The Annals of Probability
2007, Vol. 35, No. 3, 915–934
DOI: 10.1214/009117906000000980

ON THE CONTINUITY OF LOCAL TIMES OF BOREL RIGHT MARKOV PROCESSES

BY NATHALIE EISENBAUM AND HAYA KASPI

Université Paris VI-CNRS and Technion

The problem of finding a necessary and sufficient condition for the continuity of the local times for a general Markov process is still open. Barlow and Hawkes have completely treated the case of the Lévy processes, and Marcus and Rosen have solved the case of the strongly symmetric Markov processes. We treat here the continuity of the local times of Borel right processes. Our approach unifies that of Barlow and Hawkes and of Marcus and Rosen, by using an associated Gaussian process, that appears as a limit in a CLT involving the local time process.

1. Introduction. Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P_x; x \in E)$ be a Borel right process, having a reference measure m, with all states communicating and regular for themselves. Under these assumptions, a local time L_t^x exists at each point, unique up to a multiplicative constant. Let $u^\alpha(x, y)$ be the potential densities with respect to m, and normalize the local times (choose the multiplicative constant), so that for some (and all) α,

$$E_x \int_0^\infty e^{-\alpha t}\, dL_t^y = u^\alpha(x, y), \tag{1}$$

for all $x, y \in E$, where E_x is the expectation with respect to P_x. The question, under what conditions there exists a version of $(L_t^x)_{x \in E, t>0}$ so that $(x, t) \to L_t^x(\omega)$ is almost surely continuous, has occupied many researchers in the field for many years. Although, as we shall describe below, there are some very important special cases where this problem has been solved, the problem, for general Borel right processes, is still unresolved.

To put this problem in context, we would like to start by highlighting some of the most important existing results in this field. The first to address this problem was Trotter who in [29] proved that when X is the Brownian motion on the real line, it has a local time at all points and (normalized as above) there is a version of $(x, t) \to L_t^x$ that is almost surely jointly continuous. In [19] Getoor and Kesten have treated the problem for standard Markov processes that have a reference measure. They have established a sufficient condition and a necessary condition for the above joint continuity, but with a gap between the necessary and the sufficient conditions.

Received December 2005; revised October 2006.
AMS 2000 subject classifications. 60F05, 60G15, 60J25, 60J55.
Key words and phrases. Markov processes, local time, central limit theorem, Gaussian processes.

Bass and Khoshnevisan in [6], Barlow in [3, 4] and Barlow and Hawkes in [5] have treated the case of Lévy processes taking real values that have local times at all points and for which all points communicate. In [4] the necessary and sufficient conditions for the existence of an almost surely, jointly continuous version of the local time $(x,t) \to L_t^x(\omega)$ were found. Since this solution is in many ways the starting point of our approach, we shall describe it here (or rather Bertoin's [7] "translation" of it).

Let $h(a,b) = E_a(L^a_{T_b}) = E_0(L^0_{T_{b-a}}) = h(0, b-a) \stackrel{\text{def}}{=} h(b-a)$. Then one can show that $h(x) = h(-x)$ and that $d^2(a,b) = h(b-a)$ defines a distance on $\mathbb{R}$ that is equivalent to the Euclidean distance. Let $m(y) = |\{x : h(x) < y\}|$, where $|A|$ is the Lebesgue measure of the Borel set $A \subset \mathbb{R}$. Barlow's necessary and sufficient condition for the continuity of the local time is the following "majorizing measure" condition:

$$\int_{0+}^{\cdot} \sqrt{\ln \frac{1}{m(\varepsilon)}}\, d\varepsilon < \infty. \tag{2}$$

In Barlow's paper [4] the condition is stated in terms of the inverse of m, the monotone rearrangement of h. Let $\bar{h}(x) = \inf\{y : m(y) > x\}$, then $(x,t) \to L_t^x$ has a continuous version, iff

$$I(\bar{h}) = \int_{0+}^{\cdot} \frac{\bar{h}(x)}{x(\ln x)^{1/2}}\, dx < \infty. \tag{3}$$

It is easily seen that (2) and (3) are equivalent, but as was noticed by Barlow and Hawkes [5], (3) is reminiscent of Fernique's [17] and Dudley's [11] necessary and sufficient condition on the covariance function of a stationary Gaussian process (ϕ_x) to have a continuous version. This Gaussian process is precisely described in [14].

In a series of papers during the 1990s [23–27] and in their recent book [28], Marcus and Rosen study sample path properties of the local time process of strongly symmetric Markov processes. Under symmetry, the potential densities are symmetric and positive definite. Therefore, there exists a centered Gaussian process $(\phi_x)_{x \in E}$ such that $\langle \phi_x \phi_y \rangle = u(x,y)$, where $u(x,y)$ is the 0-potential density when the process is transient and $\langle \phi_x \phi_y \rangle = u^\alpha(x,y)$ for $\alpha > 0$ when the process is recurrent. Now and in the sequel, $\langle \cdot \rangle$ denotes the expectation with respect to the Gaussian measure. The main tool for their study is the celebrated Dynkin isomorphism theorem (DIT) [12, 13], which states when X is transient for any measurable function F on $\mathbb{R}^E$,

$$E_{a,b}\left\langle F\left(L^{\cdot}_{\zeta} + \frac{\phi^2_{\cdot}}{2}\right)\right\rangle = \left\langle \frac{\phi_a \phi_b}{\langle \phi_a \phi_b \rangle} F\left(\frac{\phi^2_{\cdot}}{2}\right)\right\rangle, \tag{4}$$

where ζ is the life time of the Markov process X and $E_{a,b}$ is the law of X born at a and killed at its last exit from the point b. Note that when X is recurrent, the above identity is available for X killed at an independent exponential time with parameter α. One should notice that the right-hand side of (4) is stated in terms of the Gaussian process only.

Defining a distance by $d^2(x, y) = \langle(\phi_x - \phi_y)^2\rangle = u(x, x) - 2u(x, y) + u(y, y)$, they have used the DIT to show that $(x, t) \to L_t^x$ has a jointly continuous version (in the distance d), iff the Gaussian process (ϕ_x) has a continuous version in that distance. The latter happens iff for every compact set K, in the metric d, there exists a probability measure μ on $\tilde{\mathcal{K}}$, the σ-algebra on K generated by the d-open sets, so that

$$\lim_{\delta\to 0} \sup_{x\in K} \int_0^\delta \sqrt{\ln \frac{1}{\mu(B(x, \varepsilon))}}\, d\varepsilon = 0, \tag{5}$$

where $B(x, \varepsilon)$ is a d-ball of radius ε around x. When $(x, y) \to u(x, y)$ is jointly continuous those conditions translate to a condition for the joint continuity (in the original distance) of $(x, t) \to L_t^x$. As in the case treated by Barlow, the condition for the joint continuity of the local time is identical to the condition for the continuity of a Gaussian process.

Extending these results beyond the symmetric and Lévy cases, and understanding the intriguing connection between the conditions for the continuity of local times of Markov processes and those of Gaussian processes is the objective of this paper.

We shall work under the following assumptions:

(A1) All points of E are regular for themselves.
(A2) All points of E communicate.
(A3) The process is recurrent.
(A4) There exists a Borel right dual process.

