STOCHASTIC POWER LAW FLUIDS: EXISTENCE AND UNIQUENESS OF WEAK SOLUTIONS

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We consider a stochastic partial differential equation (SPDE) which describes the velocity field of a viscous, incompressible non-Newtonian fluid subject to a random force. Here the extra stress tensor of the fluid is given by a polynomial of degree $p - 1$ of the rate of strain tensor, while the colored noise is considered as a random force. We investigate the existence and the uniqueness of weak solutions to this SPDE.

1. The power law fluids. We consider a viscous, incompressible fluid whose motion is subject to a random force. The container of the fluid is supposed to be the torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d \cong [0, 1]^d$ as a part of idealization. For a differentiable vector field $v : \mathbb{T}^d \to \mathbb{R}^d$, which is interpreted as the velocity field of the fluid, we denote the *rate of strain tensor* by

$$(1.1) \qquad e(v) = \left(\frac{\partial_i v_j + \partial_j v_i}{2} \right) : \mathbb{T}^d \to \mathbb{R}^d \otimes \mathbb{R}^d.$$

We assume that the extra stress tensor

$$\tau(v) : \mathbb{T}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$$

depends on $e(v)$ polynomially. More precisely, for $\nu > 0$ (the kinematic viscosity) and $p > 1$,

$$(1.2) \qquad \tau(v) = 2\nu \left(1 + |e(v)|^2\right)^{(p-2)/2} e(v).$$

The linearly dependent case $p = 2$ is the *Newtonian fluid* which is described by the Navier–Stokes equations, the special case of (1.3) and (1.4). On the other hand, both the *shear thinning* ($p < 2$) and the *shear thickening* ($p > 2$) cases are considered in many fields in science and engineering. For example, shear thinning fluids are used for automobile engine oil and pipeline for crude oil transportation, while applications of shear thickening fluids can be found in modeling of body armors and automobile four wheel driving systems.

Given an initial velocity $u_0 : \mathbb{T}^d \to \mathbb{R}^d$, the dynamics of the fluid are described by the following SPDE:

$$(1.3) \qquad\qquad \operatorname{div} u = 0,$$

$$(1.4) \qquad\qquad \partial_t u + (u \cdot \nabla) u = -\nabla \Pi + \operatorname{div} \tau(u) + \partial_t W,$$

where

$$(1.5) \qquad u \cdot \nabla = \sum_{j=1}^d u_j \, \partial_j \quad \text{and} \quad \operatorname{div} \tau(u) = \left(\sum_{j=1}^d \partial_j \tau_{ij}(u) \right)_{i=1}^d.$$

The unknown processes in the SPDE are the velocity field $u = u(t, x) = (u_i(t, x))_{i=1}^d$ and the pressure $\Pi = \Pi(t, x)$. The Brownian motion $W = W(t, x) = (W_i(t, x))_{i=1}^d$ with values in $L_2(\mathbb{T}^d \to \mathbb{R}^d)$ (the set of vector fields on $\mathbb{T}^d$ with L_2 components) is added as the random force. Physical interpretations of (1.3) and (1.4) are the mass conservation and the motion equation, respectively. We note that the SPDE (1.3) and (1.4) for the case $p = 2$ is the stochastic Navier–Stokes equation [2, 3].

Our motivation comes from works by Málek et al. [5], where the deterministic equation [the colored noise $\partial_t W$ in (1.3) and (1.4) is replaced by a nonrandom external force] is investigated. Let

$$(1.6) \qquad p_1(d) = \frac{3d}{d+2} \vee \frac{3d-4}{d} = \begin{cases} \dfrac{3d}{d+2}, & \text{for } d \le 4, \\[2mm] \dfrac{3d-4}{d}, & \text{for } d \ge 4, \end{cases}$$

$$(1.7) \qquad p_2(d) = \frac{2d}{d-2}, \qquad p_3(d) = \frac{3d - 8 + \sqrt{9d^2 + 64}}{2d}$$

and

$$(1.8) \qquad p \in \begin{cases} (p_1(d), \infty), & \text{if } 2 \le d \le 8, \\ (p_1(9), p_2(9)) \cup (p_3(9), \infty), & \text{if } d = 9, \\ (p_3(d), \infty), & \text{if } d \ge 10. \end{cases}$$

For example, $p_1(d) = \frac{3}{2}, \frac{9}{5}, 2, \frac{11}{5}$ for $d = 2, 3, 4, 5$. A basic existence theorem ([5], Theorem 3.4, page 222) states that the deterministic equation has a weak solution if (1.8) is satisfied, while a weak solution is unique if $p \ge 1 + \frac{d}{2}$ ([5], Theorem 4.29, page 254).

The results in the present paper (Theorems 2.1.3 and 2.2.1) confirm that the above-mentioned deterministic results are stable under the random perturbation we consider.

Let us briefly sketch the outline of the proof of our existence result.

Step 1. Set up a finite-dimensional subspace of a smooth, divergence-free vector field, say $\mathcal{V}_n$, and an approximating equation to the SPDE (1.3) and (1.4) in $\mathcal{V}_n$.

The good news here is that the approximating equation is a well posed stochastic differential equation (SDE) admitting a unique strong solution $u^n \in \mathcal{V}_n$. See Theorem 3.1.1 for detail.

Step 2. Establish some a priori bounds for the solution $u^n \in \mathcal{V}_n$ of the approximating SDE [e.g., (3.10), (3.13), (3.14) and (3.15)]. The point here is that the bounds should be *uniform in n* for them to be useful. Martingale inequalities (e.g., the Burkholder–Davis–Gundy inequality) are effectively used here, working in team with the Sobolev imbedding theorem. See, for example, the proof of (3.10) for details.

Step 3. Show that the solutions $u^n \in \mathcal{V}_n$ to the approximating SDE are tight as $n \to \infty$. This is where the a priori bounds in step 2 play their roles as the moment estimates to ensure that the tails of the solutions are thin enough in certain Sobolev norms. This tightness argument is implemented in Section 3.4.

Step 4. By step 3, u^n $(n \to \infty)$ converges in law along a subsequence to a limit. We verify that the limit is a weak solution to the SPDE (1.3) and (1.4). These will be the subjects of Section 4.1.

Here are some comments concerning the technical difference between the Navier–Stokes equations ($p = 2$) and the power law fluids. For the Navier–Stokes equations (both stochastic [2, 3] and deterministic [7]), it is reasonable to discuss solutions in the L_2-space. On the other hand, for the power law fluids given by (1.2), it is the L_p-space and its dual space that become relevant. Also, due to the extra nonlinearity introduced by (1.2), some of the arguments for $p \neq 2$ become considerably more involved than the case of $p = 2$, especially for $p < 2$. (See, e.g., proof of Lemma 3.2.2.) We will overcome this difficulty by carrying the ideas in [5] over to the framework of Itô's calculus.

1.1. *A weak formulation.* Let $\mathcal{V}$ be the set of $\mathbb{R}^d$-valued divergence free, mean-zero trigonometric polynomials, that is, the set of $v \colon \mathbb{T}^d \to \mathbb{R}^d$ of the following form:

$$(1.9) \qquad v(x) = \sum_{z \in \mathbb{Z}^d \setminus \{0\}} \widehat{v}_z \psi_z(x), \qquad x \in \mathbb{T}^d,$$

where $\psi_z(x) = \exp(2\pi \mathbf{i} z \cdot x)$ and the coefficients $\widehat{v}_z \in \mathbb{C}^d$, $z \in \mathbb{Z}^d$ satisfy

$$(1.10) \qquad \widehat{v}_z = 0 \qquad \text{except for finitely many } z,$$

$$(1.11) \qquad \overline{\widehat{v}_z} = \widehat{v}_{-z} \qquad \text{for all } z,$$

$$(1.12) \qquad z \cdot \widehat{v}_z = 0 \qquad \text{for all } z.$$

Note that (1.12) implies that

$$\operatorname{div} v = 0 \qquad \text{for all } v \in \mathcal{V}.$$

For $\alpha \in \mathbb{R}$ and $v \in \mathcal{V}$ we define

$$(1 - \Delta)^{\alpha/2} v = \sum_{z \in \mathbb{Z}^d} (1 + 4\pi^2 |z|^2)^{\alpha/2} \widehat{v}_z \psi_z.$$

We equip the torus $\mathbb{T}^d$ with the Lebesgue measure. For $p \in [1, \infty)$ and $\alpha \in \mathbb{R}$, we introduce

(1.13) $V_{p,\alpha} =$ the completion of $\mathcal{V}$ with respect to the norm $\| \cdot \|_{p,\alpha}$,

where

(1.14) $\|v\|_{p,\alpha}^p = \int_{\mathbb{T}^d} |(1 - \Delta)^{\alpha/2} v|^p.$

Then,

(1.15) $V_{p,\alpha+\beta} \subset V_{p,\alpha}$ for $1 \le p < \infty$, $\alpha \in \mathbb{R}$ and $\beta > 0$

and the inclusion $V_{p,\alpha+\beta} \to V_{p,\alpha}$ is compact if $1 < p < \infty$ ([6], (6.9), page 23).

For $v, w : \mathbb{T}^d \to \mathbb{R}^d$, with w supposed to be differentiable (for a moment), we define a vector field

(1.16) $(v \cdot \nabla)w = \sum_j v_j \, \partial_j w$

which is bilinear in (v, w). Later on, we will generalize the definition of the above vector field; cf. (1.31).

Here are integration-by-parts formulae with which we reformulate (1.3) and (1.4) into its weak formulation. In what follows, the bracket $\langle u, v \rangle$ stands for the inner product of $L_2(\mathbb{T}^d \to \mathbb{R}^d)$, or its appropriate generalization, for example, the pairing of $u \in V_{p,\alpha}$ and $u \in V_{p',-\alpha}$ ($p \in (1, \infty)$, $p' = \frac{p}{p-1}$, $\alpha \ge 0$). We let $C^r(\mathbb{T}^d \to \mathbb{R}^d)$ ($r = 1, \ldots, \infty$) denote the set of vector fields on $\mathbb{T}^d$ with C^r components.

LEMMA 1.1.1.   *For $v \in \mathcal{V}$ and $w, \varphi \in C^1(\mathbb{T}^d \to \mathbb{R}^d)$,*

(1.17) $\langle \varphi, (v \cdot \nabla)w \rangle = -\langle w, (v \cdot \nabla)\varphi \rangle.$

In particular,

(1.18) $\langle w, (v \cdot \nabla)w \rangle = 0.$

Furthermore,

(1.19) $\langle \varphi, \text{div}\, \tau(v) \rangle = -\langle \tau(v), e(\varphi) \rangle.$

PROOF.   Since $\text{div}\, v = 0$, we have that

$$\sum_j \partial_j (\varphi_i v_j) = \sum_j ((\partial_j \varphi_i) v_j + \varphi_i \, \partial_j v_j) = \sum_j (\partial_j \varphi_i) v_j.$$

Therefore,

$$\text{LHS of (1.17)} = \sum_{i,j} \langle \varphi_i, v_j \, \partial_j w_i \rangle = -\sum_{i,j} \langle \partial_j(\varphi_i v_j), w_i \rangle$$

$$\overset{(1)}{=} -\sum_{i,j} \langle (\partial_j \varphi_i) v_j, w_i \rangle = \text{RHS of (1.17)}.$$

Also, by integration by parts and the symmetry of τ_{ij},

$$\text{LHS of (1.19)} = -\sum_{i,j} \langle \partial_j \varphi_i, \tau_{ij}(v) \rangle = -\sum_{i,j} \langle e_{ij}(\varphi), \tau_{ij}(v) \rangle = \text{RHS of (1.19)}. \quad \square$$

Let us formally explain how the transformation of the problem (1.3) and (1.4) into its weak formulation is achieved. Suppose that u, Π and "$\partial_t W$" in (1.3) and (1.4) are regular enough. Then, for a test function $\varphi \in \mathcal{V}$,

$$(*) \qquad \partial_t \langle \varphi, u \rangle = -\underbrace{\langle \varphi, (u \cdot \nabla)u \rangle}_{(1)} + \underbrace{\langle \varphi, \operatorname{div} \tau(u) \rangle}_{(2)} - \underbrace{\langle \varphi, \nabla \Pi \rangle}_{(3)} + \langle \partial_t W, \varphi \rangle,$$

$$(1) \overset{(1.17)}{=} -\langle (u \cdot \nabla)\varphi, u \rangle, \ (2) \overset{(1.19)}{=} -\langle e(\varphi), \tau(u) \rangle, \ (3) = -\langle \operatorname{div} \varphi, \Pi \rangle = 0.$$

Thus, $(*)$ becomes

$$\partial_t \langle \varphi, u \rangle = \langle (u \cdot \nabla)\varphi, u \rangle - \langle e(\varphi), \tau(u) \rangle + \partial_t \langle \varphi, W \rangle.$$

By integration, we arrive at

$$(1.20) \quad \langle \varphi, u_t \rangle = \langle \varphi, u_0 \rangle + \int_0^t \left(\langle (u_s \cdot \nabla)\varphi, u_s \rangle - \langle e(\varphi), \tau(u_s) \rangle \right) ds + \langle \varphi, W_t \rangle.$$

Here $u_t = u(t, \cdot)$ and $W_t = W(t, \cdot)$. This is a standard weak formulation of (1.3) and (1.4).

