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Geometric Analysis of the Vibration of Rubber Wiper Blade

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Abstract. The purpose of this paper is to work out the theoretical aspects of the vibration problem of rubber wiper blade on convex windshield. Over the past 20 years, some 2-dimensional spring-mass models were presented in engineering science to simulate the vibration of rubber wiper blade on windshield. In this paper, we will consider the elasticity perspective on this 3-dimensional vibration problem. Our theoretical analysis suggests that there should exist two classes of vibration frequencies corresponding to "$*$-exact deformations (Class I)" and "$*$-closed deformations (Class II)". We prove mathematical theorems on the characterization of deformations of Class I. We also explain how elementary deformations of Class II can be constructed. We then deduce two mathematical formulas, for the vibration problem of rubber wiper blade on convex windshield, from our theoretical analysis. Our theoretical predictions are in almost perfect agreement with experimental data. One of the crucial steps of our analysis is a decomposition theorem motivated by the de Rham Cohomology and the Hodge Theory.

1. Introduction

The purpose of this paper is to work out the theoretical aspects of the vibration problem of rubber wiper blade on convex windshield. This problem has been actively studied in engineering science for more than 20 years. See [3–6,18–20,28,29,31–33,37]. Generally, the vibration frequencies of rubber wiper blade on convex windshield are very complicated. Figure 1.1 shows a FFT (Fast Fourier Transform) diagram for the vibration frequencies of rubber wiper on convex windshield before reversal. On this FFT diagram, frequencies below 25,600Hz can be trusted.

We try to understand the vibration frequencies of rubber wiper blade on convex windshield from the theoretical/mathematical physics point of view. Our theoretical analysis (Section 3) suggests that there should exist two classes of vibration frequencies corresponding to "$*$-exact deformations (Class I)" and "$*$-closed deformations (Class II)".

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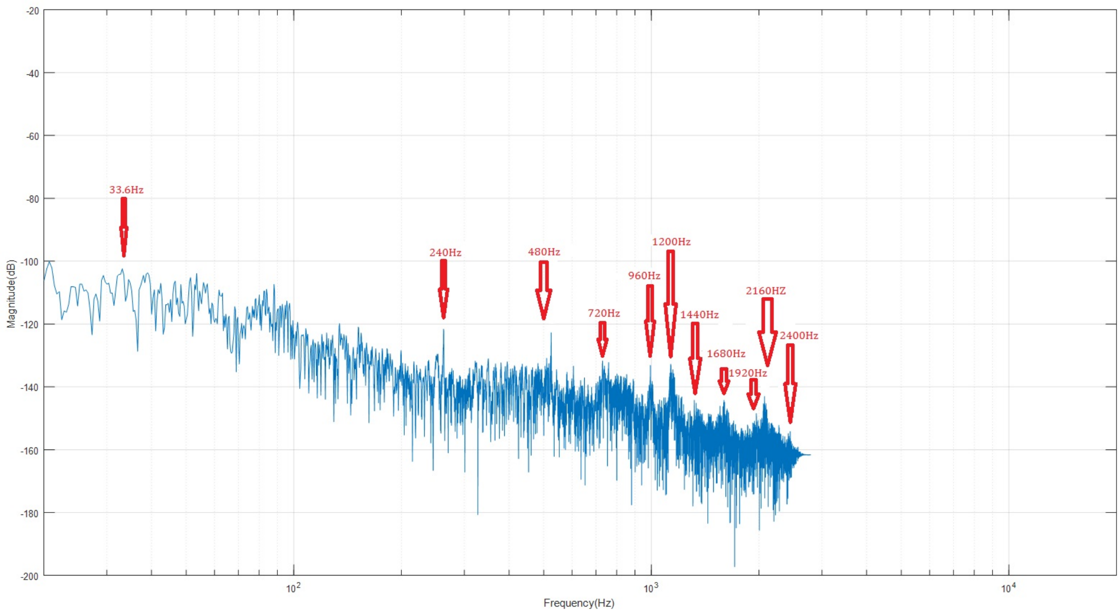

Figure 1.1: Vibration frequencies of rubber wiper blade on convex windshield.

Let ρ kg/m^3 denote the density of the rubber wiper. Let l m denote the length of the rubber wiper. Then most of the vibration frequencies of Class I should locate around

$$\sqrt{\frac{\lambda + 2\mu}{\rho}} \cdot \frac{n}{2l} \text{ Hz} \quad \text{(Class I)} \tag{1.1}$$

where n is a positive integer. Some of the vibration frequencies of Class II should appear around

$$\sqrt{\frac{\mu}{\rho}} \cdot \frac{n}{2l} \text{ Hz} \quad \text{(Class II)}. \tag{1.2}$$

However, the vibration frequencies of Class II are loosely distributed, general low vibration-frequencies of Class II may appear. Here λ and μ are the "Lame coefficients". Lame coefficients are material constants of the rubber wiper. These material constants are related to the "Young modulus E" and the "Poisson ratio σ" as follows:

$$\lambda = \frac{\sigma \cdot E}{(1+\sigma) \cdot (1-2\sigma)} \quad \text{and} \quad \mu = \frac{E}{2 \cdot (1+\sigma)}. \tag{1.3}$$

Experimental data has amazingly supported our theoretical predictions. In Figure 1.1, peaks of the specific vibration frequencies predicted by our mathematical formulas are indicated. In Figure 1.1, the strange vibration frequency around 510Hz, higher than the predicted frequency 480Hz, is caused by machine noise. For the details of comparison of experimental data with our mathematical formulas, see [10].

Somewhat surprisingly, one of the crucial steps of our analysis is a "decomposition theorem" (see Theorem 3.3) motivated by the de Rham Cohomology and the Hodge Theory. We will explain the physical aspects of this vibration problem in Section 2. Section 3 is devoted to the mathematical analysis of this vibration problem. To reduce the "wind resistance" on the windshield of a moving vehicle, the shape of windshield is generally *strictly convex* [7]. This *strict convexity* of windshield leads to the boundary condition (3.9) for our mathematical formulation. Then in Section 4, we will explain how our analysis leads to our physical predictions on the vibration frequencies of rubber wiper blade on convex windshield. The mathematical analysis presented in this article is motivated by the authors previous work on differential geometry and theoretical physics [22–26].

2. Physical aspects of the vibration problem

When a rubber wiper blade moves on the convex windshield of an automobile, there are 4 different forces acting on it: pressure on the wiper adapter, support force from the windshield, drag force from the wiper arm, and the frictional force acting on the rubber wiper blade. Figure 2.1 shows a 2-dimensional force diagram for the rubber wiper blade.

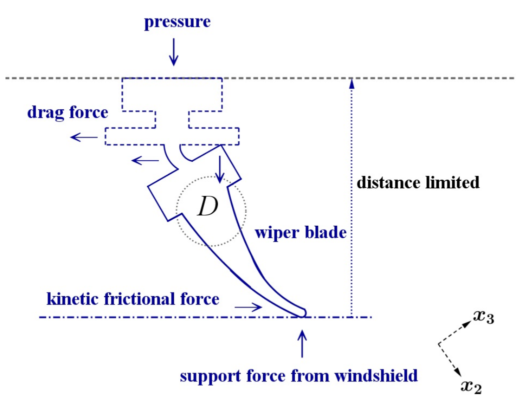

Figure 2.1: 2D Force Diagram.

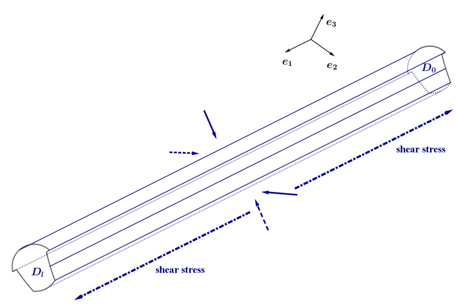

Figure 2.2: Slightly Deformed Region.

To keep rubber wiper blade working normally, the bending (deformation) of rubber wiper blade should not be severe to avoid wiper jumping or malfunctioning. Thus a suitable distance between the wiper adapter and the convex windshield must be retained. The wiper arm is always moving above the windshield with the pressure on the rubber wiper being applied by a spring under the wiper arm.

2.1. Hyperelasticity and viscoelasticity

The rubber wiper blade is made of elastic material rubber. Here we explain the physical properties of rubber wiper briefly. Rubber is a polymer material consisting of very large molecules (macromolecules). These macromolecules are linked together by the Van der Waals force (see [21, 38, 39]). When rubber is deformed slightly, it behaves like a "hyper-elastic" body [30]. Here "hyper-elastic" means that if we remove the force acting on a rubber, this rubber will return to its original shape and the "stored energy" in rubber will be released. Usually we consider the "stored energy" in rubber as a "potential function for deformation". See Appendix A of [16].