The recurrence property will simplify our arguments considerably, but it is not a very serious assumption. Indeed, by an argument due to Le Jan (see [10], Chapter XII), if X is transient, one can always "revive" it in such a way that it becomes recurrent, still keeping properties that will be used below like duality or symmetry if the original process was symmetric. Since the continuity and other fine properties of local times are local, and recurrence is a long time behavior property of the process, it has nothing to do with local properties, and therefore, using Le Jan's construction, we can extend the results to the transient case. With that in mind, and assuming that X is recurrent, let m be the unique invariant distribution for (P_t). (A1) and (A2) imply that m is actually a reference measure. Thus, the potential densities $u^\alpha(x, y)$ exist. From general theory (see [18]), we know that a dual process $\hat{X}$ exists. That is, there exists a Markov process $\hat{X}$ whose potential is given by $\hat{U}^\alpha f(x) = \int_E m(dy) f(y) u^\alpha(y, x)$. Since X is recurrent, so is $\hat{X}$, and since X has a local time at each $x \in E$, so does $\hat{X}$. In general $\hat{X}$ is not a strong Markov process. It is only a moderate Markov process, namely, it satisfies the strong Markov property only at $(\mathcal{F}_t)$ predictable stopping times. Our fourth assumption (A4) and the only serious one (beyond those needed to define the problem properly) is that $\hat{X}$ is actually a Borel right process as well, or that at least it satisfies the strong Markov property at the hitting times T_x of all $x \in E$. Note that the Lévy processes treated by Barlow satisfy this assumption (with $\hat{X} = -X$), and the symmetric processes studied by Marcus and Rosen satisfy it with $\hat{X} = X$.

To state our main results, we shall need some additional notation. Let 0 be a preassigned state in E and T_0 be its hitting time. By recurrence, $T_0 < \infty$ P_x a.s. for every $x \in E$. Let $u_{T_0}(x, y)$ be the potential densities of the process X^{T_0}, where

$$X_t^{T_0} = \begin{cases} X_t, & \text{if } t < T_0, \\ \Delta, & \text{otherwise,} \end{cases}$$

where Δ is a cemetery state. X^{T_0} is the process killed at its hitting time of the state 0. We shall show that $u_{T_0}(x, y) + u_{T_0}(y, x)$ is both symmetric and positive definite. Thus, there exists a centered Gaussian process $(\phi_x)_{x \in E}$, such that $\langle \phi_x \phi_y \rangle = u_{T_0}(x, y) + u_{T_0}(y, x)$. Using this, we now define the distance d with which we shall work:

$$d^2(x, y) = u_{T_0}(x, x) - u_{T_0}(x, y) - u_{T_0}(y, x) + u_{T_0}(y, y) = \langle (\phi_x - \phi_y)^2 \rangle. \tag{6}$$

Our first result gives a sufficient condition for the continuity of the local time process.

THEOREM 1.1. *If for every compact set K, in the d metric, there exists a probability measure μ on the Borel sets of K defined with the d-distance, so that*

$$\lim_{\delta \to 0} \sup_{x \in K} \int_0^\delta \sqrt{\ln \frac{1}{\mu(B(x, v))}}\, dv = 0,$$

where $B(x, \varepsilon)$ is a ball of d-radius ε around x, then $(t, x) \to L_t^x(\omega)$ has a jointly d-continuous version. If further $(x, y) \to u^\alpha(x, y)$ is jointly continuous, then a.s. $(x, t) \to L_t^x$ is continuous in d and the original distances. Finally, for a compact set K, set

$$\eta_K(\delta) = \sup_{z \in K} \int_0^\delta \sqrt{\ln \frac{1}{\mu(B(z, v))}}\, dv.$$

There is a positive constant C such that

$$\limsup_{\delta \to 0} \sup_{a, b \in K\, d(a,b) < \delta} \sup_{s \le t} \frac{|L_s^a - L_s^b|}{\eta_K(d(a, b))} \le C \left(\sup_{x \in K} L_t^x \right)^{1/2}.$$

Since the sufficient condition of Theorem 1.1 is actually a necessary and sufficient condition for the continuity of the Gaussian process ϕ (see [21]), Theorem 1.1 contains the following relation:
If $(\phi_x)_{x \in E}$ has a continuous version for the distance d, then $(L_t^x, x \in E, t \ge 0)$ has a jointly continuous version for the distance d.

Our next two theorems deal with a central limit theorem in $C(K)$, the space of continuous functions on a compact set K contained in E. We believe that this theorem provides the missing link for the converse of the above relation.

Let $\tau^a(s) = \inf\{t > 0 : L_t^a > s\}$. Then $\tau^a(s)$ is a process with stationary independent increments. In particular, $\tau^0(s)$ is a process with stationary independent increments, and $L^{\cdot}_{\tau^0(s)}$ is a process with stationary independent increments taking values in function space. We note further that, for any s, $L^{\cdot}_{\tau^0(s)}$ has an infinitely divisible law, and therefore, $Y_n(\cdot) = \frac{L^{\cdot}_{\tau^0(n)} - n}{\sqrt{n}}$ is an infinitely divisible random variable, taking values in the space of functions. We refer the reader to [2] and [22] for more on infinitely divisible processes taking values in Banach spaces.

THEOREM 1.2. *If $x \to Y_1(x)$ is a continuous function in the d distance, and the majorizing measure condition of Theorem 1.1 holds, then for each compact set K in the d metric, $(Y_n(x))_{x\in K}$ converges weakly in $C(K)$ to a centered Gaussian process $(\phi_x)_{x\in K}$ satisfying $\langle \phi_x \phi_y \rangle = u_{T_0}(x, y) + u_{T_0}(y, x)$.*

The characterization of continuous Gaussian processes as those for which the covariance distance satisfies the above majorizing measure condition yields the following theorem as a corollary.

THEOREM 1.3. *Assume that $u^\alpha(x, y)$ are continuous, then the following are equivalent*:

1. *$(x, t) \to L_t^x$ is jointly continuous and the above CLT holds.*
2. *The above majorizing measure condition holds.*

REMARK 1.4. We have not been able to show that the continuity of the local time process alone is a sufficient condition for the majorizing measure condition to hold. However, in view of all existing results, we conjecture that this is really the case. Theorem 1.3 allows one to replace the proof of sufficiency with a proof that the continuity of $x \to Y_n(x)$ implies its tightness in $C(K)$.

Our paper is organized as follows. In Section 2 we prove some preliminary results on the metric $d(x, y)$ defined in (6). Those will be our main tool for proving our results. Section 3 is devoted to the proof of Theorem 1.1 and in Section 4 we shall prove Theorem 1.2 and Theorem 1.3 as its corollary. We shall also recall there from [15] that in the symmetric case the tightness that is needed for the CLT follows easily from the continuity of the associated Gaussian process and the results of [16].

2. Notation and preliminary results. We adopt the basic notation of Blumenthal and Getoor [9]. We let $X, \hat{X}$ be two recurrent Borel right Markov processes in classical duality. As can be easily shown, under (A1)–(A3), the unique invariant measure m for this process is also a reference measure. Let $u^\alpha(x, y)$ be the corresponding potential densities, $U^\alpha f(x) = \int u^\alpha(x, y) f(y) m(dy)$ and $\hat{U}^\alpha f(y) = \int u^\alpha(x, y) f(x) m(dx)$. Hence, $u^\alpha(x, y)$ is the potential density of the process X starting at x and $u^\alpha(x, y)$ is the potential density of the process $\hat{X}$ starting at y. We will assume from now on that the processes have local times at each point (enough to assume that one of them has a local time at each point, the other will have it as a result), and that the local times are normalized so that

$$u^\alpha(x, y) = E_x \int e^{-\alpha t}\, dL_t^y$$

and similarly for the dual process,

$$u^\alpha(x, y) = \hat{E}_y \int e^{\alpha t}\, d\hat{L}_t^x.$$

For every state $x \in E$, let $T_x = \inf\{t > 0 : X_t = x\}$, we shall use the notation T_x for the dual process as well. Denote by $u_{T_x}(a, b)$ the potential densities of the process killed at T_x. The two resulting processes are again in duality with respect to $m(dy)$. By recurrence, $u_{T_x}(a, b)$ is finite and is equal to the increasing limit of $u_{T_x}^\alpha(a, b)$ as $\alpha \to 0$. Let ν^x be the excursion measure from x and similarly for the dual process, denote it by $\hat{\nu}^x$. All excursions from a point end at this point.

