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COMPETING PARTICLE SYSTEMS EVOLVING BY INTERACTING LÉVY PROCESSES[1]

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We consider finite and infinite systems of particles on the real line and half-line evolving in continuous time. Hereby, the particles are driven by i.i.d. Lévy processes endowed with rank-dependent drift and diffusion coefficients. In the finite systems we show that the processes of gaps in the respective particle configurations possess unique invariant distributions and prove the convergence of the gap processes to the latter in the total variation distance, assuming a bound on the jumps of the Lévy processes. In the infinite case we show that the gap process of the particle system on the half-line is tight for appropriate initial conditions and same drift and diffusion coefficients for all particles. Applications of such processes include the modeling of capital distributions among the ranked participants in a financial market, the stability of certain stochastic queueing and storage networks and the study of the Sherrington–Kirkpatrick model of spin glasses.

1. Introduction. Recently, invariant distributions for the gaps in a particle system on the real line have received much attention. In the continuous time setting such questions are motivated by the study of gaps in ordered Brownian particle systems with rank-dependent drifts and diffusion coefficients. The latter arise in the modeling of the capital distribution in a financial market (see, e.g., [3, 8, 11] and [14]) and as heavy traffic approximations of queueing networks (see Section 5 of [14] for the correspondence between the processes of gaps and reflected Brownian motions and, e.g., [12, 13] and [27] for the heavy traffic approximation of queueing networks by the latter). In the discrete time setting these questions appeared in the study of the Sherrington–Kirkpatrick model of spin glasses and were allowed to characterize the quasi-stationary states in the free energy model (see [22] for this result and [1, 24] for its extensions). In these papers the increments added to the particles in each time step were either assumed to be i.i.d. or normally distributed. The latter evolutions by Gaussian increments can be viewed as a time discretized version of the Brownian particle systems.

The present paper studies the processes of gaps in finite and infinite particle systems on the real line and half-line evolving in continuous time and in which

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particles are driven by i.i.d. Lévy processes with jumps. In the finite case the latter are endowed with rank-dependent drift and diffusion coefficients. We will refer to these evolutions as competing particle systems. More precisely, the competing particles are indexed by $i \in I$ with $I = \{1, \dots, N\}$ in the case of finitely many particles and $I = \mathbb{N}$ in the case of infinitely many particles. The evolution on the whole line will be referred to as the unregulated evolution and the evolution on the half-line $[b, \infty)$ for some barrier $b \in \mathbb{R}$ as the regulated evolution. The particle systems are defined as the unique weak solutions (see Section 2.1 and Proposition 3.1 for more details) to the following stochastic differential equations:

$$
\begin{aligned}
dX_i(t) &= \sum_{j \in I} 1_{\{X_i(t)=X_{(j)}(t)\}} \delta_j \, dt + \sum_{j \in I} 1_{\{X_i(t)=X_{(j)}(t)\}} \sigma_j \, dB_i(t) \\
&\quad + dL_i(t) + dR_i(t)
\end{aligned} \tag{1.1}
$$

for all $i \in I$ where $R_i(t) \equiv 0$, $i \in I$ in the unregulated case and

$$
R_i(t) = \sum_{0 \le s \le t} 1_{\{X_i(s-)+\Delta L_i(s)<b\}} \big(b - X_i(s-) - \Delta L_i(s)\big) + \Lambda_{(i,b)}(t), \tag{1.2}
$$

$i \in I$ in the regulated case. Hereby, $X_{(1)}(t) \le X_{(2)}(t) \le \cdots$ is the ordered vector of particles in which ties are broken according to an arbitrarily specified ordering of the initial configuration, $B(t) = (B_i(t), i \in I)$ is a collection of i.i.d. standard Brownian motions, $L(t) = (L_i(t), i \in I)$ is an independent collection of i.i.d. pure jump Lévy processes each making finitely many jumps on any finite time interval. For each $i \in I$ and $s \ge 0$ the term $\Delta L_i(s)$ denotes the jump $L_i(s) - L_i(s-)$ of the process L_i at time s and each $\Lambda_{(i,b)}(t)$ is the local time process of X_i at b. To avoid any confusion we choose the same normalization of the local time processes as in [3, 4] and [14] so that $\Lambda_{(i,b)}(t)$ is one of the summands in the semi-martingale decomposition of the process $\frac{1}{2}|X_i(t) - b|$ rather than $|X_i(t) - b|$. Moreover, we impose the initial conditions $X_1(0) \le X_2(0) \le \cdots$ and $0 = R_1(0) = R_2(0) \cdots$ and, in addition, $X_1(0) \ge b$ in the regulated case. To distinguish between the two evolutions we denote the particle configuration at a time $t \ge 0$ by $X(t) = (X_i(t) : i \in I)$ in the unregulated particle system and by $X^R(t) = (X_i^R(t) : i \in I)$ in the regulated particle system.

Heuristically, particle i in the unregulated particle system is driven by a Lévy process whose drift and diffusion coefficients change according to its rank in the particle system. The regulated particle system is defined similarly, except that whenever a particle hits the barrier b or jumps over b, it is reflected in the former case and it is placed at b in the latter case.

To simplify the notation we set $Y_i(t) = X_{(i)}(t)$ and $Y_i^R(t) = X_{(i)}^R(t)$, respectively, for all $i \in I$. Next, we can define the processes of gaps, which will be the main quantities of interest, by

$$
Z(t) \equiv (Z_1(t), Z_2(t), \dots) = \big(Y_2(t) - Y_1(t), Y_3(t) - Y_1(t), \dots\big) \tag{1.3}
$$

and

$$Z^R(t) \equiv (Z_1^R(t), Z_2^R(t), \dots) = \big(Y_1^R(t) - b, Y_2^R(t) - b, \dots\big), \tag{1.4}$$

respectively. For $I = \{1, \dots, N\}$ we can set $C_i(t) = e^{X_i(t)}$ and note that the properties of the gap process $Z(t)$ correspond to the properties of the mass partition

$$\frac{C_{(i)}(t)}{C_{(1)}(t) + \cdots + C_{(N)}(t)} = \frac{1}{\sum_{j=1}^N e^{X_{(j)}(t) - X_{(i)}(t)}}, \tag{1.5}$$

$1 \le i \le N$ and the analogous statement is true for the process $Z^R(t)$.

In models for the capital distribution in a financial market the processes $C_1(t), \dots, C_N(t)$ stand for the capitalizations of the different firms and the particles represent the logarithmic capitalizations of the firms. Due to this interpretation, we will call the diffusion coefficients from now on volatilities or volatility coefficients of the particles. In this context it is natural to allow the particles to have jumps corresponding to such events as mergers of firms, shares emissions or jumps in the stock price. Indeed, since Merton's seminal paper [17] and a vast number of works that followed, it is widely agreed that market models allowing for discontinuities in the stock price are able to better account for sudden large changes in the values of the stocks, the incompleteness of financial markets and the imperfectness of the hedging strategies for options when compared to models assuming continuous trajectories of the stock prices. Hereby, models in which the jumps originate from pure jump Lévy processes play a central role. For a recent summary of the research in this direction and a list of references we refer to the book [9].

The models on the half-line describe markets in which either firms are bailed out if their capitalization falls below a certain predetermined barrier or in which capitalizations are constrained not to overcome a certain barrier, depending on the interpretation of the model. In the absence of jumps such models can be analyzed by exploiting their relation with semi-martingale reflected Brownian motions in the sense of [20, 25] as was done in [14] for the models on the whole line [see the proof of Lemma 2.1(b) below for the connection]. In [16] the authors considered a related model for a market with two firms defaulting if their capitalization crosses a fixed threshold. They were able to determine how to split the total drift, representing the total amount of tax cuts or subsidies, between the two firms to maximize the probability that both of them survive.

The mass partitions above correspond in this context to the market weights of the ranked market participants. For this reason, all results on the gap processes in the finite particle systems translate directly into corresponding results on the processes of market weights. In particular, the invariant distributions, the existence and uniqueness of which we prove for the processes of gaps in the finite systems both on the line and on the half-line, stand for capital distributions among the ranked market participants which are not changing under the evolution. We also show that the gap processes converge to the respective unique invariant distributions in the total variation distance. This corresponds to the statement that the considered financial market approaches a stable capital allocation in a strong sense. In addition, under the invariant distributions for the gap processes, all gaps are finite by definition which implies that the limiting stable capital distributions are nondegenerate, in the sense that the market weights of all firms are positive almost surely.