When rubber is deformed seriously, part of the linking between macromolecules will be broken. In this case, this rubber will not be able to return to its original structure. This phenomenon is called "viscoelasticity". In this case, the "conservation law of mechanical energy" fails because some energy of deformation is transformed into heat. Usually it is very difficult to predict precisely the dynamics of a seriously deformed rubber.

2.2. Saint-Venant's principle

Now we discuss the dynamics of rubber wiper blade. When a rubber wiper blade moves on the windshield of an automobile, some region Ω of the rubber wiper blade is only slightly deformed. See Figures 2.1 and 2.2. This phenomenon is usually considered as the effect of "Saint-Venant's principle". The Boussinesq version of "Saint-Venant's principle" can be expressed as follows.

> "An equilibrated system of external forces applied to an elastic body, all of the points of application lying within a given sphere, produces deformations of negligible magnitude at distances from the sphere which are sufficiently large compared to its radius."

A mathematical formulation of this physical principle has been proved by Ernst [15].

Let D denote the corresponding 2-dimensional domain so that Ω is diffeomorphic to $[0, l] \times D$. See Figures 2.1 and 2.2. On this slightly deformed region Ω, the "hyperelasticity" of rubber may be assumed to be true [30]. Thus it is reasonable to predict the vibration of rubber wiper blade through mathematical analysis of the dynamics of this slightly deformed region Ω.

2.3. General physical principles on vibration

General physical principles suggest that the dominant vibration frequencies of an elastic body are related to the "standing waves" on this elastic body [34]. This principle is generally adopted in the development of Quantum Physics [9]. Physicists considered "eigenstates" in Quantum Physics. The idea of "eigenstates" becomes the cornerstone of the Molecular Orbital Theory in Quantum Chemistry [21].

Therefore we will try to find the possible "eigenstates" in our analysis of the vibration of rubber wiper blade on *convex* windshield.

3. Mathematical aspects of the vibration problem

We will start with brief discussions on 3-dimensional elasticity mechanics with emphasis on Hyper-elasticity. Basic references for the elasticity mechanics are [8, 16].

Elasticity theory is based on the framework of continuum mechanics introduced by Cauchy. The key idea of elasticity theory is to relate "the infinitesimal variation of stress tensor" with "the infinitesimal deformation" of an elastic body.

Let $T(\mathbf{B})$ and $T^*(\mathbf{B})$ respectively denote the tangent bundle and the cotangent bundle of an elastic body $\mathbf{B}$. The Cauchy stress tensor $\mathbf{S}$ on an elastic body $\mathbf{B}$ is a section of

$$T(\mathbf{B}) \otimes T^*(\mathbf{B})$$

on $\mathbf{B}$. For a given unit tangent vector $\mathbf{v} \in T_p(\mathbf{B})$, we define

$$\mathbf{v}^+ = \{\mathbf{u} \in T_p(\mathbf{B}) : \langle \mathbf{u}, \mathbf{v} \rangle = 0\}$$

to be the 2-dimensional subspace of $T_p(\mathbf{B})$ perpendicular to $\mathbf{v}$. $\mathbf{S}(\mathbf{v})$ indicates the pressure force acting on $\mathbf{v}^+$. We usually call "the component of $\mathbf{S}(\mathbf{v})$ along the $\mathbf{v}$-direction"

$$\frac{\langle \mathbf{S}(\mathbf{v}), \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \cdot \mathbf{v}$$

"the normal stress component of $\mathbf{S}(\mathbf{v})$". We call "the component of $\mathbf{S}(\mathbf{v})$ perpendicular to the $\mathbf{v}$-direction"

$$-\frac{\langle \mathbf{S}(\mathbf{v}), \mathbf{v} \rangle}{\|\mathbf{v}\|^2} \cdot \mathbf{v} + \mathbf{S}(\mathbf{v})$$

"the shear stress component of $\mathbf{S}(\mathbf{v})$".

3.1. Differential geometry of elasticity mechanics

In the following discussions, we will assume that $\mathbf{B}_t$ is a smooth family of 3-dimensional manifolds with boundary $\partial \mathbf{B}_t$ in $\mathbb{R}^3$ depending on the time variable t. $T(\mathbf{B}_t)$ and $T^*(\mathbf{B}_t)$ will respectively denote the tangent bundle and the cotangent bundle of $\mathbf{B}_t$. $\mathbf{S}$ is the $T(\mathbf{B}_t) \otimes T^*(\mathbf{B}_t)$-valued Cauchy stress tensor on $\mathbf{B}_t$. It should be noted that, according to the "conservation law of angular momentum", the Cauchy stress tensor $\mathbf{S}$ is symmetric when we identify $T^*(\mathbf{B}_t)$ with $T(\mathbf{B}_t)$ using the Euclidean metric on $\mathbb{R}^3$. See Appendix A of [16].

In the following computation, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ usually constitute an orthonormal framing field on $\mathbb{R}^3$. Since gravity is not important for the vibration of rubber wiper system, we will disregard the influence of gravity on the vibration of rubber wiper blade in this article.

According to the Hamilton principle, the motion of an elastic body must satisfy the variation formula

$$-\delta \int_{t_i}^{t_f} U_t \cdot dt + \delta \int_{t_i}^{t_f} T_t \cdot dt + \int_{t_i}^{t_f} (\delta W)_t \cdot dt = 0$$

for any admissible smooth path of virtual deformation. Here U_t is the potential corresponds to "the stored energy of elasticity" in the elastic body $\mathbf{B}_t$ at time t. See Appendix A of [16].

T_t is the kinetic energy of $\mathbf{B}_t$. $(\delta W)_t$ corresponds to "the virtual work" done by the traction force τ_t, along the boundary $\partial \mathbf{B}_t$ of $\mathbf{B}_t$, on the virtual deformation $(\delta \mathbf{x})_t$ at time t. We assume, for simplicity, that the density function ρ_t of $\mathbf{B}_t$ does not depend substantially on t so that

$$\frac{\partial \rho_t}{\partial t} \approx 0.$$

It can be shown, using the Stokes Theorem and integration by parts, that the above variation formula is equivalent to

$$\begin{aligned} 0 &= -\delta \int_{t_i}^{t_f} U_t \cdot dt + \delta \int_{t_i}^{t_f} T_t \cdot dt + \int_{t_i}^{t_f} (\delta W)_t \cdot dt \\ &= \int_{t_i}^{t_f} \left(\oint_{\partial \mathbf{B}_t} \langle -\mathbf{S}(\mathbf{n}_t) + \tau_t, (\delta \mathbf{x})_t \rangle \right) \cdot dt \\ &\quad + \int_{t_i}^{t_f} \left(\int_{\mathbf{B}_t} \left\langle -\rho_t \cdot \frac{\partial}{\partial t}\frac{\partial \mathbf{u}}{\partial t} + \sum_{k=1}^{3} (\nabla_{\mathbf{e}_k} \mathbf{S})(\mathbf{e}_k), (\delta \mathbf{x})_t \right\rangle \right) \cdot dt \end{aligned} \tag{3.1}$$

in which $(\delta \mathbf{x})_t$ is the admissible smooth path of virtual deformation depending on t. Here $\mathbf{n}_t$ is the outer normal vector field on the boundary $\partial \mathbf{B}_t$ of $\mathbf{B}_t$. In (3.1), $\mathbf{u}$ is the vector-valued position function of $\mathbf{B}_t$. $\nabla_{\mathbf{e}_k}\mathbf{S}$ is the covariant derivative of the Cauchy stress tensor $\mathbf{S}$ along the vector field $\mathbf{e}_k$. Thus the vector-valued Cauchy dynamic equation for the elastic body $\mathbf{B}_t$ is

$$-\rho \cdot \frac{\partial^2 \mathbf{u}}{\partial t^2} + \sum_{k=1}^{3} (\nabla_{\mathbf{e}_k} \mathbf{S})(\mathbf{e}_k) = \mathbf{0} \quad \text{on } \mathbf{B}_t \tag{3.2}$$

with

$$\mathbf{S}(\mathbf{n}_t) = \tau_t \text{ (traction force given at time } t) \quad \text{on } \partial \mathbf{B}_t. \tag{3.3}$$

When a hyper-elastic body stays at an equilibrium state with traction acting on its boundary, we have the following system of (usually elliptic) partial differential equations

$$\sum_{k=1}^{3}(\nabla_{\mathbf{e}_k}\mathbf{S})(\mathbf{e}_k) = \mathbf{0} \quad \text{on } \mathbf{B}_t = \mathbf{B}_0$$

with

$$\mathbf{S}(\mathbf{n}_t) = \tau_0 \quad \text{on } \partial\mathbf{B}_t = \partial\mathbf{B}_0$$

independent of t.