LEMMA 2.1. *Let x, y be two points in E. Then $u_{T_x}(y, y) = u_{T_y}(x, x)$.*

PROOF. Recall that $u_{T_x}(y, y) = E_y(L_{T_x}^y) = \lim_{\alpha \to 0} E_y \int_0^{T_x} e^{-\alpha t}\, dL_t^y$. Therefore,

$$\frac{u_{T_x}(y, y)}{u_{T_y}(x, x)} = \lim_{\alpha \to 0} \frac{E_y \int_0^{T_x} e^{-\alpha t}\, dL_t^y}{E_x \int_0^{T_y} e^{-\alpha t}\, dL_t^x}.$$

Now,

$$\begin{aligned} u^\alpha(x, x) &= E_x \int_0^\infty e^{-\alpha t}\, dL_t^x \\ &= E_x \int_0^{T_y} e^{-\alpha t}\, dL_t^x + E_x(e^{-\alpha T_y}) E_y(e^{-\alpha T_x}) u^\alpha(x, x). \end{aligned}$$

Hence,

$$E_x \int_0^{T_y} e^{-\alpha t}\, dL_t^x = u^\alpha(x, x)\big(1 - E_x(e^{-\alpha T_y}) E_y(e^{-\alpha T_x})\big) \tag{7}$$

and similarly,

$$E_y \int_0^{T_x} e^{-\alpha t}\, dL_t^y = u^\alpha(y, y)\big(1 - E_y(e^{-\alpha T_x}) E_x(e^{-\alpha T_y})\big). \tag{8}$$

Our result will follow if we can show that

$$\lim_{\alpha\to 0}\frac{u^\alpha(x,x)}{u^\alpha(y,y)}=1.$$

But,

$$\frac{u^\alpha(x,x)}{u^\alpha(y,y)}=\frac{u^\alpha(x,x)}{u^\alpha(y,x)}\frac{u^\alpha(y,x)}{u^\alpha(y,y)}=\frac{\hat{E}_x(e^{-\alpha T_y})}{E_y(e^{-\alpha T_x})}.$$

Since X is recurrent, so is the dual $\hat{X}$ and thus,

$$\lim_{\alpha\to 0}\frac{\hat{E}_x(e^{-\alpha T_y})}{E_y(e^{-\alpha T_x})}=\frac{\hat{P}_x(T_y<\infty)}{P_y(T_x<\infty)}=1. \qquad \square$$

With this result at hand we now have the following:

LEMMA 2.2. *For every x,y in E,*

$$\nu^0\big((L^x-L^y)^2\big)=2\big(u_{T_0}(x,x)-u_{T_0}(x,y)-u_{T_0}(y,x)+u_{T_0}(y,y)\big).$$

PROOF. Let (θ_t) be the usual shift operators on the state space so that $X_s(\theta_t\omega)=X_{t+s}(\omega)$ and $\hat{\theta}_t$ defined similarly for the dual process:

$$\nu^0(L^xL^y)=\nu^0\bigg(\int_0^{T_0}L^y_{T_0}(\theta_t)\,dL^x_t+\int_0^{T_0}L^x_{T_0}(\theta_t)\,dL^y_t\bigg). \tag{9}$$

By the Markov property that ν^0 satisfies, this is equal to

$$\nu^0\bigg(\int_0^{T_0}E_x(L^y_{T_0})\,dL^x_t+\int_0^{T_0}E_y(L^x_{T_0})\,dL^y_t\bigg)$$

and hence, to

$$\nu^0\big(1_{\{T_x<T_0\}}u_{T_0}(x,x)u_{T_0}(x,y)+1_{\{T_y<T_0\}}u_{T_0}(y,y)u_{T_0}(y,x)\big).$$

But

$$\nu^0(T_x<T_0)=\frac{1}{E_0(L^0_{T_x})}=\frac{1}{u_{T_x}(0,0)}=\frac{1}{u_{T_0}(x,x)}, \tag{10}$$

where the last equality follows from Lemma 2.1. Inserting this into (9) yields

$$\begin{aligned}\nu^0(L^xL^y)&=\frac{u_{T_0}(x,x)u_{T_0}(x,y)}{u_{T_0}(x,x)}\\&\quad+\frac{u_{T_0}(y,y)u_{T_0}(y,x)}{u_{T_0}(y,y)}\\&=u_{T_0}(x,y)+u_{T_0}(y,x).\end{aligned} \tag{11}$$

$\square$

COROLLARY 2.3. *$(u_{T_0}(x,y)+u_{T_0}(y,x), x, y \in E \times E)$ is symmetric, positive definite.*

PROOF. Symmetry is obvious. Let $(a_1, \dots, a_n)$ be a vector in $\mathbb{R}^n$, then

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \big(u_{T_0}(x_i, x_j) + u_{T_0}(x_j, x_i)\big)$$
$$= \nu^0\left(\left(\sum_{i=1}^{n} a_i L^{x_i}\right)^2\right) \geq 0. \qquad \square$$

We now define

$$d^2(x,y) = u_{T_0}(x,x) - u_{T_0}(x,y) - u_{T_0}(y,x) + u_{T_0}(y,y).$$

The above results prove that $d(x,y)$ defines a pseudo distance, and that there is a centered Gaussian process (ϕ_x) such that

$$\langle \phi_x, \phi_y \rangle = u_{T_0}(x,y) + u_{T_0}(y,x). \tag{12}$$

LEMMA 2.4. *Set $h(x,y) = E_x(L^x_{T_y})$, then $d^2(x,y) = h(x,y)$.*

PROOF.

$$\begin{aligned} d^2(x,y) &= u_{T_0}(x,x) - u_{T_0}(x,y) - u_{T_0}(y,x) + u_{T_0}(y,y) \\ &= h(x,0) + h(y,0) - P_x(T_y < T_0)h(y,0) - P_y(T_x < T_0)h(x,0) \\ &= h(x,0)P_y(T_x > T_0) + h(y,0)P_x(T_y > T_0) \\ &= u_{T_0}(x,x)\frac{u_{T_x}(y,0)}{u_{T_x}(0,0)} + u_{T_0}(y,y)\frac{u_{T_y}(x,0)}{u_{T_y}(0,0)}. \end{aligned}$$

But by Lemma 2.1, $u_{T_0}(x,x) = u_{T_x}(0,0)$ and $u_{T_0}(y,y) = u_{T_y}(0,0)$, and the last term is equal to

$$\begin{aligned} &u_{T_x}(y,0) + u_{T_y}(x,0) \\ &\quad = \hat{E}_0(L^y_{T_x}) + \hat{E}_0(L^x_{T_y}) \\ &\quad = \hat{P}_0(T_y < T_x)\hat{E}_y(L^y_{T_x}) + \hat{P}_0(T_x < T_y)\hat{E}_x(L^x_{T_y}) \\ &\quad = \hat{E}_y(L^y_{T_x}) = \hat{h}(y,x), \end{aligned}$$

where the one before last equality follows from Lemma 2.1 applied to the dual process. We now notice that since $d^2(x,y)$ is symmetric with respect to x and y, and with respect to the dual objects, it follows that $\hat{h}(y,x) = \hat{h}(x,y) = h(y,x) = h(x,y)$, and our result follows. $\square$

REMARK 2.5. It follows from the above result that d is a real distance on E. Indeed, for $x \neq y$, $P_x(T_y > 0) = 1$ and since x is regular for itself, this implies that $E_x(L^x_{T_y}) > 0$ and hence, that $d(x, y) > 0$. If the potential densities $u_{T_0}(x, y)$ are jointly continuous, continuity in the topology generated by this metric implies continuity in the original metric on E.

3. Sufficiency of the majorizing measure condition. Thanks to the results of the previous section, the proof of sufficiency is very close to that of Bertoin's ([7], pages 144–150).