In the context of stochastic queueing and storage networks, our model is closely related to the so-called Lévy networks which are described mathematically by Lévy processes regulated to stay in an orthant by normal projections onto the boundary and normal reflections at the boundary (see Chapter IX of [2] and the references therein for more details). In particular, the techniques we use to prove that the processes of gaps possess unique invariant distributions and converge to the latter can be used without alteration to show the same statements for an integrable Lévy process with a nondegenerate Brownian part, whose components are endowed with negative drifts and which is regulated to stay in an orthant, provided that its jumps are dominated by the drifts in an appropriate sense. The convergence in total variation of the process to its invariant distribution proves the stability of the corresponding queueing network, meaning that the joint law of the workload processes converges in the strong sense.

Throughout the paper we make the following assumptions.

ASSUMPTION 1.1. (a) In all cases we assume that the volatilities $\sigma_1, \sigma_2, \dots$ are positive and it holds that

$$\mathbb{E}[|L_i(t)|] < \infty, \qquad \mathbb{E}[L_i(t)] = 0, \qquad i \in I. \tag{1.6}$$

(b) If $I = \mathbb{N}$, the drifts and volatilities are assumed to satisfy

$$\delta_M = \delta_{M+1} = \cdots, \qquad \sigma_1 = \sigma_2 = \cdots \tag{1.7}$$

for some $M \geq 1$ in the unregulated case and, in addition, $0 > \delta_1 = \delta_2 = \cdots$ in the regulated case.

Our results for the systems with finitely many particles are summarized in the following two theorems.

THEOREM 1.2 (Invariant distributions, finite systems). (a) *If $I = \{1, \dots, N\}$ and L_1 satisfies*

$$\mathbb{E}\Bigg[\sum_{0 \leq s \leq 1} |\Delta L_1(s)|\Bigg] < \frac{1}{N} \cdot \min(\delta_1 - \delta_2, \dots, \delta_{N-1} - \delta_N), \tag{1.8}$$

then the gap process Z of the unregulated particle system has a unique invariant distribution.

(b) *If* $I = \{1, \dots, N\}$ *and* L_1 *satisfies*

$$\mathbb{E}\Big[\sum_{0 \le s \le 1} |\Delta L_1(s)|\Big] < \frac{1}{N} \cdot \min(-\delta_1, \delta_1 - \delta_2, \dots, \delta_{N-1} - \delta_N), \tag{1.9}$$

then the gap process Z^R *of the regulated particle system has a unique invariant distribution.*

THEOREM 1.3 (Convergence, finite systems). (a) *If* $I = \{1, \dots, N\}$ *and condition* (1.8) *holds, then the process* $Z(t)$, $t \ge 0$ *converges in total variation to its unique invariant distribution.*

(b) *If* $I = \{1, \dots, N\}$ *and condition* (1.9) *holds, then the same statement is true for the process* $Z^R(t)$, $t \ge 0$.

We remark that in the absence of jumps, condition (1.8) simplifies to $\delta_1 > \cdots > \delta_N$. In particular, the latter is stronger than condition (3.2) of [14] (used there to establish the existence and uniqueness of an invariant distribution for the gap process of the finite unregulated system without jumps) which in our notation reads $\frac{1}{i}\sum_{j=1}^{i} \delta_j - \frac{1}{N}\sum_{j=1}^{N} \delta_j > 0$, $1 \le i \le N-1$. This shows that one cannot expect the conditions in Theorems 1.2 and 1.3 to be sharp in general.

For the infinite regulated system we show the following proposition.

PROPOSITION 1.4 (Tightness of the infinite regulated system). *If* $I = \mathbb{N}$ *and*

$$\mathbb{E}\Big[\sum_{0 \le s \le 1} |\Delta L_1(s)|\Big] < -\delta_1, \tag{1.10}$$

then the family $Z^R(t)$, $t \ge 0$ *is tight on* $\mathbb{R}_+^{\mathbb{N}}$ *equipped with the product topology for any initial condition in*

$$W = \Big\{0 \le z_1 \le z_2 \le \cdots \le \,\Big|\, \liminf_{i \to \infty} \frac{z_i}{i} > 0\Big\}. \tag{1.11}$$

The proof of Proposition 1.4 relies heavily on the fact that under Assumption 1.1, the evolution of each particle in the infinite regulated system is given by a Lévy process with negative drift regulated to stay on $[b, \infty)$. For this reason it does not carry over to the setting of the infinite unregulated system. At this point we also note that in the special case of the infinite regulated system with equal drifts $\delta_1 = \delta_2 = \cdots$, equal volatilities $\sigma_1 = \sigma_2 = \cdots$ and no jumps, the processes $X_i^R(t) - b$ are independent reflected Brownian motions on $\mathbb{R}_+$. Due to the negativity of the drifts this implies that for each $i \in I$ the process $X_i^R(t) - b$ converges in law to an exponential random variable. Moreover, a sequence of i.i.d. exponential random variables is almost surely dense in $\mathbb{R}_+$ by the second Borel–Cantelli lemma. This shows that, in general, one cannot expect the infinite regulated system to have an invariant distribution, since already in the described special case the number of particles on each nonempty interval of the form $[b, y)$ will tend to infinity for any initial particle configuration. In contrast to this, in [22] the authors show the existence of an infinite family of invariant distributions for the gap process in the corresponding particle system on the whole line. In the more general case of nonconstant drifts or volatilities the questions of existence and uniqueness of invariant distributions for the gap process of the infinite regulated system and of the convergence of the latter to an invariant distribution are open. The approach used in the analysis of the finite systems may apply to this case as well. However, one has to establish the recurrence and irreducibility structure of the gap process in an infinite-dimensional space where the analysis of the corresponding reflected Brownian motion is more intricate.

In [19] the authors treat the infinite unregulated system without jumps with the drift sequence $\delta_1, 0, 0, \dots$ and a constant and positive volatility sequence $\sigma_1 = \sigma_2 = \cdots$ and are able to find an invariant distribution for the gap process on a subset of W. In [22] the authors characterize all invariant distributions for the process of gaps in the case of constant drift and volatility sequences. As in the regulated case the questions of existence and uniqueness of invariant distributions and of the convergence of the gap process to the latter for a general drift or volatility sequence are open. The main obstacle hereby is the same as in the case of the infinite regulated system (see the end of the previous paragraph).

It is a natural question to ask to provide an explicit description of the invariant distributions in Theorem 1.2 in the presence of jumps. However, even in the case of the regulated particle system with a single particle, where the gap process is a one-dimensional Lévy process regulated to stay nonnegative, the invariant distribution is not known explicitly. To the best of our knowledge the most explicit result in this direction is obtained in [5], Section 3. There, in the described special case with the additional assumption that the magnitude of the negative jumps of the driving Lévy process is exponentially distributed, an expression for the Fourier transform of the invariant distribution is given in terms of the characteristic exponent of the driving Lévy process and the expected accumulated local time of the gap process at zero over a unit time interval when the initial value of the gap process is chosen according to the unique invariant distribution (see [5], equation (3.11)). A completely explicit description of the invariant distribution for the gap process, even for specific examples of a regulated or an unregulated system with multiple particles in which jumps are present, seems to be out of reach at the moment.

Previous papers on Brownian systems with rank-dependent drifts and volatilities use extensively the results of [26] on reflected Brownian motions and classical results on constrained diffusion processes to obtain the existence and uniqueness of the invariant distribution and the convergence of the gap process to the latter. Due to the presence of jumps, these tools do not apply here and are replaced by more general techniques from the ergodic theory of Harris recurrent Markov processes, as in [18]. More precisely, to obtain the results of Theorems 1.2 and 1.3, we first find the dynamics of the gap processes in Lemma 2.1 below. We use it subsequently to prove the tightness of the processes of gaps by dominating each of them by an appropriate Harris recurrent Markov process. Next, we show the Feller property for the gap processes to establish the existence of the invariant distributions for the latter. Finally, to prove the uniqueness of the invariant distributions and the convergence of the processes of gaps, we employ the general theory of Harris recurrent Markov processes.

The paper is structured as follows. In Section 2 we treat the particle systems with finitely many particles. We show first their existence and explain some of their properties in Section 2.1 and then prove Theorems 1.2 and 1.3 in Section 2.2. Section 3 deals with systems of infinitely many particles. We show that the latter exist and are well defined for appropriate initial conditions in Section 3.1 and prove Proposition 1.4 in Section 3.2.

