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MEAN ERGODICITY OF REGULARIZED SOLUTION FAMILIES

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Dedicated to the Memory of Professor Sen-Yen Shaw

Abstract. We study the mean ergodicity of resolvent families and give a general theorem for nondensely defined generator. In particular, it is applied to n-times integrated semigroups.

1. Introduction

Let X be a (complex) Banach space and let $B(X)$ be the unital Banach algebra of all bounded (linear) operators on X with the identity operator I. For a linear operator T, we denote $N(T)$ and $R(T)$ the null space and the range of T, respectively. Let A be a closed linear operator on X. A net $\{S_\alpha\}_{\alpha\in D}$ of bounded operators on X is said to be an *A-ergodic net on* X [22, 23] if it satisfies the following conditions:

(A1) There is a constant $M > 0$ such that $||S_\alpha|| \le M$ for all $\alpha \in D$;

(A2) $\lim\limits_{\alpha}(S_\alpha x - x) = 0$ for all $x \in N(A)$ and $R(S_\alpha - I) \subset \overline{R(A)}$ for all $\alpha \in D$;

(A3) $R(S_\alpha) \subset D(A)$ for all α, $w\text{-}\lim\limits_{\alpha} AS_\alpha x = 0$ for all $x \in X$, and $s\text{-}\lim\limits_{\alpha} S_\alpha Ax = 0$ for all $x \in D(A)$.

The classical mean ergodic theorems had been studied and applied by many mathematicians (see [7, 10, 21-23, 25]). Abstract mean ergodic theorems applied to convergent rate can be found in [4, 24]. Recently, Kantorovitz and Piskarev [11] condidered A_t-mean stability of uniformly bounded (C_0)-semigroups and cosine operator functions for averaging methods A_t more general than the Cesáro means [11, 16]. Related references refer to [1, 4, 19]

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In [16], we had studied that the mean ergodicity of (a, k)-regularized solution families (will be defined in section 2) is possible for densely defined generator A and give some examples for semigroups and cosine functions. In this paper, we shall deal with the mean ergodicity of $(1, k)$-regularized solution families and relax the condition of the generator A (see Theorem 2.2). In particular, we apply to n-times integrated semigroups (see Corollary 3.7). First, we list an abstract mean ergodic theorem (see [22, Theorem 1.1]).

Theorem 1.1. (An Abstract Mean Ergodic Theorem). *Let* $\{S_\alpha\}_{\alpha\in D}$ *be an* A*-ergodic net of bounded linear operators on a Banach space* X. *Define a linear operator* $P : D(P)(\subset X) \to X$ *by*

$$(1.1)\qquad \begin{cases} D(P) := \{x \in X; s\text{-}\lim\limits_{\alpha} S_\alpha x \textit{ exists}\} \\ Px \quad := s\text{-}\lim\limits_{\alpha} S_\alpha x \textit{ for } x \in D(P). \end{cases}$$

Then

(1) $||P|| \le M$ *and* P *is a projection.*

(2) $N(P) = \overline{R(A)}$, $R(P) = N(A)$ *and the domain*

$$D(Q) \equiv N(A) \oplus \overline{R(A)}$$
$$= \{x \in X; \{S_\alpha x\} \textit{ contains a weakly convergent subnet.}\}$$

2. A General Convergence Theorem

Let a be a function in $L^1_{loc}([0,\infty))$ with $a(t) > 0$ on $(0,\infty)$ and let k be non-decreasing on $[0,\infty)$ such that $k(t) > 0$ for all $t > 0$. Thus $(a * k)(t) := \int_0^t a(t-s)k(s)ds$ is increasing on $[0,\infty)$. Let $A : D(A) \subset X \to X$ be a closed linear operator. A family $\{R(t); t \ge 0\}$ in $B(X)$ is called a (a, k)-*regularized resolvent family for* A [16, 17, 18, 25] if it has the following properties:

(R1) $R(\cdot)$ is strongly continuous on $[0,\infty)$ and $R(0) = I$;

(R2) $R(t)D(A) \subset D(A)$ and $AR(t)x = R(t)Ax$ for all $x \in D(A)$ and $t \ge 0$;

(R3) $R(S(t)) \subset D(A)$ and $AS(t)x = R(t)x - k(t)x$ for all $x \in X$ and for all $t \ge 0$, where $S(t)x := \int_0^t a(t-s)R(s)xds$ for $x \in X$ and $t \ge 0$.

Such A is called the generator of $R(\cdot)$. (a, k)-regularized resolvent families was first introduced in [17]. As $a \equiv 1$, $R(\cdot)$ is called a k-*convoluted semigroup* [5]. (R3) describes an important class of abstract Cauchy problem. Related references refer to [2, 13]. When $k(t) = j_\alpha(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$, $R(\cdot)$ is called an α-times integrated solution family [25]. In particular, if , in addition $a \equiv 1$ (resp. $a(t) = t$), then $R(\cdot)$ becomes a C_0-semigroup $T(\cdot)$ (resp. cosine operator function $C(\cdot)$) with generator A [10, 8, 9].

Let $\{h_\alpha\}$ be a net of complex-valued functions in $L^1[0, \infty)$ such that $h_\alpha(\cdot)R(\cdot)x$ is Bachner integrable on $[0, \infty)$ for every $x \in X$. Define, for every α, a linear operator S_α by

$$S_\alpha x := \int_0^\infty h_\alpha(t)R(t)xdt \text{ for } x \in X.$$

Applying a Hille theorem [6, Theorem II.2.6], we have $S_\alpha Ax = AS_\alpha x$ for all $x \in D(A)$. To prove the next theorem, we need the following lemma.

