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STOCHASTIC MAXIMAL L^p-REGULARITY

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In this article we prove a maximal L^p-regularity result for stochastic convolutions, which extends Krylov's basic mixed $L^p(L^q)$-inequality for the Laplace operator on $\mathbb{R}^d$ to large classes of elliptic operators, both on $\mathbb{R}^d$ and on bounded domains in $\mathbb{R}^d$ with various boundary conditions. Our method of proof is based on McIntosh's H^∞-functional calculus, R-boundedness techniques and sharp $L^p(L^q)$-square function estimates for stochastic integrals in L^q-spaces. Under an additional invertibility assumption on A, a maximal space–time L^p-regularity result is obtained as well.

1. Introduction.

Let $S = (S(t))_{t\geq 0}$ denote the heat semigroup on $L^p(\mathbb{R}^d)$,

$$S(t)f(x) = \frac{1}{\sqrt{(2\pi t)^d}} \int_{\mathbb{R}^d} e^{-(x-y)^2/2t} f(y)\, dy,$$

and let H be a Hilbert space. Generalizing the classical Littlewood–Paley inequality, Krylov [27, 30, 31] proved that for $p \in [2, \infty)$ and all $G \in L^p(\mathbb{R}_+ \times \mathbb{R}^d; H)$ one has

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^d} \left(\int_0^t \|[\nabla S(t-s)G(s,\cdot)](x)\|^2\, ds \right)^{p/2} dx\, dt \\ \leq C_p^p \|G\|^p_{L^p(\mathbb{R}_+\times\mathbb{R}^d;H)}\end{aligned} \tag{1.1}$$

and more generally, for $p, q \in [2, \infty)$ with $q \leq p$ and $G \in L^p(\mathbb{R}_+; L^q(\mathbb{R}^d; H))$,

$$\begin{aligned}\int_0^\infty \left(\int_{\mathbb{R}^d} \left(\int_0^t \|[\nabla S(t-s)G(s,\cdot)](x)\|^2\, ds \right)^{q/2} dx \right)^{p/q} dt \\ \leq C_{p,q}^p \|G\|^p_{L^p(\mathbb{R}_+;L^q(\mathbb{R}^d;H))}.\end{aligned} \tag{1.2}$$

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In both (1.1) and (1.2) we implicitly use the extension of $S(t)$ to $L^p(\mathbb{R}^d; H)$ and $L^q(\mathbb{R}^d; H)$, respectively (see the remarks preceding Theorem 4.3). These singular convolution estimates are the cornerstone of Krylov's L^q-theory of stochastic PDEs [26–31]. The proofs of (1.1) and (1.2) rely heavily on techniques from harmonic analysis, and their extension to bounded domains is a well-known open problem. The aim of the present paper is to prove a far-reaching generalization of Krylov's inequalities which, among other things, provides such an extension. Our approach is radically different from Krylov's and uses H^∞-calculus estimates, developed by McIntosh and coauthors, R-boundedness techniques and sharp $L^p(L^q)$-square function estimates for stochastic integrals in L^q-spaces.

In order to state the main result we need to introduce some terminology. Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space endowed with a filtration $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$, and let $(W_H(t))_{t\geq 0}$ be a cylindrical $\mathscr{F}$-Brownian motion on H (see Section 2.1). Furthermore let $(\mathcal{O}, \Sigma, \mu)$ be an arbitrary σ-finite measure space.

THEOREM 1.1. *Let $q \in [2, \infty)$, suppose the operator A admits a bounded H^∞-calculus of angle less than $\pi/2$ on $L^q(\mathcal{O})$, and let $(S(t))_{t\geq 0}$ denote the bounded analytic semigroup on $L^q(\mathcal{O})$ generated by $-A$. For all $\mathscr{F}$-adapted $G \in L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}; H))$ the stochastic convolution process*

$$U(t) = \int_0^t S(t-s)G(s)\,dW_H(s), \qquad t \geq 0,$$

is well defined in $L^q(\mathcal{O})$, takes values in the fractional domain $D(A^{1/2})$ almost surely and for all $2 < p < \infty$ we have the stochastic maximal L^p-regularity estimate

$$\mathbb{E}\|A^{1/2}U\|^p_{L^p(\mathbb{R}_+; L^q(\mathcal{O}))} \leq C^p \mathbb{E}\|G\|^p_{L^p(\mathbb{R}_+; L^q(\mathcal{O}; H))} \tag{1.3}$$

with a constant C independent of G. For $q = 2$ this estimate also holds with $p = 2$.

Although U also belongs to $L^p((0, T) \times \Omega; L^q(\mathcal{O}))$ for all $T \in (0, \infty)$, in general it is false that U belongs to $L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}))$ unless one makes the additional assumption that A is invertible [see Theorem 1.2(1) with $\theta = 0$].

The limiting case $p = 2$ is not allowed in Theorem 1.1 (except if $q = 2$); a counterexample is presented in Section 6. This is rather surprising, since $p = 2$ is usually the "easy" case. Theorem 1.1 is new even for $q = 2$ and $p \in (2, \infty)$.

The convolution process U is the mild solution of the abstract stochastic PDE

$$dU(t) + AU(t)\,dt = G(t)\,dW_H(t), \qquad t \geq 0,$$

and therefore Theorem 1.1 can be interpreted as maximal L^p-regularity results for such equations. As is well known [8, 14, 29] (cf. Section 7), stochastic maximal regularity estimates can be combined with fixed point arguments to obtain existence, uniqueness and regularity results for solutions to more general classes of nonlinear stochastic PDEs. This approach has proved very fruitful in the setting of deterministic PDEs, as can be seen from the surveys [16, 32]. In order to keep the present paper at a reasonable length, such applications to stochastic PDEs have been worked out in a separate paper [46]. A generalization of estimate (1.1) to the setting of stochastic integrodifferential equations has been obtained in [17]; our approach seems to be applicable in this context as well.

Let us now briefly indicate how (1.1) and (1.2) follow from Theorem 1.1 and how the corresponding estimates for bounded regular domains may be deduced. First of all, by the Itô isomorphism for $L^q(\mathcal{O})$-valued stochastic integrals (see Section 2.1), the estimate (1.3) can be rewritten as

$$\begin{aligned}(1.4)\qquad &\mathbb{E}\int_0^\infty\biggl(\int_{\mathcal{O}}\biggl(\int_0^t \|[A^{1/2}S(t-s)G(s,\cdot)](x)\|^2\,ds\biggr)^{q/2}dx\biggr)^{p/q}dt\\ &\quad\le C^p\mathbb{E}\|G\|^p_{L^p(\mathbb{R}_+;L^q(\mathcal{O};H))}.\end{aligned}$$

It is well known that the Laplace operator $-\frac{1}{2}\Delta$ admits a bounded H^∞-calculus on $L^q(\mathbb{R}^d)$, and $D((-\Delta)^{1/2})$ equals the Bessel potential space $H^{1,q}(\mathbb{R}^d)$ associated with $L^q(\mathbb{R}^d)$. As a result, (1.4) implies (1.2) without the restriction $q\le p$. By the same token, the Dirichlet Laplacian $A=-\frac{1}{2}\Delta_{\mathrm{Dir}}$ on a bounded regular domain $D\subseteq\mathbb{R}^d$ has a bounded H^∞-calculus on $L^q(D)$, and via complex interpolation (cf. (2.3), [23], Lemma 9.7, and [40], Theorem 4.3.2.2) one has

$$\begin{aligned}(1.5)\qquad D(A^{1/2}) &= [L^q(D),D(A)]_{1/2}=[L^q(D),H^{2,q}(D)\cap H_0^{1,q}(D)]_{1/2}\\ &\subseteq H^{1,q}(D)\end{aligned}$$

with $H_0^{1,q}(D)=\{f\in H^{1,q}(D): f=0 \text{ on } \partial D\}$. Noting that Δ_{Dir} is invertible, (1.3) gives $u\in L^p(\mathbb{R}_+\times\Omega;D(A^{1/2}))$. Hence by (1.5) we obtain the estimate

$$(1.6)\qquad \mathbb{E}\|U\|^p_{L^p(\mathbb{R}_+;H^{1,q}(D))}\le C^p\mathbb{E}\|G\|^p_{L^p(\mathbb{R}_+;L^q(D;H))}.$$

A similar estimate, but only on bounded time intervals, holds for Neumann Laplacian (see the remarks following Theorem 1.1 and Remark 4.5).

The main advantage of our approach is that it uses estimates from the deterministic theory of partial differential equations (e.g., the boundedness of the H^∞-calculus) directly as building blocks in the theory of stochastic partial differential equations. The boundedness of the H^∞-calculus is not a very restrictive assumption; elliptic operators typically satisfy this assumption on L^q-spaces in the range $1<q<\infty$ (see Section 2.3 for a comprehensive list of examples). For second order elliptic operators on bounded regular domains D, (1.6) holds again under mild regularity assumptions (see Example 2.6).

Under the additional assumption that the operator A is invertible, Theorem 1.1 can be strengthened to a maximal space–time L^p-regularity result; by a standard interpolation argument this also gives a sharp maximal inequality.

THEOREM 1.2. *In addition to the assumptions of Theorem 1.1 suppose that* $0 \in \varrho(A)$.