1.2. *Bounds on the nonlinear terms.* Let us prepare a couple of L_p-bounds on the nonlinear terms. They will be used to derive a priori bounds for the solutions later on.

LEMMA 1.2.1. *Let $\alpha_i \in [0, \infty)$, $p_i \in [1, \infty)$, $i = 1, 2, 3$, be such that*

$$(1.21) \qquad A \geq Bd, \qquad \text{where } A = \sum_i \alpha_i \text{ and } B = \sum_i \frac{1}{p_i} - 1.$$

(a) *Suppose* (1.21) *and that* $\frac{\alpha_i B}{A} < \frac{1}{p_i}$ *for all* $i = 1, 2, 3$. *Then, there exists* $C_1 \in (0, \infty)$ *such that*

$$(1.22) \qquad |\langle w, (v \cdot \nabla)\varphi \rangle| \leq C_1 \|v\|_{p_1, \alpha_1} \|w\|_{p_2, \alpha_2} \|\varphi\|_{p_3, 1+\alpha_3}$$

for $v, w, \varphi \in C^\infty(\mathbb{T}^d \to \mathbb{R}^d)$.

(b) *Suppose* (1.21), $\alpha_1 + \alpha_2 > 0$ *and that* $B \leq \frac{1}{p_i}$ *for all* $i = 1, 2, 3$. *Then, for any* $\theta \in (0, 1)$, *there exists* $C_2 \in (0, \infty)$ *such that*

$$(1.23) \quad |\langle w, (v \cdot \nabla)\varphi\rangle| \leq C_2 \|v\|_{p_1,\alpha_1}^\theta \|v\|_{p_1,\alpha_2}^{1-\theta} \|w\|_{p_2,\alpha_1}^{1-\theta} \|w\|_{p_2,\alpha_2}^\theta \|\varphi\|_{p_3,1+\alpha_3}.$$

PROOF.   (a) Since

$$\sum_{i,j} |w_i v_j \, \partial_j \varphi_i| \leq |w||v||\nabla\varphi|,$$

we have

$$(1) \quad |\langle w, (v \cdot \nabla)\varphi\rangle| \leq \|v\|_{q_1} \|w\|_{q_2} \|\nabla\varphi\|_{q_3} \qquad \text{whenever } \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} \leq 1.$$

Case 1. $B \leq 0$: We apply (1) with $q_i = p_i$ ($i = 1, 2, 3$) to get (1.22).

Case 2. $B > 0$: Since $\alpha \mapsto \| \cdot \|_{p_i,\alpha}$ is increasing $[(1 - \Delta)^{-\alpha/2}$ is a contraction on $L_p(T^d \to \mathbb{R}^d)$ for any $\alpha \geq 0$ and $p \geq 1]$, it is enough to prove (1.22) with α_i replaced by $\tilde{\alpha}_i = \frac{\alpha_i}{A} Bd$. Therefore, we may assume without loss of generality that

$$\max_i p_i \alpha_i < d \quad \text{and} \quad A = Bd.$$

We apply (1) to $q_i \in [p_i, \infty)$, $i = 1, 2, 3$ defined by $\frac{1}{q_i} = \frac{1}{p_i} - \frac{\alpha_i}{d}$. We then use the following Sobolev imbedding theorem (e.g., [6], formula (2.11), page 5). If $\alpha p < d$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$, then there exists $C = C(d, \alpha) \in (0, \infty)$ such that

$$(1.24) \qquad \|v\|_q \leq C\|v\|_{p,\alpha} \qquad \text{for all } v \in C^\infty(\mathbb{T}^d \to \mathbb{R}^d).$$

(b) Let us note the following interpolation inequality (e.g., [6], formula (6.5), page 23): for any $\lambda \in [0, 1]$,

$$(2) \qquad \|u\|_{p_i,\lambda\alpha_1+(1-\lambda)\alpha_2} \leq C\|u\|_{p_i,\alpha_1}^\lambda \|u\|_{p_i,\alpha_2}^{1-\lambda} \qquad \text{for } u \in V_{p_i,\alpha_1} \cap V_{p_i,\alpha_2}.$$

On the other hand, we note that the assumptions for (1.22) are satisfied if we replace (α_1, α_2) by

$$(\theta\alpha_1 + (1-\theta)\alpha_2, (1-\theta)\alpha_1 + \theta\alpha_2).$$

Thus,

$$|\langle w, (v \cdot \nabla)\varphi\rangle| \overset{(1.22)}{\leq} C_1 \|v\|_{p_1,\theta\alpha_1+(1-\theta)\alpha_2} \|w\|_{p_2,(1-\theta)\alpha_1+\theta\alpha_2} \|\varphi\|_{p_3,1+\alpha_3}$$

$$\overset{(2)}{\leq} \text{RHS of (1.23).} \qquad \qquad \square$$

LEMMA 1.2.2.   *Let* $\alpha \in (0, 1]$ *and* $p \in (\frac{2d}{d+2\alpha}, \infty)$.

(a) *Suppose that* $(d, p, \alpha) \neq (2, 2, 1)$. *Then there exists* $C_1 \in (0, \infty)$ *such that*

$$(1.25) \qquad |\langle w, (v \cdot \nabla)\varphi\rangle| \leq C_1 \|v\|_{p,\alpha} \|w\|_2 \|\varphi\|_{p,\beta(p,\alpha)}$$

for $v, w, \varphi \in C^\infty(\mathbb{T}^d \to \mathbb{R}^d)$, *where*

$$(1.26) \qquad \beta(p, \alpha) = \begin{cases} 1 + \left(\dfrac{2}{p} - \dfrac{1}{2}\right)d - \alpha > 1, & \text{if } p < \dfrac{4d}{d + 2\alpha}, \\ 1, & \text{if } p \geq \dfrac{4d}{d + 2\alpha}. \end{cases}$$

(b) *Suppose that* $d = 2$. *Then for any* $\theta \in (0, 1)$, *there exists* $C_2 \in (0, \infty)$ *such that*

$$(1.27) \qquad |\langle w, (v \cdot \nabla)\varphi\rangle| \leq C_2 \|v\|_{2,1}^\theta \|v\|_2^{1-\theta} \|w\|_{p,1}^{1-\theta} \|w\|_2^\theta \|\varphi\|_{2,1}$$

for $v, w, \varphi \in C^\infty(\mathbb{T}^d \to \mathbb{R}^d)$.

PROOF. We apply Lemma 1.2.1 to

$$(p_1, p_2, p_3) = (p, 2, p), \qquad (\alpha_1, \alpha_2) = (\alpha, 0), \qquad \alpha_3 = \left(\left(\frac{2}{p} - \frac{1}{2}\right)d - \alpha\right)^+.$$

Then $\beta(p, \alpha) = 1 + \alpha_3$, $A = \alpha + \alpha_3$, $B = \frac{2}{p} - \frac{1}{2}$. It is enough to check that the assumptions of Lemma 1.2.1(b) are satisfied if $(d, p, \alpha) = (2, 2, 1)$ and that the assumptions of Lemma 1.2.1(a) are satisfied if $(d, p, \alpha) \neq (2, 2, 1)$. In fact, the verification for the case $(d, p, \alpha) = (2, 2, 1)$ can be done by simply plugging the values. We assume $(d, p, \alpha) \neq (2, 2, 1)$ in what follows. We may assume that $B > 0$, or equivalently $p < 4$. We have $A \geq Bd$ by the choice of α_i's. Let us check that

$$(1) \qquad \frac{\alpha_1}{A}B = \frac{\alpha}{\alpha + \alpha_3}\left(\frac{2}{p} - \frac{1}{2}\right) < \frac{1}{p}.$$

If $(d, p, \alpha) \neq (2, 2, 1)$ and $p \geq \frac{4d}{d+2\alpha}$ (which implies $p > 2$), then $\alpha_3 = 0$ and (1) is satisfied. If $(d, p, \alpha) \neq (2, 2, 1)$ and $p < \frac{4d}{d+2\alpha}$ (which implies $p < \frac{d}{\alpha}$), then $\alpha_3 = (\frac{2}{p} - \frac{1}{2})d - \alpha > 0$. One then sees that (1) is equivalent to that $p < \frac{d}{\alpha}$ and hence, is satisfied. Let us check that

$$(2) \qquad \frac{\alpha_3}{A}B = \frac{\alpha_3}{\alpha + \alpha_3}\left(\frac{2}{p} - \frac{1}{2}\right) < \frac{1}{p}.$$

If $(d, p, \alpha) \neq (2, 2, 1)$ and $p \geq \frac{4d}{d+2\alpha}$, then $\alpha_3 = 0$ and (2) is satisfied. If $p < \frac{4d}{d+2\alpha}$, then $\alpha_3 = (\frac{2}{p} - \frac{1}{2})d - \alpha > 0$. One then sees that (2) is equivalent to that $p > \frac{2d}{d+2\alpha}$ and hence, is satisfied. $\square$

REMARK. We note that the following variant of (1.25) is also true:

$$(1.28) \qquad |\langle w, (v \cdot \nabla)\varphi\rangle| \leq C_1 \|v\|_2 \|w\|_{p,\alpha} \|\varphi\|_{p,\beta(p,\alpha)}.$$

This can be seen by interchanging the role of (p_1, α_1) and (p_2, α_2) in the above proof.

LEMMA 1.2.3.   *For $p \in (1, \infty)$, there exists $C_1 \in (0, \infty)$ such that*

(1.29)
$$|\langle e(\varphi), \tau(v) \rangle| \leq C_1 \big(1 + \|e(v)\|_p\big)^{p-1} \|e(\varphi)\|_p$$

for all $v \in V_{p,1}$ and $\varphi \in \mathcal{V}$.

PROOF.   Since

$$|\tau(v)| \leq C \big(1 + |e(v)|\big)^{p-1},$$

we have that

$$|\langle e(\varphi), \tau(v) \rangle| \quad \leq \quad C \int_{\mathbb{T}^d} \big(1 + |e(v)|\big)^{p-1} |e(\varphi)|$$

$$\overset{(p-1)/p+1/p=1}{\leq} \quad C \big\| 1 + |e(v)| \big\|_p^{p-1} \|e(\varphi)\|_p$$

$$\leq \quad C \big(1 + \|e(v)\|_p\big)^{p-1} \|e(\varphi)\|_p,$$

which proves (1.29).   □

Let $p \in (\frac{2d}{d+2}, \infty)$, $v, w \in V_{p,1} \cap V_{2,0}$ and $u \in V_{p,1}$. In view of Lemma 1.1.1, we think of $(v \cdot \nabla)w$ and $\operatorname{div} \tau(u)$, respectively, as the following linear functionals on $\mathcal{V}$:

$$\varphi \mapsto \langle \varphi, (v \cdot \nabla)w \rangle \overset{\text{def.}}{=} -\langle w, (v \cdot \nabla)\varphi \rangle,$$

$$\varphi \mapsto \langle \varphi, \operatorname{div} \tau(u) \rangle \overset{\text{def.}}{=} -\langle e(\varphi), \tau(u) \rangle.$$

Then, by Lemmas 1.2.2 and 1.2.3, they extend continuously, respectively, on $V_{p, \beta(p,1)}$ and on $V_{p,1}$, where

(1.30)
$$\beta(p, 1) = \begin{cases} \left(\dfrac{2}{p} - \dfrac{1}{2}\right) d > 1, & \text{if } p < \dfrac{4d}{d+2}, \\ 1, & \text{if } p \geq \dfrac{4d}{d+2}; \end{cases}$$

cf. (1.26). This way, we regard $(v \cdot \nabla)w \in V_{p', -\beta(p,1)}$ $(p' = \frac{p}{p-1})$ with

(1.31)
$$\begin{aligned} &\|(v \cdot \nabla)w\|_{p', -\beta(p,1)} \\ &\qquad \leq \begin{cases} C\|v\|_{2,1}^{\theta} \|v\|_2^{1-\theta} \|w\|_{2,1}^{1-\theta} \|w\|_2^{\theta}, & \text{if } p = d = 2, \\ C\|v\|_{p,1} \|w\|_2, & \text{if otherwise,} \end{cases} \end{aligned}$$

and $\operatorname{div} \tau(u) \in V_{p', -1}$ with

(1.32)
$$\|\operatorname{div} \tau(u)\|_{p', -1} \leq C \big(1 + \|e(u)\|_p\big)^{p-1}.$$

Finally, for $v \in V_{p,1} \cap V_{2,0}$, we define

(1.33)
$$b(v) = -(v \cdot \nabla)v + \operatorname{div} \tau(v) \in V_{p', -\beta(p,1)}.$$

With this notation, (1.20) takes the form

$$\langle \varphi, u_t \rangle = \langle \varphi, u_0 \rangle + \int_0^t \langle \varphi, b(u_s) \rangle \, ds + \langle \varphi, W_t \rangle,$$

that is,

(1.34)
$$u_t = u_0 + \int_0^t b(u_s) \, ds + W_t$$

as linear functionals on $\mathcal{V}$.