Lemma 2.1. (cf. [16, Corollary 2.4]). *Suppose k and a are nondecreasing and positive on $(0, \infty)$ such that $\lim_{t\to\infty} \frac{k(t)}{(a*k)(t)} = 0$ and $a(t) = O((a*k)(t))(t \to \infty)$. Let $R(\cdot)$ be an (a, k)-regularized solution family with generator A satisfying $||R(t)|| \le M(1 + k(t))$ for all $t \ge 0$. Define the operator Q by*

$$\begin{cases} D(Q) = \{x \in X \mid s\text{-}\lim_{t\to\infty} B_t x \textit{ exists}\} \\ \quad Qx = \lim_{t\to\infty} B_t x \textit{ for } x \in D(Q), \end{cases}$$

where $B_t x := \frac{(a*R)(t)}{(a*k)(t)} x$ for all $t > 0$ and for all $x \in X$. Then $\{B_t\}(t \to \infty)$ is an A-ergodic net and Q is a bounded projection with $||Q|| \le \sup_\alpha ||S_\alpha||$ such that $R(Q) = N(A)$, $N(Q) = \overline{R(A)}$, and the domain

$$D(Q) \equiv N(A) \oplus \overline{R(A)} = \{x \in X; \{B_t x\} \text{ contains a weakly convergent subnet}\}.$$

Proof. Since $a(t) = O((a*k)(t))(t \to \infty)$, there are some $r > 0$ and some constant $M' > 0$ such that $a(t) \le M'(a*k)(t)$ for all $t \ge r$. Therefore we have for every $t > r$ and for every $x \in X$,

$$\begin{aligned} ||B_t x|| &\le ((a*k)(t))^{-1}[||\int_0^r a(t-s)R(s)xds|| + ||\int_r^t a(t-s)R(s)xds||] \\ &\le ((a*k)(t))^{-1}[ra(t) \sup_{0\le s\le r} ||R(s|| \cdot ||x|| + M'(a*k)(t))||x||] \\ &\le M'(r \sup_{0\le s\le r} ||R(s|| + 1)||x||. \end{aligned}$$

Therefore the B_t are uniformly bounded on $[r, \infty)$ by the assumption. So, $\{B_t\}(t \ge r)$ satisfies (A1). (A2) follows from (R2) and (R3). Finally, we have

$$\begin{aligned} ||AB_t|| &= ||\frac{R(t) - k(t)}{(a*k)(t)}|| \\ &\le \frac{(M+1)k(t) + M}{(a*k)(t)} \to 0 \text{ as } t \to \infty. \end{aligned}$$

Since $B_tA \subset AB_t$, this means that the net $\{B_t\}(t \geq r)$ is an A-ergodic net. The result follows from Theorem 1.1.

Theorem 2.2. (cf. [16, Theorem 2.2]). *Let $R(\cdot)$ be an (a,k)-regularized solution family and let $\{h_\alpha\}$ be a net of complex-valued functions in $L^1[0,\infty)$ such that $h_\alpha(\cdot)R(\cdot)x$ is Bochner integrable for all $x \in X$. Define $S_\alpha x := \int_0^\infty h_\alpha(t)R(t)xdt$ for all $x \in X$ and for all α. Suppose the following coditions hold:*

(a) $\sup_\alpha ||S_\alpha|| < \infty$;

(b) $\lim_\alpha \int_0^\infty h_\alpha(t)k(t)dt = 1$;

(c) $\lim_{t\to\infty} \frac{k(t)}{(a*k)(t)} = 0$ *and* $a(t) = O((a*k)(t))(t \to \infty)$;

(d) *suppose that*

$$\lim_\alpha S_\alpha(R(t) - k(t)I)x = 0 \text{ for all } x \in X \text{ and for all } t > 0. \tag{2.1}$$

Define the operator $Q : D(Q)(\subset X) \to X$ by

$$\begin{cases} D(Q) = \{x \in X \mid s\text{-}\lim_\alpha S_\alpha x \text{ exists}\} \\ Qx = \lim_\alpha S_\alpha x \text{ for } x \in D(Q); \end{cases}$$

Then Q is a bounded projection with $||Q|| \leq \sup_\alpha ||S_\alpha||$ such that $R(Q) = N(A)$, $N(Q) = \overline{R(A)}$, and the domain

$$D(Q) \equiv N(A) \oplus \overline{R(A)} = \{x \in X; \{S_\alpha x\} \text{ contains a weakly convergent subnet}\}.$$

Proof. Clearly, $||Q|| \leq \sup_\alpha ||S_\alpha|| < \infty$. So, both $D(Q)$ and $N(Q)$ are closed. If $x \in N(A)$, (R3) implies $R(t)x = k(t)x$ for all $t \geq 0$. By (2.1), we have for every $x \in D(A)$ and for every $t > 0$, $(a*k)(t)B_tAx = R(t)x - k(t)x \in N(Q)$ and so $-Ax = \lim_{t\to\infty} B_tAx - Ax \in N(Q)$ by Lemma 2.1. Therefore $\overline{R(A)} \subset N(Q)$. If $x \in X$, then $R(t)x - k(t)x = A(a*R)(t)x \in R(A)$, so

$$S_\alpha x - \int_0^\infty h_\alpha(t)k(t)xdt = \int_0^\infty h_\alpha(t)(R(t)x - k(t)x)dt \in \overline{R(A)}.$$

If $\{S_\alpha x\}$ has a weakly convergent subnet $\{S_\beta\}$, say $y := w\text{-}\lim_\beta S_\beta x$, then

$$y - x = w\text{-}\lim_\beta (S_\beta x - x) \in \overline{R(A)} \subset N(Q). \tag{2.2}$$

This means that $Q(y-x)=0$. In particular, if $x \in D(Q)$, then $y \in D(Q)$ and $Q^2x = Qx$, that is, Q is a projection. Further, if $x \in N(Q)$, then $y = 0$ and $-x = y - x \in \overline{R(A)}$. Since $\overline{R(A)} \subset N(Q)$, this proves $N(Q) = \overline{R(A)}$. On the other hand, since $S_\alpha R(t) = R(t)S_\alpha$ for all $t > 0$ and for all α, we obtain from (2.1) that

$$[R(t) - k(t)]y = 0 \text{ for all } t > 0.$$

Thus $B_t y = y$ for all $t > 0$. This implies $y \in N(A)$ by (R2) and (R3). Since $N(A) \subset R(Q)$, this implies $R(Q) = N(A)$. Since Q is a projection, we must have $D(Q) \equiv N(A) \oplus \overline{R(A)}$. This completes the proof. ■

The assumption in Theorem 2.2 of [16] for $D(A)$ being dense X is not required here.

Remark. If a is a nonzero polynomial and $k = j_r$ for some $r > 0$, then $\lim\limits_{t\to\infty} \frac{a(t)+k(t)}{(a*k)(t)} = 0$. But, if $a \equiv 1$ and $k(t) = e^{wt}$, $t \geq 0$, for some $w > 0$, then $e^{wt} = O((a*k)(t))(t \to \infty)$. That is, $k(t)$ in Lemma 2.1 can not increase too rapidly.