(1) Space–time regularity. *For all* $\theta \in [0, \frac{1}{2})$,

$$\mathbb{E}\|U\|^p_{H^{\theta,p}(\mathbb{R}_+;D(A^{1/2-\theta}))} \leq C^p \mathbb{E}\|G\|^p_{L^p(\mathbb{R}_+;L^q(\mathcal{O};H))}.$$

(2) Maximal estimate.

$$\mathbb{E}\sup_{t\in\mathbb{R}_+}\|U(t)\|^p_{D_A(1/2-1/p,p)} \leq C^p \mathbb{E}\|G\|^p_{L^p(\mathbb{R}_+;L^q(\mathcal{O};H))},$$

where $D_A(\frac{1}{2}-\frac{1}{p}, p) := (L^q(\mathcal{O}), D(A))_{1/2-1/p,p}$ *is the real interpolation space.*

In both cases the constant C *is independent of* G.

As far as we know, Theorem 1.2 is new even for the Laplace operator on $L^2(\mathbb{R}^d)$.

The case $\theta = 0$ of part (1) easily generalizes to the more general estimate

$$\mathbb{E}\|U\|^p_{L^p(\mathbb{R}_+;D(A^{1/2+\delta}))} \leq C^p \mathbb{E}\|G\|^p_{L^p(\mathbb{R}_+;D((A\otimes I_H)^\delta))} \tag{1.7}$$

for any $\delta > 0$. A similar estimate for $\delta < 0$ can be derived by using extrapolation spaces, which is useful when dealing with space–time white noise.

A further advantage of our methods is that, with the aid of some additional tools from functional analysis, Theorems 1.1 and 1.2 and their proofs extend to more general function spaces, such as spaces which are isomorphic to closed subspaces of $L^q(\mathcal{O})$ (e.g., Sobolev and Besov spaces).

1.1. *Notation.* Unless otherwise stated, all vector spaces are real. Arguments involving spectral theory are carried out by passing to complexifications. Throughout the paper, H is a Hilbert space, and $(\mathcal{O}, \Sigma, \mu)$ is a σ-finite measure space. We use the notation $(r_n)_{n\geq 1}$ for a *Rademacher sequence*, which is a sequence of independent random variables which take the values ± 1 with equal probability. We write $a \lesssim_k b$ to express that there exists a constant c, only depending on k, such that $a \leq cb$. We write $a \eqsim_k b$ to express that $a \lesssim_k b$ and $b \lesssim_k a$.

2. Preliminaries.

2.1. *Stochastic integration.* Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space endowed with filtration $\mathscr{F} = (\mathscr{F}_t)_{t\geq 0}$. An $\mathscr{F}$*-cylindrical Brownian motion on* H is a bounded linear operator $\mathcal{W}_H : L^2(\mathbb{R}_+; H) \to L^2(\Omega)$ such that:

(i) for all $t \geq 0$ and $h \in H$ the random variable $W_H(t)h := \mathcal{W}_H(\mathbf{1}_{(0,t]} \otimes h)$ is centred Gaussian and $\mathscr{F}_t$-measurable;

(ii) for all $t_1, t_2 \geq 0$ and $h_1, h_2 \in H$ we have $\mathbb{E}(W_H(t_1)h_1 \cdot W_H(t_2)h_2) = t_1 \wedge t_2[h_1, h_2]$;

(iii) for all $t_2 \geq t_1 \geq 0$ and $h \in H$ the random variable $W_H(t_2)h - W_H(t_1)h$ is independent of $\mathscr{F}_{t_1}$.

It is easy to see that for all $h \in H$ the process $(t, \omega) \mapsto (W_H(t)h)(\omega)$ is an $\mathscr{F}$-Brownian motion (which is standard if $\|h\| = 1$).

For $0 \le a < b < \infty$, $\mathscr{F}_a$-measurable sets $F \subseteq \Omega$, $h \in H$, and $f \in L^q(\mathcal{O})$ the stochastic integral of the indicator process $(t, \omega) \mapsto 1_{(a,b]\times F}(t, \omega) f \otimes h$ with respect to W_H is defined as the $L^q(\mathcal{O})$-valued random variable

$$\int_0^t 1_{(a,b]\times F}(f \otimes h)\, dW_H := \bigl(W_H(t \wedge b)h - W_H(t \wedge a)h\bigr) 1_F f, \qquad t \ge 0.$$

By linearity, this definition extends to *adapted finite rank step processes* $G : \mathbb{R}_+ \times \Omega \to L^p(\mathcal{O}; H)$, which we define as finite linear combinations of adapted indicator processes of the above form. Recall that a process $G : \mathbb{R}_+ \times \Omega \to L^q(\mathcal{O}; H)$ is called *$\mathscr{F}$-adapted* if for every $t \in \mathbb{R}_+$, $\omega \mapsto G(t, \omega)$ is $\mathscr{F}_t$-measurable.

The next result is a special case of [43], Theorem 6.2.

PROPOSITION 2.1. *Let $p \in (1, \infty)$ and $q \in (1, \infty)$ be fixed. For all $\mathscr{F}$-adapted finite rank step processes $G : \mathbb{R}_+ \times \Omega \to L^q(\mathcal{O}; H)$ we have the "Itô isomorphism"*

$$c^p \mathbb{E}\|G\|^p_{L^q(\mathcal{O}; L^2(\mathbb{R}_+; H))} \le \mathbb{E}\left\| \int_0^\infty G\, dW_H \right\|^p_{L^q(\mathcal{O})} \le C^p \mathbb{E}\|G\|^p_{L^q(\mathcal{O}; L^2(\mathbb{R}_+; H))}$$

with constants $0 < c \le C < \infty$ independent of G.

By a standard density argument, these inequalities can be used to extend the stochastic integral to the Banach space $L^p_{\mathscr{F}}(\Omega; L^q(\mathcal{O}; L^2(\mathbb{R}_+; H)))$ of all $\mathscr{F}$-adapted processes $G : \mathbb{R}_+ \times \Omega \to L^q(\mathcal{O}; H)$ which belong to $L^p(\Omega; L^q(\mathcal{O}; L^2(\mathbb{R}_+; H)))$. In the remainder of this paper, all stochastic integrals are understood in the above sense. By Doob's inequality, the inequalities remain true if the middle term is replaced by the corresponding maximal norm. In this form, for $p = q$ they follow directly from the (real-valued) Burkholder–Davis–Gundy inequality.

By Minkowski's inequality, for $q \in [2, \infty)$ one has

$$\mathbb{E}\|G\|_{L^q(\mathcal{O}; L^2(\mathbb{R}_+; H))} \le \mathbb{E}\|G\|_{L^2(\mathbb{R}_+; L^q(\mathcal{O}; H))}.$$

In combination with Proposition 2.1, for $p \in (1, \infty)$ and $q \in [2, \infty)$ this gives the one-sided inequality

$$\mathbb{E}\left\| \int_0^\infty G\, dW_H \right\|^p_{L^q(\mathcal{O})} \le C^p \mathbb{E}\|G\|^p_{L^2(\mathbb{R}_+; L^q(\mathcal{O}; H))}. \tag{2.1}$$

REMARK 2.2. For $q \in [1, \infty)$ the space $L^q(\mathcal{O}; H)$ is canonically isomorphic to the space $\gamma(H, L^q(\mathcal{O}))$ of γ-radonifying operators from H to $L^q(\mathcal{O})$ (see [45] and the references given therein). Using this identification, Proposition 2.1 extends to arbitrary UMD Banach spaces E (see [44], Theorems 5.9 and 5.12); this class of Banach spaces includes the spaces $L^q(\mathcal{O})$ with $q \in (1, \infty)$. For Hilbert spaces

E one has the further identification $\gamma(H, E) = \mathscr{L}_2(H, E)$, the space of Hilbert–Schmidt operators from H to E.

The inequality (2.1) holds for arbitrary Banach spaces E with martingale type 2 (see [8, 9]); this class of Banach spaces includes the spaces $L^q(\mathcal{O})$ with $q \in [2, \infty)$.

2.2. *R-boundedness.* Let E_1 and E_2 be Banach spaces, and let $(r_n)_{n\geq 1}$ be a Rademacher sequence (see Section 1.1). A family $\mathscr{T}$ of bounded linear operators from E_1 to E_2 is called *R-bounded* if there exists a constant $C \geq 0$ such that for all finite sequences $(x_n)_{n=1}^N$ in E_1 and $(T_n)_{n=1}^N$ in $\mathscr{T}$ we have

$$\mathbb{E}\left\|\sum_{n=1}^N r_n T_n x_n\right\|^2 \leq C^2 \mathbb{E}\left\|\sum_{n=1}^N r_n x_n\right\|^2.$$

The least admissible constant C is called the *R-bound* of $\mathscr{T}$, notation $R(\mathscr{T})$. For Hilbert spaces E_1 and E_2, R-boundedness is equivalent to uniform boundedness and $R(\mathscr{T}) = \sup_{t\in\mathscr{T}} \|T\|$. The notion of R-boundedness has played an important role in recent progress in the regularity theory of (deterministic) parabolic evolution equations. For more information on R-boundedness and its applications we refer the reader to [11, 16, 32].