2. The stochastic power law fluids.

2.1. *The existence theorem.* We need the following definition.

DEFINITION 2.1.1. Let H be a Hilbert space and $\Gamma : H \to H$ be a self-adjoint, nonnegative definite operator of trace class. A random variable $(W_t)_{t \geq 0}$ with values in $C([0, \infty) \to H)$ is called an *H-valued Brownian motion* with the covariance operator Γ [abbreviated by BM(H, Γ) below] if, for each $\varphi \in H$ and $0 \leq s < t$,

$$E[\exp(\mathbf{i} \langle \varphi, W_t - W_s \rangle) | (W_u)_{u \leq s}] = \exp \left(-\frac{t - s}{2} \langle \varphi, \Gamma \varphi \rangle \right), \qquad \text{a.s.}$$

To introduce the notion of weak solution (Definition 2.1.2), we agree on the following standard notation and convention. For a Banach space X, we let $L_{q,\mathrm{loc}}([0, \infty) \to X)$ $(1 \leq q \leq \infty)$ denote the set of locally L_q-functions $u : [0, \infty) \to X$, with the Fréchet space metric induced by the semi-norms $\|u\|_{L_q([0,T] \to X)}$, $0 < T < \infty$, where $\|u\|_{L_q([0,T] \to X)}$ stands for the standard L_q-norm for $u|_{[0,T]} : [0, T] \to X$. We also regard $C([0, \infty) \to X)$, the set of continuous functions $u : [0, \infty) \to X$, as the Fréchet space induced by the semi-norms $\sup_{0 \leq t \leq T} \|u(t)\|_X$, $0 < T < \infty$.

We recall that the number p is from (1.2) and that $b(v) \in V_{p',-\beta(p,1)}$ for $v \in V_{p,1} \cap V_{2,0}$ is defined by (1.33).

DEFINITION 2.1.2. Suppose that:

- $\Gamma : V_{2,0} \to V_{2,0}$ is a bounded self-adjoint, nonnegative definite operator of trace class;
- μ_0 is a Borel probability measure on $V_{2,0}$;
- $(X, Y) = ((X_t, Y_t))_{t \geq 0}$ is a process defined on a probability space $(\Omega, \mathcal{F}, P)$ such that

(2.1)
$$X \in L_{p,\mathrm{loc}}([0, \infty) \to V_{p,1}) \cap L_{\infty,\mathrm{loc}}([0, \infty) \to V_{2,0})$$
$$\cap C([0, \infty) \to V_{2 \wedge p',-\beta})$$

for some $\beta > 0$ and $(Y_t)_{t \geq 0}$ is a BM$(V_{2,0}, \Gamma)$; cf. Definition 2.1.1.

Then the process (X, Y) is said to be a *weak solution* to the SDE

$$(2.2) \qquad X_t = X_0 + \int_0^t b(X_s)\,ds + Y_t$$

with the initial law μ_0 if the following conditions are satisfied:

$$(2.3) \qquad\qquad P(X_0 \in \cdot) = \mu_0;$$

(2.4) $Y_{t+\cdot} - Y_t$ and $\{\langle \varphi, X_s \rangle; s \le t, \varphi \in \mathcal{V}\}$ are independent for any $t \ge 0$;

(2.5) $\langle \varphi, X_t \rangle = \langle \varphi, X_0 \rangle + \int_0^t \langle \varphi, b(X_s) \rangle\,ds + \langle \varphi, Y_t \rangle$ for all $\varphi \in \mathcal{V}$ and $t \ge 0$.

We can now state our existence result.

THEOREM 2.1.3. *Let Γ and μ_0 be as in Definition* 2.1.2 *and suppose additionally that*:

- (1.8) *holds*;
- $\Delta\Gamma = \Gamma\Delta$ *and both* Γ, $\Delta\Gamma$ *are of trace class*;
- μ_0 *is a probability measure on* $V_{2,1}$ *and*

$$(2.6) \qquad m_\alpha = \int \|\xi\|_{2,\alpha}^2 \mu_0(d\xi) < \infty \qquad \text{for } \alpha = 0, 1.$$

Then there exists a weak solution to the SDE (2.2) *with the initial law* μ_0; *cf. Definition* 2.1.2 *such that* (2.1) *holds with* $\beta = \beta(p, 1)$; *cf.* (1.30). *Moreover, for any $T > 0$,*

$$(2.7) \qquad E\left[\sup_{t \le T} \|X_t\|_2^2 + \int_0^T \|X_t\|_{p,1}^p\,dt\right] \le (1 + T)C < \infty,$$

where $C = C(d, p, \Gamma, m_0) < \infty$.

REMARK. It would be worthwhile to mention that Theorem 2.1.3 with $p = 2$ is valid for *all* d, although it is not covered by the condition (1.8) if $d \ge 4$. In fact, Lemma 3.2.2 is the only place we need condition (1.8). For $p = 2$, however, we can avoid the use of that lemma; cf. remarks at the end of Section 3.4 and after Lemma 4.1.1.

2.2. *The uniqueness theorem.* As in the case of the deterministic equation [5], Theorem 4.29, page 254, we have the following uniqueness result:

THEOREM 2.2.1. *Suppose that*

$$(2.8) \qquad\qquad p \ge 1 + \frac{d}{2}.$$

Then the weak solution to the SDE (2.2), *subject to the a priori bound* (2.7), *is pathwise unique in the following sense*: *if* (X, Y) *and* $(\widetilde{X}, Y)$ *are two solutions on a common probability space* $(\Omega, \mathcal{F}, P)$ *with a common* $\mathrm{BM}(V_{2,0}, \Gamma)$ Y *such that* $X_0 = \widetilde{X}_0$ *a.s., then,*

$$P(X_t = \widetilde{X}_t \text{ for all } t \geq 0) = 1.$$

The above uniqueness theorem, together with the Yamada–Watanabe theorem provides us with the so-called *strong solution in the stochastic sense* to the SDE (2.2).

COROLLARY 2.2.2. *Suppose* (2.8), *in addition to all the assumptions in Theorem* 2.1.3, *and let* ξ *be a given* $V_{2,0}$-*valued random variable with the law* μ_0 *and* Y *be a given* $\mathrm{BM}(V_{2,0}, \Gamma)$ *independent of* ξ. *Then there exists a process* X *obtained as a function of* (ξ, Y), *such that* (X, Y) *is weak solution to the SDE* (2.2) *with* $X_0 = \xi$ *and with all the properties stated in Theorem* 2.1.3. *Moreover, the law of the above process* X *is unique.*

PROOF. Corollary 2.2.2 is a direct consequence of Theorems 2.1.3 and 2.2.1 via the Yamada–Watanabe theorem [1], Theorem 1.1, page 163. The Yamada–Watanabe theorem is usually stated for SDEs in finite dimensions. However, as is obvious from its proof, it applies to the present setting. □

REMARK 2.2.3. For $p \in [1 + \frac{d}{2}, \frac{2d}{d-2})$, an even stronger version of Corollary 2.2.2 is shown in [8] as a consequence of strong convergence of the Galerkin approximation; cf. Section 3.

3. The Galerkin approximation.

3.1. *The exsitence theorem for the approximations.* For each $z \in \mathbb{Z}^d \setminus \{0\}$, let $\{e_{z,j}\}_{j=1}^{d-1}$ be an orthonormal basis of the hyperplane $\{x \in \mathbb{R}^d; z \cdot x = 0\}$ and let

$$
(3.1) \quad
\begin{aligned}
&\psi_{z,j}(x) \\
&= \begin{cases} \sqrt{2} e_{z,j} \cos(2\pi z \cdot x), & j = 1, \ldots, d-1, \\ \sqrt{2} e_{z,j-d+1} \sin(2\pi z \cdot x), & j = d, \ldots, 2d-2, \end{cases} \quad x \in \mathbb{T}^d.
\end{aligned}
$$

Then

$$\{\psi_{z,j}; (z, j) \in (\mathbb{Z}^d \setminus \{0\}) \times \{1, \ldots, 2d - 2\}\}$$

is an orthonormal basis of $V_{2,0}$. We also introduce

$$
(3.2) \quad
\begin{aligned}
\mathcal{V}_n &= \text{the linear span of } \{\psi_{z,j}; (z, j) \text{ with } z \in [-n, n]^d\}; \\
\mathcal{P}_n &= \text{the orthogonal projection}: V_{2,0} \to \mathcal{V}_n.
\end{aligned}
$$

Using the orthonormal basis (3.1), we identify $\mathcal{V}_n$ with $\mathbb{R}^N$, $N = \dim \mathcal{V}_n$. Let μ_0 and Γ, $V_{2,0} \to V_{2,0}$, be as in Theorem 2.1.3. Let also ξ be a random variable such that $P(\xi \in \cdot) = \mu_0$. Finally, let W_t be a $\mathrm{BM}(V_{2,0}, \Gamma)$ defined on a probability space $(\Omega^W, \mathcal{F}^W, P^W)$. Then, $\mathcal{P}_n W_t$ is identified with an N-dimensional Brownian motion with covariance matrix $\Gamma \mathcal{P}_n$. Then we consider the following approximation of (2.5):

$$(3.3) \qquad X_t^n = X_0^n + \int_0^t \mathcal{P}_n b(X_s^n)\, ds + \mathcal{P}_n W_t, \qquad t \geq 0,$$

where $X_0^n = \mathcal{P}_n \xi$. Let

$$(3.4) \qquad X_t^{n,z,j} = \langle X_t^n, \psi_{z,j} \rangle$$

be the (z, j)-coordinate of X_t^n. Then (3.3) reads

$$(3.5) \qquad X_t^{n,z,j} = X_0^{n,z,j} + \int_0^t b^{z,j}(X_s^n)\, ds + W_t^{z,j},$$

where

$$(3.6) \qquad \begin{aligned} b^{z,j}(X_s^n) &= \langle X_s^n, (X_s^n \cdot \nabla)\psi_{z,j} \rangle - \langle \tau(X_s^n), e(\psi_{z,j}) \rangle, \\ W_t^{z,j} &= \langle W_t, \psi_{z,j} \rangle. \end{aligned}$$

Let $W.$ and ξ be as above. We then define

$$\mathcal{G}_t^{\xi, W} = \sigma(\xi, W_s, s \leq t), \qquad 0 \leq t < \infty, \qquad \mathcal{G}_\infty^{\xi, W} = \sigma\left(\bigcup_{t \geq 0} \mathcal{G}_t^{\xi, W}\right),$$

$$\mathcal{N}^{\xi, W} = \{N \subset \Omega; \exists \tilde{N} \in \mathcal{G}_\infty^{\xi, W}, N \subset \tilde{N}, P^W(\tilde{N}) = 0\}$$

and

$$(3.7) \qquad \mathcal{F}_t^{\xi, W} = \sigma(\mathcal{G}_t^{\xi, W} \cup \mathcal{N}^{\xi, W}), \qquad 0 \leq t < \infty.$$

In what follows, expectation with respect to the measure P^W will be denoted by $E^W[\cdot]$.

THEOREM 3.1.1. *Let $W.$, ξ and $\mathcal{F}_t^{\xi, W}$ be as above. Then for each $n = 1, 2, \ldots$ there exists a unique process X^n such that*:

(a) *X_t^n is $\mathcal{F}_t^{\xi, W}$-measurable for all $t \geq 0$;*
(b) *(3.3) is satisfied;*
(c) *For any $T > 0$,*

$$(3.8) \quad E^W\left[\|X_T^n\|_2^2 + 2\int_0^T \langle e(X_t^n), \tau(X_t^n) \rangle\, dt\right] = E^W[\|X_0^n\|_2^2] + \mathrm{tr}(\Gamma \mathcal{P}_n)T,$$

$$(3.9) \qquad E^W\left[\|X_T^n\|_2^2 + \frac{1}{C}\int_0^T \|X_t^n\|_{p,1}^p\, dt\right] \leq m_0 + (C + \mathrm{tr}(\Gamma))T < \infty,$$

where $C = C(d, p) \in (0, \infty)$.