Example 1. (See [16]). Let $a_r \in \mathbb{R}$, where $r := (r_1, \ldots, r_n) \in \mathbb{N}_0^n$, $|r| := \sum\limits_{j=1}^{n} r_j \leq k$, and let $A := \sum\limits_{|r|\leq k} a_r i^{|r|+1} D^r$ be the maximal differential operator on a function space X which can be any of the spaces

$$C_0(\mathbb{R}^n),\ C_b(\mathbb{R}^n),\ UC_b(\mathbb{R}^n),\ L^p(\mathbb{R}^n) \text{ for } 1 \leq p \leq \infty,$$

where $D^r := \left(\frac{\partial}{\partial x_1}\right)^{r_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{r_n}$. It is shown in [12, Theorem 4.9] that A generates an m-times integrated semigroup (i.e., an $(1, j_m)$-regularized solution family) $T(\cdot)$ satisfying $\|T(t)\| \leq M(1+t^m)$ for all $t \geq 0$, where $m = [n/2]+2$. Moreover,

$$(T(t)f)(x) := \left(\frac{1}{\sqrt{2\pi}}\right)^{n/2} (\tilde{\phi}_t * f)(x),\ f \in X,\ x \in \mathbb{R}^n,\ t \geq 0,$$

where

$$\begin{aligned}\phi_t(x) :&= \frac{1}{(m-1)!}\int_0^t (t-s)^{m-1} e^{p(x)s} ds \\ &= e^{p(x)t}/p(x)^m - \sum_{j=0}^{m-1} \frac{1}{j!} t^j / p(x)^{m-j},\ t \geq 0, x \in \mathbb{R}^n.\end{aligned}$$

Here $\tilde{\phi}_t$ denotes the inverse Fourier transform of ϕ_t.

3. Application to k-Convoluted Semigroups

In [16], we investigated the ergodic approximation for (C_0)-semigroup. In this section, we shall apply last results to r-times Integrated semigroups for $r > 0$. Let $R(\cdot)$ be an r-times integrated semigroup on X with the generator A, where $r > 0$. Suppose $k : [0, \infty) \to [0, \infty)$ is nondecreasing with $k(t) > 0$ for all $t > 0$.

Then $R(\cdot)$ is a $(1, j_r)$-regularized resolvent family for A. It is known ([14] for integer case and [15] for real case) that $R(\cdot)$ can be expressed as

$$R(t)R(s)x = \int_t^{s+t} j_{n-1}(s+t-u)R(u)xdu - \int_0^s j_{n-1}(s+t-u)R(u)xdu \tag{3.1}$$

for all $x \in X$ and for all $t, s \geq 0$. When $n = 0$, $R(\cdot)$ is a (C_0)-semigroup. It is a known fact that every n-times integrated semigroup on X is a commutative family. These still hold for $(1, k)$-regularized resolvent families. We list the result as the following:

Lemma 3.1. *Let $R(\cdot)$ be a $(1, k)$-regularized resolvent family for A. Then*

(i) $R(t)R(s) = R(s)R(t)$ *for all* $t, s \geq 0$*;*

(ii) *If $k(\cdot)$ is continuously differentiable on $[0, \infty)$, then*

$$R(t)R(s)x = (\int_0^{s+t} - \int_0^s - \int_0^t) k'(s+t-r)R(r)xdr + k(0)R(s+t)x \tag{3.2}$$

for all $t, s \geq 0$ and for all $x \in X$.

Proof. By (R2) and (R3), we have for every $t, s \geq 0$,

$$[R(t) - k(t)I](1 * R)(s) = (1 * R)(t)[R(s) - k(s)I].$$

That is,

$$R(t)(1 * R)(s) - (1 * R)(t)R(s) = k(t)(1 * R)(s) - k(s)(1 * R)(t).$$

Therefore we have for every $t, s > 0$ and for every $x \in X$,

$$\begin{aligned}
(1 * R)(t)(1 * R)(s)x &= \int_0^t \frac{\partial}{\partial r}[(1 * R)(r)(1 * R)(s+t-r)x]dr \\
&= \int_0^t R(r)(1 * R)(s+t-r)x - (1 * R)(r)R(s+t-r)xdr \\
&= \int_0^t k(r)(1 * R)(s+t-r)x - k(s+t-r)(1 * R)(r)xdr \\
&= \left(\int_0^t + \int_0^s - \int_0^{s+t}\right) k(r)(1 * R)(s+t-r)xdr.
\end{aligned} \tag{3.3}$$

By symmetry on t and s, this proves $(1*R)(t)(1*R)(s) = (1*R)(s)(1*R)(t)$ for all $t, s \geq 0$. Thus, (i) follows from differentiating to s and t, respectively.

(ii) Differentiating to t in (3.3), we get

(3.4)

$$R(t)(1*R)(s)x = \left(\int_0^t + \int_0^s - \int_0^{s+t}\right) k(r)R(s+t-r)x dr + k(t)(1*R)(s)x.$$

Using the change of variables, we have

$$R(t)(1*R)(s)x = \left(\int_0^{s+t} - \int_0^s - \int_0^t\right) k(s+t-r)R(r)x dr + k(t)(1*R)(s)x.$$

Differentiating to s again, we get

$$R(t)R(s)x = \left(\int_0^{s+t} - \int_0^s - \int_0^t\right) k'(s+t-r)R(r)x dr + k(0)R(s+t)x.$$

This proves (ii) and the proof is complete.

Remark. From Lemma 3.1(ii), if $k(\cdot) = j_n(\cdot)$, we get (3.1).

Let $\{h_n\}$ be a sequence of complex-valued functions in $L^1[0,\infty)$. We consider the following conditions:

(c1) $K := \sup\limits_{n\to\infty} \int_0^\infty |h_n(t)|k(t)dt < \infty$;

(c2) $\lim\limits_{n\to\infty} \int_0^\infty h_n(t)k(t)dt = 1$;

(c3) There is an $\delta > 0$ such that $\lim\limits_{n\to\infty} \int_0^\delta |h_n(t)|dt = 0$ and

$$\lim_{n\to\infty} \int_0^\infty |h_n(t) - h_n(t+\theta)|k(t+\theta)dt = 0 \ \text{ for every } 0 < \theta < \delta.$$

(c4) $\lim\limits_{t\to\infty} \frac{k(t+\theta)}{k(t)} = 1$ for all $\theta > 0$.