In our applications, E_1 and E_2 will always be L^q-spaces or mixed $L^p(L^q)$-spaces (possibly with values in H). All such spaces are examples of Banach function spaces which are s-concave for some $s < \infty$ [for this purpose we identify H with $\ell^2(I)$ over a suitable index set I]. For these spaces, Rademacher sums can be evaluated, up to a constant, by means of *square functions* (see [34], Theorem 1.d.6)

$$\left(\mathbb{E}\left\|\sum_{n=1}^N r_n x_n\right\|_E^2\right)^{1/2} \eqsim_E \left\|\left(\sum_{n=1}^N |x_n|^2\right)^{1/2}\right\|_E.$$

Below we shall need a continuous version of the right-hand side, for which we need to introduce some notation. Let E be a Banach function space over $(\mathcal{O}, \Sigma, \mu)$. For a Hilbert space $\mathcal{H}$, let $E(\mathcal{H})$ be the space of all strongly μ-measurable functions $G : \mathcal{O} \to \mathcal{H}$ for which $\|G(\cdot)\|_{\mathcal{H}}$ belongs to E. Typically we shall take $\mathcal{H} = L^2(\mathbb{R}_+, \nu)$ with ν a σ-finite Borel measure on $\mathbb{R}_+$.

For $E_1 = E_2 = L^q(\mathcal{O})$ the next multiplier result is due to [47]; the version below is included as a special case in a more general operator-theoretic formulation of this result, valid for arbitrary Banach spaces E_1 and E_2, in [25] (a proof is reproduced in [41]).

PROPOSITION 2.3. *Let E_1 and E_2 be Banach function spaces with finite cotype, and let ν be a σ-finite Borel measure on $\mathbb{R}_+$. Let $M : \mathbb{R}_+ \to \mathscr{L}(E_1, E_2)$ be a function with the following properties:*

(1) *for all $x \in E_1$ the function $t \mapsto M(t)x$ is strongly ν-measurable in E_2;*
(2) *the range $\mathscr{M} := \{M(t) : t \in \mathbb{R}_+\}$ is R-bounded in $\mathscr{L}(E_1, E_2)$.*

Then for all $G:\mathbb{R}_+ \to E_1$ *which satisfy* $G \in E_1(L^2(\mathbb{R}_+, \nu))$ *the function* MG: $\mathbb{R}_+ \to E_2$ *satisfies* $MG \in E_2(L^2(\mathbb{R}_+, \nu))$ *and*

$$\|MG\|_{E_2(L^2(\mathbb{R}_+,\nu))} \le R(\mathscr{M})\|G\|_{E_1(L^2(\mathbb{R}_+,\nu))}.$$

Conversely, this multiplier property characterizes R-bounded families. This fact will not be needed here.

2.3. *Operators with a bounded* H^∞*-calculus.* The H^∞-calculus was originally developed by McIntosh and his collaborators [1, 6, 12, 36] in a line of research which eventually culminated in the solution of the Kato square root problem [5]. Meanwhile, this technique has found widespread applications in harmonic analysis and PDEs. For an in-depth treatment of the theory we refer to [21, 32, 48].

Let $-A$ be the generator of a bounded strongly continuous analytic semigroup of operators on a Banach space E. As is well known (see [3], Proposition I.1.4.1), the spectrum of A is contained in the closure of a sector $\Sigma_{\sigma_0} := \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \sigma_0\}$ for some $\sigma_0 \in (0, \frac{1}{2}\pi)$, and for all $\sigma \in (\sigma_0, \pi)$ one has

$$\sup_{z\in\mathbb{C}\setminus\Sigma_\sigma} \|z(z-A)^{-1}\| < \infty.$$

Let $H^\infty(\Sigma_\sigma)$ denote the Banach space of all bounded analytic functions $\varphi : \Sigma_\sigma \to \mathbb{C}$ endowed with the supremum norm, and let $H_0^\infty(\Sigma_\sigma)$ be its linear subspace consisting of all functions satisfying an estimate

$$|\varphi(z)| \le \frac{C|z|^\varepsilon}{(1+|z|^2)^\varepsilon}$$

for some $\varepsilon > 0$. For $\varphi \in H_0^\infty(\Sigma_\sigma)$ and $\sigma' \in (\sigma_0, \sigma)$ the Bochner integral

$$\varphi(A) = \frac{1}{2\pi i}\int_{\partial\Sigma_{\sigma'}} \varphi(z)(z-A)^{-1}\,dz$$

converges absolutely and is independent of σ'. We say that A has a *bounded* $H^\infty(\Sigma_\sigma)$*-calculus* if there is a constant $C \ge 0$ such that

$$\|\varphi(A)\| \le C\|\varphi\|_\infty, \qquad \varphi \in H_0^\infty(\Sigma_\sigma). \tag{2.2}$$

The infimum of all σ such that A admits a bounded $H^\infty(\Sigma_\sigma)$-calculus is called the *angle* of the calculus.

In order to avoid unnecessary technicalities, from now on we shall always assume that A is injective and has dense range. This hardly entails any loss of generality; as for generators $-A$ of bounded analytic semigroups on reflexive Banach spaces E one has a direct sum decomposition $E = \mathsf{N}(A) \oplus \overline{\mathsf{R}(A)}$ into kernel and closure of the range of A (see, e.g., [32]). In particular, such an operator is the direct sum of a zero operator and an injective sectorial operator with dense range (see Remark 4.5 for further discussion on this issue).

If A has a bounded $H^\infty(\Sigma_\sigma)$-calculus, the mapping $\varphi \mapsto \varphi(A)$ has a unique extension to a bounded homomorphism from $H^\infty(\Sigma_\sigma)$ to $\mathscr{L}(E)$ which satisfies (2.2) with the same constant C.

Even on Hilbert spaces E there exist generators $-A$ of bounded strongly continuous analytic semigroups for which A does not admit a bounded H^∞-calculus (see [32], Example 10.17). Examples of operators which do admit such a calculus are collected below.

We shall need a generalization, taken from [25] (see also [33]), of McIntosh's square function characterization for the boundedness of H^∞-calculi in Hilbert spaces [36] (see also [12]). We use the notation of Proposition 2.3 with $d\nu = \frac{dt}{t}$.

PROPOSITION 2.4. *Let $E = L^q(\mathcal{O})$ with $q \in (1, \infty)$. Assume that A has a bounded $H^\infty(\Sigma_\sigma)$-calculus on E for some $\sigma \in (0, \pi/2)$. For each $\varphi \in H_0^\infty(\Sigma_\sigma)$ there exists a constant $C \geq 0$ such that*

$$\|t \mapsto \varphi(tA)x\|_{E(L^2(\mathbb{R}_+, dt/t))} \leq C\|x\|_E, \qquad x \in E,$$
$$\|t \mapsto \varphi(tA^*)x^*\|_{E^*(L^2(\mathbb{R}_+, dt/t))} \leq C\|x^*\|_{E^*}, \qquad x^* \in E^*.$$

Here, as before,

$$\|t \mapsto \varphi(tA)x\|_{E(L^2(\mathbb{R}_+, dt/t))} = \left\| \left(\int_{\mathbb{R}_+} |\varphi(tA)x|^2 \frac{dt}{t} \right)^{1/2} \right\|_E$$

and similarly for the expression involving A^*. Proposition 2.4 actually can be extended to arbitrary angles $\sigma \in (0, \pi)$, but we will not need this fact.

In the converse direction, if $-A$ is a generator of a bounded strongly continuous analytic semigroup on $E = L^q(\mathcal{O})$ and the above inequalities hold for some nonzero $\varphi \in H_0^\infty(\Sigma_\sigma)$, then A has a bounded $H^\infty(\Sigma_{\sigma'})$-calculus for all $\sigma' > \sigma$ [25].

We will also need the fact (combine [40], Theorem 1.15.3, and [21], Proposition 3.1.9) that if A has a bounded $H^\infty(\Sigma_\sigma)$-calculus for some $\sigma \in (0, \frac{1}{2}\pi)$, then A has bounded imaginary powers and $\sup_{s \in [-1,1]} \|A^{is}\| < \infty$. In particular this implies, for all $\theta \in (0, 1)$,

$$[E, D(A)]_\theta = D(A^\theta) \qquad \text{with equivalent norms,} \tag{2.3}$$

where $[E, D(A)]_\theta$ is the complex interpolation space of exponent θ.

2.3.1. *Examples of operators with a bounded H^∞-calculus.* Many common differential operators are known to admit a bounded H^∞-calculus (see, e.g., the lecture notes [16, 32] and the survey article [48]). In this paragraph we collect some examples illustrating this point. We always take $q \in (1, \infty)$.

EXAMPLE 2.5. The most basic example is the Laplace operator $A = -\frac{1}{2}\Delta$ on $L^q(\mathbb{R}^d)$, which has a bounded H^∞-calculus of zero angle; this follows from an application of the Mihlin multiplier theorem (see [32], Example 10.2b). For this operator one has $D(A^{1/2}) = H^{1,q}(\mathbb{R}^d)$.

Also the Laplace operator with Dirichlet boundary conditions on $L^q(D)$, where $D \subseteq \mathbb{R}^d$ is a bounded domain with C^2-boundary, has a bounded H^∞-calculus of zero angle (see [15]). In this case one has $D(A^{1/2}) \subseteq H^{1,q}(D)$ [with equality if we replace $H^{1,q}(D)$ by $H_0^{1,q}(D) = \{f \in H^{1,q}(D) : f = 0 \text{ on } \partial D\}$ (see [2], Remark 7.3, combined with [15], Theorem 2.3, for C^2-domains)].

Similar results hold under different boundary conditions.