2. Systems of finitely many particles.

2.1. *Existence and properties of the processes.* Throughout this section we deal with the two evolutions of finitely many particles, that is, we set $I = \{1, \dots, N\}$. The existence and uniqueness of a weak solution to the unregulated version of (1.1) can then be seen as follows. As remarked in Section 2 of [3], the main result of [6] implies that for any $x \in \mathbb{R}^N$ with $x_1 \leq \cdots \leq x_N$ there exists a unique weak solution to

$$dX_i^{c,x}(t) = \sum_{j=1}^{N} 1_{\{X_i^{c,x}(t)=X_{(j)}^{c,x}(t)\}}\delta_j\,dt + \sum_{j=1}^{N} 1_{\{X_i^{c,x}(t)=X_{(j)}^{c,x}(t)\}}\sigma_j\,dB_i(t), \tag{2.1}$$

$$X^{c,x}(0) = x \tag{2.2}$$

defined on some probability space $(\Omega^x, \mathcal{F}^x, \mathbb{P}^x)$. Next, let $(\Omega^{x,l}, \mathcal{F}^{x,l}, \mathbb{P}^{x,l})$, $l \in \mathbb{N}$, be copies of $(\Omega^x, \mathcal{F}^x, \mathbb{P}^x)$ such that on each of them a process $X^{c,x,l}$ of the same law as $X^{c,x}$ is defined. Moreover, let $(\Omega^L, \mathcal{F}^L, \mathbb{P}^L)$ be the probability space on which the processes $L_1(t), \dots, L_N(t)$ are defined. Then a weak solution to (1.1) can be defined on the product space

$$(\Omega, \mathcal{F}, \mathbb{P}) = \Big(\prod_{x,l} \Omega^{x,l} \times \Omega^L, \bigotimes_{x,l} \mathcal{F}^{x,l} \otimes \mathcal{F}^L, \bigotimes_{x,l} \mathbb{P}^{x,l} \otimes \mathbb{P}^L\Big). \tag{2.3}$$

Denoting by $T_1 < T_2 < \cdots$ the jump times of the $\mathbb{R}^N$-valued process $L(t) = (L_1(t), \dots, L_N(t))$ we set

$$X(0) = (X_1(0), \dots, X_N(0)), \tag{2.4}$$

$$X(t) = X^{c,X(T_k),k}(t - T_k), \qquad T_k < t < T_{k+1}, k \geq 0, \tag{2.5}$$

$$X(T_k) = X(T_k-) + \Delta L(T_k), \qquad k \geq 1, \tag{2.6}$$

where we have used the notation $T_0 = 0$. This gives a weak solution to the unregulated version of (1.1) by construction. Noting that for any other weak solution the joint distribution of its continuous part, its jump times and its jump sizes have to coincide with the corresponding quantity of the solution just constructed, we conclude that the weak solution is unique. Here we have used the uniqueness of the weak solution to (2.1), (2.2).

The regulated process X^R can be constructed on the same probability space as X as follows. We start with the desired initial value $X^R(0)$ and set

$$X_i^R(t) = X_i^{c,X^R(T_k),k}(t - T_k) + \Lambda_{(i,b)}^{c,T_k}(t), \qquad T_k < t < T_{k+1}, k \geq 0,$$

$$X_i^R(T_k) = \begin{cases} X_i^R(T_k-) + \Delta L_i(T_k), & \text{if } X_i^R(T_k-) + \Delta L_i(T_k) \geq b, \\ b, & \text{if } X_i^R(T_k-) + \Delta L_i(T_k) < b, \end{cases} \qquad k \geq 1,$$

for all $1 \leq i \leq N$ where $\Lambda_{(i,b)}^{c,T_k}(t)$, $t \geq T_k$, is the local time at b of the process $X_i^R(t)$, $t \geq T_k$. We remark that this is precisely the construction of X with the regulation and reflection at the barrier being added at the appropriate random times. Due to the definition of the processes $X^{c,x,l}$ and of the involved local time processes, X^R and the corresponding process of regulations solve the regulated system (1.1), (1.2). The uniqueness of the weak solution to (2.1), (2.2) implies that the weak solution of the regulated system is unique.

It follows immediately that the gap processes $Z(t)$, $Z^R(t)$ are well defined and unique in law. Their dynamics are given in the next lemma.

LEMMA 2.1. (a) *The components of the gap process Z in the unregulated particle system satisfy*

$$\begin{aligned} dZ_i(t) = {} & (\delta_{i+1} - \delta_1)\,dt + \sigma_{i+1}\,d\beta_{i+1}(t) - \sigma_1\,d\beta_1(t) + d\lambda_{i+1}(t) - d\lambda_1(t) \\ & + \tfrac{1}{2}\big(d\Lambda_{(i,i+1)}(t) - d\Lambda_{(i+1,i+2)}(t) + d\Lambda_{(1,2)}(t)\big) \\ & + \big(F_i(Z(t-), \Delta\lambda(t)) - \big(Z_i(t-) + \Delta\big(\lambda_{i+1}(t) - \lambda_1(t)\big)\big)\big). \end{aligned}$$

(b) *The components of the gap process Z^R in the regulated particle system are governed by*

$$\begin{aligned} dZ_i^R(t) = {} & \delta_i\,dt + \sigma_i\,d\beta_i(t) + d\lambda_i(t) + \tfrac{1}{2}\big(d\Lambda_{(i-1,i)}^R(t) - d\Lambda_{(i,i+1)}^R(t)\big)1_{\{i\neq 1\}} \\ & + \big(d\Lambda_{(0,1)}^R(t) - \tfrac{1}{2}\,d\Lambda_{(1,2)}^R(t)\big)1_{\{i=1\}} \\ & + \big(F_i^R(Z^R(t-), \Delta\lambda(t)) - \big(Z_i^R(t-) + \Delta\lambda_i(t)\big)\big). \end{aligned}$$

Hereby, $\beta_1(t), \ldots, \beta_N(t)$ are i.i.d. standard Brownian motions, $\lambda(t) = (\lambda_1(t), \ldots, \lambda_N(t))$ is a Lévy process whose components are i.i.d. pure jump Lévy processes with the same distribution as $L_1(t)$ and independent of $\beta_1(t), \ldots, \beta_N(t)$, $\Lambda_{(i,i+1)}(t)$ and $\Lambda_{(i,i+1)}^R(t)$ are the local times at 0 of the processes $Y_{i+1}(t) - Y_i(t)$ and $Y_{i+1}^R(t) - Y_i^R(t)$, respectively, with $\Lambda_{(N,N+1)}(t) \equiv \Lambda_{(N,N+1)}^R(t) \equiv 0$, and

$\Lambda^R_{(0,1)}(t)$ *is the local time of the process* $Y^R_1(t)$ *at* b. *Finally,* F *and* F^R *describe the value of the gap process after a jump as a function of its value before the jump and the jump of* λ. *More explicitly,*

$$F_i(z,\eta) = \min_{1\le j_1<\cdots<j_{i+1}\le N} \max(z_{j_1-1}+\eta_{j_1},\dots,z_{j_{i+1}-1}+\eta_{j_{i+1}}) - \min_{1\le j\le N}(z_{j-1}+\eta_j)$$

for all $1 \le i \le N-1$, $z=(z_1,\dots,z_{N-1}) \in \mathbb{R}^{N-1}_+$ *and* $\eta=(\eta_1,\dots,\eta_N)\in\mathbb{R}^N$ *with the convention* $z_0 = 0$ *and*

$$F^R_i(z,\eta) = \min_{1\le j_1<\cdots<j_i\le N} \max\bigl(\max(z_{j_1}+\eta_{j_1},0),\dots,\max(z_{j_i}+\eta_{j_i},0)\bigr)$$

for all $1\le i\le N$, $z=(z_1,\dots,z_N)\in\mathbb{R}^N_+$ *and* $\eta=(\eta_1,\dots,\eta_N)\in\mathbb{R}^N$. *Moreover, the state spaces the processes are given by*

$$W^{N-1} = \{z_1,\dots,z_{N-1} | 0\le z_1\le z_2\le\cdots\le z_{N-1}\} \subset \mathbb{R}^{N-1}_+$$

and W^N, *respectively.*

REMARK. The contribution of the terms

$$\bigl(F_i(Z(t-),\Delta\lambda(t)) - \bigl(Z_i(t-)+\Delta\bigl(\lambda_{i+1}(t)-\lambda_1(t)\bigr)\bigr)\bigr), \qquad 1\le i\le N-1,$$

and

$$\bigl(F^R_i(Z^R(t-),\Delta\lambda(t)) - \bigl(Z^R_i(t-)+\Delta\lambda_i(t)\bigr)\bigr), \qquad 1\le i\le N,$$

in the respective dynamics can be understood as follows. The contribution is nonzero if and only if one of the particles jumps and this jump changes the ranks of the particles. If this jump does not change the left-most particle in the unregulated system or does not involve a regulation in the regulated system, then these terms correspond to consecutive normal reflections of the process of gaps at faces of W^{N-1} or W^N, respectively. More precisely, the gap process is normally reflected at the faces $\{z_{j_1} = z_{j_1+1}\},\dots,\{z_{j_2-1} = z_{j_2}\}$, if the inequalities $z_{j_1}\le z_{j_1+1},\dots,z_{j_1}\le z_{j_2}$ are violated by the jump and at the faces $\{z_{j_2} = z_{j_2-1}\},\dots,\{z_{j_1+1} = z_{j_1}\}$, if the inequalities $z_{j_1}\le z_{j_2},\dots,z_{j_2-1}\le z_{j_2}$ are violated by the jump. If the jump changes the left-most particle in the unregulated system, then the particles are relabeled and this term gives the change of the gaps due to relabeling. If a particle jumps below the barrier in the regulated evolution, then the particle configuration is regulated and the particles are relabeled. The term then gives the change of gaps due to both these operations.