If $\delta > 0$ is such that $\lim\limits_{t\to\infty} \frac{k(t+\theta)}{k(t)} = 1$ for all $0 < \theta < \delta$, then we have for any $0 < \theta < \delta$,

$$\lim_{t\to\infty} \frac{k(t+2\theta)}{k(t)} = \lim_{t\to\infty} \frac{k(t+2\theta)}{k(t+\theta)} \lim_{t\to\infty} \frac{k(t+\theta)}{k(t)} = 1.$$

Therefore we have $\lim\limits_{t\to\infty} \frac{k(t+\theta)}{k(t)} = 1$ for all $\theta > 0$.

Lemma 3.2 *Let $\{h_n\}$ be a sequence of complex-valued functions in $L^1[0,\infty)$ satisfying* (c1) *and* (c3)*.*

(i) $\lim\limits_{n\to\infty}\int_0^\infty |h_n(t)|(k(s+t)-k(t))dt = 0$ *for all* $0 < s < \delta$*;*

(ii) $\lim\limits_{n\to\infty}\int_0^N |h_n(t)|dt = 0$ *for all* $N > 0$*;*

(iii) *If* (c4) *holds, then*
$$\lim_{n\to\infty}\int_0^\infty |h_n(t)-h_n(t+\theta)|k(t+\theta)dt = 0 \text{ for every } \theta > 0.$$

(iv) *If* $f : [0,\infty) \to X$ *is strongly measurable such that* $||f(t)|| \le M(1+k(t))$ *for all* $t \ge 0$ *and some constant* $M > 0$*. Then*
$$\lim_{n\to\infty}\int_0^\infty h_n(t)[f(t+\theta)-f(t)]dt = 0 \text{ for all } 0 < \theta < \delta.$$

Proof. (i) Since $k(\cdot)$ is nondecreasing and $k(t) > 0$ on $(0,\infty)$, by (c1) and (c3) we have for every $0 < s < \delta$,
$$\begin{aligned}
0 &\le \int_0^\infty |h_n(t)|(k(s+t)-k(t))dt \\
&= \int_0^\infty |h_n(t)|k(s+t)dt - \int_0^\infty |h_n(t+s)|k(t+s)dt - \int_0^s |h_\alpha(t)|k(t)dt \\
&\le \int_0^\infty ||h_n(t)|-|h_n(t+s)||k(s+t)dt + k(s)\int_0^s |h_\alpha(t)|dt \\
&\le \int_0^\infty |h_n(t)-h(t+s)|k(s+t)dt + k(s)\int_0^s |h_n(t)|dt \to 0 \text{ as } n\to\infty.
\end{aligned}$$
This proves (i).

(ii) Let $N > 0$ be arbitrary and let m be a positive integer so that $\theta := \frac{N}{m} < \frac{\delta}{2}$. Then we have
$$\begin{aligned}
&\int_0^N |h_n(t)|dt = \sum_{\ell=0}^{m-1}\int_{\ell\theta}^{(\ell+1)\theta} |h_n(t)|dt = \sum_{\ell=0}^{m-1}\int_0^\theta |h_n(t+\ell\theta)|dt \\
&\le \sum_{\ell=2}^{m-1}\int_0^\theta \left\{\left[\sum_{j=2}^{\ell} |h_n(t+j\theta)-h_n(t+(j-1)\theta)| + |h_n(t+\theta)|\right]\right\} dt + \int_0^{2\theta} |h_n(t)|dt \\
&\le \sum_{\ell=2}^{m-1}\sum_{j=2}^{\ell}\int_{(j-1)\theta}^{j\theta} |h_n(t+\theta)-h_n(t)|dt + (m-1)\int_0^{2\theta} |h_n(t)|dt \\
&= \sum_{\ell=2}^{m-1}\int_\theta^{\ell\theta} |h_n(t+\theta)-h_n(t)|dt + (m-1)\int_0^{2\theta} |h_n(t)|dt \\
&\le \frac{m-2}{k(\theta)}\int_\theta^\infty |h_n(t+\theta)-h_n(t)|g(t+\theta)dt + (m-1)\int_0^{2\theta} |h_n(t)|dt \\
&\quad \to 0 \text{ as } n\to\infty \text{ by (c3)}.
\end{aligned}$$
This proves (ii).

(iii) Assume (c4). Let $c > 0$ be arbitrary. Then $\theta := \frac{c}{m} < \delta$ for some positive integer m. By (c4), there is an $N > 0$ such that $1 \le \frac{k(t+c)}{k(t)} \le 1+\varepsilon$ for all $t \ge N$. Since g is nondecreasing and positive on $(0,\infty)$, this implies $1 \le \frac{k(t+\ell\theta)}{k(t+j\theta)} \le 1+\varepsilon$ for all $0 \le j < \ell \le m$ and for all $t \ge N$. Thus, we have

$$
\begin{aligned}
&\int_0^\infty |h_n(t) - h_n(t+c)|k(t+c)dt \\
=& \int_0^\infty |h_n(t) - h_n(t+m\theta)|k(t+m\theta)dt \\
\le& \sum_{\ell=1}^{m} \int_0^\infty |h_n(t+\ell\theta) - h_n(t+(\ell-1)\theta)|k(t+m\theta)dt \\
\le& \sum_{\ell=1}^{m} \int_0^N |h_n(t+\ell\theta) - h_n(t+(\ell-1)\theta)|k(N+m\theta)dt \\
&+ \sum_{\ell=1}^{m} \int_N^\infty |h_n(t+\ell\theta) - h_n(t+(\ell-1)\theta)|(1+\varepsilon)k(t+\ell\theta)dt \\
&\to 0 + 0 \text{ as } n \to \infty \text{ by part (ii) and (c3).}
\end{aligned}
$$

This proves (iii).