EXAMPLE 2.6. Let $D \subseteq \mathbb{R}^d$ be a bounded domain with C^2-boundary. Consider the closed and densely defined operator A in $L^q(D)$ defined by

$$-Af(x) = \sum_{i,j=1}^{d} a_{ij}(x)\,\partial_{x_i x_j} f(x) + \sum_{j=1}^{d} b_j(x)\,\partial_{x_j} f(x) + c(x) f(x)$$

on the domain $D(A) = H^{2,q}(D) \cap H_0^{1,q}(D)$. We assume that the coefficient $a_{ij} = a_{ji}$ and b_j, c_j are bounded and measurable and that $-A$ is uniformly elliptic, that is, there is a constant $\nu > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \nu|\xi|^2, \qquad x \in D,\ \xi \in \mathbb{R}^d.$$

It is shown in [4, 15] that if the coefficients a_{ij} are Hölder conditions on $\overline{D}$, then $w + A$ admits a bounded H^∞-calculus of angle less than $\pi/2$ for all $w \in \mathbb{R}$ large enough, and one has $D((w + A)^{1/2}) \subseteq H^{1,q}(D)$. An analogous result holds for $D = \mathbb{R}^d$; in this case one can weaken the Hölder continuity assumption on a_{ij} to a VMO assumption (see [18]).

Similar results hold for higher-order parameter-elliptic systems on smooth domains satisfying the Lopatinksii–Shapiro conditions (see [15], Theorem 2.3, and [23]).

EXAMPLE 2.7. Let $D \subseteq \mathbb{R}^d$ be a bounded domain with C^2-boundary. An important operator arising in the context of the Navier–Stokes equations is the Stokes operator $A = -P\Delta$, where P is the Helmholtz projection of $[L^q(D)]^d$ onto the Helmholtz space $L^q_\sigma(D)$, with domain $D(A) = [H^{2,q}(D)]^d \cap [H_0^{1,q}(D)]^d \cap L^q_\sigma(D)$. The operator A has a bounded H^∞-calculus of angle less than $\pi/2$ on $L^q_\sigma(D)$ (see [23], Theorem 9.17, and the references therein). For $w \in \mathbb{R}$ large enough $D((w + A)^{1/2}) = [H_0^{1,q}(D)]^d \cap L^q_\sigma(D)$.

EXAMPLE 2.8. Let $-A$ be an injective operator with dense range which generates a positive contraction semigroup $S = (S(t))_{t\geq 0}$ on $L^q(\mathcal{O})$. If S extends to a bounded analytic semigroup on $L^q(\mathcal{O})$, then A has a bounded H^∞-calculus of angle less than $\pi/2$ (see [24], Corollary 5.2).

Some of the above results have extensions to domains D with $C^{1,1}$-boundary. Further important examples of operators with a bounded H^∞-calculus can be obtained by considering kernels bounds (see [7] and the references therein). Finally, we note that also operators of Ornstein–Uhlenbeck type and operators of Schrödinger type $A = -\Delta + V$ have a bounded H^∞-calculus (see [23]).

3. *R*-boundedness of stochastic convolutions. In this section we will prove the R-boundedness of a certain family of stochastic convolution operators. This result plays a key role in the proofs of Theorems 1.1 and 1.2.

Fix $p \in [2,\infty)$ and $q \in [2,\infty)$ for the moment, and let $\mathcal{K}$ be the set of all absolutely continuous functions $k:\mathbb{R}_+ \to \mathbb{R}$ such that $\lim_{t\to\infty} k(t) = 0$ and

$$\int_0^\infty \sqrt{t}|k'(t)|\,dt \leq 1.$$

For $k \in \mathcal{K}$ and $\mathscr{F}$-adapted finite rank step processes $G:\mathbb{R}_+ \times \Omega \to L^q(\mathcal{O};H)$ we define the process $I(k)G:\mathbb{R}_+ \times \Omega \to L^q(\mathcal{O})$ by

$$I(k)G(t) := \int_0^t k(t-s)G(s)\,dW_H(s), \qquad t \geq 0. \tag{3.1}$$

Since G is an $\mathscr{F}$-adapted finite rank step process, the Itô isometry for scalar-valued processes shows that these stochastic integrals are well defined for all $t \geq 0$. From (2.1) it follows that $I(k)$ extends to a bounded operator from $L^p_{\mathscr{F}}(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O};H))$ into $L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}))$, and that the family

$$\mathcal{I} := \{I(k) : k \in \mathcal{K}\}$$

is uniformly bounded. Indeed, for any $k \in \mathcal{K}$ we can write

$$k(s) = -\int_s^\infty k'(r)\,dr, s \in \mathbb{R}_+.$$

Now, since G is an $\mathscr{F}$-adapted finite rank step process, the stochastic Fubini theorem may be applied (see [14, 42]), and for all $t > 0$ we have

$$\begin{aligned} I(k)G(t) &= -\int_0^\infty k'(r)\int_0^\infty \mathbf{1}_{0<s<t}\mathbf{1}_{t-s<r}G(s)\,dW_H(s)\,dr \\ &= -\int_0^\infty \sqrt{r}k'(r)J(r)G(t)\,dr. \end{aligned} \tag{3.2}$$

Here for $r > 0$ the process $J(r)G:\mathbb{R}_+ \times \Omega \to L^q(\mathcal{O})$ is defined by

$$J(r)G(t) := \frac{1}{\sqrt{r}}\int_{(t-r)\vee 0}^t G\,dW_H.$$

By (2.1) the operators $J(r)$ are bounded from $L^p_{\mathscr{F}}(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}; H))$ to $L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}))$, and the family

$$\mathcal{J} := \{J(r) : r > 0\}$$

is uniformly bounded. Now the uniform boundedness of $\mathcal{I}$ follows from (3.2).

THEOREM 3.1. *For all $p \in (2, \infty)$ and $q \in [2, \infty)$ the family $\mathcal{I}$ is R-bounded from $L^p_{\mathscr{F}}(\mathbb{R}_+ \times \Omega, L^q(\mathcal{O}; H))$ to $L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}))$. The same result holds when $p = q = 2$.*

The case $p = q = 2$ follows from the general fact that a family of Hilbert spaces is R-bounded if and only if it is uniformly bounded. In what follows we shall concentrate ourselves on the cases $p \in (2, \infty)$ and $q \in [2, \infty)$.

By the same reasoning as before the problem of R-boundedness of $\mathcal{I}$ can be reduced to that of the family $\mathcal{J}$.

PROPOSITION 3.2. *If $\mathcal{J}$ is R-bounded, then $\mathcal{I}$ is R-bounded and $R(\mathcal{I}) \leq R(\mathcal{J})$.*

PROOF. This follows from (3.2), convexity and density (see [32], Corollary 2.14). □

The remainder of this section is devoted to the proof that $\mathcal{J}$ is R-bounded from $L^p_{\mathscr{F}}(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}; H))$ to $L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}))$ for the indicated ranges of p and q.

We begin with a duality lemma which is a straightforward generalization from the scalar case presented in [35], Proposition 8.12. Here the absolute values are to be taken in the pointwise sense.

LEMMA 3.3. *Let $(T(\delta))_{\delta>0}$ be a strongly continuous one-parameter family of positive linear operators on $L^r(\mathcal{N}; L^s(\mathcal{O}))$, where $r, s \in [1, \infty]$ and $(\mathcal{N}, \nu)$ is another σ-finite measure space, and suppose the maximal function*

$$T_\star(g) := \sup_{\delta>0} |T(\delta)g| \tag{3.3}$$

is measurable and $L^r(\mathcal{N}; L^s(\mathcal{O}))$-bounded by some constant $C \geq 0$. Let $\frac{1}{r} + \frac{1}{r'} = 1$, $\frac{1}{s} + \frac{1}{s'} = 1$. Then, for all $N \geq 1$, $f_1, \ldots, f_N \in L^{r'}(\mathcal{N}; L^{s'}(\mathcal{O}))$ and $\delta_1, \ldots, \delta_N > 0$,

$$\left\| \sum_{n=1}^N T^*(\delta_n)|f_n| \right\|_{L^{r'}(\mathcal{N}; L^{s'}(\mathcal{O}))} \leq C \left\| \sum_{n=1}^N |f_n| \right\|_{L^{r'}(\mathcal{N}; L^{s'}(\mathcal{O}))}.$$

For functions $f \in L^r(\mathbb{R}_+)$ we define the *one-sided Hardy–Littlewood maximal function* $M(f): \mathbb{R}_+ \to [0, \infty]$ by

$$M(f)(t) := \sup_{\delta>0} \frac{1}{\delta} \int_t^{t+\delta} |f(\tau)|\, d\tau.$$

Similarly, for functions $f \in L^r(\mathbb{R}_+; L^s(\mathcal{O}))$ we define

$$\widetilde{M}(f)(t)(a) := \sup_{\delta>0} \frac{1}{\delta} \int_t^{t+\delta} |f(\tau)(a)|\, d\tau, \qquad a \in \mathcal{O}.$$

PROPOSITION 3.4 (Fefferman–Stein). *For all $r \in (1, \infty)$ and $s \in (1, \infty]$ the one-sided Hardy–Littlewood maximal function $\widetilde{M}$ is bounded on $L^r(\mathbb{R}_+; L^s(\mathcal{O}))$.*

PROOF. This follows from the usual (discrete) formulation of the Fefferman–Stein inequality (see [39], Section II.1) by approximation (the cases $r = s$ and $s = \infty$ are easy consequences of the Hardy–Littlewood maximal inequality). $\square$

PROOF OF THEOREM 3.1. It remains to prove the R-boundedness of $\mathcal{J}$.