LEMMA 3.1. *For $a \in E$, set $\tau^a_t = \inf\{s : L^a_s > t\}$. Then for every $a, b \in E$,*

$$P\{\exists s \le \tau^b_y : L^b_s - L^a_s > x\} \le \exp\left(-\frac{x^2}{4yh(a,b)}\right). \tag{13}$$

PROOF. By the Markov property, we may start our process at b. $t \to L^a_{\tau^b_t}$ is a subordinator (which may have a jump at 0 if our process does not start at b). $L^a_{\tau^b_t}$ stays at 0 and performs its first jump at time $L^b_{T_a}$. It has therefore no drift. $L^b_{T_a}$ under P_b has an exponential distribution with expectation $E_b(L^b_{T_a}) = h(a, b)$. Next let $R = \inf\{t > T_a : X_t = b\}$ and note that $R = \tau^b_{L^b_{T_a}}$. Since $L^a_{T_a} = 0$, L^a_R has again an exponential distribution with expectation $E_a(L^a_{T_b}) = h(a, b)$. Hence, the Lévy measure of $L^a_{\tau^b_t}$ is equal to $\frac{1}{h^2(a,b)} \exp(-\frac{1}{h(a,b)}x)$, its Lévy exponent $\Psi(\lambda)$ is equal to $\frac{\lambda}{\lambda h(a,b)+1}$, and $\exp(-\lambda L^a_{\tau^b_s} + \frac{s\lambda}{\lambda h(a,b)+1})$ is a martingale. By the optional sampling theorem applied to $T \wedge y$ where $T = \inf\{s : s - L^a_{\tau^b_s} > x\}$, we can show that

$$P\{T \le y\} \le \exp\left(-\lambda x + \lambda y\left(1 - \frac{1}{\lambda h(a,b) + 1}\right)\right).$$

Taking now $\lambda = \frac{x}{2yh(a,b)}$ gives us the required upper bound. $\square$

Define now

$$Y_a(q)(t) = q \wedge L^a_t. \tag{14}$$

Then for every $q > 0, a, b \in E$, if $q \wedge L^b_t - q \wedge L^a_t > x$ at some time $t \ge 0$, then the time when this occurs is bounded by τ^b_q. Hence,

$$\begin{aligned}&\{\exists s : |Y_a(q)(s) - Y_b(q)(s)| > x\}\\ &\quad = \{\exists s \le \tau^b_q : L^b_s - L^a_s > x\} \cup \{\exists s \le \tau^a_q : L^a_s - L^b_s > x\}.\end{aligned}$$

Therefore,

$$P\{|Y_a(q) - Y_b(q)|_u > x\} \le 2\exp\left(-\frac{x^2}{4qh(a,b)}\right), \tag{15}$$

where $|\ |_u$ is the uniform bound with respect to time. It now follows that, for every $c > 0$,

$$\begin{aligned} &E\big(\exp(|Y_b(q) - Y_a(q)|_u^2/c) - 1\big) \\ &\quad = \frac{1}{c}\int_0^\infty \exp\Big(\frac{x}{c}\Big) P\{|Y_b(q) - Y_a(q)|_u^2 > x\}\, dx \\ &\quad \le \frac{2}{c}\int_0^\infty \exp\Big(\frac{x}{c}\Big) \exp\Big(\frac{x}{4qh(a,b)}\Big)\, dx = 2\Big(\frac{c}{4qh(a,b)} - 1\Big)^{-1}. \end{aligned}$$

Taking now $c > 12qh(a,b)$, we get

$$E\big(\exp(|Y_b(q) - Y_a(q)|_u^2) - 1\big) < 1.$$

In the language of Ledoux and Talagrand (page 298 in [21]),

$$\|Y_b(q) - Y_a(q)\|_\psi \le \tilde{d}(a,b),$$

where $\tilde{d}^2(a,b) = 12qh(a,b)$ and the Young function $\psi(x) = \exp(x^2) - 1$. We shall fix now $q > 1$ and abuse the notation by denoting $Y_a(q)(t)$ by $Y_a(t)$.

PROOF OF THEOREM 1.1. *Step* 1. By Theorem 11.14 of [21], if for a compact set K there exists a probability measure μ on $(K, \tilde{d})$ such that for

$$\lim_{\eta\to 0} \sup_{x\in K} \int_0^\eta \sqrt{\ln \frac{1}{\mu(B(x,\varepsilon))}}\, d\varepsilon = 0,$$

where $B(x,\varepsilon)$ is a ball of radius ε in the distance $\tilde{d}$ around x, then for each $x \in E$, $(a \to Y_a)_{a\in K}$ has a P_x almost surely continuous version with respect to the distance $\tilde{d}$. That is, there is a process$(\tilde{Y}_a)_{a\in K}$ with continuous sample paths in the distance $\tilde{d}$ (and therefore d), so that for every $a \in K$, $Y_a = \tilde{Y}_a$ P_x almost surely.

Let $(\tilde{Y}_a)_{a\in K}$ be that version and define $\bar{Y}_*(t) = \sup\{\tilde{Y}_a(t) : a \in K\}$ and $\Theta(q) = \inf\{t : \bar{Y}_*(t) = q\}$. Since $a \to \tilde{Y}_a$ is continuous, $\bar{Y}_*(t) = \sup\{\tilde{Y}_r(t) : r \in \Gamma\}$, where Γ is a countable dense set in K with respect to the distance $\tilde{d}$. Thus, $\bar{Y}_*(t)$ is a nondecreasing adapted process. Hence, $\Theta(q)$ is an $(\mathcal{F}_t)$ stopping time. By the Blumenthal 0–1 law, $\{\Theta(q) = 0\}$ is a probability 0 or 1 event for every P_z.

Step 2. $\Theta(q) > 0$ *for* q *large enough*. We shall show that, for q large enough, $\Theta(q) > 0$, P_x a.s. and that $\Theta(q) \to \infty$ as $q \to \infty$. To do that, we shall use Proposition 1 of [20]. Indeed, let

$$f(q,\omega,a,b) = |\tilde{Y}_a(\tau(1)) - \tilde{Y}_b(\tau(1))|$$

and

$$\tilde{f}(q,\omega,a,b) = \frac{f(q,\omega,a,b)}{\tilde{d}(q,a,b)},$$

where

$$\tau(1) = \inf\{s : \tilde{Y}_0(s) \geq 1\},$$

where 0 is a preassigned state, that we assume is in K. Then

$$\begin{aligned}
&P_x\{\exp(\tilde{f}^2(q,\omega,a,b)) - 1 > \alpha\} \\
&\quad = P_x\{\exp(\tilde{f}^2(q,\omega,a,b)) > 1+\alpha\} \\
&\quad = P_x\{f^2(q,\omega,a,b) > 12\ln(1+\alpha)qd^2(a,b)\} \\
&\quad = P_x\{f(q,\omega,a,b) > \sqrt{12q\ln(1+\alpha)}d(a,b)\} \\
&\quad \leq P_x\Big\{\sup_{s\leq\tau_1^0} |L_s^a \wedge q - L_s^b \wedge q| > \sqrt{12q\ln(1+\alpha)}d(a,b)\Big\} \\
&\quad \leq 2\exp\biggl(-\frac{12q\ln(1+\alpha)d^2(a,b)}{4qh(a,b)}\biggr) \\
&\quad \leq 2\exp\biggl(-\frac{12q\ln(1+\alpha)}{4q}\biggr) \\
&\quad = 2\biggl(\frac{1}{1+\alpha}\biggr)^3,
\end{aligned}$$

where the first inequality follows from (15) and the third from Lemma 2.2. It follows that, for all q,

$$E_x(\exp(\tilde{f}^2(q,\omega,a,b))) \leq 3. \tag{16}$$

Define now

$$C(q) = \int_K \int_K \exp(\tilde{f}^2(q,\omega,a,b))\mu(da)\mu(db).$$

Then $E_x(C(q)) \leq 3$, and therefore, $C(q) < \infty$, P_x a.s. Since $a \to \tilde{Y}_a(\tau(1))$ is continuous in the $\tilde{d}(=\tilde{d}(q))$ distance, we can use Heinkel's formula to deduce that, for all $(a,b) \in K$,