Suppose, in addition, that $p \geq \frac{2d}{d+2}$, where p is from (1.2). Then, for any $T > 0$,

$$
(3.10) \qquad E^W\left[\sup_{t \leq T} \|X_t^n\|_2^2 + \int_0^T \|X_t^n\|_{p,1}^p \, dt\right] \leq (1+T)C' < \infty,
$$

where $C' = C'(d, p, \Gamma, m_0) \in (0, \infty)$.

PROOF. We fix the accuracy n of the approximation introduced above and suppress the superscript "n" from the notation $X = X^n$. We write the summation over $z \in [-n, n]^d$ and $j = 1, \ldots, 2d - 2$ simply by $\sum_{z,j}$. Since $v \mapsto \mathcal{P}_n b(v) : \mathcal{V}_n \to \mathcal{V}_n$ is locally Lipschitz continuous [see (3.6)] and

$$
(1) \qquad \langle v, b(v) \rangle \stackrel{(1.18)}{=} -\langle e(v), \tau(v) \rangle \leq C - \frac{1}{C}\|v\|_{p,1}^p,
$$

where we have used [5], formula (1.11), page 196, and formula (1.20)$_2$, page 198, to see the second inequality. This implies that there exists a unique process $X_{\cdot}$ with the properties (a)–(b) above, as can be seen from standard existence and uniqueness results for the SDE, for example, [1], Theorem 2.4, page 177, and Theorem 3.1, pages 178–179; cf. the remark after the proof. Note that for $\alpha = 0, 1, 2, \ldots,$

$$
\|\nabla^\alpha v\|_2^2 = \langle v, (-\Delta)^\alpha v \rangle = \sum_{z,j} (-4\pi^2 |z|^2)^\alpha \langle v, \psi_{z,j} \rangle^2, \qquad v \in \mathcal{V}_n.
$$

On the other hand, we have by Itô's formula that

$$
|X_t^{z,j}|^2 = |X_0^{z,j}|^2 + 2\int_0^t X_s^{z,j} \, dW_s^{z,j} + 2\int_0^t X_s^{z,j} b_s^{z,j}(X_s) \, ds + \langle \psi_{z,j}, \Gamma \psi_{z,j} \rangle t.
$$

Therefore,

$$
(3.11) \qquad \begin{aligned}
\|\nabla^\alpha X_t\|_2^2 = {} & \|\nabla^\alpha X_0\|_2^2 + 2M_t + 2\int_0^t \langle (-\Delta)^\alpha X_s, b(X_s) \rangle \, ds \\
& + \mathrm{tr}(\Gamma(-\Delta)^\alpha \mathcal{P}_n)t,
\end{aligned}
$$

where

$$
(3.12) \qquad M_t = \sum_{z,j} \int_0^t (-\Delta)^\alpha X_s^{z,j} \, dW_s^{z,j}.
$$

Here we will use (3.11) only for $\alpha = 0$. The case $\alpha = 1$ will be used in the proof of Lemma 3.2.3 later on. By (3.11) with $\alpha = 0$,

$$
(2) \qquad \|X_t\|_2^2 + \frac{2}{C}\int_0^t \|X_s\|_{p,1}^p \, ds \leq \|X_0\|_2^2 + 2M_t + (C + \mathrm{tr}(\Gamma))t,
$$

where M_t in (2) is defined by (3.12) with $\alpha = 0$. Since it is not difficult to see that the above M_t is a martingale (cf. [2], proof of (10), page 60), we get (3.8) by taking expectation of the equality (3.11). Similarly, we obtain (3.9) by taking expectation

of the inequality (2). To see (3.10), it is enough to show that there exists $\delta \in (0, 1]$ such that

$$(3) \qquad E^W\Big[\sup_{t \leq T} \|X_t\|_2^2\Big] \leq (1 + T)C + CE^W\Big[\Big(\int_0^T \|X_t\|_{p,1}^p \, dt\Big)^\delta\Big].$$

To see this, we start with a bound on the quadratic variation of the martingale $M_\cdot$,

$$(4) \qquad \langle M \rangle_t = \int_0^t \langle \Gamma X_s, X_s \rangle \, ds \leq \|\Gamma\|_{2 \to 2} \int_0^t \|X_s\|_2^2 \, ds,$$

where $\|\Gamma\|_{2 \to 2}$ denotes the operator norm of $\Gamma : V_{2,0} \to V_{2,0}$. We now recall the Burkholder–Davis–Gundy inequality ([1], Theorem 3.1, page 110),

$$(5) \qquad E^W\Big[\sup_{t \leq T} |M_t|^q\Big] \leq CE^W[\langle M \rangle_T^{q/2}] \quad \text{for } q \in (0, \infty).$$

We then observe that

$$E^W\Big[\sup_{t \leq T} \|X_t\|_2^2\Big] \overset{(2)}{\leq} (1 + T)C + 2E^W\Big[\sup_{t \leq T} |M_t|\Big]$$

$$(6)$$

$$\overset{(4)-(5)}{\leq} (1 + T)C + C'E^W\Big[\Big(\int_0^T \|X_s\|_2^2 \, ds\Big)^{1/2}\Big].$$

This proves (3) for $p \geq 2$. We assume $p < 2$ in what follows. We have

$$e_\ell \overset{\text{def.}}{=} \inf\{t; \|X_t\|_2 \geq \ell\} \nearrow \infty, \qquad \text{as } \ell \nearrow \infty,$$

since the process X_t does not explode. On the other hand, it is clear that the following variant of (6) is true:

$$(6') \qquad E^W\Big[\sup_{t \leq T \wedge e_\ell} \|X_t\|_2^2\Big] \leq (1 + T)C + CE^W\Big[\Big(\int_0^{T \wedge e_\ell} \|X_s\|_2^2 \, ds\Big)^{1/2}\Big].$$

We have by Sobolev embedding that for $v \in V_{p,1}$,

$$(7) \qquad \|v\|_2 \leq C\|v\|_{p,1}, \qquad \text{since } p \geq \frac{2d}{d+2}.$$

Let $\varepsilon > 0$, $r = \frac{4}{2-p} \in (4, \infty)$ and $r' = \frac{r}{r-1} = \frac{4}{2+p} \in (1, 4/3)$. Then,

$$\Big(\int_0^{T \wedge e_\ell} \|X_s\|_2^2 \, ds\Big)^{1/2}$$

$$(8)$$

$$\leq \sup_{s \leq T \wedge e_\ell} \|X_s\|_2^{(2-p)/2} \Big(\int_0^{T \wedge e_\ell} \|X_s\|_2^p \, ds\Big)^{1/2}$$

$$\overset{(7)}{\leq} C \sup_{s \leq T \wedge e_\ell} \|X_s\|_2^{(2-p)/2} \Big(\int_0^{T \wedge e_\ell} \|X_s\|_{p,1}^p \, ds\Big)^{1/2}$$

$$\overset{\text{Young}}{\leq} \frac{\varepsilon^r C}{r} \sup_{s \leq T \wedge e_\ell} \|X_s\|_2^2 + \frac{\varepsilon^{-r'} C}{r'} \Big(\int_0^{T \wedge e_\ell} \|X_s\|_{p,1}^p \, ds\Big)^{2/(2+p)}.$$

Since $E^W[\sup_{t \leq T \wedge e_\ell} \|X_t\|_2^2] \leq \ell^2 < \infty$, we have by (6) and (8) that

$$E^W\left[\sup_{t \leq T \wedge e_\ell} \|X_t\|_2^2\right] \leq (1+T)C + CE^W\left[\left(\int_0^{T \wedge e_\ell} \|X_t\|_{p,1}^p \, dt\right)^{2/(2+p)}\right].$$

Letting $\ell \nearrow \infty$, we obtain (3). $\square$

REMARK. Unfortunately, the SDE (3.3) does not satisfy the condition (2.18) imposed in the existence theorem ([1], Theorem 2.4, page 177). However, we easily see from the proof of the existence theorem that (2.18) there can be replaced by

$$\|\sigma(x)\|^2 + x \cdot b(x) \leq K(1 + |x|^2).$$

We have applied [1], Theorem 2.4, page 177, with this modification.

3.2. *Further a priori bounds.* We first prove the following general estimates which apply both to the weak solution X to (2.2) and to the unique solution to (3.3).

LEMMA 3.2.1. *Let $T > 0$ and $X = (X_t)_{t \geq 0}$ be a process on a probability space $(\Omega, \mathcal{F}, P)$ such that*

$$X \in L_p([0,T] \to V_{p,1}) \cap L_\infty([0,T] \to V_{2,0}), \qquad a.s.$$

and

$$A_T = E\left[\int_0^T \|X_s\|_{p,1}^p \, ds\right] < \infty, \qquad B_T = E\left[\sup_{s \in [0,T]} \|X_s\|_2^2\right] < \infty.$$

(a) *For $p \in [\frac{2d}{d+2}, \infty)$,*

$$(3.13) \qquad E\left[\left(\int_0^T \|(X_s \cdot \nabla)X_s\|_{p',-\beta(p,1)}^p \, ds\right)^\delta\right] \leq C A_T^\delta B_T^{1-\delta} < \infty,$$

where $\delta = \frac{p}{p+2}$, $p' = \frac{p}{p-1}$, $\beta(p,1)$ is defined by (1.30) and $C = C(d,p) \in (0,\infty)$.

(b)

$$(3.14) \qquad E\left[\int_0^T \|\operatorname{div}\tau(X_s)\|_{p',-1}^{p'} \, ds\right] \leq (T + A_T)C' < \infty,$$

where $C' = C'(p,v) \in (0,\infty)$.

PROOF. (a) We have by (1.31) that

$$(1) \qquad \|(v \cdot \nabla)v\|_{p',-\beta(p,1)} \leq C \|v\|_{p,1} \|v\|_2 \qquad \text{for } v \in V_{p,1} \cap V_{2,0}.$$

We then use (1) to see that

$$I \overset{\text{def.}}{=} \int_0^T \|(X_s \cdot \nabla)X_s\|_{p',-\beta(p,1)}^p \, ds \overset{(1)}{\leq} C \int_0^T \|X_s\|_{p,1}^p \|X_s\|_2^p \, ds$$

$$\leq C \sup_{s \in [0,T]} \|X_s\|_2^p \int_0^T \|X_s\|_{p,1}^p \, ds.$$

Finally, noting that $\frac{p\delta}{1-\delta} = 2$, we conclude that

$$E[I^\delta] \leq CE\left[\sup_{s \in [0,T]} \|X_s\|_2^{p\delta} \left(\int_0^T \|X_s\|_{p,1}^p \, ds \right)^\delta \right]$$

$$\leq CE\left[\sup_{s \in [0,T]} \|X_s\|_2^2 \right]^{1-\delta} E\left[\int_0^T \|X_s\|_{p,1}^p \, ds \right]^\delta = CB_T^{1-\delta} A_T^\delta.$$

(b)

$$\| \operatorname{div} \tau(X_s) \|_{p',-1} \overset{(1.29)}{\leq} C(1 + \|e(X_s)\|_p)^{p-1}$$

which implies that

$$\| \operatorname{div} \tau(X_s) \|_{p',-1}^{p'} \leq C + C \|e(X_s)\|_p^p$$

and hence, that

$$E\left[\int_0^T \| \operatorname{div} \tau(X_s) \|_{p',-1}^{p'} \, ds \right]$$

$$\leq CT + CE\left[\int_0^T \|e(X_s)\|_p^p \, ds \right] \leq (T + A_T)C. \qquad \square$$

Let $X^n = (X_t^n)_{t \geq 0} \in \mathcal{V}$ be the unique solution of (3.3) for the Galerkin approximation.

LEMMA 3.2.2. *Suppose* (1.8). *Then, there exist* $\tilde{p} \in (1, p)$ *and* $\tilde{\alpha} \in (1, \infty)$ *such that for each* $T > 0$

$$(3.15) \qquad E^W\left[\int_0^T \|X_t^n\|_{\tilde{p},\tilde{\alpha}}^{\tilde{p}} \, dt \right] \leq C_T < \infty,$$

where the constant C_T *is independent of* n.

We will have slightly better than is stated in Lemma 3.2.2 in the course of the proof. For (i) $d = 2$ and $p \geq 2$ and (ii) $d \geq 3$ and $p > p_3(d)$, we have that

$$(3.16) \qquad E^W\left[\int_0^T \|\Delta X_t^n\|_2^{2p/(p+2\lambda)} \, dt \right] \leq C_T < \infty,$$

where $\lambda \geq 0$ is defined by (3.18) below. For $p < \frac{2d}{d-2}$, we have that

$$(3.17) \qquad E^W\left[\int_0^T \|X_t^n\|_{p,\tilde{\alpha}}^{\tilde{p}}\,dt\right] \leq C_T < \infty$$

for *any* $\tilde{p} \in (1, p)$ with some $\tilde{\alpha} = \tilde{\alpha}(\tilde{p}) > 1$.