Let $N \geq 1$, $\delta_1, \ldots, \delta_N > 0$ and $G_1, \ldots, G_N \in L^p_{\mathscr{F}}(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}; H))$ be arbitrary and fixed. Note that the functions $f_n := \|G_n\|_H^2$ belong to $L^{p/2}(\mathbb{R}_+ \times \Omega; L^{q/2}(\mathcal{O}))$.

Let $(r_n)_{n=1}^N$ be a Rademacher sequence on a probability space $(\Omega_r, \mathscr{F}_r, \mathbb{P}_r)$. Using the inequalities of Proposition 2.1 applied pointwise with respect to $(\omega, t) \in \Omega_r \times \mathbb{R}_+$ [in (i)] and the Kahane–Khintchine inequalities [in (ii) and (iii)], we may estimate as follows (with implicit constants independent of the choice of N, δ_n, and G_n):

$$\begin{aligned}
&\mathbb{E}_r \left\| \sum_{n=1}^N r_n J(\delta_n) G_n \right\|^p_{L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}))} \\
&\quad = \mathbb{E}_r \left\| t \mapsto \int_0^\infty \sum_{n=1}^N \frac{r_n}{\sqrt{\delta_n}} \mathbf{1}_{((t-\delta_n)\vee 0, t)} G_n \, dW_H \right\|^p_{L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}))} \\
&\quad \overset{\text{(i)}}{\eqsim}_{p,q} \mathbb{E}_r \left\| t \mapsto \left(\int_0^\infty \left\| \sum_{n=1}^N \frac{r_n}{\sqrt{\delta_n}} \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. \times \mathbf{1}_{((t-\delta_n)\vee 0, t)}(s) G_n(s) \right\|_H^2 ds \right)^{1/2} \right\|^p_{L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}))} \\
&\quad = \mathbb{E} \int_0^\infty \mathbb{E}_r \left\| \sum_{n=1}^N \frac{r_n}{\sqrt{\delta_n}} \mathbf{1}_{((t-\delta_n)\vee 0, t)} G_n \right\|^p_{L^q(\mathcal{O}; L^2(\mathbb{R}_+; H))} dt
\end{aligned}$$

$$
\begin{aligned}
&\overset{\text{(ii)}}{\eqsim}_{p,q} \mathbb{E}\int_0^\infty \biggl(\mathbb{E}_r \biggl\|\sum_{n=1}^N \frac{r_n}{\sqrt{\delta_n}}\mathbf{1}_{((t-\delta_n)\vee 0,t)} G_n\biggr\|_{L^q(\mathcal{O};L^2(\mathbb{R}_+;H))}^q\biggr)^{p/q} dt \\
&= \mathbb{E}\int_0^\infty \biggl(\int_{\mathcal{O}} \mathbb{E}_r \biggl\|\sum_{n=1}^N \frac{r_n}{\sqrt{\delta_n}}\mathbf{1}_{((t-\delta_n)\vee 0,t)} G_n\biggr\|_{L^2(\mathbb{R}_+;H)}^q d\mu\biggr)^{p/q} dt \\
&\overset{\text{(iii)}}{\eqsim}_{q} \mathbb{E}\int_0^\infty \biggl(\int_{\mathcal{O}} \biggl(\mathbb{E}_r \biggl\|\sum_{n=1}^N \frac{r_n}{\sqrt{\delta_n}}\mathbf{1}_{((t-\delta_n)\vee 0,t)} G_n\biggr\|_{L^2(\mathbb{R}_+;H)}^2\biggr)^{q/2} d\mu\biggr)^{p/q} dt \\
&= \mathbb{E}\int_0^\infty \biggl(\int_{\mathcal{O}} \biggl(\int_0^\infty \mathbb{E}_r \biggl\|\sum_{n=1}^N \frac{r_n}{\sqrt{\delta_n}}\mathbf{1}_{((t-\delta_n)\vee 0,t)}(s) G_n(s)\biggr\|_{H}^2 ds\biggr)^{q/2} d\mu\biggr)^{p/q} dt \\
&= \mathbb{E}\int_0^\infty \biggl(\int_{\mathcal{O}} \biggl(\int_0^\infty \sum_{n=1}^N \frac{1}{\delta_n}\mathbf{1}_{((t-\delta_n)\vee 0,t)}(s) f_n(s)\, ds\biggr)^{q/2} d\mu\biggr)^{p/q} dt \\
&= \mathbb{E}\biggl\|\sum_{n=1}^N T^*(\delta_n) f_n\biggr\|_{L^{p/2}(\mathbb{R}_+;L^{q/2}(\mathcal{O}))}^{p/2},
\end{aligned}
$$

where the positive linear operators $T^*(\delta)$ on $L^{p/2}(\mathbb{R}_+; L^{q/2}(\mathcal{O}))$ are defined by

$$
T^*(\delta)\phi(t) := \frac{1}{\delta}\int_{(t-\delta)\vee 0}^t \phi(s)\, ds, \qquad \phi \in L^{p/2}(\mathbb{R}_+; L^{q/2}(\mathcal{O})).
$$

Let $\frac{2}{p} + \frac{1}{r} = 1$ and $\frac{2}{q} + \frac{1}{s} = 1$. Then $T^*(\delta)$ is the adjoint of the operator $T(\delta)$ on $L^r(\mathbb{R}_+; L^s(\mathcal{O}))$ given by

$$
T(\delta)\psi(t) = \frac{1}{\delta}\int_t^{t+\delta} \psi(s)\, ds, \qquad \psi \in L^r(\mathbb{R}_+; L^s(\mathcal{O})).
$$

Since $\sup_{\delta>0}|T(\delta)\psi| \le \widetilde{M}(\psi)$ and the latter is bounded on $L^r(\mathbb{R}_+; L^s(\mathcal{O}))$ by Proposition 3.4, by Fubini's theorem we find that $T_\star$ is bounded on $L^r(\mathbb{R}_+ \times \Omega; L^s(\mathcal{O}))$. Hence we may apply Lemma 3.3 to conclude that

$$
\begin{aligned}
\mathbb{E}\biggl\|\sum_{n=1}^N T^*(\delta_n) f_n\biggr\|_{L^{p/2}(\mathbb{R}_+;L^{q/2}(\mathcal{O}))}^{p/2} &\lesssim_{p,q} \mathbb{E}\biggl\|\sum_{n=1}^N f_n\biggr\|_{L^{p/2}(\mathbb{R}_+;L^{q/2}(\mathcal{O}))}^{p/2} \\
&\eqsim_{p,q} \mathbb{E}_r\biggl\|\sum_{n=1}^N r_n G_n\biggr\|_{L^p(\mathbb{R}_+\times\Omega;L^q(\mathcal{O};H)))}^{p},
\end{aligned}
$$

where the last step follows by reversing the computation above. $\square$

REMARK 3.5. The above proof uses the right-hand side inequality in Proposition 2.1 in an essential way; it seems that the simpler inequality (2.1) is insufficient for this purpose.

4. Proof of Theorem 1.1. We start with a Poisson representation formula.

LEMMA 4.1. *Let $\alpha \in (0, \pi/2)$ and $\alpha' \in (\alpha, \pi]$ be given, let E be a Banach space and let $f : \Sigma_{\alpha'} \to E$ be a bounded analytic function. Then for all $s > 0$ we have*

$$f(s) = \sum_{j \in \{-1,1\}} \frac{j}{2\alpha} \int_0^\infty k_\alpha(u, s) f(ue^{ij\alpha})\, du,$$

where $k_\alpha : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is given by

$$k_\alpha(u, t) = \frac{(t/u)^{\pi/(2\alpha)}}{(t/u)^{\pi/\alpha} + 1} \frac{1}{u}. \tag{4.1}$$

PROOF. If $g : \Sigma_{1/2\pi+\varepsilon} \to E$ is analytic and bounded for some $\varepsilon > 0$, then

$$g(t) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{t}{t^2 + v^2} g(iv)\, dv$$

by the Poisson formula on the half-space (see [22], Chapter 8). For small $\varepsilon > 0$ let $\phi : \Sigma_{(1/2)\pi+\varepsilon} \to \Sigma_{\alpha'}$ be defined by $\phi(z) := z^{2\alpha/\pi}$. Then ϕ is analytic, and taking $g = f \circ \phi$ gives

$$f(t^{2\alpha/\pi}) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{t}{t^2 + v^2} f\big(|v|^{2\alpha/\pi} e^{i\, \mathrm{sign}(v)\alpha}\big)\, dv.$$

The required result is obtained by taking $s = t^{2\alpha/\pi}$ and $u = |v|^{2\alpha/\pi}$. $\square$

The next lemma isolates an elementary property of the functions k_α.

LEMMA 4.2. *For $\alpha \in (0, \pi)$ and $\theta \in [0, 1]$ put*

$$k_{\alpha,\theta}(u, t) := \sqrt{u}(u/t)^\theta k_\alpha(u, t),$$

where k_α is given by (4.1). Then

$$\sup_{u>0} \int_0^\infty \sqrt{t} \left| \frac{\partial k_{\alpha,\theta}}{\partial t}(u, t) \right| dt < \infty.$$

PROOF. This is an easy consequence of the identity

$$\frac{\partial k_{\alpha,\theta}}{\partial t}(u, t) = u^{-3/2} h'(t/u),$$

where $h(x) = x^{\pi/(2\alpha)-\theta}/(x^{\pi/\alpha} + 1)$. $\square$

Note that by Lemma 4.2, small enough multiples of $k_{\alpha,\theta}(u, \cdot)$ belong to the set $\mathcal{K}$ defined in Section 3.