Since the manifold $\mathbf{B}_t$ usually changes with t, it should be noted that the system (3.2) is not defined on a fixed region of $\mathbb{R}^3$. This is the fundamental reason why the system (3.2) is nonlinear in nature.

Now we explain how to transform the vector-valued Cauchy dynamic equation (3.2) into a vector-valued dynamic equation defined on the fixed region $\mathbf{B}_0$.

Let ϕ^t denote the diffeomorphism from $\mathbf{B}_0$ to $\mathbf{B}_t$ defined by the solution of the Cauchy dynamic equation (3.2). Let ψ^t denote the corresponding inverse diffeomorphism from $\mathbf{B}_t$ to $\mathbf{B}_0$. We may define a family of stress tensors $\mathfrak{S}_t$ on $\mathbf{B}_0$, depending on t, by setting

$$\mathfrak{S}_t = \psi^t_* \circ \mathbf{S} \circ \phi^t_*.$$

Here ϕ^t_* and ψ^t_* are respectively the differential maps associated with the diffeomorphisms ϕ^t and ψ^t. $\mathfrak{S}_t$ is usually called the "second Piola-Kirchhoff stress tensor" associated with the Cauchy stress tensor $\mathbf{S}$ on $\mathbf{B}_t = \phi^t(\mathbf{B}_0)$. See Appendix A of [16].

Note that

$$\mathbf{S} = \phi^t_* \circ \mathfrak{S}_t \circ \psi^t_*.$$

We may parameterize $\mathbf{B}_t = \phi^t(\mathbf{B}_0)$ by the points on $\mathbf{B}_0$. By doing so, we obtain the "first Piola-Kirchhoff stress tensor"

$$\mathbf{S}_{\phi^t}$$

associated with the Cauchy stress tensor $\mathbf{S}$ on $\mathbf{B}_t = \phi^t(\mathbf{B}_0)$.

We may express (3.2) as a system of equations defined on the fixed region $\mathbf{B}_0$ of $\mathbb{R}^3$ as follows. Let

$$\mathbf{F} = \sum_{k=1}^{3}(\nabla_{\mathbf{e}_k}\mathbf{S})(\mathbf{e}_k) = \sum_{k=1}^{3}\left[\nabla_{\mathbf{e}_k}(\phi^t_* \circ \mathfrak{S}_t \circ \psi^t_*)\right](\mathbf{e}_k)$$

denote the force field on $\mathbf{B}_t$ associated with the Cauchy stress tensor $\mathbf{S}$. Then we may express (3.2) as

$$-\rho \cdot \frac{\partial^2 \phi^t}{\partial t^2} + \mathbf{F}_{\phi^t} = \mathbf{0} \quad \text{on } \mathbf{B}_0 \tag{3.4}$$

in which $\mathbf{F}_{\phi^t}$ is the vector field on $\mathbf{B}_t = \phi^t(\mathbf{B}_0)$ parameterized by the points on $\mathbf{B}_0$. It can be observed readily that the system (3.4) is naturally nonlinear. Note that the boundary condition (3.3) can be expressed as

$$\mathfrak{S}_t \circ \psi_*^t(\mathbf{n}_t) = \psi_*^t(\tau_t) \quad \text{on } \partial\mathbf{B}_0. \tag{3.5}$$

3.2. Linearization of the nonlinear Cauchy dynamic equation

To tackle the vibration problem of rubber wiper blade on convex windshield, we will consider the linearization of the nonlinear Cauchy dynamic equation on a slightly deformed region Ω of the rubber wiper blade. See Figures 2.1 and 2.2.

Usually Ω is diffeomorphic to $[0, l] \times D$. Note that Ω is exactly $[0, l] \times D$ when the rubber wiper blade is at a static equilibrium state. According to the general principles of perturbation theory, it is reasonable to consider the following simplified problem: the linearization of the nonlinear Cauchy dynamic equation on $\Omega = [0, l] \times D$. We assume that the rubber material is homogeneous so that the following "Isotropy" condition (see Appendix D of [16]) is satisfied:

$$\begin{aligned} \langle (\delta\mathbf{S})(\mathbf{e}_j), \mathbf{e}_k \rangle &= \lambda \cdot \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) \cdot \delta_{jk} + 2\mu \cdot \epsilon_{jk} \\ &= \lambda \cdot (\operatorname{div} \mathbf{v}) \cdot \delta_{jk} + 2\mu \cdot \epsilon_{jk} \end{aligned} \tag{3.6}$$

with

$$\epsilon_{jk} = \frac{1}{2} \left(\frac{\partial v_k}{\partial x_j} + \frac{\partial v_j}{\partial x_k} \right). \tag{3.7}$$

Here $(\delta\mathbf{S})$ is "the infinitesimal variation of Cauchy stress tensor" corresponding to "the infinitesimal deformation" $\mathbf{v}(t, \mathbf{x}) = \big(v_1(t, \mathbf{x}), v_2(t, \mathbf{x}), v_3(t, \mathbf{x})\big)$ on the hyper-elastic body $\mathbf{B}_0$. Here $\mathbf{x} = (x_1, x_2, x_3) \in \mathbf{B}_0$.

The linearization of (3.4) at a static equilibrium state can be expressed as the vector-valued Lame equation:

$$\rho \cdot \frac{\partial^2 \mathbf{v}}{\partial t^2} = (\lambda + \mu) \cdot \nabla \left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \right) + \mu \cdot \left(\frac{\partial^2 \mathbf{v}}{\partial x_1^2} + \frac{\partial^2 \mathbf{v}}{\partial x_2^2} + \frac{\partial^2 \mathbf{v}}{\partial x_3^2} \right)$$

in which ρ is the (constant) density of the hyper-elastic body $\mathbf{B}_0$. Linearization of the boundary condition (3.5) can be expressed as

$$(\delta\mathbf{S})(\mathbf{n}_0) = (\delta\tau).$$

Here the coefficients λ and μ are Lame coefficients. See (1.3).

3.3. Lame equation with specific physical boundary conditions

We assume, for simplicity, that D is a 2-dimensional manifold with boundary on $\mathbb{R}^2$. Let

$$\mathfrak{B}_\Omega = \{(x_1, x_2, x_3) \in \Omega : 0 \leq x_1 \leq l \text{ and } (x_2, x_3) \in \partial D\}.$$

Then the boundary $\partial\Omega$ of $\Omega = [0, l] \times D$ can be decomposed as follows:

$$\partial\Omega = D_0 \cup \mathfrak{B}_\Omega \cup D_l$$

in which $D_0 = \{0\} \times D$ and $D_l = \{l\} \times D$. Note that $\Omega = [0, l] \times D$ is not a 3-dimensional manifold with boundary because $\partial\Omega$ contains the one-dimensional corner

$$\text{Corner}_\Omega = (\mathfrak{B}_\Omega \cap D_0) \cup (\mathfrak{B}_\Omega \cap D_l).$$

When a rubber wiper blade moves on windshield, there is no stress or traction force acting along the outer normal vector field on $D_0 \cup D_l$. This physical condition can be expressed as the boundary condition

$$\langle(\delta\mathbf{S})(\mathbf{n}_0), \mathbf{n}_0\rangle = 0 \quad \text{on } D_0 \cup D_l \tag{3.8}$$

at time t.

We assume that the rubber wiper blade moves on windshield at nearly a constant speed. To reduce the "wind resistance" on the windshield of a moving vehicle, the shape of windshield is generally *strictly convex* [7]. Thus we may assume that there is *no* frictional force acting on the ends $D_0 \cup D_l$ of $\Omega = [0, l] \times D$. We may express this physical condition as the boundary condition

$$\operatorname{div} \mathbf{v} = 0 \quad \text{on } D_0 \cup D_l \tag{3.9}$$

at time t. See Section 4 for the explanation of (3.9).