The rest of this section is devoted to the proof of Lemma 3.2.2. We suppress the superscript n from the notation. We write the summation over $z \in [-n, n]^d$ and $j = 1, \ldots, 2d - 2$ simply by $\sum_{z,j}$. We first establish the following bounds.

LEMMA 3.2.3. *Suppose that* $p \in (\frac{3d-4}{d}, \infty)$ *if* $d \geq 3$ *and let*

$$(3.18) \qquad \lambda = \begin{cases} 0, & \text{if } d = 2, \\ \dfrac{2(3-p)^+}{dp - 3d + 4}, & \text{if } d \geq 3, \end{cases}$$

cf. [5], *formula* (3.47), *page* 236,

$$(3.19) \qquad \mathcal{J}_t = \begin{cases} \dfrac{\|\Delta X_t\|_2^2}{(1 + \|\nabla X_t\|_2^2)^\lambda}, & \text{if } p \geq 2, \\ \dfrac{\|\Delta X_t\|_p^2}{(1 + \|\nabla X_t\|_2^2)^\lambda (1 + \|\nabla X_t\|_p)^{2-p}}, & \text{if } 1 < p < 2. \end{cases}$$

Then, for any $T > 0$,

$$(3.20) \qquad E^W\left[\int_0^T \mathcal{J}_t\,dt\right] \leq C_T < \infty,$$

where $C_T = C(T, d, p, \Gamma, m_1)$.

PROOF. By (3.11) with $\alpha = 1$,

$$(1) \qquad \frac{1}{2}\|\nabla X_t\|_2^2 = \frac{1}{2}\|\nabla X_0\|_2^2 + M_t + \int_0^t K_s\,ds,$$

where

$$M_t = -\sum_{z,j}\int_0^t \Delta X_s^{z,j}\,dW_s^{z,j}, \qquad K_s = \langle -\Delta X_s, b(X_s)\rangle + \frac{1}{2}\operatorname{tr}(-\Gamma \Delta \mathcal{P}_n).$$

Step 1. We will prove that

$$(2) \qquad K_s + c_1 \mathcal{I}_s \leq \begin{cases} 0, & \text{if } d = 2, \\ C_1(1 + \|\nabla X_t\|_2^2)^\lambda (1 + \|\nabla X_t\|_p)^p, & \text{if } d \geq 3, \end{cases}$$

where $c_1, C_1 \in (0, \infty)$ are constants and

$$\mathcal{I}_s = \int_{\mathbb{T}^d} (1 + |e(X_s)|^2)^{(p-2)/2}|\nabla e(X_s)|^2.$$

To show (2), note that

$$\langle -\Delta X_s, b(X_s)\rangle = \langle -\Delta X_s, (X_s \cdot \nabla)X_s\rangle - \langle \tau(X_s), e(-\Delta X_s)\rangle.$$

We see from the argument in [5], proof of (3.19), page 225, that

$$(3) \qquad \langle \tau(X_s), e(-\Delta X_s)\rangle \ge 2c_1 \mathcal{I}_s.$$

On the other hand, we have by integration by parts and Hölder's inequality that

$$\langle -\Delta X_s, (X_s \cdot \nabla)X_s\rangle = \sum_{i,j,k} \int_{\mathbb{T}^d} \partial_k X_s^j \, \partial_j X_s^i \, \partial_k X_s^i \le \|\nabla X_s\|_3^3,$$

where $X_s^j = \sum_{z\in[-n,n]^d} X_s^{z,j} \psi_{z,j}$. It is also well known that the inner product on the LHS vanishes if $d = 2$ ([5], formula (3.20), page 225). By the argument in [5], proof of (3.46), pages 234–235 (this is where the choice of λ is used), we get

$$\|\nabla X_s\|_3^3 \le C_1(1 + \|\nabla X_t\|_2^2)^\lambda (1 + \|\nabla X_t\|_p)^p + c_1 \mathcal{I}_s.$$

These imply that

$$(4) \qquad \begin{aligned} &\langle -\Delta X_s, (X_s \cdot \nabla)X_s\rangle \\ &\times \begin{cases} = 0, & \text{if } d = 2, \\ \le C_1(1 + \|\nabla X_t\|_2^2)^\lambda (1 + \|\nabla X_t\|_p)^p + c_1 \mathcal{I}_s, & \text{if } d \ge 3. \end{cases} \end{aligned}$$

We get (2) by (3)–(4).

Step 2. Proof of (3.20). By [5], formulas (3.25) and (3.26), page 227, $\mathcal{J}_t$ and $\mathcal{I}_t$ are related as

$$\mathcal{J}_t \le C \frac{\mathcal{I}_t}{(1 + \|\nabla X_t\|_2^2)^\lambda}.$$

Therefore, it is enough to prove that

$$(5) \qquad E^W\left[\int_0^t \frac{\mathcal{I}_s \, ds}{(1 + \|\nabla X_s\|_2^2)^\lambda}\right] \le C_T < \infty,$$

where $C_T = C(T, d, p, \Gamma, m_0, m_1) \in (0, \infty)$.

To see this, we introduce the following concave function of $x \ge 0$:

$$f(x) = \begin{cases} \dfrac{1}{1-\lambda}(1+x)^{1-\lambda}, & \text{if } \lambda \neq 1, \\ \ln(1+x), & \text{if } \lambda = 1. \end{cases}$$

Then we have by (1) and Itô's formula that

$$f(\|\nabla X_t\|_2^2) \le f(\|\nabla X_0\|_2^2) + \int_0^t \frac{dM_s}{(1 + \|\nabla X_s\|_2^2)^\lambda} + 2\int_0^t \frac{K_s \, ds}{(1 + \|\nabla X_s\|_2^2)^\lambda},$$

where we have omitted the term with $f'' \leq 0$. Moreover, by (2)

$$\frac{K_s}{(1 + \|\nabla X_s\|_2^2)^\lambda} \leq -\frac{c_1 \mathcal{I}_s}{(1 + \|\nabla X_s\|_2^2)^\lambda} + C_1(1 + \|\nabla X_s\|_p)^p,$$

$$0 \leq f(x) \leq C_2(1 + x) \qquad \text{if } \lambda \in [0, 1]$$

and

$$-\frac{1}{\lambda - 1} \leq f(x) \leq 0 \qquad \text{if } \lambda > 1.$$

Putting these together, we get

$$-C_3 + 2c_1 E^W \left[\int_0^t \frac{\mathcal{I}_s \, ds}{(1 + \|\nabla X_s\|_2^2)^\lambda} \right]$$

$$\leq C_2(1 + E[\|\nabla X_0\|_2^2]) + C_1 E^W \left[\int_0^t (1 + \|\nabla X_s\|_p)^p \, ds \right]$$

$$\overset{(3.10)}{\leq} C(T, d, p, \Gamma, m_0, m_1) < \infty,$$

where $C_3 = 0$ if $\lambda \in (0, 1]$ and $C_3 = \frac{1}{\lambda - 1}$ if $\lambda > 1$. This proves (5). $\quad\square$

PROOF OF LEMMA 3.2.2. We note that

$$p_1(d) < p_3(d) < p_2(d) \qquad \text{for } d \leq 8,$$

$$p_1(9) = 2.555\ldots < p_2(9) = 2.5714\ldots < p_3(9) = 2.620\ldots,$$

$$p_2(d) < p_1(d) \qquad \text{for } d \geq 10.$$

Thus, condition (1.8) takes the following form in any $d \geq 2$:

$$(3.21) \qquad\qquad p \in (p_1(d), p_2(d)) \cup (p_3(d), \infty).$$

We consider the following four cases separately:

Case 1. $d = 2$ and $p \geq 2$;
Case 2. $d \geq 3$ and $p > p_3(d)$;
Case 3. $p \in (p_1(d), p_2(d))$ and $p \geq 2$;
Case 4. $p \in (p_1(d), 2)$ (this case appears only if $d = 2, 3$).

The first two cases cover the interval $(p_3(d), \infty)$ in (3.21). [Note that $p_3(2) = 2$, while the last two cases cover the interval $(p_1(d), p_2(d))$.]

Case 1. By (3.20), (3.15) has already been shown with $\tilde{p} = \tilde{\alpha} = 2$.

Case 2. Note that $p > p_3(d) > 2$ and that $\beta \overset{\text{def.}}{=} \frac{p}{p+2\lambda} > 1/2$. We prove (3.16). Since $\lambda\beta = \frac{p}{2}(1-\beta)$,

$$E^W\left[\int_0^T \|\Delta X_s\|_2^{2\beta}\,ds\right]$$

$$= E^W\left[\int_0^T \mathcal{J}_s^\beta(1+\|\nabla X_s\|_2^2)^{\lambda\beta}\,ds\right]$$

(1)
$$\overset{\beta+(1-\beta)=1}{\leq} E^W\left[\int_0^T \mathcal{J}_s\,ds\right]^\beta E^W\left[\int_0^T (1+\|\nabla X_s\|_2^2)^{p/2}\,ds\right]^{1-\beta}$$

$$\overset{(3.10),\,(3.20)}{\leq} C_T < \infty,$$

where we used (3.20) for $p \geq 2$.

Case 3. We prove (3.17) for given $\tilde{p} \in (1, p)$ with some $\tilde{\alpha} = \tilde{\alpha}(\tilde{p}) \in (1, 2)$. Let $\beta = \frac{p}{p+2\lambda} \in (0, 1)$. Then the bound (1) from case 2 is still valid, although it may no longer be the case that $2\beta > 1$ here. On the other hand, it is not difficult to see via the interpolation and the Sobolev imbedding that for any $\tilde{p} \in (1, p)$, there exist $\tilde{\alpha} \in (1, 2)$ and $\theta \in (0, 1)$ such that

$$\int_0^T \|X_s\|_{p,\tilde{\alpha}}^{\tilde{p}}\,ds \leq C\left(\int_0^T \|X_s\|_{p,1}^p\,ds\right)^\theta\left(\int_0^T \|X_s\|_{2,2}^{2\beta}\,ds\right)^{1-\theta};$$

cf. [5], proof of (3.58), page 238. This is where the restriction $p < \frac{2d}{d-2}$ is necessary. Thus,

$$E^W\left[\int_0^T \|X_s\|_{p,\tilde{\alpha}}^{\tilde{p}}\,ds\right] \leq CE^W\left[\int_0^T \|X_s\|_{p,1}^p\,ds\right]^\theta E^W\left[\int_0^T \|X_s\|_{2,2}^{2\beta}\,ds\right]^{1-\theta}$$

$$\overset{(3.10),\,(1)}{\leq} C_T < \infty.$$

Case 4. We prove (3.17) for given $\tilde{p} \in (1, p)$ and with some $\tilde{\alpha} = \tilde{\alpha}(\tilde{p}) \in (1, 2)$. We recall that $p > \frac{3d}{d+2}$ and set

$$\beta = \frac{((d+2)p - 3d)p}{2((d+5)p - 3d - p^2)} \in \left(0, \frac{1}{2}\right).$$

Then,

(2)
$$\rho \overset{\text{def.}}{=} \frac{(2-p)d\lambda}{2(1-\beta)p} \in [0, 1) \quad \text{and} \quad \frac{(2-p)\beta}{1-\beta} \in (0, p).$$

As a result of applications of Hölder's inequality, the interpolation and the Sobolev imbedding (cf. [5], formulas (3.60)–(3.63), pages 239–240), we arrive at the following bound:

(3)
$$\int_0^T \|\Delta X_s\|_p^{2\beta}\,ds \leq C\left(\int_0^T \mathcal{J}_s\,ds\right)^\beta(I_1 + I_2)^{1-\beta},$$

where

$$I_1 = \int_0^T (1 + \|\nabla X_s\|_p)^{(2-p)\beta/(1-\beta)} \, ds,$$

$$I_2 = \left(\int_0^T \|\Delta X_s\|_p^{2\beta} \, ds\right)^\rho \left(\int_0^T \|\nabla X_s\|_p^p \, ds\right)^{1-\rho}.$$

We first prove that

$$(4) \qquad E^W\left[\int_0^T \|\Delta X_s\|_p^{2\beta} \, ds\right] \le C_T < \infty.$$

We first assume $d = 3$, where $\rho > 0$. Let $r = \frac{1}{\rho} \in (1, \infty)$ and $r' = \frac{r}{r-1} = \frac{1}{1-\rho} \in (1, \infty)$. Then, for $\varepsilon > 0$,

$$E^W\left[\int_0^T \|\Delta X_s\|_p^{2\beta} \, ds\right] \overset{(3)}{\le} CE^W\left[\left(\int_0^T \mathcal{J}_s \, ds\right)^\beta (I_1 + I_2)^{1-\beta}\right]$$

$$\overset{\beta+(1-\beta)=1}{\le} CE^W\left[\int_0^T \mathcal{J}_s \, ds\right]^\beta E^W[I_1 + I_2]^{1-\beta}$$

$$\overset{(3.20)}{\le} C_T E[1 + I_1 + I_2],$$

$$E^W[I_1] \overset{(3.10),(2)}{\le} C_T < \infty,$$

$$E^W[I_2] \overset{\text{Young}}{\le} \frac{\varepsilon^r}{r} E^W\left[\int_0^T \|\Delta X_s\|_p^{2\beta} \, ds\right]$$

$$+ \frac{\varepsilon^{-r'}}{r'} E^W\left[\int_0^T \|\nabla X_s\|_p^p \, ds\right]$$

$$\overset{(3.10)}{\le} \frac{\varepsilon^r}{r} E^W\left[\int_0^T \|\Delta X_s\|_p^{2\beta} \, ds\right] + C_T.$$

Putting things together, with ε small enough, we arrive at (4) for $d = 3$. If $d = 2$ and hence, $\rho = 0$, then we have $E^W[I_2] \le C_T$ directly from (3.10). Therefore, the proof of (4) is even easier than the above.