In the proof of Theorem 1.2 we shall need a small generalization of Theorem 1.1, stated next as Theorem 4.3. Theorem 1.1 corresponds to the special case $\theta = 0$.

It will be useful to introduce the notation

$$F \diamond G(t) := \int_0^t F(t-s)G(s)\,dW_H(s), \qquad t \geq 0,$$

whenever $F : \mathbb{R}_+ \to \mathscr{L}(L^q(\mathcal{O}))$ is a function for which these stochastic integrals are well-defined in $L^q(\mathcal{O})$. In order to see that the integrand is well defined as an adapted $L^q(\mathcal{O}; H)$-valued process we note that every bounded operator T on $L^q(\mathcal{O})$ extends to a bounded operator on $L^q(\mathcal{O}; H)$ of the same norm (see [39], Section I.8.24); on the dense subspace $L^q(\mathcal{O}) \otimes H$ this extension is given by $T(f \otimes h) = Tf \otimes h$.

THEOREM 4.3. *Let $q \in [2, \infty)$ and $\sigma \in (0, \pi/2)$, and suppose that A has a bounded $H^\infty(\Sigma_\sigma)$-calculus on $L^q(\mathcal{O})$. Let S denote the bounded analytic semigroup generated by $-A$. Set*

$$S_\theta(t) := \frac{t^{-\theta}}{\Gamma(1-\theta)} S(t).$$

For all $p \in (2, \infty)$ and $\theta \in [0, \frac{1}{2})$ there exists a constant $C \geq 0$ such that for all $G \in L^p_{\mathscr{F}}(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}; H))$ we have $S_\theta \diamond G(t) \in D(A^{1/2-\theta})$ almost surely for almost all $t \geq 0$ and

$$\|A^{1/2-\theta} S_\theta \diamond G\|_{L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}))} \leq C \|G\|_{L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}; H))}.$$

This estimate also holds when $p = q = 2$.

PROOF. By a density argument it suffices to consider $\mathscr{F}$-adapted finite step processes $G : \mathbb{R}_+ \times \Omega \to D(A_H)$, where $A_H = A \otimes I_H$ is the generator of the bounded analytic semigroup $S(t)$ viewed as acting on $L^q(\mathcal{O}; H)$. For such G, the process $A_H^{1/2} G$ takes values in $L^q(\mathcal{O}; H)$. By Hölder's inequality and (2.1), $S_\theta \diamond G(t)$ and $A^{1/2-\theta} S_\theta \diamond G(t)$ are well defined in $L^q(\mathcal{O})$ for each $t \in \mathbb{R}_+$, and both processes are jointly measurable on $\mathbb{R}_+ \times \Omega$ (see [45], Proposition A.1).

The idea of the proof is to reduce the estimation of $A^{1/2-\theta} S_\theta \diamond G$ to an estimation of $k_{\alpha,\theta}(u, \cdot) \diamond (V(u)G)$, where $k_{\alpha,\theta}$ are the scalar kernels introduced in Section 3 and $V(u)$ is a suitable operator depending on u and A. The latter is then estimated using the H^∞-calculus of A.

Fix $\theta \in [0, \frac{1}{2})$. We proceed in two steps.

Step 1. First we shall rewrite $A^{1/2-\theta} S_\theta \diamond G$ using Lemma 4.1. Fix $0 < \alpha < \alpha' < \frac{1}{2}\pi - \sigma$. Since $z \mapsto S(z)$ is analytic and bounded on $\Sigma_{\alpha'}$, it follows from Lemma 4.1 that for all $x \in D(A^{1/2})$,

$$\frac{1}{\Gamma(1-\theta)}(t-s)^{-\theta} A^{1/2-\theta} S(t-s)x = \int_0^\infty k_{\alpha,\theta}(u, t-s) V(u)x \,\frac{du}{u}, \tag{4.2}$$

where

$$V(u) := \frac{1}{\Gamma(1-\theta)} \sum_{j\in\{-1,1\}} \frac{j}{2\alpha} (\varphi_j(uA))^2$$

and $\varphi_j \in H_0^\infty(\Sigma_{(1/2)\pi-\alpha'})$ is given by $\varphi_j(u) = u^{1/4-(1/2)\theta}\exp(-\frac{1}{2}ue^{ij\alpha})$. We write $I_{\alpha,\theta} = I(k_{\alpha,\theta})$ for the operator as defined by (3.1) with $k = k_{\alpha,\theta}$ as in Lemma 4.2. By (4.2) and the stochastic Fubini theorem we obtain, for all $t \geq 0$,

$$\begin{aligned} A^{1/2-\theta} S_\theta \diamond G(t) &= \int_0^t \int_0^\infty V(u) k_{\alpha,\theta}(u, t-s) G(s) \frac{du}{u} \, dW_H(s) \\ &= \int_0^\infty V(u) I_{\alpha,\theta}(u) G(t) \frac{du}{u}. \end{aligned}$$

Step 2. Next we prove the estimate. Let $E_1 = L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}; H))$ and $E_2 = L^p(\mathbb{R}_+ \times \Omega; L^q(\mathcal{O}))$. The space $L^q(\mathcal{O})$ is reflexive and therefore $E_2^* = L^{p'}(\mathbb{R}_+ \times \Omega; L^{q'}(\mathcal{O}))$ isometrically. For all $\zeta^* \in L^{p'}(\mathbb{R}_+ \times \Omega; L^{q'}(\mathcal{O}))$ with $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$,

$$\begin{aligned} &\langle A^{1/2-\theta} S_\theta \diamond G, \zeta^* \rangle_{E_2} \\ &\quad = \frac{1}{\Gamma(1-\theta)} \\ &\qquad \times \sum_{j\in\{-1,1\}} \frac{j}{2\alpha} \mathbb{E} \int_0^\infty \Big\langle \int_0^\infty (\varphi_j(uA))^2 I_{\alpha,\theta}(u) G(t) \frac{du}{u}, \zeta^*(t) \Big\rangle_{L^q(\mathcal{O})} dt \\ &\quad = \frac{1}{\Gamma(1-\theta)} \sum_{j\in\{-1,1\}} \frac{j}{2\alpha} \int_0^\infty \langle \varphi_j(uA) I_{\alpha,\theta}(u) G, \varphi_j(uA^*)\zeta^* \rangle_{E_2} \frac{du}{u} \\ &\quad \overset{(*)}{=} \frac{1}{\Gamma(1-\theta)} \sum_{j\in\{-1,1\}} \frac{j}{2\alpha} \langle \varphi_j(uA) I_{\alpha,\theta}(u) G, \varphi_j(uA^*)\zeta^* \rangle_{E_2(L^2(\mathbb{R}_+, du/u))}, \end{aligned}$$

where $\langle \cdot, \cdot \rangle_F$ denotes the duality pairing between a Banach space F and its dual F^*; the identity $(*)$ follows by writing out the duality between the Banach function spaces E_2 and E_2^* as an integral over $\mathbb{R}_+ \times \Omega \times \mathcal{O}$ and then using the Fubini theorem. It follows that

$$\begin{aligned} |\langle A^{1/2-\theta} S_\theta \diamond G, \zeta^* \rangle| \leq \frac{1}{\Gamma(1-\theta)} \sum_{j\in\{-1,1\}} \frac{1}{2\alpha} &\|\varphi_j(uA) I_{\alpha,\theta}(u) G\|_{E_2(L^2(\mathbb{R}_+, du/u))} \\ &\times \|\varphi_j(uA^*)\zeta^*\|_{E_2^*(L^2(\mathbb{R}_+, du/u))}. \end{aligned}$$

By Proposition 2.4 (applied "pointwise" in $\mathbb{R}_+ \times \Omega$),

$$\|\varphi_j(uA^*)\zeta^*\|_{E_2^*(L^2(\mathbb{R}_+, du/u))} \leq C_1 \|\zeta^*\|_{E_2^*}.$$

Since $\varphi_j(uA)I_{\alpha,\theta}(u)G = I_{\alpha,\theta}(u)\varphi_j(uA_H)G$, from Proposition 2.3 and another pointwise application of Proposition 2.4 (this time for $A_H = A \otimes I_H$, noting that A_H satisfies the assumptions of the proposition if A does) we obtain

$$\begin{aligned}
&\|\varphi_j(uA)I_{\alpha,\theta}(u)G\|_{E_2(L^2(\mathbb{R}_+,du/u))}\\
&\qquad\le R\big(I_{\alpha,\theta}(u):u\in\mathbb{R}\big)\|\varphi_j(uA_H)G\|_{E_1(L^2(\mathbb{R}_+,du/u))}\\
&\qquad\le R\big(I_{\alpha,\theta}(u):u\in\mathbb{R}\big)C_2\|G\|_{E_1}.
\end{aligned}$$

By Lemma 4.2 the R-bound can be estimated by

$$R\big(I_{\alpha,\theta}(u):u\in\mathbb{R}\big)\le C_3R(\mathcal{I}),$$

and the latter is finite by Theorem 3.1. We conclude that

$$|\langle A^{1/2-\theta}S_\theta\diamond G,\zeta^*\rangle|\le\frac{1}{\alpha\Gamma(1-\theta)}C_1C_2C_3R(\mathcal{I})\|G\|_{E_1}\|\zeta^*\|_{E_2^*}.$$

Taking the supremum over all $\|\zeta^*\|_{E_2^*}\le 1$ it follows that

$$\|A^{1/2-\theta}S_\theta\diamond G\|_{E_2}\le\frac{1}{\alpha\Gamma(1-\theta)}C_1C_2C_3R(\mathcal{I})\|G\|_{E_1}.$$ □

REMARK 4.4. As in [27], Remark 2.1, one shows that the inequality in Theorem 4.3 fails for $p=q\in[1,2)$. In Section 6 we prove that Theorem 4.3 also fails for $p=2$ and $q\in(2,\infty)$.