In the following analysis, we will discuss the solutions of the Lame equation

$$\begin{aligned} \frac{\partial^2 \mathbf{v}}{\partial t^2} &= \frac{\lambda + \mu}{\rho} \cdot \nabla \left[\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right] + \frac{\mu}{\rho}\left[\frac{\partial^2 \mathbf{v}}{\partial x_1^2} + \frac{\partial^2 \mathbf{v}}{\partial x_2^2} + \frac{\partial^2 \mathbf{v}}{\partial x_3^2}\right] \\ &= \frac{(\lambda + \mu)}{\rho} \cdot \nabla(\operatorname{div} \mathbf{v}) + \frac{\mu}{\rho} \cdot \Delta \mathbf{v} \end{aligned} \tag{3.10}$$

on $\Omega = [0, l] \times D$, depending on t, satisfying the boundary conditions (3.8) and (3.9). Here we adopt the notation

$$\Delta \equiv \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right)$$

frequently used in PDE. Thus the operator $-\Delta$ is positive-definite.

Proposition 3.1. *Let I be an open interval for the time variable t. We have the following results.*

(A) *Assume that $\widetilde{w}(t, \mathbf{x})$ is a C^∞ function defined on $I \times \Omega$ satisfying the wave equation*

$$\frac{\partial^2 \widetilde{w}}{\partial t^2} - \frac{\lambda + 2\mu}{\rho} \cdot \Delta \widetilde{w} = 0.$$

Then the deformation $\nabla \widetilde{w}(t, \mathbf{x})$ satisfies the Lame equation (3.10)*:*

$$\frac{\partial^2 (\nabla \widetilde{w})}{\partial t^2} = \frac{\lambda + \mu}{\rho} \cdot \nabla(\operatorname{div} \nabla \widetilde{w}) + \frac{\mu}{\rho} \cdot \Delta(\nabla \widetilde{w}).$$

(B) *Assume that $\mathbf{u}(t, \mathbf{x})$ is a C^∞ vector-valued function defined on $I \times \Omega$ satisfying*

$$\operatorname{div} \mathbf{u} = 0$$

and the wave equation

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \frac{\mu}{\rho} \cdot \Delta \mathbf{u} = 0.$$

Then the deformation $\mathbf{u}(t, \mathbf{x})$ satisfies the Lame equation (3.10)*:*

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = \frac{\lambda + \mu}{\rho} \cdot \nabla(\operatorname{div} \mathbf{u}) + \frac{\mu}{\rho} \cdot \Delta \mathbf{u}.$$

Proof. This proposition can be checked readily using the equality $\Delta = \operatorname{div} \circ \nabla$. □

We will show that a decomposition theorem (see Theorem 3.3) for the solutions $\mathbf{v}(t, \mathbf{x})$ of the Lame equation (3.10) is possible. This means that we may express $\mathbf{v}(t, \mathbf{x})$ as the sum of $\nabla \widetilde{w}(t, \mathbf{x})$ and $\mathbf{u}(t, \mathbf{x})$ mentioned in Proposition 3.1. We start with the following preliminary result.

Lemma 3.2. *Let I be an open interval for the time variable t. Assume that $\mathbf{v}(t, \mathbf{x})$ is a C^∞ vector-valued function defined on $I \times \Omega = I \times [0, l] \times D$ satisfying the Lame equation* (3.10)*. Then there exists a unique function $w \in C^\infty(I, W_0^{1,2}(\Omega))$ satisfying*

$$\Delta w = \operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}. \tag{3.11}$$

Here $W_0^{1,2}(\Omega)$ is the closure of $C_0^1(\Omega)$ in the Hilbert space $W^{1,2}(\Omega)$ consisting of weakly differentiable functions f satisfying $\|f\|_{L^2(\Omega)} + \|\nabla f\|_{L^2(\Omega)} < +\infty$.

Proof. The unique existence of the function $w \in C^\infty(I, W_0^{1,2}(\Omega))$ satisfying (3.10) follows from the general theory of elliptic partial differential equations. See Chapters 7 and 8 of [17]. □

Remark 3.1. According to the general theory of elliptic PDE, Chapter 8 of [17], the function w of (3.11) can be extended smoothly across the 2-dimensional portion of $\partial\Omega$ for each $t \in I$. However, the regularity of w around the corner

$$\mathrm{Corner}_{\Omega} = (\mathfrak{B}_{\Omega} \cap D_0) \cup (\mathfrak{B}_{\Omega} \cap D_l)$$

of $\partial\Omega$ might be worse than usually expected. See [12]. Fortunately, this regularity problem is not very important for the applications of our theory.

Theorem 3.3 (Decomposition Theorem)**.** *Let I be an open interval for the time variable t. Assume that $\mathbf{v}(t, \mathbf{x})$ is a C^∞ vector-valued function defined on $I \times \Omega$ satisfying the Lame equation* (3.10)*. Then there exists a function $\widetilde{w}(t, \mathbf{x}) \in C^\infty(I, W_0^{1,2}(\Omega))$ satisfying*

$$\Delta\widetilde{w} = \operatorname{div} \mathbf{v} \tag{3.12}$$

and the wave equation

$$\frac{\partial^2 \widetilde{w}}{\partial t^2} - \frac{\lambda + 2\mu}{\rho} \cdot \Delta\widetilde{w} = 0 \tag{3.13}$$

such that $\mathbf{v}(t, \mathbf{x})$ can be expressed as

$$\mathbf{v}(t, \mathbf{x}) = \mathbf{u}(t, \mathbf{x}) + \nabla\widetilde{w}(t, \mathbf{x})$$

in which $\mathbf{u}(t, \mathbf{x})$ is a vector-valued function on $I \times \Omega$ satisfying

$$\operatorname{div} \mathbf{u} = 0 \tag{3.14}$$

and the wave equation

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \frac{\mu}{\rho} \cdot \Delta\mathbf{u} = 0. \tag{3.15}$$

When the boundary condition (3.9)

$$\operatorname{div} \mathbf{v} = 0 \quad \text{on } I \times (D_0 \cup D_l)$$

is satisfied, we may require that $\widetilde{w}(t, \mathbf{x})$ satisfies the boundary condition

$$\widetilde{w} = 0 \quad \text{on } I \times (D_0 \cup D_l). \tag{3.16}$$

Proof. Let $w \in C^\infty(I, W_0^{1,2}(\Omega))$ denote the function, stated in Proposition 3.1, satisfying (3.11)

$$\Delta w = \operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}.$$

Let $\theta = \operatorname{div}\mathbf{v}$ so that $\Delta w = \operatorname{div}\mathbf{v} = \theta$. By taking the divergence of both sides of the Lame equation (3.10), we obtain

$$\frac{\partial^2(\operatorname{div}\mathbf{v})}{\partial t^2} = \operatorname{div}\circ\frac{\partial^2\mathbf{v}}{\partial t^2} = \frac{(\lambda+\mu)}{\rho}\cdot\operatorname{div}\circ\nabla(\operatorname{div}\mathbf{v}) + \frac{\mu}{\rho}\cdot(\operatorname{div}\circ\Delta\mathbf{v})$$

and so

$$\frac{\partial^2\theta}{\partial t^2} = \frac{(\lambda+\mu)}{\rho}\cdot\operatorname{div}\circ\nabla\theta + \frac{\mu}{\rho}\cdot(\Delta\circ\operatorname{div}\mathbf{v}) = \frac{(\lambda+\mu)}{\rho}\cdot\Delta\theta + \frac{\mu}{\rho}\cdot\Delta\theta.$$

This means that $\theta = \operatorname{div}\mathbf{v}$ satisfies the wave equation

$$\frac{\partial^2\theta}{\partial t^2} = \frac{(\lambda+2\mu)}{\rho}\cdot\Delta\theta.$$

Since $\Delta w = \operatorname{div}\mathbf{v} = \theta$, it follows that

$$\Delta\left(\frac{\partial^2 w}{\partial t^2} - \frac{\lambda+2\mu}{\rho}\cdot\Delta w\right) = \frac{\partial^2\Delta w}{\partial t^2} - \frac{\lambda+2\mu}{\rho}\cdot\Delta\circ\Delta w = \frac{\partial^2\theta}{\partial t^2} - \frac{\lambda+2\mu}{\rho}\cdot\Delta\theta = 0$$

and so the function

$$h \equiv \frac{\partial^2 w}{\partial t^2} - \frac{\lambda+2\mu}{\rho}\cdot\Delta w = \frac{\partial^2 w}{\partial t^2} - \frac{\lambda+2\mu}{\rho}\cdot(\operatorname{div}\mathbf{v}) \tag{3.17}$$

is a C^∞ family of space-harmonic functions depending on time $t \in I$.