(iv) By part (ii), we have

$$\lim_{n\to\infty} \int_0^N |h_n(t)|dt = 0 \text{ for any } N > 0.$$

Since k is nondecreasing and positive on $(0,\infty)$, there is a constant $M' > 0$ such that $||f(t)|| \le M'k(t)$ for $t \ge \frac{\delta}{2}$. By (c1), we have

$$
\begin{aligned}
&\int_0^\infty ||h_n(t)f(t)||dt \\
\le& M(1+k(\frac{\delta}{2})) \int_0^{\frac{\delta}{2}} |h_n(t)|dt + \int_{\frac{\delta}{2}}^\infty M'|h_n(t)|k(t)dt \\
\le& M(1+k(\frac{\delta}{2})) \int_0^{\frac{\delta}{2}} |h_n(t)|dt + K.
\end{aligned}
$$

By the first part of (c3), this implies $\sup\limits_{n\ge1} \int_0^\infty ||h_n(t)f(t)||dt < \infty$.

On the other hand, we have for every $0 < \theta < \frac{\delta}{2}$

$$
\begin{aligned}
&\int_0^\infty h_n(t)[f(t+\theta) - f(t)]dt \\
=& \int_{\frac{\delta}{2}}^\infty [h_n(t) - h_n(t+\theta)]f(t+\theta)dt \\
&+ \int_0^{\frac{\delta}{2}} [h_n(t) - h_n(t+\theta)]f(t+\theta)dt - \int_0^\theta h_n(t)f(t)dt
\end{aligned}
$$

Since $||f(t)|| \leq M'k(t)$ for all $t \geq \frac{\delta}{2}$, we have

$$\begin{aligned}
&\left\|\int_0^\infty h_n(t)[f(t+\theta)-f(t)]dt\right\| \\
&\leq M'\int_\delta^\infty |h_n(t)-h_n(t+\theta)|k(t+\theta)dt \\
&\quad +M(1+k(\theta+\frac{\delta}{2}))\int_0^{\frac{\delta}{2}} |h_n(t)-h_n(t+\theta)|dt + M(1+k(\theta))\int_0^\theta |h_n(t)|dt \\
&\quad \to 0 \text{ as } n\to\infty \text{ by (c3).}
\end{aligned}$$

This proves (iv) and the proof is complete. ■

The proof of Lemma 3.2(ii) had applied the proof of [16, Lemma 2.5] but Lemma 3.2(iv) is without using the condition $\lim_{t\to\infty} k(t) = \infty$ which is an important assumption of [16, Lemma 2.5]. Under the conditions (c1)-(c4), if $||R(t)|| \leq M(1+k(t))\,(t \geq 0)$ for some constant $M > 0$, then the linear operators S_n, $n \geq 1$, defined by $S_n x := \int_0^\infty h_n(t)R(t)x dt$ for $x \in X$ are uniformly bounded linear operators on X by the proof of Lemma 3.2(iv). The following lemma gives a sufficient condition for the condition (d) in Theorem 2.2.

Lemma 3.3. *Suppose* $\{h_n\}_{n=1}^\infty$ *is a sequence of complex-valued functions in* $L^1[0,\infty)$ *satisfying* (c1)-(c3) *Let* $R(\cdot)$ *be a* $(1,k)$*-regularized resolvent family on* X *with generator* A*. Suppose there is a constant* $M > 0$ *such that* $||R(t)|| \leq M(1+k(t))$ *for all* $t \geq 0$*. If* $k(\cdot)$ *is continuously differentiable on* $(0,\infty)$*, then*

$$\lim_\alpha S_n(R(s)x - k(s)x) = 0 \text{ for all } x \in X \text{ and for all } 0 < s < \delta.$$

Proof. Let $0 < s < \delta$ and let $x \in X$ be arbitrary. By Lemma 3.2(i) and (c1), we have for every $0 \leq r \leq s$,

$$\int_0^\infty |h_n(t)|\cdot||R(t+r)||dt \leq K' := \sup_{n\geq 1}\int_0^\infty |h_n(t)|k(t+s)dt < \infty. \tag{3.5}$$

By Lemma 3.2(iv), we have

$$\lim_{n\to\infty}\int_0^\infty h_n(t)[R(t+s)x - R(t)x]dt = 0 \text{ for all } x \in X.$$

Since $||R(t)|| \leq M(1+k(t))$ for all $t \geq 0$ and $k(\cdot) > 0$ is nondecreasing on $(0,\infty)$, we get from (3.5) that $||\int_0^\infty h_n(t)[R(t+r)x - R(t)x]dt|| \leq 2K'||x||$ for all $0 \leq r \leq s$. From Lemma 3.1(ii), we have for every $t, s \geq 0$,

$$\begin{aligned}
R(t)R(s)x &= \left(\int_t^{s+t} - \int_0^t\right) k'(s+t-r)R(r)xdr + k(0)R(s+t)x \\
&= \int_0^s k'(s-r)[R(r+t)x - R(t)x]dr - \int_0^s k'(s+t-r)R(r)xdr \\
&\quad + k(0)[R(s+t)x - R(t)x] + k(s)R(t)x.
\end{aligned}$$

By the Fubini's theorem for Bochner integration, we have

$$\begin{aligned}
&\int_0^\infty h_n(t)R(t)[R(s)x-k(s)x]dt \\
=&\int_0^\infty h_n(t)\int_0^s k'(s-r)[R(r+t)x-R(t)x]drdt \\
&-\int_0^\infty h_n(t)\int_0^s k'(s+t-r)R(r)xdrdt \\
&+k(0)\int_0^\infty h_n(t)[R(s+t)x-R(t)x]dt \\
=&\int_0^s k'(s-r)\int_0^\infty h_n(t)[R(r+t)x-R(t)x]dtdr \\
&-\int_0^\infty h_n(t)\int_0^s k'(s+t-r)R(r)xdrdt \\
&+k(0)\int_0^\infty h_n(t)[R(s+t)x-R(t)x]dt.
\end{aligned}$$

Since $k(\cdot)$ is nondecreasing and continuously differentiable, $k'(t)\geq 0$ on $[0,\infty)$. Thus, we have

$$\begin{aligned}
&\left\|\int_0^\infty h_n(t)R(t)[R(s)x-k(s)x]dt\right\| \\
\leq&\int_0^s k'(s-r)\left\|\int_0^\infty h_n(t)[R(r+t)x-R(t)x]dt\right\|dr \\
&+\int_0^\infty |h_n(t)|\int_0^s k'(s+t-r)drdt\cdot\sup_{0\leq r\leq s}||R(r)x|| \\
&+k(0)\left\|\int_0^\infty h_n(t)[R(s+t)x-R(t)x]dt\right\| \\
\leq&\int_0^s k'(s-r)\left\|\int_0^\infty h_n(t)[R(r+t)x-R(t)x]dt\right\|dr \\
&+\int_0^\infty |h_n(t)|[k(s+t)-k(t)]dt\cdot\sup_{0\leq r\leq s}||R(r)x|| \\
&+k(0)\left\|\int_0^\infty h_n(t)[R(s+t)x-R(t)x]dt\right\|.
\end{aligned}$$

Applying Lebesgue dominated convergence theorem and Lemma 3.2(iv), this inequality implies that

$$\lim_{n\to\infty}\left\|\int_0^\infty h_n(t)R(t)[R(s)x-k(s)x]dt\right\|=0 \text{ for all } x\in X.$$

This completes the proof. ■

The following lemma is useful to find adaptive functions h_n satisfying (c1)-(c3).