REMARK 4.5. Stochastic maximal L^p-regularity on bounded time intervals may be deduced from Theorem 4.3 by considering processes G with support in $(0,T)\times\Omega$. In this situation it suffices to know that $w+A$ has a bounded H^∞-calculus of angle less than $\pi/2$ for some $w\in\mathbb{R}$ large enough, and we obtain the inequality

$$\|S\diamond G\|_{L^p((0,T)\times\Omega;D((w+A)^{1/2}))}\le Ce^{wT}\|G\|_{L^p((0,T)\times\Omega;L^q(\mathcal{O};H))}$$

with the constant C independent of G and T. In particular, injectivity of A is not needed for this estimate. We leave the easy details to the reader.

REMARK 4.6. In the proof of Theorem 4.3 we used both inequalities of Proposition 2.4. It is an open problem whether only the first one suffices. For $p=q=2$ this is indeed the case. To see this, take $\varphi(z)=z^{1/2}\exp(-z)$ and assume that

$$\|\varphi(tA)f\|_{L^2(\mathcal{O};L^2(\mathbb{R}_+,dt/t))}\le C\|f\|,\qquad f\in L^2(\mathcal{O}).$$

Let $(h_n)_{n\geq 1}$ be an orthonormal basis for H. Then by the Itô isometry, Fubini's theorem and the Plancherel formula,

$$\begin{aligned}\|A^{1/2}S \diamond G\|^2_{L^2(\mathbb{R}_+\times\Omega;L^2(\mathcal{O}))} &= \int_0^\infty \mathbb{E}\int_0^t \sum_{n\geq 1} \|A^{1/2}S(t-s)G(s)h_n\|^2_{L^2(\mathcal{O})}\,ds\,dt \\ &= \int_0^\infty \mathbb{E}\sum_{n\geq 1}\int_0^\infty \|A^{1/2}S(t)G(s)h_n\|^2_{L^2(\mathcal{O})}\,dt\,ds \\ &\leq C^2\int_0^\infty \mathbb{E}\sum_{n\geq 1}\|G(s)h_n\|^2_{L^2(\mathcal{O})}\,ds \\ &= C^2\|G\|^2_{L^2(\mathbb{R}_+\times\Omega;L^2(\mathcal{O};H))}.\end{aligned}$$

5. Proof of Theorem 1.2. To prepare for the proof of Theorem 1.2 we start by collecting some results on sums of closed operators on a UMD Banach space E; below we shall only need the case $E = L^q(\mathcal{O})$ with $q \in (1,\infty)$.

Let A have a bounded H^∞-calculus on E of angle less than $\pi/2$. Let $\mathscr{A}$ be the closed and densely defined operator on $L^p(\mathbb{R}_+;E)$ with domain $D(\mathscr{A}) := L^p(\mathbb{R}_+;D(A))$ defined by

$$(\mathscr{A}f)(t) := Af(t).$$

Let $\mathscr{B}$ be the closed and densely defined operator on $L^p(\mathbb{R}_+;E)$ with domain $D(\mathscr{B}) := H_0^{1,p}(\mathbb{R}_+;E)$ given by

$$\mathscr{B}f := f'.$$

Here $H_0^{\theta,p}(\mathbb{R}_+;E) = \{f \in H^{\theta,p}(\mathbb{R}_+;E) : f(0) = 0\}$, where $H^{\theta,p}(\mathbb{R}_+;E) = [L^p(\mathbb{R}_+;E), H^{1,p}(\mathbb{R}_+;E)]_\theta$ is the Bessel potential space defined by complex interpolation.

The operators $\mathscr{A}$ and $\mathscr{B}$ have bounded imaginary powers (see, e.g., [3], Lemma III.4.10.5), and by [38], Theorems 4 and 5), the operator

$$\mathscr{C} := \mathscr{A} + \mathscr{B}, \qquad D(\mathscr{C}) := D(\mathscr{A}) \cap D(\mathscr{B}),$$

is closed and has bounded imaginary powers as well. Furthermore, $\mathscr{C}$ is injective and has dense range, and for all $\theta \in (0,1)$ one has (see, e.g., [9], Proposition 3.1)

$$(\mathscr{C}^{-\theta}f)(t) = \frac{1}{\Gamma(\theta)}\int_0^t (t-s)^{\theta-1}S(t-s)f(s)\,ds. \tag{5.1}$$

Moreover, for all $\theta \in (0,1]$ one has (see (2.3) and [37], Corollary 1)

$$D(\mathscr{C}^\theta) = L^p(\mathbb{R}_+;D(A^\theta)) \cap H_0^{\theta,p}(\mathbb{R}_+;E). \tag{5.2}$$

PROOF OF THEOREM 1.2. By a density argument, it suffices to consider an arbitrary $\mathscr{F}$-adapted finite rank step process $G : \mathbb{R}_+ \times \Omega \to D(A_H)$, where $A_H = A \otimes I_H$.

(1) By the Da Prato–Kwapień–Zabczyk factorization argument (see [9] and [14], Section 5.3, and references therein), using (5.1), the stochastic Fubini theorem and the equality

$$\frac{1}{\Gamma(\theta)\Gamma(1-\theta)}\int_r^t (t-s)^{\theta-1}(s-r)^{-\theta}\,ds = 1$$

one obtains, for all $t \in \mathbb{R}_+$,

$$\mathscr{C}^{-\theta}(A^{1/2-\theta}S_\theta \diamond G)(t) = A^{1/2-\theta}S \diamond G(t) \qquad \text{almost surely,}$$

and hence, by (5.2) and Theorem 4.3,

$$\begin{aligned}\|A^{1/2-\theta}S \diamond G\|_{L^p(\Omega;H^{\theta,p}(\mathbb{R}_+;L^q(\mathcal{O})))} &\le \|\mathscr{C}^\theta(A^{1/2-\theta}S \diamond G)(t)\|_{L^p(\mathbb{R}_+\times\Omega;L^q(\mathcal{O}))}\\ &= \|(A^{1/2-\theta}S_\theta \diamond G)(t)\|_{L^p(\mathbb{R}_+\times\Omega;L^q(\mathcal{O}))}\\ &\le C\|G\|_{L^p(\mathbb{R}_+\times\Omega;L^q(\mathcal{O};H))}.\end{aligned}$$

(2) Let $\theta \in (\frac{1}{p},\frac{1}{2})$. By [49], Theorem 3.6 (see also [3], Theorem III.4.10.2, and [37], Proposition 3) there is a continuous embedding

$$H^{\theta,p}(\mathbb{R}_+;L^q(\mathcal{O})) \cap L^p(\mathbb{R}_+;D(A^\theta)) \hookrightarrow BUC\Big(\mathbb{R}_+;D_A\Big(\theta-\frac{1}{p},p\Big)\Big)$$

of norm K. Here $BUC(\mathbb{R}_+;L^q(\mathcal{O}))$ denotes the Banach space of all bounded uniformly continuous functions from $\mathbb{R}_+$ to $L^q(\mathcal{O})$. Combining this with the result of part (1), noting that $\|S \diamond G\|_{L^p(\mathbb{R}_+\times\Omega;D(A^{1/2}))} \le C\|G\|_{L^p(\mathbb{R}_+\times\Omega;L^q(\mathcal{O};H))}$ by Theorem 1.1 and the fact that $0 \in \varrho(A)$,

$$\begin{aligned}&\|A^{1/2-\theta}S \diamond G\|_{L^p(\Omega;BUC(\mathbb{R}_+;D_A(\theta-1/p,p)))}\\ &\quad\le K\max\{\|A^{1/2-\theta}S \diamond G\|_{L^p(\Omega;H^{\theta,p}(\mathbb{R}_+;L^p(\mathcal{O})))},\\ &\qquad\qquad \|A^{1/2-\theta}S \diamond G\|_{L^p(\Omega;L^p(\mathbb{R}_+;D(A^\theta)))}\}\\ &\quad\le CK\|G\|_{L^p(\mathbb{R}_+\times\Omega;L^q(\mathcal{O};H))}.\end{aligned}$$

Hence by [40], Theorem 1.15.2(e),

$$\begin{aligned}&\|S \diamond G\|_{L^p(\Omega;BUC(\mathbb{R}_+;D_A(1/2-1/p,p)))}\\ &\quad\eqsim_{A,\theta,p} \|A^{1/2-\theta}S \diamond G\|_{L^p(\Omega;BUC(\mathbb{R}_+;D_A(\theta-1/p,p)))}\\ &\quad\le CK\|G\|_{L^p(\mathbb{R}_+\times\Omega;L^q(\mathcal{O};H))}.\end{aligned}$$

□

REMARK 5.1. A standard stopping time argument (see, e.g., [14], Proposition 4.16) shows that Theorem 1.2 can be localized. For instance, from Theorem 1.2(2) one can infer that for all $G \in L^0_{\mathscr{F}}(\Omega;L^p(\mathbb{R}_+;L^q(\mathcal{O};H)))$ one has

$$S \diamond G \in L^0\Big(\Omega;BUC\Big(\mathbb{R}_+;D_A\Big(\frac{1}{2}-\frac{1}{p},p\Big)\Big)\Big).$$

Here $L^0(\Omega; E)$ denotes the space of strongly measurable functions on Ω with values in a Banach space E.