Now we define a C^∞ family $H(t,\mathbf{x})$ of space-harmonic functions depending on time $t \in I$ as follows:

$$H(t,\mathbf{x}) \equiv \int_0^t \left(\int_0^s h(r,\mathbf{x})\cdot dr\right)\cdot ds. \tag{3.18}$$

Since h is a C^∞ family of space-harmonic functions depending on time $t \in I$, it is clear that

$$\Delta H = 0 \quad \text{and} \quad \frac{\partial^2 H(t,\mathbf{x})}{\partial t^2} = h(t,\mathbf{x}). \tag{3.19}$$

We define $\widetilde{w}(t,\mathbf{x}) = w(t,\mathbf{x}) - H(t,\mathbf{x})$. Then we obtain (3.12):

$$\Delta\widetilde{w} = \Delta w - \Delta H = (\operatorname{div}\mathbf{v}) - 0 = \operatorname{div}\mathbf{v}.$$

It can be inferred from (3.19) and (3.17) that

$$\frac{\partial^2 H}{\partial t^2} - \frac{\lambda+2\mu}{\rho}\cdot\Delta H = h - 0 = h \equiv \frac{\partial^2 w}{\partial t^2} - \frac{\lambda+2\mu}{\rho}\cdot\Delta w.$$

It can be inferred readily from this equality that (3.13)

$$\frac{\partial^2\widetilde{w}}{\partial t^2} - \frac{\lambda+2\mu}{\rho}\cdot\Delta\widetilde{w} = \frac{\partial^2(w-H)}{\partial t^2} - \frac{\lambda+2\mu}{\rho}\cdot\Delta(w-H) = 0$$

is true.

When the boundary condition (3.9)

$$\operatorname{div}\mathbf{v} = 0 \quad \text{on } I \times (D_0 \cup D_l)$$

is satisfied, it can be observed readily from the definition (3.17) of $h(t, \mathbf{x})$ that

$$h = \frac{\partial^2 w}{\partial t^2} - \frac{\lambda + 2\mu}{\rho} \cdot (\operatorname{div}\mathbf{v}) = 0 \quad \text{on } D_0 \cup D_l$$

for each $t \in I$. Thus it follows from the definition (3.18) of $H(t, \mathbf{x})$ that

$$H = 0 \ \text{ and so } \ \widetilde{w} = w - H = 0 \quad \text{on } I \times (D_0 \cup D_l).$$

This proves (3.16).

Now we define

$$\mathbf{u}(t, \mathbf{x}) \equiv \mathbf{v}(t, \mathbf{x}) - \nabla\widetilde{w}(t, \mathbf{x})$$

so that $\mathbf{v} = \mathbf{u} + \nabla\widetilde{w}$. It is clear that (3.14) is true:

$$\operatorname{div}\mathbf{u} \equiv \operatorname{div}\mathbf{v} - \operatorname{div}\nabla\widetilde{w} = \operatorname{div}\mathbf{v} - \Delta\widetilde{w} = 0.$$

Since $\mathbf{v}$ satisfies the Lame equation (3.10), we have

$$\frac{\partial^2(\mathbf{u} + \nabla\widetilde{w})}{\partial t^2} = \frac{\lambda + \mu}{\rho} \cdot \nabla(\operatorname{div}\mathbf{u} + \operatorname{div}\nabla\widetilde{w}) + \frac{\mu}{\rho} \cdot \Delta(\mathbf{u} + \nabla\widetilde{w})$$

in which $\mathbf{v} = \mathbf{u} + \nabla\widetilde{w}$. Since $\operatorname{div}\mathbf{u} = 0$, we have

$$\begin{aligned}\frac{\partial^2(\mathbf{u} + \nabla\widetilde{w})}{\partial t^2} &= \frac{\lambda + \mu}{\rho} \cdot \nabla(0 + \operatorname{div}\nabla\widetilde{w}) + \frac{\mu}{\rho} \cdot \Delta(\mathbf{u} + \nabla\widetilde{w}) \\ &= \frac{\lambda + \mu}{\rho} \cdot \nabla(\Delta\widetilde{w}) + \frac{\mu}{\rho} \cdot \Delta(\mathbf{u} + \nabla\widetilde{w}) = \frac{\mu}{\rho} \cdot \Delta\mathbf{u} + \frac{\lambda + 2\mu}{\rho} \cdot \nabla(\Delta\widetilde{w}).\end{aligned}$$

Since $\widetilde{w}$ satisfies the wave equation (3.13), we infer from the above equality that

$$\frac{\partial^2\mathbf{u}}{\partial t^2} - \frac{\mu}{\rho} \cdot \Delta\mathbf{u} = 0.$$

This proves (3.15). □

Remark 3.2. The earliest version of the decomposition, using "grad" and "curl", of a vector field on $\mathbb{R}^3$ is due to H. von Helmholtz. Helmholtz Decomposition for vector fields on $\mathbb{R}^3$ was used by Maxwell in his theory of classical Electromagnetic Field. Helmholtz's work motivated the earlier development of Cohomology Theory and Hodge Theory.

It should be noted that the condition (3.16) of Theorem 3.3 is crucial for our theory. Usually the choice of boundary condition for PDE depends on the nature of the problem considered.

Recently, an existence theorem for the Lame equation, with nonlinear input force satisfying "null condition", was proved by Sideris and Klainerman under strict assumptions on the smallness of initial data. See [27,36]. The elasticity problem, considered by Klainerman and Sideris, is defined on $\mathbb{R}^3$ without specific boundary condition imposed.

Remark 3.3. In elasticity theory, there are "Saint-Venant compatibility conditions" related to the "overdeterminedness of stress distribution". Recently, it is known that these compatibility conditions are related to de Rham cohomology [1,2,13,14].

We will call deformations, mentioned in Theorem 3.3, of the form

$$\nabla\widetilde{w}(t,\mathbf{x}) \text{ with } \widetilde{w} \text{ satisfying (3.12) and (3.13)}$$

"$*$-exact deformations". We will call deformations, mentioned in Theorem 3.3, of the form

$$\mathbf{u}(t,\mathbf{x}) \text{ with } \operatorname{div}\mathbf{u} = 0 \text{ (3.14) satisfying (3.15)}$$

"$*$-closed deformations".

Our next goal is to characterize the "$*$-exact deformations". We start with the following preliminary result.

Proposition 3.4. *Let I be an open interval for the time variable t. Assume that $f(t,x_2,x_3)$ is a C^∞ function defined on $I\times D$ satisfying the following wave equation*

$$\frac{\partial^2 f}{\partial t^2} - \frac{\lambda+2\mu}{\rho}\cdot\left(\frac{-n^2\cdot\pi^2}{l^2}\cdot f + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}\right) = 0 \tag{3.20}$$

in which n is a positive integer. Then the function $\widetilde{w}(t,\mathbf{x})$ defined by

$$\widetilde{w}(t,x_1,x_2,x_3) \equiv f(t,x_2,x_3)\cdot\sqrt{\frac{2}{l}}\cdot\sin\left(\frac{n\cdot\pi\cdot x_1}{l}\right) \tag{3.21}$$

is a solution of the wave equation (3.13)

$$\frac{\partial^2\widetilde{w}}{\partial t^2} - \frac{\lambda+2\mu}{\rho}\cdot\Delta\widetilde{w} = 0$$

satisfying the boundary condition (3.16)

$$\widetilde{w} = 0 \quad \text{on } I\times(D_0\cup D_l).$$

Let $\mathbf{v}(t,\mathbf{x}) = \nabla\widetilde{w}(t,\mathbf{x})$. Then the deformation $\mathbf{v}(t,\mathbf{x}) = \nabla\widetilde{w}(t,\mathbf{x})$ is a solution of the Lame equation (3.10) *satisfying the boundary conditions* (3.8) *and* (3.9) *at time $t\in I$.*

Proof. It can be inferred readily from (3.20) that $\widetilde{w}(t, x_1, x_2, x_3)$ satisfies the wave equation (3.13). Since

$$\sin\left(\frac{n \cdot \pi \cdot 0}{l}\right) = \sin\left(\frac{n \cdot \pi \cdot l}{l}\right), \tag{3.22}$$

it is clear that $\widetilde{w}(t, x_1, x_2, x_3)$ satisfies the boundary condition (3.16). Since

$$\operatorname{div} \mathbf{v} = \Delta\widetilde{w} = \left(\frac{-n^2 \cdot \pi^2}{l^2} \cdot f + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}\right) \cdot \sin\left(\frac{n \cdot \pi \cdot x_1}{l}\right),$$

it is clear that the boundary condition (3.9) is satisfied by $\mathbf{v}(t, \mathbf{x}) = \nabla\widetilde{w}(t, \mathbf{x})$ because of (3.22).