PROOF OF LEMMA 2.1. (a) For any fixed $t\ge 0$ let $\pi_t:\{1,\dots,N\}\to\{1,\dots,N\}$ be a bijection such that

$$X_{\pi_t^{-1}(1)}(t-) \le X_{\pi_t^{-1}(2)}(t-) \le \cdots \le X_{\pi_t^{-1}(N)}(t-). \tag{2.7}$$

In Section 3 of [3] the authors show that in the absence of jumps, that is, when $L(t) \equiv 0$, it holds

$$dY_i(t) = \sum_{j=1}^{N} 1_{\{\pi_t(j)=i\}}\, dX_j(t) + \frac{1}{2}\bigl(d\Lambda_{(i-1,i)}(t) - d\Lambda_{(i,i+1)}(t)\bigr), \tag{2.8}$$

where $\Lambda_{(i,i+1)}(t)$ are defined as in the statement of the lemma for $1 \le i \le N$ and we have set $\Lambda_{(0,1)}(t) \equiv 0$. Moreover, equation (3.3) of [3] states that there exist i.i.d. standard Brownian motions $\beta_1(t), \dots, \beta_N(t)$ such that

$$dY_i(t) = \delta_i\, dt + \sigma_i\, d\beta_i(t) + \tfrac{1}{2}\bigl(d\Lambda_{(i-1,i)}(t) - d\Lambda_{(i,i+1)}(t)\bigr). \tag{2.9}$$

This yields immediately the claim of part (a) of the lemma if $L(t) \equiv 0$. In the presence of jumps we define the pure jump processes $\lambda_1(t), \dots, \lambda_N(t)$ by $\Delta\lambda_i(t) = \Delta L_{\pi_t^{-1}(i)}(t)$. The latter are i.i.d. pure jump Lévy processes with the same law as $L_1(t)$. Indeed, the jump times of $\lambda(t)$ and $L(t)$ coincide and the law of $\Delta\lambda(t)$ conditional on $\Delta\lambda(t) \neq 0$ is the same as the law of $\Delta L(t)$ conditional on $\Delta L(t) \neq 0$. Next, we recall that T_k, $k \ge 0$ were defined by $T_0 = 0$ and as the jump times of the process $L(t)$ for $k \ge 1$. Since $X(t)$ coincides with the corresponding process in the absence of jumps for $T_{k-1} < t < T_k$ and any $k \ge 1$ by its construction, it remains to verify that $\Delta Z_i(T_k)$ coincides with the jump given by the right-hand side of the equation in part (a) of the lemma for all $1 \le i \le N-1$ and $k \ge 1$. But this follows directly from the definition of F.

(b) As in (a), we first treat the case $L(t) \equiv 0$. In this case we will show that the processes $\widetilde{Z}_i^R(t) = Y_i^R(t) - Y_{i-1}^R(t)$, $1 \le i \le N$, with $Y_0^R(t) \equiv b$ follow dynamics corresponding to the dynamics in part (b) of the lemma. To this end, we set

$$N_j(t) = \bigl|\{1 \le i \le N \,|\, X_i^R(t) = X_{(j)}^R(t)\}\bigr| \tag{2.10}$$

for all $1 \le j \le N$, $t \ge 0$. Next, we observe that Theorem 2.3 of [4] applied to the continuous semi-martingales $X_1^R(t), \dots, X_N^R(t)$ yields for $1 \le j \le N$,

$$\begin{aligned} dY_j^R(t) &= \sum_{i=1}^{N} (N_j(t))^{-1} 1_{\{X_{(j)}^R(t) = X_i^R(t)\}}\, dX_i^R(t) + (N_j(t))^{-1} \sum_{k=1}^{j-1} d\Lambda_{(k,j)}^R(t) \\ &\quad - (N_j(t))^{-1} \sum_{k=j+1}^{N} d\Lambda_{(j,k)}^R(t), \end{aligned}$$

where $\Lambda_{(j_1,j_2)}^R(t)$ is the local time of $Y_{j_2}^R(t) - Y_{j_1}^R(t)$ at zero for $1 \le j_1 < j_2 \le N$. Plugging (1.1) and (1.2) into the latter equation and applying the strong Markov property of $(\widetilde{Z}_1^R(t), \dots, \widetilde{Z}_N^R(t))$ to the entrance times of the set

$$\partial_\varepsilon \mathbb{R}_+^N = \{z \in \mathbb{R}_+^N \,|\, \mathrm{dist}(z, \partial\mathbb{R}_+^N) \ge \varepsilon\}$$

for a fixed $\varepsilon > 0$, one shows that $\widetilde{Z}^R(t) = (\widetilde{Z}_1^R(t), \dots, \widetilde{Z}_N^R(t))$ evolves as a Brownian motion

$$\big(\sigma_1\beta_1(t), \sigma_2\beta_2(t) - \sigma_1\beta_1(t), \dots, \sigma_N\beta_N(t) - \sigma_{N-1}\beta_{N-1}(t)\big)$$

with constant drift vector $(\delta_1, \delta_2 - \delta_1, \dots, \delta_N - \delta_{N-1})$ between an entrance time of the set $\partial_\varepsilon \mathbb{R}_+^N$ and the first hitting time of $\partial \mathbb{R}_+^N$ after that. This is due to an application of Knight's theorem in the form of [21], page 183, to the martingale parts of $Y_j^R(t)$, $1 \le j \le N$. Consequently, letting ε tend to zero, we observe that $\widetilde{Z}^R(t)$ is a semi-martingale reflected Brownian motion in the orthant $\mathbb{R}_+^N$ in the sense of [20, 25]. Now, Lemma 2.1 of [25] shows that the Lebesgue measure of the set $\{t \ge 0 | \widetilde{Z}^R(t) \in \partial \mathbb{R}_+^N\}$ is zero almost surely and Theorem 1 of [20] implies that the times for which either $N_j(t) \ge 3$ or $N_j(t) = 2$ and $\widetilde{Z}_1(t) = 0$ do not contribute to the dynamics of $Y_j^R(t)$ for all $1 \le j \le N$. Hence, the dynamics simplifies to

$$\begin{aligned} dY_j^R(t) &= \delta_j\, dt + \sigma_j\, d\beta_j(t) + \big(d\Lambda_{(0,1)}^R(t) - \tfrac{1}{2}\, d\Lambda_{(1,2)}^R(t)\big) \cdot 1_{\{j=1\}} \\ &\quad + \tfrac{1}{2}\big(d\Lambda_{(j-1,j)}^R(t) - d\Lambda_{(j,j+1)}^R(t)\big) \cdot 1_{\{j \ne 1\}} \end{aligned}$$

for all $1 \le j \le N$. This yields immediately the statement of part (b) of the lemma in the absence of jumps. The general case follows by defining $\lambda(t)$ in the same way as in the proof of part (a) of the lemma and by making the next two observations. First, the dynamics of the process $Z^R(t)$ between the jump times of the process $L(t)$ coincides with the dynamics of the system in the absence of jumps as a consequence of the construction of the process $X^R(t)$. Second, the jumps of the process $Z^R(t)$ coincide with the jumps of the right-hand side of the equation in part (b) of the lemma due to the definition of F^R. □

2.2. *Invariant distributions and convergence.* In this section we investigate the existence and uniqueness of invariant distributions of the gap processes in the two finite particle systems, as well as the convergence of the processes of gaps to the respective invariant distributions. We start with the proof of Theorem 1.2.