Lemma 3.4. *Let* $h \in L^1[0,\infty)$ *satisfy* $h(\cdot)k(\cdot) \in L^1[0,\infty)$ *and* $\int_0^\infty h(t)k(t)dt = 1$. *Suppose* k *satisfies* (c4). *Define for every* $\lambda > 1$,

$$h_\lambda(t) = \begin{cases} \textit{arbitrary value}, t = 0 \\ \lambda^{-1} h(\frac{t}{\lambda}) \frac{k(\frac{t}{\lambda})}{k(t)} \qquad \textit{for all} t > 0. \end{cases}$$

Then $\{h_\lambda\}$ *satisfies* (c1)-(c3) *for* $\delta = \infty$.

Proof. Since k is positive and nondecreasing on $(0,\infty)$, we have $|h_\lambda(t)| \leq \lambda^{-1}|h(\frac{t}{\lambda})|$ for all $t > 0$ and for all $\lambda > 1$. So, we have for every $\lambda > 1$, $h_\lambda \in L^1[0,\infty)$,

$$\int_0^\infty h_\lambda(t)k(t)dt = \int_0^\infty \lambda^{-1} h(\frac{t}{\lambda})k(\frac{t}{\lambda})dt = \int_0^\infty h(t)k(t)dt = 1$$

and

$$\left|\int_0^\infty h_\lambda(t)k(t)dt\right| \leq \int_0^\infty \lambda^{-1}|h(\frac{t}{\lambda})|k(\frac{t}{\lambda})dt = \int_0^\infty |h(t)|k(t)dt < \infty.$$

This proves that $\{h_\lambda\}$ satisfies (c1) and (c2).

We show that $\{h_\lambda\}$ satisfies the condition (c3). Since k is positive and nondecreasing on $(0,\infty)$, we have for every $\lambda > 1$

$$\begin{aligned} \textstyle\int_0^\delta |h_\lambda(t)|dt &= \int_0^\delta \lambda^{-1}|h(\frac{t}{\lambda})|\frac{k(\frac{t}{\lambda})}{k(t)}dt \\ &\leq \int_0^\delta \lambda^{-1}|h(\frac{t}{\lambda})|dt = \int_0^{\frac{\delta}{\lambda}} |h(t)|dt \to 0 \text{ as } \lambda \to \infty. \end{aligned}$$

This proves the first part of (c3).

Now, let $\varepsilon > 0$ and $\theta > 0$ be arbitrary. By (c4), there is an $N > 1$ such that

$$(*) \qquad 0 \leq \frac{k(t+\theta)}{k(t)} - 1 < \varepsilon \text{ for all } t \geq N.$$

Using the change of variables, we have for every $\lambda > 1$,

$$\begin{aligned}
&\int_0^\infty |h_\lambda(t) - h_\lambda(t+\theta)|k(t+\theta)dt \\
&= \lambda^{-1}\int_0^\infty |h(\frac{t}{\lambda})\frac{k(\frac{t}{\lambda})}{k(t)} - h(\frac{t+\theta}{\lambda})\frac{k(\frac{t+\theta}{\lambda})}{k(t+\theta)}|k(t+\theta)dt \\
&= \lambda^{-1}\int_0^\infty |h(\frac{t}{\lambda})k(\frac{t}{\lambda})\frac{k(t+\theta)}{k(t)} - h(\frac{t+\theta}{\lambda})k(\frac{t+\theta}{\lambda})|dt \\
&= \int_0^\infty |h(t)k(t)\frac{k(\lambda t+\theta)}{k(\lambda t)} - h(t+\frac{\theta}{\lambda})k(t+\frac{\theta}{\lambda})|dt \\
&\le \int_0^\infty |h(t)k(t)\frac{k(\lambda t+\theta)}{k(\lambda t)} - h(t)k(t)|dt \\
&\quad + \int_0^\infty |h(t)k(t) - h(t+\frac{\theta}{\lambda})k(t+\frac{\theta}{\lambda})|dt \\
&= I_1 + I_2.
\end{aligned}$$

Since $k(\cdot)$ is nondecreasing, we get from (*) that

$$\begin{aligned}
I_1 \; &= \int_0^{\frac{N}{\lambda}} |h(t)k(t)\frac{k(\lambda t+\theta)}{k(\lambda t)} - h(t)k(t)|dt \\
&\quad + \int_{\frac{N}{\lambda}}^\infty |h(t)k(t)\frac{k(\lambda t+\theta)}{k(\lambda t)} - h(t)k(t)|dt \\
&= \int_0^{\frac{N}{\lambda}} |h(t)|\frac{k(t)}{k(\lambda t)}k(N+\theta)dt + \varepsilon\int_{\frac{N}{\lambda}}^\infty |h(t)k(t)|dt \\
&\le k(N+\theta)\int_0^{\frac{N}{\lambda}} |h(t)|dt + \varepsilon\int_0^\infty |h(t)|k(t)dt \\
&\quad \to 0 + \varepsilon\int_0^\infty |h(t)|k(t)dt \;\text{ as } \lambda \to \infty.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this means that $I_1 \to 0$ as $\lambda \to \infty$.

On the other hand, $I_2 \to 0$ as $\lambda \to \infty$ is a known fact (cf. [3, exercise 43]). These means that h_λ satisfies the second part of (c3). The proof is complete. ∎

Examples.

(i) If $k(\cdot)$ is a nonzero polynomial, it satisfies (c4).