Now we consider the boundary condition (3.8). Since the boundary condition (3.9) is satisfied, it follows from (3.6) and (3.7) that

$$\begin{aligned}\langle(\delta\mathbf{S})(\mathbf{e}_j), \mathbf{e}_k\rangle &= \lambda \cdot (\operatorname{div} \mathbf{v}) \cdot \delta_{jk} + 2\mu \cdot \epsilon_{jk} \\ &= 0 + \mu \cdot \left(\frac{\partial v_k}{\partial x_j} + \frac{\partial v_j}{\partial x_k}\right) = 2\mu \cdot \frac{\partial^2 \widetilde{w}}{\partial x_j \partial x_k}\end{aligned}$$

on $I \times (D_0 \cup D_l)$. Since $\mathbf{n}_0 = \mp\mathbf{e}_1$ on $D_0 \cup D_l$, we have

$$\langle(\delta\mathbf{S})(\mathbf{n}_0), \mathbf{n}_0\rangle = 2\mu \cdot \frac{\partial^2 \widetilde{w}}{\partial x_1 \partial x_1} = 0$$

because of (3.22). □

Since the wave equation (3.13) and the pair of boundary conditions (3.8) and (3.9) are all linear, it is clear that any convergent linear combination of functions of the form (3.21) will give rise to a solution of the wave equation (3.13) satisfying the pair of boundary conditions (3.8) and (3.9). Our next main theorem shows that these are exactly the only solutions for the wave equation (3.13) satisfying the pair of boundary conditions (3.8) and (3.9).

Theorem 3.5 (characterization of $*$-exact deformations)**.** *Let I be an open interval for the time variable t. Assume that $\widetilde{w}(t, \mathbf{x})$ is a C^∞ solution of the wave equation* (3.13)

$$\frac{\partial^2 \widetilde{w}}{\partial t^2} - \frac{\lambda + 2\mu}{\rho} \cdot \Delta\widetilde{w} = 0$$

on $I \times \Omega$ satisfying the boundary condition (3.16)

$$\widetilde{w} = 0 \quad \text{on } I \times (D_0 \cup D_l)$$

mentioned in Theorem 3.3*. Assume that the boundary condition* (3.9)

$$\operatorname{div} \nabla\widetilde{w} = 0 \quad \text{on } D_0 \cup D_l$$

and the boundary condition (3.8)

$$\langle (\delta \mathbf{S})(\mathbf{n}_0), \mathbf{n}_0 \rangle = 0 \quad \textit{on } D_0 \cup D_l$$

are satisfied by the deformation $\nabla \widetilde{w}(t, \mathbf{x})$ *at any time* $t \in I$. *Then the function* $\widetilde{w}(t, \mathbf{x})$ *can be expressed as*

$$\widetilde{w}(t, \mathbf{x}) = \sum_{n=1}^{\infty} \Gamma_n(t, x_2, x_3) \cdot \sqrt{\frac{2}{l}} \cdot \sin\Big(\frac{n \cdot \pi \cdot x_1}{l}\Big) \tag{3.23}$$

in which the function

$$\Gamma_n(t, x_2, x_3) = \int_0^l \widetilde{w}(t, s, x_2, x_3) \cdot \sqrt{\frac{2}{l}} \cdot \sin\Big(\frac{n \cdot \pi \cdot s}{l}\Big) \cdot ds$$

satisfies the wave equation (3.20)

$$\frac{\partial^2 \Gamma_n}{\partial t^2} - \frac{\lambda + 2\mu}{\rho} \cdot \left(\frac{-n^2 \cdot \pi^2}{l^2} \cdot \Gamma_n + \frac{\partial^2 \Gamma_n}{\partial x_2^2} + \frac{\partial^2 \Gamma_n}{\partial x_3^2} \right) = 0.$$

Proof. Since $\operatorname{div} \nabla \widetilde{w} = 0$ on $D_0 \cup D_l$, it follows from (3.6) and (3.7) that

$$\langle (\delta \mathbf{S})(\mathbf{e}_j), \mathbf{e}_k \rangle = \lambda \cdot (\operatorname{div} \mathbf{v}) \cdot \delta_{jk} + 2\mu \cdot \epsilon_{jk} = 0 + 2\mu \cdot \frac{\partial^2 \widetilde{w}}{\partial x_j \partial x_k}$$

on $D_0 \cup D_l$ at any time $t \in I$. Since $\mathbf{n}_0 = \mp \mathbf{e}_1$ on $D_0 \cup D_l$, the condition $\langle (\delta \mathbf{S})(\mathbf{n}_0), \mathbf{n}_0 \rangle = 0$ on $D_0 \cup D_l$ can be expressed as

$$0 = \langle (\delta \mathbf{S})(\mathbf{n}_0), \mathbf{n}_0 \rangle = 2\mu \cdot \frac{\partial^2 \widetilde{w}}{\partial x_1^2} = 0 \quad \text{on } D_0 \cup D_l.$$

Thus we have

$$\frac{\partial^2 \widetilde{w}}{\partial x_1^2}(t, 0, x_2, x_3) = 0 = \frac{\partial^2 \widetilde{w}}{\partial x_1^2}(t, l, x_2, x_3)$$

at any time $t \in I$. It follows from the spectral theory that we have

$$\frac{\partial^2 \widetilde{w}}{\partial x_1^2}(t, \mathbf{x}) = \sum_{n=1}^{\infty} \gamma_n(t, x_2, x_3) \cdot \sqrt{\frac{2}{l}} \cdot \sin\Big(\frac{n \cdot \pi \cdot x_1}{l}\Big) \tag{3.24}$$

in which

$$\gamma_n(t, x_2, x_3) = \int_0^l \frac{\partial^2 \widetilde{w}}{\partial x_1^2}(t, s, x_2, x_3) \cdot \sqrt{\frac{2}{l}} \cdot \sin\Big(\frac{n \cdot \pi \cdot s}{l}\Big) \cdot ds$$

is the L^2 projection of $\frac{\partial^2 \widetilde{w}}{\partial x_1^2}$ on the eigenfunction

$$\sqrt{\frac{2}{l}} \cdot \sin\Big(\frac{n \cdot \pi \cdot s}{l}\Big)$$

for the Dirichlet problem of one-dimensional Laplace operator on $[0, l]$. It should be noted that the convergence of the Fourier series (3.24) is uniform. See Corollary 10.4 of [35].

Since $\widetilde{w} = 0$ on $D_0 \cup D_l$ at any time $t \in I$, we have

$$\widetilde{w}(t, \mathbf{x}) = \sum_{n=1}^{\infty} \Gamma_n(t, x_2, x_3) \cdot \sqrt{\frac{2}{l}} \cdot \sin\left(\frac{n \cdot \pi \cdot x_1}{l}\right)$$

in which

$$\Gamma_n(t, x_2, x_3) = \int_0^l \widetilde{w}(t, s, x_2, x_3) \cdot \sqrt{\frac{2}{l}} \cdot \sin\left(\frac{n \cdot \pi \cdot s}{l}\right) \cdot ds.$$

Note, by using integration by parts twice, that

$$\gamma_n = -\frac{n^2\pi^2}{l^2} \cdot \int_0^l \widetilde{w}(t, s, x_2, x_3) \cdot \sqrt{\frac{2}{l}} \cdot \sin\left(\frac{n\pi \cdot s}{l}\right) \cdot ds = -\frac{n^2\pi^2}{l^2} \cdot \Gamma_n. \tag{3.25}$$