$$|\tilde{Y}_a(\tau(1)) - \tilde{Y}_b(\tau(1))| \leq 20 \sup_{z\in K} \int_0^{\sqrt{12q}d(a,b)/2} \biggl(\ln\biggl(\frac{C(q)}{\mu^2(\tilde{B}(z,u))}\biggr)\biggr)^{1/2} du,$$

where, as before, $\tilde{B}(x,v)$ is a ball of radius v in the $\tilde{d}$ distance. This after the change of variable $v = \frac{u}{\sqrt{3q}}$ is equal to

$$20\sqrt{3q} \sup_{z\in K} \int_0^{d(a,b)} \biggl(\ln\biggl(\frac{C(q)}{\mu^2(B(z,v/2))}\biggr)\biggr)^{1/2} dv,$$

where $B(z, v)$ is a ball of radius v in the d distance. This is bounded above by

$$40\sqrt{3q}\sup_{z\in K}\int_0^{d(a,b)}\left(\ln\left(\frac{C(q)}{\mu^2(B(z,v))}\right)\right)^{1/2}dv,$$

which is easily shown to be bounded by

$$40\sqrt{3q}(\ln C(q))^{1/2}d(a,b)+\sqrt{2}\sup_{z\in K}\int_0^{d(a,b)}\left(\ln\left(\frac{1}{\mu(B(z,v))}\right)\right)^{1/2}dv.$$

For $\delta > 0$, define

$$\eta(\delta)=\sup_{z\in K}\int_0^{\delta}\left(\ln\left(\frac{1}{\mu(B(z,v))}\right)\right)^{1/2}dv. \tag{17}$$

By our assumptions, $\eta(D) < \infty$, where D is the diameter of the compact set K, and $\lim_{\delta\to 0}\eta(\delta) = 0$. Returning to our computation,

$$\tilde{Y}_a(\tau(1)) \le \tilde{Y}_0(\tau(1)) + c\sqrt{q}\big((\ln C(q))^{1/2}D + \sqrt{2}\eta(D)\big),$$

where c is a constant and D is the diameter of K in the d distance. We now recall that P_x a.s. $\tilde{Y}_0(\tau(1)) = 1$, and so, on $\{\Theta(q) \le \tau(1)\}$,

$$q \le 1 + c\sqrt{q}\big((\ln C(q))^{1/2}D + \sqrt{2}\eta(D)\big).$$

Using, as in [3], the fact that $y^2 \le A + By$ implies that $y^2 \le 2A + B^2$, we see that, on $\{\Theta(q) \le \tau(1)\}$,

$$q \le 2 + \big(c((\ln C(q))^{1/2}D + \sqrt{2}\eta(D))\big)^2.$$

Thus,

$$\begin{aligned}
P_x\{\Theta(q) \le \tau(1)\} &\le P_x\{\big(c((\ln C(q))^{1/2}D + \sqrt{2}\eta(D))\big)^2 \ge q-2\}\\
&\le P_x\left\{2((\ln C(q))^{1/2}D)^2 + 4\eta^2(D) \ge \frac{q-2}{c^2}\right\}\\
&= P_x\left\{\ln C(q) \ge \frac{q-2}{2c^2D^2} - \frac{2\eta^2(D)}{D^2}\right\}\\
&= P_x\left\{C(q) \ge \exp\left(\frac{q-2}{2c^2D^2} - \frac{2\eta^2(D)}{D^2}\right)\right\}\\
&\le 3\exp\left(-\frac{q-2}{2c^2D^2} + \frac{2\eta^2(D)}{D^2}\right),
\end{aligned}$$

where the last inequality follows from the fact that $E_x(C(q)) \le 3$, this last term is smaller than 1 for q large enough. It now follows that $P_x\{\Theta(q) = 0\} \le P_x\{\Theta(q) \le \tau(1)\} < 1$, so that $P_x\{\Theta(q) = 0\} = 0$ for q large enough.

Step 3. $\lim_{n\to\infty} \Theta(n)$. Repeating the above computation with $q(n) = 3n$ and $\tau(n) = \inf\{t > 0 : Y_0(t) > n\}$, instead of $q = 1$ and $\tau(1)$, we similarly obtain

$$P_x\{\Theta(q(n)) \le \tau(n)\} \le 3\exp\left(-\frac{3n-2n}{2c^2D^2} + \frac{2\eta^2(D)}{D^2}\right) \\ \le A\exp\left(-\frac{n}{B}\right),$$

where A and B are constants. It now follows from the fact that $\tau(n) \to \infty$ as $n \to \infty$, and the Borel–Cantelli lemma that $\Theta(3n) \to \infty$, P_x a.s., and so, the local time has a jointly continuous version (with respect to P_x) in $\mathbb{R}_+ \times K$ for all compact K, and thus a jointly continuous version.

Step 4. *Jointly continuous potential densities*. To prove the last assertion of Theorem 1.1, we need to show that when $u^\alpha(x, y)$ is jointly continuous, then a.s. $(t, x) \to L^x(t, \omega)$ is jointly continuous. First note that by Remark 3.4.4 of [28] $u^\beta(x, y)$ is jointly continuous for any $\beta > 0$, and thus, it can be easily shown that $u_{T_0}(x, y)$ is jointly continuous as well. Hence, the continuity in the d distance implies continuity in the original distance [Actually, as was noted by Bertoin ([7], page 147), this will make the distances equivalent in the sense that their induced topologies will be the same.] We fix now a q and prove that, P_x almost surely, $\tilde{Y}_a(q)(t) = L^a_t$ simultaneously for all $a \in K$ and $t < \Theta(q)$. Specifically, for each $a \in K$, P_x almost surely, $L^a_t = \tilde{Y}_a(q)(t)$ for all $t < \Theta(q)$. By Fubini's theorem and the occupation time density formula [8], for each continuous f with support in K,

$$\int_0^t f(X_s)\,ds = \int_K f(a)\tilde{Y}_a(q)(t)m(da)$$

for all $t < \Theta(q)$ P_x almost surely. Letting f range over a countable dense family in $C(K)$ (the continuous functions with support in K), we see that the identity above holds for all bounded measurable functions with support in K. In particular, P_x almost surely for all $a \in K$, $\varepsilon > 0$ and $t < \Theta(q)$,

$$\int_0^t f_{\varepsilon,a}(X_s)\,ds = \int_K f_{\varepsilon,a}(x)\tilde{Y}_x(q)(t)m(dx),$$

where $f_{\varepsilon,a}$ is an approximating delta function that defines the local time at $a \in K$, (see Theorem 3.6.3 and the discussion preceding it in Marcus and Rosen's recent book [28], that can be easily adapted to the nonsymmetric situation). Since $x \to \tilde{Y}_x(q)(t)$ is continuous, the right-hand side converges to $\tilde{Y}_a(q)(t)$ as $\varepsilon \to 0$, and $\lim_{\varepsilon\to 0} \int_0^t f_{\varepsilon,a}(X_s)\,ds = L^a_t$ uniformly in $[0, \Theta(q))$ by Theorem 3.6.3 of [28]. Thus, $\tilde{Y}_a(q)(t) = L^a_t$, P_x almost surely for all $a \in K$ and $t < \Theta(q)$, so that $(a, t) \to L^a_t$ is P_x a.s. continuous on $K \times [0, \Theta(q)))$. Since we have seen that P_x a.s. $\Theta(3n) \to \infty$, as $n \to \infty$, it follows that $(a, t) \to L^a_t$ is P_x a.s. continuous on $K \times [0, \infty)$ and therefore, in $E \times [0, \infty)$. Since this is true for every $x \in E$, it follows that a.s. $(a, t) \to L^a_t$ is continuous.

