21 [in particular, we use that $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$] and the fact that p divides z_1 and z_2, we have that each of the sequences $\lambda^{-z_1}|\mathrm{Per}_{z_1}(G_n)|$ and $\lambda^{-z_2}|\mathrm{Per}_{z_2}(G_n)|$ tends to a finite, nonzero limit as n tends to infinity (and in fact the limit is p). For two sequences x_n and y_n of positive real numbers, let $x_n \sim y_n$ denote the statement that their ratio tends to a finite, nonzero limit as n tends to infinity. Then we have that $\mathbb{E}_\alpha f_n \sim (\alpha\lambda)^{z_1} \sim \mathrm{Var}(f_n)$ and $\mathbb{E}_\alpha g_n \sim (\alpha\lambda)^{z_2} \sim \mathrm{Var}(g_n)$. Note that $\mathbb{E}_\alpha f_n \geq \mathrm{Var}(f_n)$ and $\mathbb{E}_\alpha g_n \geq \mathrm{Var}(g_n)$. A simple application of Chebyshev's Inequality implies that

$$\begin{aligned}\mathbb{P}_\alpha(f_n \leq 0) &\leq \mathbb{P}_\alpha\big(f_n - \mathbb{E}_\alpha f_n \leq -\mathrm{Var}(f_n)\big)\\ &\leq \left(\frac{1}{\mathrm{Var}(f_n)^{1/2}}\right)^2 \sim \left(\frac{1}{\alpha\lambda}\right)^{z_1} \leq \left(\frac{1}{\alpha\lambda}\right)^{am-i}\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}_\alpha(g_n \leq 0) &\leq \mathbb{P}_\alpha\big(g_n - \mathbb{E}_\alpha g_n \leq -\mathrm{Var}(g_n)\big)\\ &\leq \left(\frac{1}{\mathrm{Var}(g_n)^{1/2}}\right)^2 \sim \left(\frac{1}{\alpha\lambda}\right)^{z_2} \leq \left(\frac{1}{\alpha\lambda}\right)^{am-i-p}.\end{aligned}$$

We have shown that the probability that there is no periodic orbit of period z_1 tends to 0 exponentially in m as n tends to infinity, and the probability that there exists no periodic orbit of period z_2 tends to 0 exponentially in m as n tends to infinity.

In summary, we have shown that the following events occur with probability tending to 1 exponentially in m as n tends to infinity:

- there exists a periodic point of period $z - i$;
- there exists a periodic point of period $z - i - p$;
- any two periodic points of period greater than $z - 2p$ lie in the same irreducible component (of necessarily positive entropy);
- there is at most one irreducible component of positive entropy.

We conclude that with probability tending to 1 exponentially in m as n tends to infinity, there exists a unique irreducible component of positive entropy, and the induced edge shift on that component has period p. □

6. Remarks.

REMARK 6.1. The proofs of Theorems 3.1 and 4.2 do not require all of the Standing Assumptions 2.21. In fact, these proofs only use that $\mathrm{Sp}_\times(G_n) = \mathrm{Sp}_\times(G_1)$ for each n and that $z(G_n)$ tends to infinity as n tends to infinity.

REMARK 6.2. Theorem 3.1 states that at the critical threshold $\alpha = 1/\lambda$, the probability of emptiness tends to zero. Using the fact that entropy is a monotone increasing random variable (as defined in Section 2.3), one may deduce from Theorem 5.13 that for $\alpha = 1/\lambda$, the probability that the random SFT has zero entropy tends to 1. It might be interesting to know more about the behavior of typical random SFTs at the critical threshold.

REMARK 6.3. We have considered only random $\mathbb{Z}$-SFTs, but one may also consider random $\mathbb{Z}^d$-SFTs for any d in $\mathbb{N}$ by adapting the construction of Ω_n and $\mathbb{P}_\alpha$ in the obvious way. It appears that most of the proofs presented above may not be immediately adapted for $d > 1$, but there is one exception, which we state below. Let X be a nonempty $\mathbb{Z}^d$-SFT. For $d > 1$, there are various zeta functions for X (for a definition distinct from ours, see [39]); we consider

$$\zeta_X(t) = \exp\left(\sum_{p=1}^{\infty} \frac{N_p}{p} t^p\right),$$

where N_p is the number of periodic points x in X such that the number of points in the orbit of x divides p. The function ζ_X has radius of convergence $1/\rho$, where $\log \rho = \limsup_p p^{-1} \log(N_p)$. For example, for a full $\mathbb{Z}^d$ shift on a symbols, $\rho = a$, regardless of d. Using exactly the same proof as presented in Section 3, we obtain that

$$\limsup_{n\to\infty} \mathbb{P}_\alpha(\mathcal{E}_n) \le \begin{cases} (\zeta_X(\alpha))^{-1}, & \text{if } \alpha \in [0, 1/\rho), \\ 0, & \text{if } \alpha \in [1/\rho, 1]. \end{cases}$$

For $\alpha \ge 1/\rho$, this bound implies that the limiting probability of emptiness is 0. In this context, we note that there is no algorithm, which, given a $\mathbb{Z}^d$-SFT X defined by a finite list of finite forbidden configurations, will decide whether X is empty [6]. Nonetheless, we may be able to compute the limiting probability of emptiness. For example, if X is a full shift on a symbols, then for $\alpha \ge 1/a$, we have that the limiting probability of emptiness is 0.

REMARK 6.4. One may also consider more general random subshifts. Recall that a set $X \subset \mathcal{A}^{\mathbb{Z}}$ is a subshift if it is closed and shift-invariant. For a nonempty subshift X and a natural number n, we may consider the (finite) set of subshifts obtained by forbidding words of length n from X. After defining a probability measure $\mathbb{P}_\alpha$ on this space as in Section 2, we obtain random subshifts of X. Now we may investigate the asymptotic probability of properties of these random subshifts. Recall that any subshift X can be written as $\bigcap X_n$, where (X_n) is a sequence of SFTs (called the Markov approximations of X) and $\lim_n h(X_n) = h(X)$. A subshift X is called *almost sofic* [48] if there exists a sequence (X_n) of irreducible SFTs such that $X_n \subset X$ and $\lim_n h(X_n) = h(X)$. Using this inner and outer approximation by SFTs, the conclusion of Theorem 1.3 still holds if the system X is only assumed to be an almost sofic subshift.

REMARK 6.5. Theorem 5.15 asserts the existence of a constant $c > 0$, but we are left with several questions about this constant. Fix a sequence (G_n) satisfying the hypotheses of Theorem 5.15. Let $\alpha_* = \inf\{\alpha > 0 : \lim_n \mathbb{P}_\alpha(\mathcal{U}_n) = 1\}$. What is α_*? What is α_* in the case that (G_n) is the sequence of n-block graphs of a mixing SFT of positive entropy (or even a full shift)?

Acknowledgments. The author expresses great thanks to his advisor, Mike Boyle, for his guidance and encouragement in general, and also for many helpful conversations and comments regarding this work in particular. Further, the author thanks the anonymous referee for reading the paper carefully and making helpful suggestions.

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