(ii) If $k(\cdot) = j_r(\cdot)$, it satisfies (c4). In case, we can take $h(t) = e^{-t}$, $t \ge 0$. Then $h_\lambda(t) = \lambda^{-r-1}e^{t/\lambda}$, $t \ge 0$ and $\lambda > 1$, satisfy (c1)-(c3) by Lemma 3.4.

(iii) The exponential functions $e^{\varepsilon t}$, $\varepsilon > 0$, do not satisfy (c4).

In fact, (c4) implies the condition (d) in Theorem 2.2.

Lemma 3.5 *If* $k : [0,\infty) \to [0,\infty)$ *is nondecreasing on* $[0,\infty)$ *and is positive on* $(0,\infty)$ *satisfying* $\lim_{t\to\infty} \frac{k(t+\theta)}{k(t)} = 1$ *for some* $\theta > 0$*, then*

(i) *for any* $\varepsilon > 0$*, there is a constant* $M > 0$ *such that* $k(t) \leq Me^{\varepsilon t}$ *for all* $t \geq 0$*;*

(ii) $\lim\limits_{t\to\infty} \frac{k(t)}{(1*k)(t)} = 0$.

Proof. Let $\varepsilon' > 0$ and let $s > \theta$ be arbitrary. Let n be the smallest integer greater than or equal to $\frac{s}{\theta}$. Then

$$n \geq \frac{s}{\theta} > n - 1 \geq 0.$$

(i) It suffices to show $\lim\limits_{t\to\infty} \frac{k(t)}{e^{\varepsilon t}} = 0$ for any $\varepsilon > 0$. Let $\varepsilon \in (0, 1)$ be arbitrary and choose an $\varepsilon > 0$ such that $\frac{\ln(1+\varepsilon)}{\theta} < \frac{\varepsilon'}{2}$. By the assumption, there is an $N > 0$ such that

$$1 \leq \frac{k(t+\theta)}{k(t)} \leq 1 + \varepsilon \text{ for all } t \geq N.$$

Therefore we have

$$\begin{aligned} &k(N+s) \leq k(N+n\theta) \\ &\leq (1+\varepsilon)k(N+(n-1)\theta) \leq \cdots \leq (1+\varepsilon)^n k(N) \\ &\leq (1+\varepsilon)^{1+\frac{s}{\theta}} k(N) = e^{(1+\frac{s}{\theta})\ln(1+\varepsilon)} k(N) \\ &\leq (1+\varepsilon)k(N)e^{\frac{s\varepsilon'}{2}}. \end{aligned}$$

This implies

$$\limsup_{s\to\infty} \frac{k(N+s)}{e^{(N+s)\varepsilon'}} \leq (1+\varepsilon) \limsup_{s\to\infty} k(N)e^{-N\varepsilon' - \frac{s\varepsilon'}{2}} = 0.$$

Thus we have $\lim\limits_{s\to\infty} \frac{k(N+s)}{e^{(N+s)\varepsilon'}} = 0$. This proves (i).

(ii) Let $\varepsilon > 0$ be arbitrary. Then there is an integer $N > 0$ such that

$$1 \leq \frac{k(t+\frac{s}{n})}{k(t)} \leq 1 + \varepsilon \text{ for all } t \geq N.$$

Thus, we have

$$\begin{aligned} &(1*k)(N+s) = \int_0^N k(u)du + \sum_{j=1}^n \int_{N+(j-1)\frac{s}{n}}^{N+j\frac{s}{n}} k(u)du \\ &\geq \sum_{j=1}^n \int_0^{\frac{s}{n}} k(N+(j-1)\frac{s}{n}+u)du \\ &\geq \sum_{j=1}^n \int_0^{\frac{s}{n}} (1+\varepsilon)^{-(n+1-j)} k(N+n\frac{s}{n}+u)du \\ &\geq \sum_{j=1}^n \frac{s}{n}(1+\varepsilon)^{-(n+1-j)} k(N+s) \\ &= \frac{s}{n} k(N+s)\varepsilon^{-1}(1-(1+\varepsilon)^{-n}). \end{aligned}$$

Since $\frac{s}{n} \to 1$ as $s \to \infty$, this means that

$$\limsup_{s\to\infty} \frac{k(N+s)}{(1*k)(N+s)} \le \varepsilon.$$

This proves (ii) and the proof is complete. ∎

Lemma 3.5 shows that (c4) implies $k(t) = o(e^{\varepsilon t})(t \to \infty)$ for any $\varepsilon > 0$ and the condition (d) in Theorem 2.2 with $a \equiv 1$. Combining Theorem 2.2 and these results of this section, we have the following main result.

Theorem 3.6. *Let $R(\cdot)$ be an $(1, k)$-regularized solution family with generator A and let $h \in L^1[0,\infty)$ be such that $hk \in L^1[0,\infty)$. Suppose k satisfies* (c4) *and $||R(t)|| \le M(1 + k(t))$, $t \ge 0$ for some constant $M \ge 0$. Define the functions h_λ, $\lambda > 1$ by*

$$h_\lambda(t) = \begin{cases} \text{arbitrary value, for } t = 0 & \\ \lambda^{-1} h(\dfrac{t}{\lambda}) \dfrac{k(\frac{t}{\lambda})}{k(t)} & \text{for all } t > 0. \end{cases}$$

(i) *If $x \in \overline{R(A)}$, then $\lim\limits_{\lambda\to\infty} \int_0^\infty h_\lambda(t)R(t)xdt = 0$;*

(ii) *If $x \in N(A)$, then $\lim\limits_{\lambda\to\infty} \int_0^\infty h_\lambda(t)R(t)xdt = \int_0^\infty h(t)k(t)dtx$;*

(iii) *If $\int_0^\infty h(t)k(t)dt \ne 0$ and $\{\int_0^\infty h_\lambda(t)R(t)xdt\}(\lambda \to \infty)$ has a weakly convergent subsequence for some $x \in X$, then $x \in N(A) \oplus \overline{R(A)}$.*

Proof. By Lemma 3.5, (c4) implies the condition (c) in Theorem 2.2. Let $c := \int_0^\infty h(t)k(t)dt$. If $c \ne 0$, then the net $\{c^{-1}h_\lambda\}(\lambda \to \infty)$ satisfies (c1)-(c3) with $\delta = \infty$ by Lemma 3.4. Therefore $\{\int_0^\infty c^{-1}h_\lambda(t)R(t)xdt\}(\lambda \to \infty)$ satisfies the conditions (a)-(d) by Lemmas 3.3 and 3.5. It follows from Theorem 2.2 that (i)-(iii) hold for $c \ne 0$.