PROOF OF THEOREM 1.2. (a) (1) We first prove that the family $Z(t)$, $t \ge 0$, is tight for any initial value z in W^{N-1}. To do this, it suffices to show that the family $\widetilde{Z}_i(t) \equiv Z_i(t) - Z_{i-1}(t)$, $t \ge 0$, is tight for all $1 \le i \le N-1$ where we have set $Z_0(t) \equiv 0$. To this end, we fix a $1 \le i \le N-1$, set $y = \widetilde{Z}_i(0)$ and define the process D_i^y on the same probability space as X by

$$\begin{aligned} D_i^y(t) &= y + (\delta_{i+1} - \delta_i)t + \sigma_{i+1}\beta_{i+1}(t) - \sigma_i\beta_i(t) \\ &\quad + \sum_{j=1}^N \sum_{0 \le s \le t} |\Delta\lambda_j(s)| + \Lambda^{D_i^y}(t), \end{aligned} \tag{2.11}$$

where $\Lambda^{D_i^y}(t)$ is the local time at 0 of the process $D_i^y(t)$ and $\beta_1(t),\dots,\beta_N(t)$ and $\lambda_1(t),\dots,\lambda_N(t)$ are the same as in the dynamics of $Z(t)$ given in Lemma 2.1(a). We note that $\Delta\widetilde{Z}_i(t)\le\Delta D_i^y(t)$ for all $t\ge 0$ and that after each time $t\ge 0$ with $\widetilde{Z}_i(t)=D_i^y(t)$ the processes $\widetilde{Z}_i$ and D_i^y evolve in the same way until either the $(i-1)$st ranked particle and the ith ranked particle collide, or the $(i+1)$st ranked particle and the $(i+2)$nd ranked particle collide, or there is a jump of $(\lambda_1(t),\dots,\lambda_N(t))$. In particular, for any $t\ge 0$ the accumulated local time at 0 on the set $\{0\le s\le t|\widetilde{Z}_i(s)=D_i^y(s)=0\}$ is the same for the processes $\widetilde{Z}_i$ and D_i^y. Putting these observations together, we conclude that $\widetilde{Z}_i(t)\le D_i^y(t)$ almost surely for all $t\ge 0$. Hence, to prove that the family $\widetilde{Z}_i(t)$, $t\ge 0$ is tight, it suffices to show that the family $D_i^y(t)$, $t\ge 0$ is tight.

(2) Next, we fix an $\varepsilon>0$ and find a $C=C(\varepsilon)>0$ such that

$$\mathbb{P}\Big(\sup_{0\le u\le\varepsilon} D_i^0(u)\le C\Big)>0. \tag{2.12}$$

We claim that the process $D_i^y(n\varepsilon)$, $n\in\mathbb{N}$, is a recurrent Harris chain on $\mathbb{R}_+$ with respect to the set $[0,C]$ in the sense of Section 5.6 of [10]. Indeed, the Harris property follows immediately from the Harris property of the corresponding process in the absence of jumps. Moreover, the law of large numbers for Lévy processes and condition (1.8) show that almost surely

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}\Bigg((\delta_{i+1}-\delta_i)t+\sigma_{i+1}\beta_{i+1}(t)-\sigma_i\beta_i(t)+\sum_{j=1}^N\sum_{0\le s\le t}|\Delta\lambda_j(s)|\Bigg)\\&\qquad=\delta_{i+1}-\delta_i+\sum_{j=1}^N\mathbb{E}\Big[\sum_{0\le s\le 1}|\Delta\lambda_j(s)|\Big]<0.\end{aligned}$$

Hence, there exist stopping times $0<\tau_1<\tau_2<\cdots$ tending to infinity with $D_i^y(\tau_k)=0$ for all $k\in\mathbb{N}$ with probability 1. This follows by setting

$$\begin{aligned}\tau_1&=\inf\{u\ge 1|D_i^y(u)=0\},\\ \tau_{k+1}&=\inf\{u\ge\tau_k+1|D_i^y(u)=0\},\qquad k\ge 1,\end{aligned}$$

and coupling $D_i^y(t)$, $t\ge s$ with the Lévy process

$$\begin{aligned}&D_i^y(s)+(\delta_{i+1}-\delta_i)(t-s)+\sigma_{i+1}\big(\beta_{i+1}(t)-\beta_{i+1}(s)\big)-\sigma_i\big(\beta_i(t)-\beta_i(s)\big)\\&\qquad+\sum_{j=1}^N\sum_{s<u\le t}|\Delta\lambda_j(u)|,\qquad t\ge s,\end{aligned}$$

to conclude

$$\mathbb{P}\big(\{D_i^y(s)=0\}\cup\{\exists t>s|D_i^y(t)=0\}\big)=1$$

for all $s \geq 0$. In addition, we let $\tau_k(\varepsilon)$ be the integer multiple of ε which is closest to τ_k from above for all $k \in \mathbb{N}$ and observe that the strong Markov property of D_i applied to the stopping times τ_k, $k \in \mathbb{N}$, (2.12) and the second Borel–Cantelli lemma imply that $D_i^y(\tau_k(\varepsilon)) \leq C$ for infinitely many $k \in \mathbb{N}$ almost surely. This shows the recurrence of $D_i^y(n\varepsilon)$, $n \in \mathbb{N}$. Moreover, by the results of Section 5.6c of [10], the chain is aperiodic and converges in total variation to its unique invariant distribution which we denote by ν_ε. Furthermore, the uniqueness of invariant distributions of recurrent Harris chains implies that $\nu_1 = \nu_{1/2} = \nu_{1/4} = \cdots$. Next, we fix a $\zeta > 0$ and a $w > 0$ and define the function $f_w : \mathbb{R}_+ \to \mathbb{R}_+$ by $f_w(v) = e^{-w\cdot v}$. Since D_i is a Feller process [this can be shown along the lines of step (3) below], its semi-group of transition operators is strongly continuous on the space of continuous functions on $\mathbb{R}_+$ vanishing at infinity (see, e.g., Theorem 17.6 in [15]). Thus, denoting the semi-group of transition operators corresponding to $D_i(t)$, $t \geq 0$, by $P^{D_i}(t)$, $t \geq 0$, we can find a $Q \in \mathbb{N}$ such that

$$\forall 0 \leq t \leq 2^{-Q} : \| P^{D_i}(t) f_w - f_w \|_\infty < \zeta. \tag{2.13}$$

Moreover, for each $t \geq 0$ we let $n(t)$ be the largest integer such that $n(t) \cdot 2^{-Q} \leq t$. All in all, setting $\mu_{y,t} = \delta_y P^{D_i}(t)$ and writing $\mu(f)$ for $\int f \, d\mu$, we get for any $t \geq 0$,

$$\begin{aligned}
&|\mu_{y,t}(f_w) - \nu_1(f_w)| \\
&\quad \leq |\mu_{y,t}(f_w) - \mu_{y,n(t)2^{-Q}}(f_w)| + |\mu_{y,n(t)2^{-Q}}(f_w) - \nu_1(f_w)| \\
&\quad = |\mu_{y,n(t)2^{-Q}}(P^{D_i}(t - n(t)2^{-Q}) f_w - f_w)| + |\mu_{y,n(t)2^{-Q}}(f_w) - \nu_1(f_w)| \\
&\quad \leq \zeta + |\mu_{y,n(t)2^{-Q}}(f_w) - \nu_1(f_w)|.
\end{aligned}$$

Taking first the limit $t \to \infty$ and then the limit $\zeta \downarrow 0$, we conclude that $\mu_{y,t}(f_w)$ converges to $\nu_1(f_w)$. Since $w > 0$ was arbitrary, it follows that the Laplace transforms of the measures $\delta_y P^{D_i}(t)$ converge point-wise to the Laplace transform of ν_1, hence, the measures $\delta_y P^{D_i}(t)$ converge weakly to ν_1. Thus, the family $D_i^y(t)$, $t \geq 0$, converges in law to ν_1 and is, therefore, tight as claimed.