We finally turn to (3.15). It is not difficult to see via the interpolation (cf. [5], proof of (3.65), pages 240–241) that for any $\tilde{p} \in (1, p)$, there exist $\tilde{\alpha} \in (1, 2)$ and $\theta \in (0, 1)$ such that

$$\int_0^T \|X_s\|_{p,\tilde{\alpha}}^{\tilde{p}} \, ds \le C\left(\int_0^T \|X_s\|_{p,1}^p \, ds\right)^\theta \left(\int_0^T \|X_s\|_{p,2}^{2\beta} \, ds\right)^{1-\theta}.$$

Thus,

$$E^W\left[\int_0^T \|X_s\|_{p,\tilde{\alpha}}^{\tilde{p}} \, ds\right] \le CE^W\left[\int_0^T \|X_s\|_{p,1}^p \, ds\right]^\theta E^W\left[\int_0^T \|X_s\|_{p,2}^{2\beta} \, ds\right]^{1-\theta}$$

$$\overset{(3.10),(4)}{\le} C_T < \infty. \qquad \square$$

3.3. *Compact imbedding lemmas*. We will need some compact imbedding lemmas from [3]. We first introduce the following definition.

DEFINITION 3.3.1.   Let $p \in [1, \infty)$, $T \in (0, \infty)$ and E be a Banach space.

(a) We let $L_{p,1}([0, T] \to E)$ denote the Sobolev space of all $u \in L_p([0, T] \to E)$ such that

$$u(t) = u(0) + \int_0^t u'(s) \, ds \qquad \text{for almost all } t \in [0, T]$$

with some $u(0) \in E$ and $u'(\cdot) \in L_p([0, T] \to E)$. We endow the space $L_{p,1}([0, T] \to E)$ with the norm $\|u\|_{L_{p,1}([0,T]\to E)}$ defined by

$$\|u\|^p_{L_{p,1}([0,T]\to E)} = \int_0^T \left(|u(t)|^p_E + |u'(t)|^p_E \right) dt.$$

(b) For $\alpha \in (0, 1)$, we let $L_{p,\alpha}([0, T] \to E)$ denote the Sobolev space of all $u \in L_p([0, T] \to E)$ such that

$$\int_{0 < s < t < T} \frac{|u(t) - u(s)|^p_E}{|t - s|^{1+\alpha p}} \, ds \, dt < \infty.$$

We endow the space $L_{p,\alpha}([0, T] \to E)$ with the norm $\|u\|_{L_{p,\alpha}([0,T]\to E)}$ defined by

$$\|u\|^p_{L_{p,\alpha}([0,T]\to E)} = \int_0^T |u(t)|^p \, dt + \int_{0 < s < t < T} \frac{|u(t) - u(s)|^p_E}{|t - s|^{1+\alpha p}} \, ds \, dt.$$

To introduce the compact imbedding lemmas, we agree on the following standard convention. Let X be a vector space and $X_i \subset X$ be a subspace with the norm $\|\cdot\|_i$ ($i = 1, 2$). Then we equip $X_0 \cap X_1$ and $X_0 + X_1$, respectively, with the norms

$$\|u\|_{X_0 \cap X_1} = \|u\|_0 + \|u\|_1,$$
$$\|u\|_{X_0 + X_1} = \inf\{\|u_0\|_0 + \|u_1\|_1; u = u_0 + u_1, u_i \in X_i\}.$$

The following lemmas will be used in Section 3.4.

LEMMA 3.3.2 ([3], Theorem 2.2, page 370).   *Let*:

- $E_1, \ldots, E_n$ *and* E *be Banach spaces such that each* $E_i \overset{\text{compact}}{\hookrightarrow} E$, $i = 1, \ldots, n$.
- $p_1, \ldots, p_n \in (1, \infty)$, $\alpha_1, \ldots, \alpha_n > 0$ *are such that* $p_i \alpha_i > 1$, $i = 1, \ldots, n$.

Then, for any $T > 0$,

$$L_{p_1,\alpha_1}([0, T] \to E_1) + \cdots + L_{p_n,\alpha_n}([0, T] \to E_n) \overset{\text{compact}}{\hookrightarrow} C([0, T] \to E).$$

LEMMA 3.3.3 ([3], Theorem 2.1, page 372). *Let*

$$E_0 \overset{\text{compact}}{\hookrightarrow} E \hookrightarrow E_1$$

be Banach spaces such that the first embedding is compact and E_0, E_1 are reflexive. Then, for any $p \in (1, \infty)$, $\alpha \in (0, 1)$ and $T > 0$,

$$L_p([0, T] \to E_0) \cap L_{p,\alpha}([0, T] \to E_1) \overset{\text{compact}}{\hookrightarrow} L_p([0, T] \to E).$$

3.4. *Convergence of the approximations.* Let $X^n = (X_t^n)_{t \geq 0} \in \mathcal{V}$ be the unique solution to (3.3) for the Galerkin approximation. We write

(3.22) $$p' = \frac{p}{p-1}, \qquad p'' = p \wedge p'.$$

Let $\beta(p, 1)$ be defined by (1.30) and let $\tilde{p} > 1$ be the one from Lemma 3.2.2. We may assume that $\tilde{p} \in (1, p'']$. We also agree on the following standard convention. Let S be a set and ρ_i be a metric on $S_i \subset S$ ($i = 1, 2$). Then we tacitly consider the metric $\rho_1 + \rho_2$ on the set $S_1 \cap S_2$; cf. (3.23).

PROPOSITION 3.4.1. *Let $\beta > \beta(p, 1)$. Then there exist a process X and a sequence $(\widetilde{X}^k)_{k \geq 1}$ of processes defined on a probability space $(\Omega, \mathcal{F}, P)$ such that the following properties are satisfied*:

(a) *The process X takes values in*

(3.23) $$C\big([0, \infty) \to V_{2 \wedge p', -\beta}\big) \cap L_{\tilde{p}, \text{loc}}\big([0, \infty) \to V_{\tilde{p}, 1}\big).$$

(b) *For some sequence $n(k) \nearrow \infty$, $\widetilde{X}^k$ has the same law as $X^{n(k)}$ and*

(3.24) $$\lim_{k \to \infty} \widetilde{X}^k = X \text{ in the metric space (3.23), } P\text{-a.s.}$$

REMARKS. (1) Due to Skorohod's representation theorem used in Lemma 3.4.5 below, the probability space $(\Omega, \mathcal{F}, P)$ in the above proposition may not be the same as $(\Omega^W, \mathcal{F}^W, P^W)$, where we have solved the SDE (3.3).

(2) See (4.4) below for additional information on the convergence (3.24).

We divide the Proposition 3.4.1 into Lemmas 3.4.3–3.4.5. To prepare the proofs of these lemmas, we write (3.3) as

(3.25) $$X_t^n = X_0^n + I_t^n + J_t^n + W_t^n,$$

with

$$I_t^n = \int_0^t \mathcal{P}_n\big((X_s^n \cdot \nabla) X_s^n\big) ds, \qquad J_t^n = \int_0^t \mathcal{P}_n(\text{div } \tau(X_s^n)) \, ds,$$

$$W_t^n = \mathcal{P}_n W_t.$$

It is elementary to obtain the following regularity bound of the noise term W_t^n [2], Corollary 4.2, page 92: for any $p \in [1, \infty)$, $\alpha \in [0, 1/2)$ and $T > 0$, there exists $C_T = C_{\alpha,p,T} \in (0, \infty)$ such that

$$(3.26) \qquad \sup_{n \geq 0} E^W [\|W_\cdot^n\|_{L_{p,\alpha}([0,T] \to V_{2,0})}^p] \leq C_T \operatorname{tr}(\Gamma)^{p/2}.$$

We will control $I_\cdot^n$ and $J_\cdot^n$ by (3.13) and (3.14). However, to be able to do so, we have to get rid of the projection $\mathcal{P}_n$. This is the content of the following:

LEMMA 3.4.2. *Let $T \in (0, \infty)$. Then,*

$$(3.27) \qquad \sup_{n \geq 1} E^W [\|I_\cdot^n\|_{L_{p,1}([0,T] \to V_{p',-\beta(p,1)})}^\gamma] \leq C_T < \infty,$$

where $\gamma = \frac{p^2}{p+2}$. Also,

$$(3.28) \qquad \sup_{n \geq 1} E^W [\|J_\cdot^n\|_{L_{p',1}([0,T] \to V_{p',-\beta(p,1)})}^{p'}] \leq C_T < \infty.$$

PROOF. For any $p \in (1, \infty)$, there exists $A_p \in (0, \infty)$ such that

$$\|\mathcal{P}_n v\|_p \leq A_p \|v\|_p \qquad \text{for all } v \in V_{p,0}.$$

(See, e.g., [4], Theorem 3.5.7, page 213.) This implies that $\|\mathcal{P}_n v\|_{p,\alpha} \leq A_p \|v\|_{p,\alpha}$ and hence, $\|\mathcal{P}_n v\|_{p',-\alpha} \leq A_p \|v\|_{p',-\alpha}$ for any $p \in (1, \infty)$ and $\alpha \geq 0$. We combine this and (3.13) and (3.14) to obtain (3.27) and (3.28). □

LEMMA 3.4.3. *For $\beta > \beta(p, 1)$, the laws $\{P^W (X^n \in \cdot)\}_{n=1}^\infty$ are tight on $C([0, \infty) \to V_{2 \wedge p', -\beta})$.*

PROOF. As is easily seen, it is enough to prove the following:
(1) The laws $\{P^W ((X_t^n)_{t \leq T} \in \cdot)\}_{n=1}^\infty$ are tight on $C([0, T] \to V_{2 \wedge p', -\beta})$ for each fixed $T > 0$. To see (1), we set

$$\mathcal{S} = L_{p,1}([0, T] \to V_{p',-\beta(p,1)}) + L_{p',1}([0, T] \to V_{p',-1})$$
$$+ L_{2/\gamma,\gamma}([0, T] \to V_{2,0}), \qquad \text{with } \gamma \in (0, 1/2).$$

We then see from Lemma 3.3.2 that

$$(2) \qquad \mathcal{S} \overset{\text{compact}}{\hookrightarrow} C([0, T] \to V_{2 \wedge p', -\beta}).$$

On the other hand, we have that

$$(3) \qquad \sup_n E^W [\|I_\cdot^n\|_{L_{p,1}([0,T] \to V_{p',-\beta(p,1)})}^\delta] \overset{(3.27)}{\leq} C_T < \infty \qquad \text{for some } \delta \in (0, 1];$$

$$(4) \qquad \sup_n E^W [\|J_\cdot^n\|_{L_{p',1}([0,T] \to V_{p',-1})}] \overset{(3.28)}{\leq} C_T < \infty;$$

$$(5) \quad \sup_n E^W [\|X_0^n + W_\cdot^n\|_{L_{2/\gamma,\gamma}([0,T] \to V_{2,0})}] \overset{(3.26)}{\leq} C_T < \infty.$$

We conclude from (3)–(5) and (3.25) that

$$\sup_n E^W[\|X_.^n\|_{\mathcal{S}}^\delta] \le C_T < \infty$$

and hence, that for $R > 0$,

(6)
$$\sup_n P^W(\|X_.^n\|_{\mathcal{S}} > R) \le \frac{1}{R^\delta} \sup_n E^W[\|X_.^n\|_{\mathcal{S}}^\delta]$$

$$\le \frac{C_T}{R^\delta} \longrightarrow 0 \qquad \text{as } R \longrightarrow \infty.$$

We see from (2) that the set

$$\{X_.; \|X_.^n\|_{\mathcal{S}} \le R\}$$

is relatively compact in $C([0, T] \to V_{2 \wedge p', -\beta})$. Hence, by (6), we have the tightness (1). $\quad\square$

LEMMA 3.4.4. *The laws $\{P^W(X^n \in \cdot)\}_{n=1}^\infty$ are tight on $L_{\tilde{p},\mathrm{loc}}([0, \infty) \to V_{\tilde{p},1})$.*

PROOF. Let $\tilde{p} > 1$ and $\tilde{\alpha} > 1$ be from Lemma 3.2.2. We may assume that $\tilde{p} \in (1, p'']$. It is enough to prove the following:

(1)
The laws $\{P^W((X_t^n)_{t \le T} \in \cdot)\}_{n=1}^\infty$ are tight on $L_{\tilde{p}}([0, T] \to V_{\tilde{p},1})$
for each fixed $T > 0$.