Since $\widetilde{w} = 0$ on $D_0 \cup D_l$ at any time $t \in I$, we have

$$\frac{\partial^2 \widetilde{w}}{\partial x_2^2} = \frac{\partial^2 \widetilde{w}}{\partial x_3^2} = 0 = \frac{\partial^2 \widetilde{w}}{\partial t^2} \quad \text{on } D_0 \cup D_l$$

at any time $t \in I$. Thus we have

$$\begin{aligned} \frac{\partial^2 \widetilde{w}}{\partial x_2^2}(t, \mathbf{x}) &= \sum_{n=1}^{\infty} \sqrt{\frac{2}{l}} \sin\left(\frac{n\pi x_1}{l}\right) \cdot \int_0^l \frac{\partial^2 \widetilde{w}}{\partial x_2^2}(t, s, x_2, x_3) \cdot \sqrt{\frac{2}{l}} \sin\left(\frac{n\pi s}{l}\right) \cdot ds \\ &= \sum_{n=1}^{\infty} \sqrt{\frac{2}{l}} \cdot \sin\left(\frac{n \cdot \pi \cdot x_1}{l}\right) \cdot \frac{\partial^2 \Gamma_n}{\partial x_2^2}(t, x_2, x_3) \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} \frac{\partial^2 \widetilde{w}}{\partial x_3^2}(t, \mathbf{x}) &= \sum_{n=1}^{\infty} \sqrt{\frac{2}{l}} \sin\left(\frac{n\pi x_1}{l}\right) \cdot \int_0^l \frac{\partial^2 \widetilde{w}}{\partial x_3^2}(t, s, x_2, x_3) \cdot \sqrt{\frac{2}{l}} \sin\left(\frac{n\pi s}{l}\right) \cdot ds \\ &= \sum_{n=1}^{\infty} \sqrt{\frac{2}{l}} \cdot \sin\left(\frac{n \cdot \pi \cdot x_1}{l}\right) \cdot \frac{\partial^2 \Gamma_n}{\partial x_3^2}(t, x_2, x_3). \end{aligned} \tag{3.27}$$

Similarly we have

$$\begin{aligned} \frac{\partial^2 \widetilde{w}}{\partial t^2}(t, \mathbf{x}) &= \sum_{n=1}^{\infty} \sqrt{\frac{2}{l}} \sin\left(\frac{n\pi x_1}{l}\right) \cdot \int_0^l \frac{\partial^2 \widetilde{w}}{\partial t^2}(t, s, x_2, x_3) \cdot \sqrt{\frac{2}{l}} \sin\left(\frac{n\pi s}{l}\right) \cdot ds \\ &= \sum_{n=1}^{\infty} \sqrt{\frac{2}{l}} \cdot \sin\left(\frac{n \cdot \pi \cdot x_1}{l}\right) \cdot \frac{\partial^2 \Gamma_n}{\partial t^2}(t, x_2, x_3). \end{aligned} \tag{3.28}$$

It can be inferred readily from (3.25), (3.26), (3.27) and (3.28) that the wave equation (3.13) can be expressed as

$$\begin{aligned} 0 &= \frac{\partial^2 \widetilde{w}}{\partial t^2} - \frac{\lambda + 2\mu}{\rho} \cdot \Delta \widetilde{w} \\ &= \sum_{n=1}^{\infty} \sqrt{\frac{2}{l}} \sin\left(\frac{n\pi x_1}{l}\right) \cdot \left[\frac{\partial^2 \Gamma_n}{\partial t^2} - \frac{\lambda + 2\mu}{\rho} \left(\frac{-n^2\pi^2 \Gamma_n}{l^2} + \frac{\partial^2 \Gamma_n}{\partial x_2^2} + \frac{\partial^2 \Gamma_n}{\partial x_3^2}\right)\right]. \end{aligned}$$

In particular, the function $\Gamma_n(t, x_2, x_3)$ must satisfy the wave equation (3.20)

$$\frac{\partial^2 \Gamma_n}{\partial t^2} - \frac{\lambda + 2\mu}{\rho} \cdot \left(\frac{-n^2 \cdot \pi^2 \cdot \Gamma_n}{l^2} + \frac{\partial^2 \Gamma_n}{\partial x_2^2} + \frac{\partial^2 \Gamma_n}{\partial x_3^2} \right) = 0. \qquad \square$$

Remark 3.4. Theorem 3.5 shows us that the solutions of Class I may be classified by the "shear index" n. We say that a solution of Class I has shear index n if and only if this solution can be expressed as

$$\Gamma_n(t, x_2, x_3) \cdot \sqrt{\frac{2}{l}} \cdot \sin\left(\frac{n \cdot \pi \cdot x_1}{l} \right)$$

with the function $\Gamma_n(t, x_2, x_3)$ satisfying the wave equation (3.20).

Now we discuss "$*$-closed deformations". Unlike "$*$-exact deformations", simple and useful characterization of "$*$-closed deformations" does not seem possible. In fact, a cohomological characterization of "$*$-closed deformations" is possible but unlikely to be useful, because we are most concerned about the "vibration frequencies" of such deformations. Our next few propositions will be useful to help us to understand that "$*$-closed deformations" have diverse vibration frequencies.

Proposition 3.6. *Let I be an open interval for the time variable t. Assume that $\mathbf{u}(t, \mathbf{x}) = (u_1(t, \mathbf{x}), u_2(t, \mathbf{x}), u_3(t, \mathbf{x}))$ is a C^∞ vector-valued function defined on $I \times \Omega$ with* (3.14)

$$\operatorname{div} \mathbf{u} = 0$$

satisfying the wave equation (3.15)

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \frac{\mu}{\rho} \cdot \Delta \mathbf{u} = 0.$$

Then the boundary condition (3.8)

$$\langle (\delta \mathbf{S})(\mathbf{n}_0), \mathbf{n}_0 \rangle = 0 \quad \textit{on } D_0 \cup D_l$$

for the deformation $\mathbf{u}(t, \mathbf{x})$ is satisfied if and only if

$$\frac{\partial u_1}{\partial x_1}(t, 0, x_2, x_3) = 0 = \frac{\partial u_1}{\partial x_1}(t, l, x_2, x_3). \tag{3.29}$$

When the boundary condition (3.8) *is satisfied by $\mathbf{u}(t, \mathbf{x})$, the function $u_1(t, \mathbf{x})$ can be expressed as a uniformly convergent series*

$$u_1(t, x_1, x_2, x_3) = \widetilde{c}_0(t, x_2, x_3) + \sum_{n=1}^{\infty} \widetilde{c}_n(t, x_2, x_3) \cdot \cos\left(\frac{n \cdot \pi \cdot x_1}{l} \right). \tag{3.30}$$

Proof. Since $\operatorname{div}\mathbf{u} = 0$, it follows from (3.6) and (3.7) that

$$\langle(\delta\mathbf{S})(\mathbf{e}_j), \mathbf{e}_k\rangle = \lambda \cdot (\operatorname{div}\mathbf{u}) \cdot \delta_{jk} + 2\mu \cdot \epsilon_{jk} = 0 + \mu \cdot \left(\frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k}\right)$$

on $D_0 \cup D_l$ at any time $t \in I$. Since $\mathbf{n}_0 = \mp\mathbf{e}_1$ on $D_0 \cup D_l$, the boundary condition (3.8) $\langle(\delta\mathbf{S})(\mathbf{n}_0), \mathbf{n}_0\rangle = 0$ on $D_0 \cup D_l$ can be expressed as

$$0 = \langle(\delta\mathbf{S})(\mathbf{n}_0), \mathbf{n}_0\rangle = 2\mu \cdot \frac{\partial u_1}{\partial x_1} \quad \text{on } D_0 \cup D_l$$

at any time $t \in I$. Thus the boundary condition (3.8) for the deformation $\mathbf{u}(t, \mathbf{x})$ is satisfied if and only if (3.29)

$$\frac{\partial u_1}{\partial x_1}(t, 0, x_2, x_3) = 0 = \frac{\partial u_1}{\partial x_1}(t, l, x_2, x_3)$$

is true. When the boundary condition (3.8) for $\mathbf{u}(t, \mathbf{x})$ is satisfied, it follows from the spectral theory that we have

$$\frac{\partial u_1}{\partial x_1}(t, \mathbf{x}) = \sum_{n=1}^{\infty} c_n(t, x_2, x_3) \cdot \sqrt{\frac{2}{l}} \cdot \sin\left(\frac{n \cdot \pi \cdot x_1}{l}\right) \tag{3.31}$$

in which

$$c_n(t, x_2, x_3) = \int_0^l \frac{\partial u_1}{\partial x_1}(t, s, x_2, x_3) \cdot \sqrt{\frac{2}{l}} \cdot \sin\left(\frac{n \cdot \pi \cdot s}{l}\right) \cdot ds.$$