(3) Next, denote by $P(t)$, $t \geq 0$, the Markov semi-group of operators corresponding to the Markov process Z. The tightness of $Z(t)$, $t \geq 0$ implies that the family of probability measures $\frac{1}{t}\int_0^t \delta_z P(s)\, ds$, $t > 0$, is tight. Hence, we can find a sequence $t_1 < t_2 < \cdots$ tending to infinity such that the weak limit

$$\nu \equiv \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} (\delta_z P(s))\, ds \tag{2.14}$$

exists and is a probability measure on W^{N-1}. We claim that ν is an invariant distribution of Z. To this end, we first remark that Z is a Feller process. For the evolution without jumps this is a consequence of the Feller property of the reflected Brownian motion (see Theorem 1.1 and the following remark in [26]) and the observation that the process $(\widetilde{Z}_1(t), \dots, \widetilde{Z}_{N-1}(t))$ can be viewed as a reflected Brownian motion in the sense of [26] (see [14], Section 5). In our case the Feller property can be seen as follows. Let z^0 be an arbitrary point in W^{N-1} and $(z^n)_{n=1}^\infty$ be a sequence in W^{N-1} converging to it. Moreover, let Z^{z^0} and Z^{z^n}, $n \ge 1$, be the gap processes with initial values z^0 and z^n, $n \ge 1$, respectively. We need to show that for any fixed $t \ge 0$ the random vectors $Z^{z^n}(t)$ converge in law to $Z^{z^0}(t)$. To this end, we note that the gap process Z can be constructed by first generating the sequence of its jump times $T_1 < T_2 < \cdots$ and the corresponding jumps $\Delta\lambda(T_1), \Delta\lambda(T_2), \dots$ of $\lambda(t) = (\lambda_1(t), \dots, \lambda_N(t))$ and then defining Z conditional on these choices by

$$Z(t) = \sum_{k=0}^{\infty} 1_{\{T_k \le t < T_{k+1}\}} Z^{c, F(Z(T_k-), \Delta\lambda(T_k)), k}(t - T_k), \tag{2.15}$$

where we have set $T_0 = 0$, $Z(0-) = Z(0)$, F is the continuous function as in part (a) of the Lemma 2.1 and $Z^{c, F(Z(T_k-), \Delta\lambda(T_k)), k}$, $k \ge 0$, are independent gap processes of unregulated evolutions without jumps with respective initial values $F(Z(T_k-), \Delta\lambda(T_k))$, $k \ge 0$. Due to the Dominated Convergence theorem it suffices to show that the law of $Z^{z^n}(t)$ conditional on a realization of λ converges weakly to the law of $Z^{z^0}(t)$ conditional on the same realization of λ. But in view of the representation (2.15), this can be shown by using induction over the unique value of k for which $t \in [T_k, T_{k+1})$ and the Feller property of $Z^{c,\cdot,k}$.

Moreover, for any $t \ge 0$ we have

$$\lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} (\delta_z P(s))\, ds = \lim_{n\to\infty} \frac{1}{t_n} \int_t^{t_n+t} (\delta_z P(s))\, ds, \tag{2.16}$$

where the limits are taken in the weak sense. This is a consequence of the fact that the total variation norm of $\frac{1}{t_n}\int_0^{t_n}(\delta_z P(s))\, ds - \frac{1}{t_n}\int_t^{t_n+t}(\delta_z P(s))\, ds$ is bounded above by $\frac{2t}{t_n}$ for all n with $t_n \ge t$. Hence, by the Feller property of Z

$$\begin{aligned}
\int_{W^{N-1}} f\, d(\nu P(t)) &= \lim_{n\to\infty} \int_{W^{N-1}} P(t) f\, d\left(\frac{1}{t_n}\int_0^{t_n} (\delta_z P(s))\, ds\right) \\
&= \lim_{n\to\infty} \int_{W^{N-1}} f\, d\left(\frac{1}{t_n}\int_t^{t_n+t} (\delta_z P(s))\, ds\right) \\
&= \int_{W^{N-1}} f\, d\nu
\end{aligned}$$

for all continuous bounded functions f. Thus, ν is an invariant distribution of Z.

(4) We now prove that ν is the only invariant distribution. To this end, consider the process $Z(n)$, $n \in \mathbb{N}$. We claim that it is a recurrent Harris chain in the sense of Section 5.6 in [10]. Indeed, in the absence of jumps the set $\{t \ge 0 | Z(t) \in \partial W^{N-1}\}$ has Lebesgue measure zero almost surely and the process Z evolves as a Brownian motion $(\sigma_2\beta_2(t) - \sigma_1\beta_1(t), \dots, \sigma_N\beta_N(t) - \sigma_1\beta_1(t))$ with constant drift vector $(\delta_2 - \delta_1, \dots, \delta_N - \delta_1)$ in the interior of W^{N-1} as we have seen in the proof of

Lemma 2.1(a). Since the covariance matrix of the latter Brownian motion is nondegenerate and there is a positive probability that Z has no jumps in the time interval [0, 1], we conclude that $Z(n)$, $n \in \mathbb{N}$, is a Harris chain on W^{N-1}. Noting that ν is an invariant distribution for this chain, we conclude that the chain must be recurrent. Thus, it has a unique invariant distribution (see [10], Section 5.6). Since any other invariant distribution of Z is an invariant distribution of $Z(n)$, $n \in \mathbb{N}$, it has to coincide with ν.

(b) (1) Part (b) of the theorem can be established by using the technique of the proof of part (a). However, we present here a softer argument based on a monotonicity property special to the regulated system. The latter was inspired by the proof of Lemma 1 on page 162 of [7] which deals with a similar discrete time problem. The crucial idea is to introduce a family of processes indexed by $\alpha \geq 0$ such that each of them evolves as the regulated system after time $-\alpha$ and for any two indices α_1, α_2 the corresponding processes are driven by the same Brownian motions and pure jump Lévy processes after time $\max(-\alpha_1, -\alpha_2)$. To construct such a family we start by defining auxilliary independent standard Brownian motions $\beta_{m,1}(t), \dots, \beta_{m,N}(t)$ and independent i.i.d. pure jump Lévy processes $\lambda_{m,1}(t), \dots, \lambda_{m,N}(t)$ of the same law as $\lambda_1(t)$ for all $m \in \mathbb{Z}_-$ on an extension of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which X^R was defined. Moreover, we let $m(t)$ be the largest integer which is less than or equal to t and $l(t) = t - m(t)$ for all $t \in \mathbb{R}$. Finally, we can use the notation to define a family of processes V^α on W^N indexed by $\alpha \geq 0$ with the desired properties by setting $V^\alpha(t) = b \cdot \mathbf{1} = (b, \dots, b)$ for $t \leq -\alpha$,

$$\begin{aligned} dV_i^\alpha(t) &= \delta_i\,dt + \sigma_i\,d\beta_{m(t),i}(l(t)) + d\lambda_{m(t),i}(l(t)) \\ &\quad + \tfrac{1}{2}\big(d\Lambda^\alpha_{(i-1,i)}(t) - d\Lambda^\alpha_{(i,i+1)}(t)\big) \cdot 1_{\{i\neq 1\}} \\ &\quad + \big(d\Lambda^\alpha_{(0,1)}(t) - \tfrac{1}{2}\,d\Lambda^\alpha_{(1,2)}(t)\big) \cdot 1_{\{i=1\}} \\ &\quad + \big(F_i^R\big(V^\alpha(t-), \Delta\lambda_{m(t)}(l(t))\big) - \big(V_i^\alpha(t-) + \Delta\lambda_{m(t),i}(l(t))\big)\big) \end{aligned}$$

for $-\alpha \leq t < 0$ and

$$\begin{aligned} dV_i^\alpha(t) &= \delta_i\,dt + \sigma_i\,d\beta_i(t) + d\lambda_i(t) + \tfrac{1}{2}\big(d\Lambda^\alpha_{(i-1,i)}(t) - d\Lambda^\alpha_{(i,i+1)}(t)\big) \cdot 1_{\{i\neq 1\}} \\ &\quad + \big(d\Lambda^\alpha_{(0,1)}(t) - \tfrac{1}{2}\,d\Lambda^\alpha_{(1,2)}(t)\big) \cdot 1_{\{i=1\}} \\ &\quad + \big(F_i^R(V^\alpha(t-), \Delta\lambda(t)) - \big(V_i^\alpha(t-) + \Delta\lambda_i(t)\big)\big) \end{aligned}$$

for $t \geq 0$. Hereby, notation is as in Lemma 2.1(b) with the local times defined with respect to V^α instead of Y^R. We note that the processes V^α are defined in such a way that the law of $V^\alpha(t)$ for $t \geq -\alpha$ is the law of the ordered particle configuration in the regulated evolution at time $t + \alpha$, started in $b \cdot \mathbf{1}$.