To see (1), we set

$$\mathcal{I} = L_{\tilde{p}}([0, T] \to V_{\tilde{p},\tilde{\alpha}}) \cap L_{\tilde{p},\gamma}([0, T] \to V_{\tilde{p},-\beta(p,1)}) \qquad \text{with } \gamma \in (0, 1/2).$$

Note that

$$V_{\tilde{p},\tilde{\alpha}} \overset{\text{compact}}{\hookrightarrow} V_{\tilde{p},1} \hookrightarrow V_{\tilde{p},-\beta(p,1)}$$

and hence, by Lemma 3.3.3, that

(2)
$$\mathcal{I} \overset{\text{compact}}{\hookrightarrow} L_{\tilde{p}}([0, T] \to V_{\tilde{p},1}).$$

On the other hand,

(3)
$$\sup_n E^W\big[\|X_.^n\|_{L_{\tilde{p}}([0,T] \to V_{\tilde{p},\tilde{\alpha}})}\big] \overset{(3.15)}{\le} C_T < \infty.$$

Moreover, for some $\delta \in (0, 1]$,

$$\sup_n E^W\big[\|X_.^n\|_{L_{\tilde{p},\gamma}([0,T] \to V_{\tilde{p},-\beta(p,1)})}^\delta\big]$$

$$\le \sup_n E^W\big[\|X_0^n + I_.^n + J_.^n\|_{L_{\tilde{p},\gamma}([0,T] \to V_{\tilde{p},-\beta(p,1)})}^\delta\big]$$

$$+ \sup_n E^W\big[\|W_.^n\|_{L_{\tilde{p},\gamma}([0,T] \to V_{2,0})}^\delta\big]$$

$$\overset{(3.26)–(3.28)}{\le} C_T < \infty.$$

We conclude from (2) and (3) that

$$\sup_n E^W[\|X^n_\cdot\|^\delta_{\mathcal{I}}] \le C_T < \infty$$

and hence, that for $R > 0$,

(4)
$$\sup_n P^W(\|X^n_\cdot\|_{\mathcal{I}} > R) \le \frac{1}{R^\delta} \sup_n E^W[\|X^n_\cdot\|^\delta_{\mathcal{I}}]$$

$$\le \frac{C_T}{R^\delta} \longrightarrow 0 \qquad \text{as } R \longrightarrow \infty.$$

We will see from this and (2) that the set

$$\{X_\cdot; \|X^n_\cdot\|_{\mathcal{I}} \le R\}$$

is relatively compact in $L_{\tilde{p}}([0, T] \to V_{\tilde{p},1})$. Hence, by (4) we have the tightness (1). $\square$

Finally, Proposition 3.4.1 follows from Lemmas 3.4.3, 3.4.4 and the following:

LEMMA 3.4.5. *Suppose that*:

- *(S_j, ρ_j) $(j = 1, \dots, m)$ are complete separable metric spaces such that all of S_j $(j = 1, \dots, m)$ are subsets of a set S;*
- *$(X_n)_{n \in \mathbb{N}}$ is a sequence of random variables with values in $\bigcap_{j=1}^m S_j$ defined on a probability space $(\Omega, \mathcal{F}, P)$;*
- *$(X_n)_{n \in \mathbb{N}}$ is tight in each of (S_j, ρ_j), $j = 1, \dots, m$, separately.*
  Then, there exists a sequence $n(k) \to \infty$, random variables $X, \tilde{X}_k$, $k = 1, 2, \dots$, with values in $\bigcap_{j=1}^m S_j$ defined on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ such that

$$\tilde{P}(\tilde{X}_k \in \cdot) = P(X_{n(k)} \in \cdot) \qquad \text{for all } k = 1, 2, \dots;$$

$$\lim_{k \to \infty} \sum_{j=1}^m \rho_j(X, \tilde{X}_k) = 0 \qquad \tilde{P}\text{-a.s.}$$

PROOF.   By induction, it is enough to consider the case of $m = 2$. Let $\varepsilon > 0$ be arbitrary. Then, for $j = 1, 2$, there exists a compact subset K_j of S_j such that

$$P(X_n \in K_j) \ge 1 - \varepsilon \qquad \text{for all } j = 1, 2 \text{ and } n = 1, 2, \dots.$$

Now a very simple but crucial observation is that $K_1 \cap K_2$ is compact in $S_1 \cap S_2$ with respect to the metric $\rho_1 + \rho_2$. Also,

$$P(X_n \in K_1 \cap K_2) \ge 1 - 2\varepsilon \qquad \text{for all } j = 1, 2 \text{ and } n = 1, 2, \dots.$$

These imply that (X_n) is tight in $S_1 \cap S_2$ with respect to the metric $\rho_1 + \rho_2$. Thus, the lemma follows from Prohorov's theorem ([1], Theorem 2.6, page 7) and Skorohod's representation theorem ([1], Theorem 2.7, page 9). $\square$

REMARK. This remark, together with the one after Lemma 4.1.1, concerns the validity of Theorem 2.1.3 with $p = 2$ for all d. Let $\alpha < 1$. Then we can also prove that

$$(3.29) \quad \text{the laws } \{P^W(X^n \in \cdot)\}_{n=1}^{\infty} \text{ are tight on } L_{p'',\mathrm{loc}}([0, \infty) \to V_{p,\alpha}).$$

This can be seen as follows. We set

$$\mathcal{I} = L_{p''}([0, T] \to V_{p,1}) \cap L_{p'',\gamma}([0, T] \to V_{p'',-\beta(p,1)}), \qquad \text{with } \gamma \in (0, 1/2).$$

Since

$$V_{p,1} \overset{\text{compact}}{\hookrightarrow} V_{p,\alpha} \hookrightarrow V_{p'',-\beta(p,1)},$$

we have by Lemma 3.3.3 that

$$\mathcal{I} \overset{\text{compact}}{\hookrightarrow} L_{p''}([0, T] \to V_{p,\alpha}).$$

Then we get (3.29) by similar argument as in Lemma 3.4.4.

By the tightness (3.29), Lemmas 3.4.3 and 3.4.5, we obtain a variant of Proposition 3.4.1 in which the convergence $\widetilde{X}^k \to X$, P-a.s. takes place in the metric space

$$(3.30) \qquad C([0, \infty) \to V_{2 \wedge p', -\beta}) \cap L_{p'',\mathrm{loc}}([0, \infty) \to V_{p,\alpha})$$

instead of (3.23). We note that this modification of Proposition 3.4.1 is valid for $p \in [\frac{2d}{d+2}, \infty)$ since we did not use Lemma 3.2.2.

4. Proof of Theorems 2.1.3 and 2.2.1.

4.1. *Proof of Theorem* 2.1.3. Let X and $\widetilde{X}^k$ be as in Proposition 3.4.1. We will verify (2.1) [with $\beta = \beta(p, 1)$] as well as (2.3)–(2.5) and (2.7) for X. (2.3) can easily be seen. In fact,

$$\widetilde{X}_0^k \to X_0 \qquad \text{a.s. in } V_{2 \wedge p', -\beta},$$

$$\widetilde{X}_0^k \overset{\text{law}}{=} X_0^{n(k)} = \mathcal{P}_{n(k)}\xi \to \xi \qquad \text{in } V_{2,0}.$$

Thus, the laws of X_0 and ξ are identical.

$$\widetilde{X}_0^k \overset{\text{law}}{=} X_0^{n(k)} = \mathcal{P}_{n(k)}\xi \to \xi \qquad \text{in } V_{2,0}.$$

Note that the function

$$v_\cdot \mapsto \sup_{t \leq T} \|v_t\|_2^2 + \int_0^T \|v_t\|_{p,1}^p \, dt$$

is lower semi-continuous on the metric space (3.23). Thus, (2.7) follows from (3.10) and Proposition 3.4.1 via Fatou's lemma.

To show (2.4) and (2.5), we prepare the following:

LEMMA 4.1.1.   *Let $\varphi \in \mathcal{V}$ and $T > 0$. Then,*

$$(4.1) \quad \lim_{k \to \infty} \int_0^T |\langle \varphi, (\widetilde{X}_t^k \cdot \nabla) \widetilde{X}_t^k - (X_t \cdot \nabla) X_t \rangle| \, dt = 0 \qquad \text{in probability } (P),$$

$$(4.2) \quad \lim_{k \to \infty} \int_0^T |\langle e(\varphi), \tau(\widetilde{X}_t^k) - \tau(X_t) \rangle| \, dt = 0 \qquad \text{in } L_1(P),$$

$$(4.3) \quad \lim_{k \to \infty} \int_0^T \langle \varphi, \mathcal{P}_{n(k)} b(\widetilde{X}_t^k) - b(X_t) \rangle \, dt = 0 \qquad \text{in probability } (P).$$

PROOF.   We write $Z_t^k = \widetilde{X}_t^k - X_t$ to simplify the notation. We start by proving that

$$(4.4) \qquad \lim_{k \to \infty} E\left[\int_0^T \|Z_t^k\|_{p_1,1}^{p_1} \, dt \right] = 0, \qquad \text{if } p_1 < p.$$

By Proposition 3.4.1,

$$I_k \stackrel{\text{def.}}{=} \int_0^T \|Z_t^k\|_{1,1} \, dt \stackrel{k \to \infty}{\longrightarrow} 0, \qquad P\text{-a.s.}$$

Moreover, the random variables $\{I_k\}_{k \geq 1}$ are uniformly integrable since

$$E[I_k^p] \stackrel{(3.10)}{\leq} C_T < \infty.$$

Therefore,

$$(2) \qquad\qquad\qquad\qquad \lim_{k \to \infty} E[I_k] = 0.$$

Let $k(m) \nearrow \infty$ be such that

$$(3) \qquad \Phi_{m,t} \stackrel{\text{def.}}{=} |Z_t^{k(m)}| + |\nabla Z_t^{k(m)}| \stackrel{m \to \infty}{\longrightarrow} 0, \qquad dt|_{[0,T]} \times dx \times P\text{-a.e.},$$

where $dt|_{[0,T]} \times dx$ denotes the Lebesgue measure on $[0, T] \times \mathbb{T}^d$. Such a sequence $k(m)$ exists by (2). The sequence $\{\Phi_{m,\cdot}\}_{m \geq 1}$ is uniformly integrable with respect to $dt|_{[0,T]} \times dx \times P$. In fact,

$$E\left[\int_0^T \int_{\mathbb{T}^d} \Phi_{m,t}^p \, dt \right] \stackrel{(3.10)}{\leq} C_T < \infty.$$

Therefore, (3), together with this uniform integrability, implies (4.4) along the subsequence $k(m)$. Finally, we get rid of the subsequence, since the subsequence as $k(m)$ above can be chosen from any subsequence of k given in advance. We now prove (4.1). Since

$$(\widetilde{X}_t^k \cdot \nabla) \widetilde{X}_t^k - (X_t \cdot \nabla) X_t = (Z_t^k \cdot \nabla) \widetilde{X}_t^k + (X_t \cdot \nabla) Z_t^k,$$

we have

$$\int_0^T |\langle \varphi, (\widetilde{X}_t^k \cdot \nabla) \widetilde{X}_t^k - (X_t \cdot \nabla) X_t \rangle| \, dt \leq J_1 + J_2,$$

where

$$J_1 = \int_0^T |\langle \varphi, (Z_t^k \cdot \nabla) \widetilde{X}_t^k \rangle| \, dt \quad \text{and} \quad J_2 = \int_0^T |\langle \varphi, (X_t \cdot \nabla) Z_t^k \rangle| \, dt.$$