Step 5. *Modulous of continuity*. To get the modulous result, we follow [3], with Heinkel's inequality replacing the Gracia, Rodemich, Rumsey inequality that appears there. We now return to Heinkel's inequality with L_t^x, which we now know is continuous in the d distance. Let K be a compact set in the d metric. For a fixed $t > 0$, let q be such that $\sup_{x\in K} L_t^x(\omega) \le q \le 2\sup_{x\in K} L_t^x(\omega)$. Note that we have

$$\liminf_{\delta\downarrow 0} \frac{1}{\delta}\eta_K(\delta) \ge \lim_{\delta\downarrow 0}\left(\ln\frac{1}{\mu(B(z,\delta))}\right)^{1/2}$$

for every z in K. Hence, unless μ charges all the points of K and K is finite, we obtain

$$\lim_{\delta\downarrow 0} \frac{1}{\delta}\eta_K(\delta) = +\infty$$

[we have used the fact that for every infinite countable subset A of K, $\sum_{z\in A}\mu(\{z\}) \le 1$]. Of course, in the case when K is finite, the question of the modulous of continuity is meaningless. Hence, we can choose $\delta(\omega) > 0$ small enough such that $(\ln(C(q, K, \omega)))^{1/2}\delta(\omega) < \sqrt{2}\eta_K(\delta(\omega))$. If $\varepsilon_t(\omega)$ is chosen to be smaller than $\delta(\omega)$, then, as in the computation of Step 2, with $(L_t^a)_{a\in K}$ replacing $(\tilde{Y}_a(\tau(1))_{a\in K})$ that appears there, one can show that for $a, b \in K$,

$$|L_s^a - L_s^b| \le C\left(\left(\sup_{x\in K} L_t^x\right)^{1/2}\eta_K(d(a,b))\right)$$

for all $0 \le s \le t$ and all $a, b \in K$ such that $d(a,b) < \varepsilon_t(\omega)$. C is a constant, which by our computations is smaller than 80, and is by no means the best possible (see [3, 4, 6] in the Lévy case and [23] in the symmetric case). We shall not pursue this issue any further here. □

4. The central limit theorem for local times. Trying to understand the true reason why in all existing results the conditions for the joint continuity of the local time are identical to those for the continuity of Gaussian processes, one is led to seek the explanation in a suitable CLT. Indeed, let τ_t^0 be the inverse of the local time at 0, then $(L^x_{\tau_t^0})_{x\in E}$, is a process with stationary independent increments with values that are functions on E_0. By Lemma 2.2 and its proof, $E(L^x_{\tau_n^0}) = n$, and $E_0((L^x_{\tau_n^0})^2) = 2u_{T_0}(x,x)n$. It follows that $\frac{L^x_{\tau_n^0}-n}{\sqrt{n}}$ converges in distribution to a centered normal random variable with variance $2u_{T_0}(x,x)$. The following lemma will show that the process $(\frac{L^x_{\tau_n^0}-n}{\sqrt{n}})_{x\in E}$ converges in finite-dimensional distributions to a Gaussian process with covariance $u_{T_0}(x,y) + u_{T_0}(y,x)$.

LEMMA 4.1.

$$E_0\left(\frac{(L^x_{\tau_n^0} - n)(L^y_{\tau_n^0} - n)}{n}\right) = u_{T_0}(x,y) + u_{T_0}(y,x).$$

PROOF. Denote by G the set of left endpoints of excursions from $\{t : X_t = 0\}$. Then we have

$$E_0(L^x_{\tau^0_n}) = E_0\Biggl(\sum_{s\in G, s\le\tau^0_n} L^x_{T_0}\circ\theta_s\Biggr) = E_0\int_0^{\tau^0_n} \nu^0(L^x)\,dL^0_t = n, \tag{18}$$

where the second equality follows from excursion theory (compensating the sum of jumps), and the third by a change of variable $s = L^0_t$, and the fact that $\nu^0(L^x) = 1$ for all $x \in E$, which we have shown in the proof of Lemma 2.2. Similarly,

$$\begin{aligned}
E_0(L^x_{\tau^0_n} L^y_{\tau^0_n}) &= E_0 \sum_{s\in G, s\le\tau^0_n} L^x_{T_0}\circ\theta_s \sum_{t\in G, t\le\tau^0_n} L^y_{T_0}\circ\theta_t \\
&= E_0 \sum_{s\in G, s\le\tau^0_n} (L^x_{T_0} L^y_{T_0})\circ\theta_s \\
&\quad + E_0 \sum_{s\in G, s\le\tau^0_n} L^x_{T_0}\circ\theta_s \sum_{t\in G, t\le\tau^0_n, t\ne s} L^y_{T_0}\circ\theta_t.
\end{aligned} \tag{19}$$

Using excursion theory as above, the first sum of (19) is equal to

$$E_0\int_0^{\tau^0_n} \nu^0(L^x L^y)\,dL^0_t = n\nu^0(L^x L^y) = n\bigl(u_{T_0}(x,y) + u_{T_0}(y,x)\bigr), \tag{20}$$

where the last equality follows as in the proof of Lemma 2.2. Using excursion theory again, the second term of (19) is composed of two sums. The first is equal to

$$E_0\int_0^{\tau^0_n} \nu^0(L^x) E_0(L^y_{\tau^0_n - t})\,dL^0_t$$

and the second is identical to the above with x and y interchanged. Since $\nu^0(L^x) = 1$, this integral is equal to $E_0\int_0^n E_0(L^y_{\tau^0_{n-u}})\,du = E_0\int_0^n E_0(L^y_{\tau^0_u})\,du = \frac{n^2}{2}$. Since the value of this term is independent of x and y, it remains the same when interchanging x and y. Thus,

$$\begin{aligned}
&E_0\biggl(\frac{(L^x_{\tau^0_n} - n)(L^y_{\tau^0_n} - n)}{n}\biggr) \\
&\quad = \frac{n(u_{T_0}(x,y) + u_{T_0}(y,x)) + n^2 - n^2 - n^2 + n^2}{n} \\
&\quad = u_{T_0}(x,y) + u_{T_0}(y,x). \qquad \square
\end{aligned}$$

PROOF OF THEOREM 1.2. All one needs to prove is tightness in $C(K)$ which, since $L^0_{\tau^0_n} - n = 0$, amounts to showing that for every $\eta > 0, \varepsilon > 0, \exists\delta > 0$,

$\exists n_0 \in \mathbf{N}$, so that for all $n \geq n_0$,

$$P\left\{\sup_{a,b\in K, d(a,b)<\delta} \frac{|L^a_{\tau^0_n} - L^b_{\tau^0_n}|}{\sqrt{n}} > \eta\right\} < \varepsilon.$$

To prove this, we shall use here again Proposition 1.1 of [20], but in view of the computations in the last section, with some steps abridged. We shall split the proof into a few steps.

Step 1. *Definition of objects appearing in Heinkel's inequality*:

$$Y_n(q,a,b) = \sup_{s\leq \tau^0_n} |q \wedge L^a_s - q \wedge L^b_s|, \tag{21}$$

$$\tilde{Y}_n(q,a,b) = \frac{Y_n(q,a,b)}{\rho(a,b)}, \tag{22}$$

where $\rho(a,b) = \sqrt{12q}\, d(a,b)$. Then as in the computations preceding (16), for all $y \in E$,

$$P_y\{\exp(\tilde{Y}^2_n(q,a,b)) - 1 > \alpha\} \leq 2\left(\frac{1}{1+\alpha}\right)^3. \tag{23}$$

Hence, for all n, $E_y(\exp(\tilde{Y}^2_n(q,a,b))) \leq 3$.

Define now

$$C(n,q) = \int_K \int_K \exp(\tilde{Y}^2_n(q,a,b))\mu(da)\mu(db) \tag{24}$$

Then $E_y(C(n,q)) \leq 3$ and therefore, $C(n,q) < \infty$, P_y a.s. for all $y \in E$.