(2) Moreover, we have for V^{α_1}, V^{α_2} with $\alpha_1 < \alpha_2$ the inequalities

$$V_1^{\alpha_1}(t) \leq V_1^{\alpha_2}(t), \dots, V_N^{\alpha_1}(t) \leq V_N^{\alpha_2}(t) \tag{2.17}$$

for all $t \geq -\alpha_1$. Indeed, this is clear for $t = -\alpha_1$. Furthermore, the inequalities are preserved under the jumps of the two processes because the jump times and the jump sizes before the relabeling and the regulation of the particle configuration are the same for both processes. In addition, started with each stopping time at which a nonempty set $J \subset \{1, \dots, N\}$ of coordinate processes of V^{α_1} and V^{α_2} coincide, these coordinate processes evolve in the same way until one of the particles with rank in J collides with a particle with rank not in J or until there is a jump which changes the rank of at least one particle whose rank was originally in J. By distinguishing the two cases, one concludes that the regulated evolution preserves the component-wise order $\leq$ on configurations in W^N. Hence, for all $1 \leq i \leq N$ we may define

$$V_i^{\infty}(t) = \uparrow \lim_{\alpha \to \infty} V_i^{\alpha}(t) \tag{2.18}$$

as an element of $[b, \infty]$.

(3) Next, we claim that the random vector $V^{\infty}(0)$ is finite almost surely. To this end, we observe that the process $Z^R(t)$, $t \geq 0$, started at $Z^R(0) = 0$ converges in distribution to $V^{\infty}(0) - b \cdot \mathbf{1}$ for $t \to \infty$, because

$$Z^R(t) \stackrel{d}{=} \big(V_1^t(0) - b, \dots, V_N^t(0) - b\big) \tag{2.19}$$

and the right-hand side converges to $V^{\infty}(0) - b \cdot \mathbf{1}$ almost surely. In addition, we have for all $1 \leq i \leq N$ and all $n \in \mathbb{N}$ that

$$Z_i^R(n) = \sum_{j=1}^{i} \widetilde{Z}_j^R(n) \leq \sum_{j=1}^{i} D_j^R(n) \tag{2.20}$$

with $\widetilde{Z}_j^R(t) = Z_j^R(t) - Z_{j-1}^R(t)$ for all $t \geq 0$, $1 \leq j \leq N$, $Z_0^R(t) \equiv 0$,

$$D_1^R(t) = \delta_1 t + \sigma_1 \beta_1(t) + \sum_{k=1}^{N} \sum_{0 \leq s \leq t} |\Delta\lambda_k(s)| + \Lambda^{D_1^R}(t),$$

$$D_j^R(t) = (\delta_j - \delta_{j-1})t + \sigma_j \beta_j(t) - \sigma_{j-1}\beta_{j-1}(t) + \sum_{k=1}^{N} \sum_{0 \leq s \leq t} |\Delta\lambda_k(s)| + \Lambda^{D_j^R}(t),$$

$2 \leq j \leq N$ and $\Lambda^{D_j^R}(t)$ being the local time of $D_j^R(t)$ at zero for $1 \leq j \leq N$. We now use condition (1.9) and follow the lines of step (2) of the proof of part (a) to conclude that $D_j^R(t)$ converges in distribution to an almost surely finite random variable in the limit $t \to \infty$ for each $1 \leq j \leq N$. Thus, for all $1 \leq i \leq N$, the sequence $\sum_{j=1}^{i} D_j^R(n)$, $n \in \mathbb{N}$, is tight. Hence, the sequence $Z_i^R(n)$, $n \in \mathbb{N}$, is also tight, so the limit in distribution of $Z_i^R(n)$, $n \in \mathbb{N}$, which is $V_i^{\infty}(0) - b$, must be almost surely finite for all $1 \leq i \leq N$.

(4) Next, we prove that Z^R is a Feller process. To this end, it suffices to prove the Feller property in the absence of jumps and to follow the lines of step (3) in the

proof of part (a) to extend it to the general case. But in the absence of jumps, the process Z^R is a reflected Brownian motion in W^N in the sense of [26] as we have shown in the proof of part (b) of the Lemma 2.1. Thus, by Theorem 1.1 and the following remark in [26], the process Z^R is a Feller process in the absence of jumps. The Feller property and the convergence in law of $Z^R(t)$, $t \geq 0$ to $V^\infty(0) - b \cdot \mathbf{1}$ imply that the law of $V^\infty(0) - b \cdot \mathbf{1}$ is an invariant distribution of the process Z^R. The uniqueness of the invariant distribution can be deduced from the uniqueness of the invariant distribution of the chain $Z^R(n)$, $n \in \mathbb{N}$ by following step (4) of the proof of part (a). □

Now, we are able to prove Theorem 1.3 which deals with the convergence of the gap processes to the respective invariant distributions.

PROOF OF THEOREM 1.3. In the course of the proof of Lemma 2.1 we have seen that in the absence of jumps, the processes Z and Z^R are reflected Brownian motions in the sense of [26]. Moreover, as observed in the proof of Theorem 1.2, their respective covariance matrices are nondegenerate. In the presence of jumps this shows the Harris property of the processes $Z(t)$, $t \geq 0$, and $Z^R(t)$, $t \geq 0$, as well as the Harris property and the irreducibility of the chains $Z(n)$, $n \in \mathbb{N}$, and $Z^R(n)$, $n \in \mathbb{N}$. The existence of invariant distributions (Theorem 1.2) implies that the processes $Z(t)$, $t \geq 0$, and $Z^R(t)$, $t \geq 0$, are positive Harris recurrent in the sense of [18]. Thus, Theorem 6.1 of [18] shows that $Z(t)$, $t \geq 0$, and $Z^R(t)$, $t \geq 0$, converge in total variation to their respective invariant distributions. □

3. Systems of infinitely many particles.

3.1. *Existence and uniqueness in law of the processes.* From now on we let $I = \mathbb{N}$. Due to the fact that the processes $L_1(t), L_2(t), \ldots$ are i.i.d., an application of the second Borel–Cantelli lemma shows immediately that in both evolutions infinitely many jumps of the particles will occur almost surely on each nonempty time interval, if $L_1(t)$ is not identically equal to zero. Hence, the proof of existence of the finite particle systems does not carry over to the infinite case. Also, the proof in [19] for the case of the unregulated system without jumps, which uses Girsanov's theorem for Brownian motion, cannot be applied here, since it would prove the existence of the solution to the unregulated version of (1.1) in the case of a certain nontrivial dependence structure between the processes $B_1(t), B_2(t), \ldots$ and $L_1(t), L_2(t), \ldots$. Instead, we prove the existence and the uniqueness in law of the infinite particle systems by a bound on the tail of the distribution of the running supremum of an integrable Lévy process.

PROPOSITION 3.1. *If the initial configuration $X_1(0), X_2(0), \ldots$ of the particles satisfies*

$$\sum_{i=1}^{\infty} \mathbb{P}\Big(\sup_{0 \leq s \leq t} \big(-\sigma_1 B_1(s) - L_1(s)\big) \geq X_i(0) - y \Big) < \infty \tag{3.1}$$

for all $t \geq 0$ *and* $y \in \mathbb{R}$, *then the unregulated version of* (1.1) *has a unique weak solution and the corresponding ordered particle system* $Y_1(t), Y_2(t), \dots$ *is well defined for all* $t \geq 0$.

In particular, condition (3.1) *is satisfied if there are constants* $\gamma_1 > 0$, $\gamma_2 \in \mathbb{R}$ *with*

$$X_i(0) \geq \gamma_1 i + \gamma_2, \qquad i \geq 1. \tag{3.2}$$

The same statements are true for the regulated system (1.1), (1.2).