We may take p_1 in (4.4) as bigger than $\frac{3d}{d+2}$ so that there exists $0 < \alpha < 1$ such that $\frac{2d}{d+2\alpha} < p_1$. Then by (1.25), we have that

$$|\langle \varphi, (Z_t^k \cdot \nabla) \widetilde{X}_t^k \rangle| \le C \|Z_t^k\|_{p_1,\alpha} \|\widetilde{X}_t^k\|_2 \|\varphi\|_{p_1,\beta(p_1,\alpha)}$$

and hence that

$$J_1 \le C \|\varphi\|_{p_1,\beta(p_1,\alpha)} \sup_{t \le T} \|\widetilde{X}_t^k\|_2 \int_0^T \|Z_t^k\|_{p_1,\alpha} \, dt.$$

By (3.10) and (4.4),

$$\sup_{k \ge 1} E\Big[\sup_{t \le T} \|\widetilde{X}_t^k\|_2^2\Big] < \infty \quad \text{and} \quad \lim_{k \to \infty} \int_0^T \|Z_t^k\|_{p_1,\alpha} \, dt = 0 \qquad P\text{-a.s.}$$

Thus, $\lim_{k \to \infty} J_1 = 0$ in probability. On the other hand, we have by (1.28) that

$$|\langle \varphi, (X_t \cdot \nabla) Z_t^k \rangle| \le C \|Z_t^k\|_{p_1,\alpha} \|X_t\|_2 \|\varphi\|_{p_1,\beta(p_1,\alpha)}$$

and hence that

$$J_2 \le C \|\varphi\|_{p_1,\beta(p_1,\alpha)} \sup_{t \le T} \|X_t\|_2 \int_0^T \|Z_t^k\|_{p_1,\alpha} \, dt.$$

By (2.7) and (4.4),

$$E\Big[\sup_{t \le T} \|X_t\|_2^2\Big] < \infty \quad \text{and} \quad \lim_{k \to \infty} \int_0^T \|Z_t^k\|_{p_1,\alpha} \, dt = 0 \qquad P\text{-a.s.}$$

Thus, $\lim_{k \to \infty} J_2 = 0$ in probability.

We now turn to (4.2). It is enough to prove that

$$(4) \qquad \lim_{k \to \infty} E\Big[\int_0^T \|\tau(\widetilde{X}_t^k) - \tau(X_t)\|_1 \, dt\Big] = 0.$$

Again, let $k(m)$ be such that (3) holds. Then,

$$(5) \qquad \lim_{m \to \infty} \tau\big(\widetilde{X}_t^{k(m)}\big) = \tau(X_t), \qquad dt|_{[0,T]} \times dx \times P\text{-a.e.}$$

On the other hand, we have for $p' = \frac{p}{p-1}$ that

$$E\Big[\int_0^T dt \int_{\mathbb{T}^d} |\tau(\widetilde{X}_t^k)|^{p'}\Big] \le C E\Big[\int_0^T dt \int_{\mathbb{T}^d} (1 + |e(\widetilde{X}_t^k)|)^p\Big] \overset{(3.10)}{\le} C_T < \infty,$$

which implies that $\tau(\widetilde{X}_t^k), k \in \mathbb{N}$ are uniformly integrable with respect to $dt|_{[0,T]} \times dx \times P$. Therefore, (5), together with this uniform integrability, implies (4) along

the subsequence $k(m)$. Finally, we get rid of the subsequence, since the subsequence as $k(m)$ above can be chosen from any subsequence of k given in advance.

Equation (4.3) follows from (4.1) and (4.2). Since $\varphi \in \mathcal{V}$ is fixed and k is tending to ∞, we do not have to care about $\mathcal{P}_{n(k)}$ here. $\quad\square$

REMARK. If $p = 2$, then Lemma 4.1.1 is valid for all d. This is for the following reason. By inspection of the proof above, we see immediately that (4.1) follows also from the modification of Proposition 3.4.1 mentioned at the end of Section 3.4. Also, for $p = 2$, (4.2) is equivalent to

$$\lim_{k \to \infty} \int_0^T \langle \Delta\varphi, \widetilde{X}_t^k - X_t \rangle \, dt = 0 \qquad \text{in } L_1(P),$$

which also follows from the modification of Proposition 3.4.1 mentioned at the end of Section 3.4.

LEMMA 4.1.2. *Let*

$$(4.5) \qquad Y_t = Y_t(X) = X_t - X_0 - \int_0^t b(X_s) \, ds, \qquad t \ge 0.$$

Then, $Y.$ is a $\mathrm{BM}(V_{2,0}, \Gamma)$. *Moreover, $Y_{t+.} - Y_t$ and $\{\langle \varphi, X_s \rangle; s \le t, \varphi \in \mathcal{V}\}$ are independent for any $t \ge 0$.*

PROOF. It is enough to prove that for each $\varphi \in \mathcal{V}$ and $0 \le s < t$,

$$(1) \qquad E[\exp(\mathbf{i}\langle \varphi, Y_t - Y_s \rangle)|\mathcal{G}_s] = \exp\left(-\frac{t-s}{2}\langle \varphi, \Gamma\varphi \rangle\right), \qquad \text{a.s.,}$$

where $\mathcal{G}_s = \sigma(\langle \varphi, X_u \rangle; u \le s, \varphi \in \mathcal{V})$. We set

$$F(X) = f(\langle \varphi_1, X_{u_1} \rangle, \ldots, \langle \varphi_n, X_{u_n} \rangle),$$

where $f \in C_{\mathrm{b}}(\mathbb{R}^n)$, $0 \le u_1 < \cdots < u_n \le s$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{V}$ are chosen arbitrarily in advance. Then (1) can be verified by showing that

$$(2) \qquad E[\exp(\mathbf{i}\langle \varphi, Y_t - Y_s \rangle)F(X)] = \exp\left(-\frac{t-s}{2}\langle \varphi, \Gamma\varphi \rangle\right)E[F(X)].$$

Let

$$Y_t^k = \widetilde{X}_t^k - \widetilde{X}_0^k - \int_0^t \mathcal{P}_{n(k)}b(\widetilde{X}_s^k) \, ds, \qquad t \ge 0.$$

Then we see from Theorem 3.1.1 that

$$(3) \qquad \begin{aligned} &E[\exp(\mathbf{i}\langle \varphi, Y_t^k - Y_s^k \rangle)F(\widetilde{X}^k)] \\ &\qquad = \exp\left(-\frac{t-s}{2}\langle \varphi, \Gamma\mathcal{P}_{n(k)}\varphi \rangle\right)E[F(\widetilde{X}^k)]. \end{aligned}$$

Moreover, we have

$$\lim_{k \to \infty} \langle \varphi, Y_t^k - Y_s^k \rangle \overset{(3.24),(4.3)}{=} \lim_{k \to \infty} \langle \varphi, Y_t - Y_s \rangle \qquad \text{in probability}$$

and hence,

$$\lim_{k \to \infty} \text{ LHS of (3)} = \text{ LHS of (2)}.$$

On the other hand,

$$\lim_{k \to \infty} \text{ RHS of (3)} \overset{(3.24)}{=} \text{ RHS of (2)}.$$

These prove (2). $\quad \square$

Finally, we prove (2.1) with $\beta = \beta(p, 1)$. It follows from (2.7) that

$$X \in L_{p,\text{loc}}\big([0, \infty) \to V_{p,1}\big) \cap L_{\infty,\text{loc}}\big([0, \infty) \to V_{2,0}\big).$$

Thus, it remains to show that $X \in C\big([0, \infty) \to V_{2 \wedge p', -\beta(p,1)}\big)$. But this follows from Lemma 3.2.1 and that $Y \in C\big([0, \infty) \to V_{2,0}\big)$.

4.2. *Proof of Theorem* 2.2.1. Here we can follow the argument of [5], Theorem 4.29, page 254, almost verbatim. We will present it for the convenience of the readers.

We need two technical lemmas.

LEMMA 4.2.1. *Let H be a Hilbert space and V be a Banach space such that*

$$V \hookrightarrow H \hookrightarrow V^*.$$

Suppose that $f \in L_p([0, T] \to V)$ $(p \in (1, \infty),\ T > 0)$ *has derivative* f' *in* $L_{p'}([0, T] \to V^*)$. *Then,*

$$(4.6) \qquad \frac{d}{dt} |f|_H^2 = 2_V \langle f, f' \rangle_{V^*}$$

in the distributional sense on $(0, T)$.

PROOF. The case of $p = 2$ can be found in [7], Lemma 1.2, pages 60–61. The extension to general p is straightforward. $\quad \square$

LEMMA 4.2.2 ([5], Lemma 4.35, page 255). *Let* $q \in (2, \infty)$ *if* $d = 2$ *and* $q \in [2, \frac{2d}{d-2}]$ *if* $d \geq 3$. *Then there exists* $c \in (0, \infty)$ *such that*

$$(4.7) \qquad \|v\|_q \leq c \|v\|_2^{\theta} \|\nabla v\|_2^{1-\theta} \qquad \text{with } \theta = \frac{2d - q(d-2)}{2q}$$

for all $v \in V_{2,1}$ *with* $\int_{\mathbb{T}^d} v = 0$.

Let X and $\widetilde{X}$ be as in the assumptions of Theorem 2.2.1 and

$$Z_t = X_t - \widetilde{X}_t = \int_0^t \bigl(b(X_s) - b(\widetilde{X}_s)\bigr)\,ds.$$

Then,

(1) $$Z_. \in L_{p,\mathrm{loc}}\bigl([0,\infty) \to V_{p,1}\bigr)$$

and by Lemma 3.2.1,

(2) $$\partial_t Z_. = b(X_.) - b(\widetilde{X}_.) \in L_{p,\mathrm{loc}}\bigl([0,\infty) \to V_{p',-\beta(p,1)}\bigr).$$

Since $p \geq p'$ and $\beta(p,1) = 1$ for $p \geq 1 + \frac{d}{2}(\geq \frac{4d}{d+2})$, we see from (2) and Lemma 4.2.1 (applied to $f = Z_.$ and $V = V_{p,1}$) that

(3) $$\frac{1}{2}\frac{d}{dt}\|Z_t\|_2^2 \overset{(4.6)}{=} \langle Z_t, b(X_t) - b(\widetilde{X}_t)\rangle = -I_t - J_t$$

in the distributional sense, where

$$I_t = \langle Z_t, (X_t \cdot \nabla)X_t - (\widetilde{X}_t \cdot \nabla)\widetilde{X}_t\rangle \quad \text{and} \quad J_t = \langle e(Z_t), \tau(X_t) - \tau(\widetilde{X}_t)\rangle.$$

We have by [5], formula (1.25), page 198 and formula (1.11), page 196, that

(4) $$J_t \geq c_1\|e(Z_t)\|_2^2 \geq c_2\|\nabla Z_t\|_2^2.$$

On the other hand, since $\widetilde{X}_t = X_t - Z_t$, we see that

$$\langle Z_t, (\widetilde{X}_t \cdot \nabla)\widetilde{X}_t\rangle \overset{(1.18)}{=} \langle Z_t, (\widetilde{X}_t \cdot \nabla)X_t\rangle = \bigl\langle Z_t, \bigl((X_t - Z_t) \cdot \nabla\bigr)X_t\bigr\rangle,$$

and hence that

$$I_t = \langle Z_t, (Z_t \cdot \nabla)X_t\rangle.$$

Therefore,

$$|I_t| \overset{1/p+(p-1)/(2p)+(p-1)/(2p)=1}{\leq} \|\nabla X_t\|_p\|Z_t\|_{2p/(p-1)}^2$$

(5) $$\overset{(4.7)}{\leq} C_3\|\nabla Z_t\|_2^{d/p}\|\nabla X_t\|_p\|Z_t\|_2^{(2p-d)/p}$$

$$\overset{d/(2p)+(2p-d)/(2p)=1}{\leq} c_2\|\nabla Z_t\|_2^2 + C_4\|\nabla X_t\|_p^{2p/(2p-d)}\|Z_t\|_2^2.$$

We see from (3)–(5) that

$$\frac{1}{2}\frac{d}{dt}\|Z_t\|_2^2 \leq C_4\|\nabla X_t\|_p^{2p/(2p-d)}\|Z_t\|_2^2.$$

Since $\frac{2p}{2p-d} \leq p$, this implies via Gronwall's lemma (we need an appropriate generalization since the derivative above is in the distributional sense) that

$$\|Z_t\|_2^2 \leq \|Z_0\|_2^2 \exp\Bigl(C_4\int_0^t \|\nabla X_s\|_p^{2p/(2p-d)}\,ds\Bigr).$$

This proves that $\|Z_t\|_2 \equiv 0$.

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