REMARK 5.2. Arguing as in Remark 4.5, also Theorem 1.2 admits a version for bounded time intervals.

REMARK 5.3. As has been pointed out in Remark 2.2, the role of $L^q(\mathcal{O})$ in Proposition 2.1 can be taken over by an arbitrary UMD Banach space E. We do not know, however, whether Theorems 3.1 and Proposition 3.4 can be extended to UMD Banach spaces E, say with (martingale) type 2 in order to rule out the spaces $L^q(\mathcal{O})$ with $q \in (1, 2)$ for which Theorem 1.1 is known to be false (see Remark 4.4).

6. A counterexample to stochastic maximal L^2-regularity. We show next that Theorem 1.1 is not valid with $p = 2$ and $q \in (2, \infty)$, even when $H = \mathbb{R}$ and G is deterministic. Stated differently, analytic generators on $L^q(\mathcal{O})$ do not always enjoy stochastic maximal L^2-regularity for $q \in (2, \infty)$. This is rather surprising, since stochastic maximal L^2-regularity for Hilbert spaces [in particular, for $L^2(\mathcal{O})$] is easy to prove (see Remark 4.6).

In the next theorem, W denotes a real-valued Brownian motion.

THEOREM 6.1. *Let $q \in (2, \infty)$ and fix an increasing sequence $0 < \lambda_1 < \lambda_2 < \cdots$ diverging to ∞. Let A be the diagonal operator on ℓ^q defined by $Ae_k := \lambda_k e_k$ with its maximal domain. Then A has a bounded H^∞-calculus of zero angle, but there does not exist a constant C such that for all $G \in L^2(\mathbb{R}_+; \ell^q)$,*

$$\int_0^\infty \mathbb{E}\left\| \int_0^t A^{1/2} S(t-s) G(s)\, dW(s) \right\|_{\ell^q}^2 dt \le C^2 \int_0^\infty \|G(t)\|_{\ell^q}^2\, dt. \tag{6.1}$$

PROOF. The verification that A has a bounded H^∞-calculus of zero angle is routine.

By Proposition 2.1, the estimate (6.1) is equivalent to

$$\begin{aligned} &\mathbb{E}\int_0^\infty \left(\sum_{k\ge 1} \left(\int_0^t \lambda_k e^{-2\lambda_k (t-s)} |g_k(s)|^2\, ds \right)^{q/2} \right)^{2/q} dt \\ &\qquad \le C_1^2 \int_0^\infty \left(\sum_{k \ge 1} |g_k(t)|^q \right)^{2/q} dt, \end{aligned} \tag{6.2}$$

where C_1 is a different constant independent of $G = (g_k)_{k\ge 1}$. We claim that this inequality implies deterministic maximal L^1-regularity for the operator $B = 2A$ on the space $\ell^{q/2}$, by which we mean that there is a constant C_2 such that

$$\left\| \int_0^t B e^{-(t-s)B} f(s)\, ds \right\|_{L^1(\mathbb{R}_+; \ell^{q/2})} \le C_2 \|f\|_{L^1(\mathbb{R}_+; \ell^{q/2})}$$

for all $f = (f_k)_{k\geq 1}$ in $L^1(\mathbb{R}_+; \ell^{q/2})$. This inequality is equivalent to

$$\begin{aligned}\int_0^\infty \biggl(\sum_{k\geq 1}\biggl|\int_0^t 2\lambda_k e^{-2\lambda_k(t-s)} f_k(s)\,ds\biggr|^{q/2}\biggr)^{2/q} dt \\ \leq C_2 \int_0^\infty \biggl(\sum_{k\geq 1} |f_k(s)|^{q/2}\biggr)^{2/q} ds.\end{aligned} \tag{6.3}$$

To see that (6.3) follows from (6.2), we may reduce to nonnegative f by considering positive and negative parts of each f_k separately. Then (6.3) follows by taking $g_k = \sqrt{f_k^{\pm}}$ in (6.2).

Now the theorem follows from [20], where it is shown that B fails maximal L^1-regularity on $\ell^{q/2}$ with $q \in (2,\infty)$. $\square$

Of course, by Theorem 1.1 the operator A of this example has stochastic maximal L^p-regularity for $p \in (2,\infty)$.

7. Discussion. We have already compared Theorem 1.1 with Krylov's inequalities (1.1) and (1.2) in the Introduction. Theorem 1.1 also extends various other regularity results in the literature.

7.1. *Hilbert spaces.* For generators $-A$ of bounded strongly continuous analytic semigroups on Hilbert spaces, stochastic maximal L^2-regularity was proved by Da Prato (see [14], Section 6.3, and references therein) under the assumption $D(A^\theta) = D_A(\theta, 2)$ for all $\theta \in (0,1)$. This condition is equivalent to the existence of a bounded H^∞-calculus of angle less than $\frac{1}{2}\pi$ for A (see [21], Remark 6.6.10, and [32], Theorem 11.9). Thus, the case $p = q = 2$ of Theorem 1.1 contains Da Prato's result (see also Remark 4.6). For $p \in (2,\infty)$, Theorem 1.1 seems to be new even in the Hilbert space case, that is, $q = 2$. Similarly, only the case $p = 2$ of Theorem 1.2(2) is known in the Hilbert space case (see [14], Section 6.2, and note that $D_A(\frac{1}{2}, 2) = D(A^{1/2})$ when E is a Hilbert space; here we should mention the fact that analytic contraction semigroups on Hilbert spaces have a bounded H^∞-calculus of angle less than $\pi/2$ [32], Theorem 11.13). As observed in [19], the above mentioned assumption in Da Prato's result can be weakened to $D(A^{1/2}) \supseteq D_A(\frac{1}{2}, 2)$.

7.2. *Martingale type* 2 *spaces.* For $p = 2$, the related estimate

$$\mathbb{E}\|U\|^2_{L^p(0,T;D(A))} \leq C^2 \mathbb{E}\|G\|^2_{L^p(0,T;D_{A_H}(1/2,2))} \tag{7.1}$$

was obtained by Brzeźniak [8] for so-called M-type 2 Banach spaces E [this class includes the L^q-spaces for $q \in [2,\infty)$]. If A is a second order elliptic operator on a space $E = L^q(\mathbb{R}^d)$, one typically has

$$D_{A_H}(\tfrac{1}{2}, 2) = B^1_{q,2}(\mathbb{R}^d; H) \subseteq H^{1,q}(\mathbb{R}^d; H) = D(A_H^{1/2}),$$

where the middle inclusion, being a consequence of [40], Remark 2.3.3/4 and Theorem 4.6.1, is strict for $q \in (2, \infty)$. Similar reasoning applies in the case of bounded regular domains in $\mathbb{R}^d$.

As a consequence, the inequality (7.1) is weaker than the one which follows from (1.7) (with $\delta = \frac{1}{2}$),

$$\mathbb{E}\|U\|^2_{L^p(0,T;D(A))} \le C^2 \mathbb{E}\|G\|^2_{L^p(0,T;D(A_H^{1/2}))}. \tag{7.2}$$

More importantly, the fact that the real interpolation spaces $D_{A_H}(\frac{1}{2}, 2)$ are Besov spaces causes difficulties in the treatment of nonlinear problems (as was noted in [8, 29]). Such problems can be avoided if one uses the inequality (7.2) instead.

7.3. *Real interpolation spaces.* For analytic generators $-A$ on M-type 2 spaces E, stochastic maximal L^p-regularity for $p \in [2, \infty)$ in the real interpolation spaces $D_A(\theta, p)$ for $\theta \in [0, 1)$ was proved by Da Prato and Lunardi [13] (see also [10]); the solution U then belongs to $L^p(0, T; D_A(\theta + \frac{1}{2}, p))$. With $\theta = \frac{1}{2}$ this gives the estimate

$$\mathbb{E}\|U\|^2_{L^p(0,T;D_A(1,p))} \le C^2 \mathbb{E}\|G\|^2_{L^p(0,T;D_{A_H}(1/2,p))}. \tag{7.3}$$

Comparing with (7.2), this time the applicability is limited by the observation that the solution space $D_A(1, p)$ may be larger than $D(A)$ when $E = L^q(\mathcal{O})$ with $q \ge p$. This happens, for instance, in the special case where A is a second order elliptic operator on $E = L^q(\mathbb{R}^d)$ with $q \in (2, \infty)$. Taking $p = q$ one has

$$D_A(1, q) = B^2_{q,q}(\mathbb{R}^d) \supsetneq H^{2,q}(\mathbb{R}^d) = D(A).$$

Again similar reasoning applies in the case of bounded regular domains in $\mathbb{R}^d$.

When comparing the results of [13] with ours, it should be noted that if $-A$ is invertible and generates a bounded strongly continuous analytic semigroup on a Banach space E, then by a result of Dore (see, e.g., [21], Corollary 6.5.8) A admits a bounded H^∞-functional calculus of angle less that $\pi/2$ on the spaces $D_A(\theta, p)$ for all $\theta \in (0, 1)$ and $p \in (1, \infty)$. Hence, at least for $E = L^q(\mathcal{O})$, estimate (7.3) is also contained in Theorem 1.1.

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