Step 2. *Application of Heinkel's inequality*. It follows from Proposition 1.1 of [20] that

$$|q \wedge L^a_{\tau^0_n} - q \wedge L^b_{\tau^0_n}| \leq 20 \sup_{z\in K} \int_0^{\sqrt{12q}\, d(a,b)/2} \left(\ln\left(\frac{C(n,q)}{\mu^2(B(z,u))}\right)\right)^{1/2} du.$$

Following again the same arguments as in the previous section, the right-hand side of the above inequality is bounded above by

$$40\sqrt{3q}\left((\ln C(n,q))^{1/2} d(a,b) + \sqrt{2} \sup_{z\in K} \int_0^{d(a,b)} \left(\ln\left(\frac{1}{\mu(B(z,v))}\right)\right)^{1/2} dv\right).$$

Recall the definition of $\eta(\delta)$ from (17). By our assumption, $\eta(\delta) \to 0$ as $\delta \to 0$.

Step 3. With the above result at hand, we now take $q = n + \lambda\sqrt{n}$ to obtain

$$\begin{aligned}
&\frac{|(n+\lambda\sqrt{n}) \wedge L^a_{\tau^0_n} - (n+\lambda\sqrt{n}) \wedge L^b_{\tau^0_n}|}{\sqrt{n}} \\
&\quad\leq \frac{1}{\sqrt{n}} 40\sqrt{3}(n+\lambda\sqrt{n})^{1/2}\left[\left(\ln\left(C\left(n, n+\lambda\sqrt{n}\right)\right)\right)^{1/2} d(a,b) + \sqrt{2}\eta(d(a,b))\right] \\
&\quad= 40\sqrt{3}\left(1 + \frac{\lambda}{\sqrt{n}}\right)^{1/2}\left[\left(\ln\left(C\left(n, n+\lambda\sqrt{n}\right)\right)\right)^{1/2} d(a,b) + \sqrt{2}\eta(d(a,b))\right].
\end{aligned}$$

Returning now to the proof of tightness,

$$P\left(\sup_{d(a,b)<a,b\in K,\delta} \frac{|L^a_{\tau^0_n} - L^b_{\tau^0_n}|}{\sqrt{n}} > \eta\right) \tag{25}$$

$$\leq P\left(\sup_{a,b\in K,d(a,b)<\delta} \frac{|(n+\lambda\sqrt{n}) \wedge L^a_{\tau^0_n} - (n+\lambda\sqrt{n}) \wedge L^b_{\tau^0_n}|}{\sqrt{n}} > \eta\right) \tag{26}$$

$$+ P\left(\sup_{x\in K} L^x_{\tau^0_n} > n + \lambda\sqrt{n}\right). \tag{27}$$

Step 4. Starting with (27) and using the above inequality with $a = x$, $b = 0$, and recalling that $L^0_{\tau^0_n} = n$, we get

$$(n+\lambda\sqrt{n}) \wedge L^x_{\tau^0_n} \leq n + 40\sqrt{3}(n+\lambda\sqrt{n})^{1/2}\left[\left(\ln\left(C\left(n, n+\lambda\sqrt{n}\right)\right)\right)^{1/2} D + \sqrt{2}\eta(D)\right],$$

where D is the diameter of K with respect to the distance d which, by our assumption, is finite as is $\eta(D)$. Now on $\{\sup_{x\in K} L^x_{\tau^0_n} > n + \lambda\sqrt{n}\}$,

$$n + \lambda\sqrt{n} \leq n + 40\sqrt{3}(n+\lambda\sqrt{n})^{1/2}\left[\left(\ln\left(C\left(n, n+\lambda\sqrt{n}\right)\right)\right)^{1/2} D + \sqrt{2}\eta(D)\right],$$

so that

$$\lambda\sqrt{n} \leq 40\sqrt{3}(n+\lambda\sqrt{n})^{1/2}\left[\left(\ln\left(C\left(n, n+\lambda\sqrt{n}\right)\right)\right)^{1/2} D + \sqrt{2}\eta(D)\right],$$

which is equivalent to

$$\frac{\lambda}{(1+\lambda/\sqrt{n})^{1/2}} \leq 40\sqrt{3}\left[\left(\ln\left(C\left(n, n+\lambda\sqrt{n}\right)\right)\right)^{1/2} D + \sqrt{2}\eta(D)\right].$$

Thus,

$$P\left\{\sup_{x\in K} L^x_{\tau^0_n} > n + \lambda\sqrt{n}\right\}$$
$$\leq P\left\{40\sqrt{3}\left[\left(\ln\left(C\left(n, n+\lambda\sqrt{n}\right)\right)\right)^{1/2} D + \sqrt{2}\eta(D)\right] \geq \frac{\lambda}{(1+\lambda/\sqrt{n})^{1/2}}\right\}.$$

This last probability is equal to

$$P\left\{\ln\left(C\left(n, n+\lambda\sqrt{n}\right)\right) \geq \left(\frac{\lambda}{(1+\lambda/\sqrt{n})^{1/2} 40\sqrt{3}D} - \frac{\sqrt{2}\eta(D)}{D}\right)^2\right\}. \tag{28}$$

For λ big enough, so that the first term on the right-hand side of the inequality of (28) is larger than 3 times the second, this is smaller or equal to

$$P\left\{\ln\left(C\left(n, n+\lambda\sqrt{n}\right)\right) \geq \frac{\lambda^2}{3(1+\lambda/\sqrt{n})4800D^2}\right\}$$
$$\leq 3\exp\left(-\frac{\lambda^2 C}{D^2(1+\lambda/\sqrt{n})}\right) \leq A\exp\left(-\frac{\lambda^2}{1+\lambda}B\right),$$

where the first inequality follows from the fact that we have shown that $E(C(n, q)) \leq 3$ for all q and n, and A, B, C are some constants.

We now choose λ big enough to satisfy all the above inequalities and make this last bound smaller than $\varepsilon/2$. Note that this λ is chosen independently of n. With this λ, we return to (26).

Step 5.

$$P\left\{\sup_{d(a,b)<\delta} \frac{|(n+\lambda\sqrt{n}) \wedge L^a_{\tau^0_n} - (n+\lambda\sqrt{n}) \wedge L^b_{\tau^0_n}|}{\sqrt{n}} > \eta\right\}$$
$$\leq P\left\{40\sqrt{3}\left(1+\frac{\lambda}{\sqrt{n}}\right)^{1/2}\left[\left(\ln\left(C(n, n+\lambda\sqrt{n})\right)\right)^{1/2}\delta + \sqrt{2}\eta(\delta)\right] > \eta\right\}$$
$$= P\left\{\left(\ln\left(C(n, n+\lambda\sqrt{n})\right)\right)^{1/2}\delta + \sqrt{2}\eta(\delta) > \frac{\eta}{40\sqrt{3}(1+\lambda/\sqrt{n})^{1/2}}\right\}.$$

Choose now δ^* small enough so that $\sqrt{2}\eta(\delta) < \frac{\eta}{80\sqrt{3}(1+\lambda)^{1/2}}$ for all $\delta \leq \delta^*$. For $\delta \leq \delta^*$, the above probability is smaller or equal to

$$P\left\{\left(\ln\left(C(n, n+\lambda\sqrt{n})\right)\right)^{1/2}\delta > \frac{\eta}{80\sqrt{3}(1+\lambda/\sqrt{n})^{1/2}}\right\},$$

which is equal to

$$P\left\{\left(\ln\left(C(n, n+\lambda\sqrt{n})\right)\right) > \frac{\eta^2 c}{(1+\lambda/\sqrt{n})\delta^2}\right\}$$
$$\leq P\left\{\left(\ln\left(C(n, n+\lambda\sqrt{n})\right)\right) > \frac{\eta^2 c}{(1+\lambda)\delta^2}\right\}$$
$$\leq 3\exp\left(-\frac{\eta^2 c}{(1+\lambda)\delta^2}\right),$$

where c is a constant. Note that this bound is independent of n and one can choose δ small enough to satisfy all the above inequalities and make it smaller than $\varepsilon/2$, which proves the desired tightness. $\square$

REMARK 4.3. Both K and $C(K)$ are defined with respect to the metric d. If the potential densities are jointly continuous, this will imply a corresponding CLT with respect to the original metric.