PROOF. (1) We prove the proposition only for the unregulated system, since the assertion for the regulated system can be shown in the same way by setting $M = 1$. To this end, we assume (3.1) and introduce the probability space

$$\Bigl(\prod_{(N,x)} \Omega^{(N,x)} \times \Omega^{B,L}, \bigotimes_{(N,x)} \mathcal{F}^{(N,x)} \otimes \mathcal{F}^{B,L}, \bigotimes_{(N,x)} \mathbb{P}^{(N,x)} \otimes \mathbb{P}^{B,L}\Bigr).$$

Hereby, the products are over $\{(N,x)|N \geq 1, x \in \mathbb{R}^N, x_1 \leq \dots \leq x_N\}$, on each $(\Omega^{(N,x)}, \mathcal{F}^{(N,x)}, \mathbb{P}^{(N,x)})$ the unregulated system $X^{N,x_1,\dots,x_N}$ with N particles, initial configuration $x_1 \leq x_2 \leq \dots \leq x_N$ and parameters $\delta_1, \dots, \delta_N, \sigma_1, \dots, \sigma_1$ are defined and $(\Omega^{B,L}, \mathcal{F}^{B,L}, \mathbb{P}^{B,L})$ is a probability space on which the i.i.d. Brownian motions $B_1(t), B_2(t), \dots$ and the i.i.d. pure jump Lévy processes $L_1(t), L_2(t), \dots$ independent of the Brownian motions are defined. We call the product space above $(\Omega^\infty, \mathcal{F}^\infty, \mathbb{P}^\infty)$. Next, we recall the definition of the constant M introduced in Assumption 1.1, set $G_i(t) = \sigma_1 B_i(t) + L_i(t)$ for all $t \geq 0$ and $i \in I$ for the sake of shorter notation and define the process X, the sets $A_0 \subset A_1 \subset \cdots$ and the stopping times $0 = \rho_0 \leq \rho_1 \leq \cdots$ inductively by

$$\begin{aligned}
&A_k = \{i \geq 1 | \exists 1 \leq j \leq M, 0 \leq s \leq \rho_k : X_i(s) = X_{(j)}(s)\},\\
&X_i(s) = X_i^{|A_k|, X_{(1)}(\rho_k), \dots, X_{(|A_k|)}(\rho_k)}(s - \rho_k), \qquad i \in A_k, \rho_k \leq s < \rho_{k+1},\\
&X_i(s) = X_i(\rho_k) + \delta_M(s - \rho_k) + G_i(s) - G_i(\rho_k), \qquad i \notin A_k, \rho_k \leq s < \rho_{k+1},\\
&\rho_{k+1} = \inf\{s \geq \rho_k | \exists 1 \leq j \leq M, i \notin A_k : X_i(s) = X_{(j)}(s)\}.
\end{aligned}$$

Fixing a $t > 0$ we observe that by condition (3.1) and the first Borel–Cantelli lemma there are almost surely only finitely many particles which visit the interval $(-\infty, \widetilde{y}]$ up to time t for any fixed $\widetilde{y} \in \mathbb{R}$. This follows by setting $y = \widetilde{y} + \max_{i \geq 1} |\delta_i| \cdot t$ in (3.1) and recalling that the set of collision times of distinct particles in the finite unregulated particle system has Lebesgue measure zero almost surely (see the proof of Lemma 2.1). Furthermore, we note that by choosing a large enough $\widetilde{y}$ we can make

$$\sum_{j=1}^{M} \mathbb{P}\Bigl(X_j(0) + \sup_{0 \leq s \leq t} \bigl(\sigma_1 B_1(s) + L_1(s)\bigr) + \max_{i \geq 1} |\delta_i| \cdot t > \widetilde{y}\Bigr)$$

as small as we want. We also note that in the regulated case this expression should be replaced by

$$\mathbb{P}\Big(\sup_{0\le s\le t}\big(X_1(0)+\delta_1 s+\sigma_1 B_1(s)+L_1(s)+\Lambda_{(1,b)}(s)\big)>\widetilde{y}\Big).$$

This observation implies that there exists a $K=K(\omega)$ such that $K<\infty$, $\rho_K\le t<\rho_{K+1}$ and the sets $A_0,\dots,A_K$ are finite almost surely. Thus, X is well defined on $[0,t]$ for almost every $\omega\in\Omega^\infty$. Since $t>0$ was arbitrary, we have shown that X is well defined on $[0,\infty)$ for almost every $\omega\in\Omega^\infty$. By its construction, X is a weak solution to the unregulated version of the system (1.1). Moreover, at any time $t\ge 0$ the ordered particle system $Y_1(t),Y_2(t),\dots$ is well defined, since there are finitely many particles on each interval of the form $(-\infty,\widetilde{y}]$ almost surely [due to (3.1) with $y=\widetilde{y}+\max_{i\ge1}|\delta_i|\cdot t$ and the first Borel–Cantelli lemma].

(2) For the uniqueness part let X' be a different weak solution to the unregulated version of (1.1). Then we can define inductively the sets $A'_0\subset A'_1\subset\cdots$ and the stopping times $0=\rho'_0\le\rho'_1\le\cdots$ by

$$A'_k=\{i\ge 1|\exists 1\le j\le M,0\le s\le\rho'_k:X'_i(s)=X'_{(j)}(s)\},\tag{3.3}$$

$$\rho'_{k+1}=\inf\{s\ge\rho'_k|\exists 1\le j\le M,i\notin A_k:X'_i(s)=X'_{(j)}(s)\}.\tag{3.4}$$

Due to the uniqueness of the weak solution to the unregulated system (1.1) in the case of finitely many particles, the joint distribution of $\rho'_0,\rho'_1,\dots$ and X' on $[\rho'_0,\rho'_1],[\rho'_1,\rho'_2],\dots$ has to coincide with the joint distribution of $\rho_0,\rho_1,\dots$ and X on $[\rho_0,\rho_1],[\rho_1,\rho_2],\dots$. Thus, the law of X' is the same as the law of X.

(3) To prove that (3.2) implies (3.1) we first observe that

$$\mathbb{E}\big[\max\big(|\sigma_1 B_1(t)+L_1(t)|,1\big)\big]<\infty\tag{3.5}$$

for any fixed $t\ge 0$. Noting that $\max(|x|,1)$ is a nonnegative continuous submultiplicative function (see Proposition 25.4 of [23]), we conclude from the Theorem 25.18 of [23] that

$$\mathbb{E}\Big[\sup_{0\le s\le t}|\sigma_1 B_1(s)+L_1(s)|\Big]<\infty\tag{3.6}$$

for any fixed $t\ge 0$. Thus, (3.2) implies (3.1). $\square$

3.2. *Tightness of the infinite regulated system.* We can now prove Proposition 1.4 which guarantees the tightness of the gap process in the infinite regulated evolution.

PROOF OF PROPOSITION 1.4. We start with the observation that for each $i\in I$ the process $X_i^R(t),t\ge 0$ and the corresponding process of regulations $R_i(t),t\ge 0$, solve the system of stochastic differential equations

$$dX_i^R(t)=\delta_1\,dt+\sigma_1\,dB_i(t)+dL_i(t)+dR_i(t),$$
$$dR_i(t)=\big(b-X_i^R(t-)-\Delta L_i(t)\big)1_{\{X_i^R(t-)+\Delta L_i(t)<b\}}+d\Lambda_{(i,b)}(t).$$

This is due to the fact that the drift and the volatility sequences are constant and that the set of collision times of distinct particles has Lebesgue measure zero almost surely. The latter statement is a consequence of the corresponding property of the regulated system with finitely many particles [see the proof of part (b) of Lemma 2.1] and the construction of the infinite particle systems (see the proof of Proposition 3.1). Next, for each $i \in I$ we introduce the process

$$H_i^R(t) = X_i^R(0) + \delta_1 t + \sigma_1 B_i(t) + \sum_{0 \le s \le t} |\Delta L_i(s)| + \Lambda^{H_i^R}(t), \tag{3.7}$$

where $\Lambda^{H_i^R}(t)$ is the local time of H_i^R at b. Using condition (1.10) and arguing as in the steps (1) and (2) of the proof of Theorem 1.2(a), we conclude that $X_i^R(t) \le H_i^R(t)$ for all $t \ge 0$ and $i \in I$ almost surely and, in addition, that for each $i \in I$ the process $H_i^R(t)$, $t \ge 0$, converges in law to an almost surely finite random variable $H_i^R(\infty)$ whose law does not depend on $X_i^R(0)$ and i. Moreover, due to the independence of the processes H_i^R, $i \in I$, the random vector $(H_1^R(t), \dots, H_j^R(t))$ converges in distribution to $(H_1^R(\infty), \dots, H_j^R(\infty))$ for any $j \in I$ where $H_i^R(\infty)$, $i \ge 2$, are chosen as independent copies of $H_1^R(\infty)$. Finally, the chain of inequalities

$$0 \le Z_j^R(t) \le \max_{1 \le i \le j} X_i^R(t) - b \le \max_{1 \le i \le j} H_i^R(t) - b \tag{3.8}$$

for all $t \ge 0$ shows that the family $Z_j^R(t)$, $t \ge 0$, is tight for all $j \in I$. This yields the tightness of $Z^R(t)$, $t \ge 0$, on $\mathbb{R}_+^{\mathbb{N}}$ with the product topology. □

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