If $c = 0$, we choose a function $g \in L^1[0,\infty)$ such that $\int_0^\infty g(t)k(t)dt = 1$. Define

$$g_\lambda(t) = \begin{cases} \text{arbitrary value, for } t = 0 & \\ \lambda^{-1} g(\frac{t}{\lambda}) \frac{k(\frac{t}{\lambda})}{k(t)} & \text{for all } t > 0. \end{cases}$$

By above arguments, we have for every $s > 0$,

$$\lim_{\lambda\to\infty} \int_0^\infty [h_\lambda(t) + sg_\lambda(t)]R(t)xdt = 0 \qquad \text{for all } x \in \overline{R(A)}$$

and

$$\lim_{\lambda\to\infty} \int_0^\infty [h_\lambda(t) + sg_\lambda(t)]R(t)xdt = sx \qquad \text{for all } x \in N(A).$$

These prove that $\lim\limits_{\lambda\to\infty}\int_0^\infty h_\lambda(t)R(t)xdt = 0$ for $x \in N(A)\oplus\overline{R(A)}$ Therefore (i) and (ii) hold for case $c = 0$ and the proof is complete. ■

Take $h(t) = e^{-t}$, $t \geq 0$ in Theorem 3.6, we have the following special case.

Corollary 3.7. *Let $n \geq 0$ be an integer. Suppose that $R(\cdot)$ is an n-times integrated semigroup with generator A and suppose that $||R(t)|| \leq M(1 + j_n(t))$, $t \geq 0$ for some constant $M \geq 0$. Then*

(i) *If $x \in \overline{R(A)}$, then $\lim\limits_{\lambda\downarrow 0}\lambda^{n+1}\int_0^\infty e^{-\lambda t}R(t)xdt = 0$;*

(ii) *If $x \in N(A)$, then $\lim\limits_{\lambda\downarrow 0}\lambda^{n+1}\int_0^\infty e^{-\lambda t}R(t)xdt = x$;*

(iii) *If $\{\lambda^{n+1}\int_0^\infty e^{-\lambda t}R(t)xdt\}(\lambda \to 0+)$ has a weakly convergent subsequence for some $x \in X$, then $x \in N(A)\oplus\overline{R(A)}$.*

References

1. W. Arendt and C. J. K. Batty, Tauberian theorems and stability of one-parameter semigroups, *Trans. Amer. Math. Soc.*, **306** (1988), 837-852.
2. W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics, Vol. 96, Birkh-Lauser Verlag, 2001.
3. G. de Barra, *Measure Theory and Integration*, Ellis Horwood series in mathematics and its applications, Halsted Press, New York, 1981.
4. J.-C. Chang and S.-Y. Shaw, Rates of approximation and ergodic limits of resolvent families, *Arch. Math.*, **66** (1996), 320-330.
5. I. Cioranescu and G. Lumer, On $K(t)$-convoluted semigroups, *Pitman Research Notes in Mathematics*, **324** (1995), 86-93.
6. J. diestel and J. J. Uhl, JR., *Vector Measures*, Providence, R. I.: American Mathematical Society, 1977.
7. K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, 194, Springer-Verlag, 2000.
8. B. Jefferies and S. Piskarev, Tauberian theorems for semigroups, *Rend. Del. Circ. Mat. Di Palermo (2) Suppl.*, **68** (2002), 513-521.
9. B. Jefferies and S. Piskarev, Tauberian theorems for cosine operator functions, *Tr. Mat. Inst. Steklova*, **236** (2002), Differ. Uravn. i Din. Sist., 474-480.
10. J. Goldstein, *Semigroups of Linear Operators and Applications*, Oxford University Press, 1985.

11. S. Kantorovitz and S. Piskarev, Mean stability of semigroups, *Taiwanese J. Math.*, **6** (2002), 89-103.
12. H. Kellerman and M. Hieber, Integrated semigroups, *J. Funct. Anal.*, **84** (1989), 160-180.
13. C.-C. Kuo and S.-Y. Shaw, On strong and weak solutions of abstract Cauchy problems, *J. Concrete and Applicable Math.*, **2(3)** (2004), 191-212.
14. Y.-C. Li and S.-Y. Shaw, N-Times integrated C-semigroups and the absract Cauchy problem, *Taiwaness J. Math.*, **1(1)** (1997), 75-102.
15. Y.-C. Li and S.-Y. Shaw, Perturbation of non-exponentially-bounded α-times integrated C-semigroups, *J. Math. Soc. Japan*, **55** (2003), 1115-1136.
16. Y.-C. Li and S.-Y. Shaw, Mean ergodicity and mean stability of regularized solution families, *Mediterr. J. Math.*, **1** (2004), 175-193.
17. C. Lizama, Regularized solutions for abstract Volterra equations, *J. Math. Anal. Appl.*, **243** (2000), 278-292.
18. C. Lizama and J. Sanchez, On perturbation of k-regularized resolvent families, *Taiwanese J. Math.*, **7** (2003), 217-227.
19. Y. I. Lyubich and Q. P. Vu, Asymptotic stability of linear differential equations in Banach spaces, *Studia Math.*, **88** (1988), 37-42.
20. J. Prüss, Evolutionary Integral Equations and Applications, in: *Monographs in Mathematics*, Vol. 87, Birkhäuser, Verlag, 1993.
21. S.-Y. Shaw, Mean and pointwise ergodic theorems for cosine operator functions, *Math. J. Okayama Univ.*, **27** (1985), 197-203.
22. S.-Y. Shaw, Mean ergodic theorems and linear functional equations, *J. Funct. Anal.*, **87** (1989), 428-441.
23. S.-Y. Shaw, Abstract ergodic theorems, *Rend. Del. Circ. Mat. Di Palermo, Ser. II, Suppl.*, **52** (1998), 141-155.
24. S.-Y. Shaw, Non-optimal rates of ergodic limits and approximate solutions, *J. Approx. Theory*, **94** (1998), 285-299.
25. S.-Y. Shaw, Ergodic theorems with rates for r-times integrated solution families, in: *Operator Theory and Related Topics*, Vol II, Oper. Theory Adv. Appl., 118, Birkhäuser, Basel, 2000, pp. 359-371.

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