REMARK 4.4. The following was done in [15] when the process X is symmetric; we present it here again for the sake of completeness. It has been shown in [16] that

$$\left(L^x_{\tau_n} + \tfrac{1}{2}\phi_x^2; x \in E\right) \stackrel{\text{law}}{=} \left(\tfrac{1}{2}(\phi_x + \sqrt{n})^2; x \in E\right). \tag{29}$$

Subtracting n from both sides and dividing them by $\sqrt{n}$,

$$\left(\frac{L^x_{\tau_n} - n}{\sqrt{n}} + \frac{\phi_x^2}{2\sqrt{n}}; x \in E\right) \stackrel{\text{law}}{=} \left(\frac{\phi_x^2}{2\sqrt{n}} + \sqrt{2}\phi_x; x \in E\right)$$

and our tightness in $C(K)$ follows directly from the tightness of the Gaussian law. See also [1] for tightness in the symmetric case using the DIT directly.

PROOF OF THEOREM 1.3. Under the assumption that $u^\alpha(x, y)$ are continuous, Theorems 1.1 and 1.2 show that the existence of a majorizing measure is sufficient for both the continuity and the tightness in $C(K)$ of the local time process both with respect to the metric d and then, by the above continuity, with respect to the original distance on E. The necessity follows from the characterization of Gaussian processes that are continuous in the metric d, as those for which the majorizing measure conditions are satisfied for each compact set K. □

REFERENCES

[1] ADLER, R. J., MARCUS, M. and ZINN, J. (1990). Central limit theorems for the local times of certain Markov processes and the squares of Gaussian processes. *Ann. Probab.* **18** 1126–1140. MR1062061

[2] ARAUJO, A. and GINE, E. (1980). *The Central Limit Theorem for Real and Banach Valued Random Variables.* Wiley, New York. MR0576407

[3] BARLOW, M. T. (1985). Continuity of local times for Lévy processes. *Z. Wahrsch. Verw. Gebiete* **69** 23–35. MR0775850

[4] BARLOW, M. T. (1988). Necessary and sufficient conditions for the continuity of local times for Lévy processes. *Ann. Probab.* **16** 1389–1427. MR0958195

[5] BARLOW, M. T. and HAWKES, J. (1985). Application de l'entropie metrique a la continuite des temps locaux des processus de Lévy. *C. R. Acad. Sci. Paris Ser. I Math.* **301** 237–239. MR0801971

[6] BASS, R. and KHOSHNEVISAN, D. (1992). Stochastic calculus and the continuity of local times of Lévy processes. *Séminaire de Probabilités XXVI*. *Lecture Notes in Math.* **1526** 1–10. Springer, Berlin. MR1231979

[7] BERTOIN, J. (1996). *Lévy Processes.* Cambridge Univ. Press. MR1406564

[8] BLUMENTHAL, R. M. and GETOOR, R. K. (1963). Local times for Markov processes. *Z. Wahrsch. Verw. Gebiete* **3** 50–74. MR0165569

[9] BLUMENTHAL, R. M. and GETOOR, R. K. (1968). *Markov Processes and Potential Theory.* Academic Press, New York. MR0264757

[10] DELLACHERIE, C. and MEYER, P. A. (1987). *Probabilités et Potentiel. D*. Herman, Paris. MR0488194

[11] DUDLEY, R. M. (1973). Sample functions of the Gaussian process. *Ann. Probab.* **1** 66–103. MR0346884

[12] DYNKIN, E. B. (1983). Local times and quantum fields. In *Seminar on Stochastic Processes 1983* 64–84. Birkhäuser, Boston. MR0902412

[13] DYNKIN, E. B. (1984). Gaussian and non-Gaussian random fields associated with Markov processes. *J. Funct. Anal.* **55** 344–376. MR0734803

[14] EISENBAUM, N. (2003). On the infinite divisibility of squared Gaussian processes. *Probab. Theory Related Fields* **125** 381–392. MR1964459

[15] EISENBAUM, N. and KHOSHNEVISAN, D. (2002). On the most visited sites of symmetric Markov processes. *Stochastic Process. Appl.* **101** 241–256. MR1931268

[16] EISENBAUM, N., KASPI, H., MARCUS, M., ROSEN, J. and SHI, Z. (2000). A Ray–Knight theorem for symmetric Markov processes. *Ann. Probab.* **28** 1781–1796. MR1813843

[17] FERNIQUE, X. (1975). Régularité des trajectoires des fonctions aleatoires Gaussiennes. *École d'Été de Probabilités de Saint Flour IV. Lecture Notes in Math.* **480** 1–95. Springer, Berlin. MR0413238

[18] FITZSIMMONS, P. J. (1987). Homogeneous random measures and a weak order for excessive functions of a Markov process. *Trans. Amer. Math. Soc.* **303** 217–244. MR0902778

[19] GETOOR, R. K. and KESTEN, H. (1972). Continuity of local times of Markov processes. *Compositio Math.* **24** 277–303. MR0310977

[20] HEINKEL, B. (1977). Mesures majorantes et théorèmes de la limites centrale dans $C(S)$. *Z. Wahrsch. Verw. Gebiete* **38** 339–351. MR0440667

[21] LEDOUX, M. and TALAGRAND, M. (1991). *Probability in Banach Spaces.* Springer, Berlin. MR1102015

[22] LINDE, W. (1986). *Probability in Banach Spaces—Stable and Infinitely Divisible Distributions*, 2nd ed. Wiley, Chichester. MR0874529

[23] MARCUS, M. B. and ROSEN, J. (1992). Moduli of continuity of local times of symmetric Markov processes via Gaussian processes. *J. Theoret. Probab.* **5** 791–825. MR1182681

[24] MARCUS, M. B. and ROSEN, J. (1992). p-variation of the local times of symmetric stable processes and Gaussian processes with stationary distributions. *Ann. Probab.* **20** 1685–1713. MR1188038

[25] MARCUS, M. B. and ROSEN, J. (1992). Sample path properties of the local times of strongly symmetric Markov processes via Gaussian processes. *Ann. Probab* **20** 1603–1684. MR1188037

[26] MARCUS, M. B. and ROSEN, J. (1993). ϕ-variation of the local times of symmetric Lévy processes and stationary Gaussian processes. In *Seminar on Stochastic Processes 1992* 209–220. Birkhäuser, Boston. MR1278084

[27] MARCUS, M. B. and ROSEN, J. (2001). Gaussian processes and the local times of symmetric Lévy processes. In *Lévy Processes—Theory and Applications* 67–89. Birkhäuser, Boston. MR1833693

[28] MARCUS, M. B. and ROSEN, J. (2006). *Markov Processes*, *Gaussian Processes and Local Times*. Cambridge Univ. Press. MR2250510

[29] TROTTER, H. (1958). A property of Brownian motion paths. *Ill. J. Math.* **2** 425–433. MR0096311

LABORATOIRE DE PROBABILITÉS
UNIVERSITÉ PARIS VI—CNRS
4 PLACE JUSSIEU
75252 PARIS CEDEX 05
FRANCE
E-MAIL: nae@ccr.jussieu.fr

INDUSTRIAL ENGINEERING
AND MANAGEMENT
TECHNION
TECHNION CITY
HAIFA 32000
ISRAEL
E-MAIL: iehaya@tx.technion.ac.il