Note that the Hölder continuity of $\frac{\partial u_1}{\partial x_1}(t, \mathbf{x})$ in the variable x_1 suffices to ensure that the series (3.31) converges uniformly. See Corollary 10.4 of [35]. Thus we have

$$\begin{aligned} u_1(t, x_1, x_2, x_3) &= u_1(t, 0, x_2, x_3) + \int_0^{x_1} \frac{\partial u_1}{\partial x_1}(t, s, x_2, x_3) \cdot ds \\ &= u_1(t, 0, x_2, x_3) + \sum_{n=1}^{\infty} c_n(t, x_2, x_3) \cdot \sqrt{\frac{2}{l}} \cdot \int_0^{x_1} \sin\left(\frac{n \cdot \pi \cdot s}{l}\right) \cdot ds \\ &= \widetilde{c}_0(t, x_2, x_3) + \sum_{n=1}^{\infty} \widetilde{c}_n(t, x_2, x_3) \cdot \cos\left(\frac{n \cdot \pi \cdot x_1}{l}\right) \end{aligned}$$

in which

$$\widetilde{c}_0(t, x_2, x_3) = u_1(t, 0, x_2, x_3) + \sum_{n=1}^{\infty} c_n(t, x_2, x_3) \cdot \sqrt{\frac{2}{l}} \cdot \frac{l}{n \cdot \pi}$$

and

$$\widetilde{c}_n(t, x_2, x_3) = -\sqrt{\frac{2}{l}} \cdot \frac{l}{n \cdot \pi} \cdot c_n(t, x_2, x_3)$$

for each positive integer n. This completes the proof of (3.30). $\square$

Remark 3.5. In Theorem 3.5, we know that the functions $\Gamma_n(t, x_2, x_3)$ must satisfy specific wave equations. Unlike Theorem 3.5, Proposition 3.6 does not tell us what the functions $\widetilde{c}_n(t, x_2, x_3)$ should be.

Our next two propositions will show us how elementary "$*$-closed deformations" can be constructed. These elementary "$*$-closed deformations", constructed in Propositions 3.7 and 3.8, may help us understand the vibration frequencies of rubber wiper blade on convex windshield.

Proposition 3.7. *Let I be an open interval for the time variable t. Assume that $f(t, x_2, x_3)$ is a C^∞ function defined on $I \times D$ satisfying the following wave equation*

$$\frac{\partial^2 f}{\partial t^2} - \frac{\mu}{\rho} \cdot \left(\frac{-n^2 \cdot \pi^2}{l^2} \cdot f + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} \right) = 0$$

in which n is a positive integer. Let $\mathbf{u}(t, \mathbf{x}) = (u_1(t, \mathbf{x}), u_2(t, \mathbf{x}), u_3(t, \mathbf{x}))$ denote the C^∞ vector-valued function defined on $I \times \Omega$ as follows:

$$u_1(t, \mathbf{x}) = \frac{l}{n \cdot \pi} \cdot \cos\left(\frac{n \cdot \pi \cdot x_1}{l}\right) \cdot \left(\frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}\right)$$

with

$$u_2(t, \mathbf{x}) = \sin\left(\frac{n \cdot \pi \cdot x_1}{l}\right) \cdot \frac{\partial f}{\partial x_2} \quad \textit{and} \quad u_3(t, \mathbf{x}) = \sin\left(\frac{n \cdot \pi \cdot x_1}{l}\right) \cdot \frac{\partial f}{\partial x_3}.$$

Then this deformation $\mathbf{u}(t, \mathbf{x})$ satisfies (3.14)

$$\operatorname{div} \mathbf{u} = 0$$

and the wave equation (3.15)

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \frac{\mu}{\rho} \cdot \Delta \mathbf{u} = 0.$$

Moreover, it satisfies the boundary condition (3.8)

$$\langle (\delta \mathbf{S})(\mathbf{n}_0), \mathbf{n}_0 \rangle = 0 \quad \textit{on } D_0 \cup D_l$$

because

$$\frac{\partial u_1}{\partial x_1}(t, 0, x_2, x_3) = 0 = \frac{\partial u_1}{\partial x_1}(t, l, x_2, x_3).$$

Proof. This proposition can be checked directly. □

Proposition 3.8. *Let I be an open interval for the time variable t. Assume that $g(t, \mathbf{x})$ is a C^∞ function on $I \times \Omega$ satisfying the wave equation*

$$\frac{\partial^2 g}{\partial t^2} - \frac{\mu}{\rho} \cdot \Delta g = 0.$$

Then the deformation $\mathbf{u}(t, \mathbf{x}) = (u_1(t, \mathbf{x}), u_2(t, \mathbf{x}), u_3(t, \mathbf{x}))$ *defined by*

$$\mathbf{u}(t, \mathbf{x}) = \left(0, -\frac{\partial g}{\partial x_3}(t, \mathbf{x}), \frac{\partial g}{\partial x_2}(t, \mathbf{x})\right)$$

satisfies (3.14)

$$\operatorname{div} \mathbf{u} = 0$$

and the wave equation (3.15)

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} - \frac{\mu}{\rho} \cdot \Delta \mathbf{u} = 0.$$

Moreover, it satisfies the boundary condition (3.8)

$$\langle (\delta \mathbf{S})(\mathbf{n}_0), \mathbf{n}_0 \rangle = 0 \quad \text{on } D_0 \cup D_l$$

because $u_1(t, \mathbf{x}) = 0$.

Proof. This proposition can be checked directly. □

4. Physical explanation

The pressure on the rubber wiper blade comes from the Wiper-Arm. This force only acts on some sites of the rubber wiper blade. To reduce the "wind resistance" on the windshield of a moving vehicle, the shape of windshield is generally *strictly convex* [7]. Since the windshield is strictly convex with (usually small) *positive* sectional curvature, the ends of rubber wiper blade usually do *not* touch the convex windshield. See Figure 4.1.

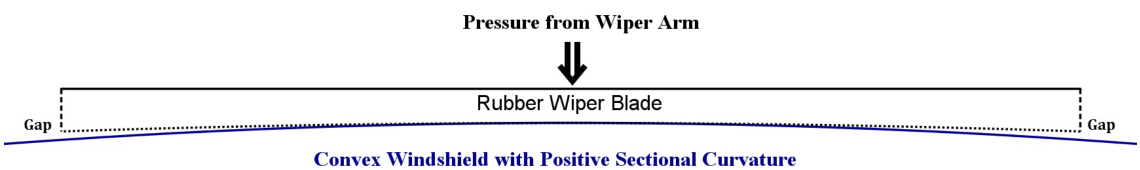

Figure 4.1: Rubber wiper blade on strictly convex windshield.

Thus we may deduce from the Saint-Venant principle, mentioned in Section 2, that "the infinitesimal variation of volume", associated with the infinitesimal deformation $\mathbf{v}$, around the *ends* of rubber wiper blade is zero (3.9):

$$\operatorname{div} \mathbf{v} = 0 \quad \text{on } D_0 \cup D_l.$$

More explanation on the mechanics of rubber wiper blade on *convex* windshield can be found in [11].

Now we explain why Theorem 3.5 (characterization of $*$-exact deformations) leads to the prediction that most of the vibration frequencies of Class I ($*$-exact deformations) should locate around (1.1):

$$\sqrt{\frac{\lambda+2\mu}{\rho}} \cdot \frac{n}{2l} \text{ Hz} \quad \text{(Class I)}$$

where n is a positive integer. Theorem 3.5 tells us that each general solution $\widetilde{w}(t,\mathbf{x})$ of Class I can be expressed as (3.23):

$$\widetilde{w}(t,\mathbf{x}) = \sum_{n=1}^{\infty} \Gamma_n(t,x_2,x_3) \cdot \sqrt{\frac{2}{l}} \cdot \sin\left(\frac{n\cdot\pi\cdot x_1}{l}\right)$$

in which the function $\Gamma_n(t,x_2,x_3)$ satisfies the wave equation (3.20)

$$\frac{\partial^2\Gamma_n}{\partial t^2} - \frac{\lambda+2\mu}{\rho} \cdot \left(\frac{-n^2\cdot\pi^2}{l^2} \cdot \Gamma_n + \frac{\partial^2\Gamma_n}{\partial x_2^2} + \frac{\partial^2\Gamma_n}{\partial x_3^2}\right) = 0.$$

When a rubber wiper blade moves on the windshield of an automobile, the kinetic frictional force, acting on the lower part of rubber wiper blade, causes the rubber wiper blade to vibrate. From the physical point of view, it is reasonable to consider eigenfunctions $\Gamma_n(t,x_2,x_3) = e^{i\cdot k\cdot t} \cdot \gamma_n(x_2,x_3)$ satisfying

$$\frac{\partial^2\Gamma_n}{\partial x_2^2} + \frac{\partial^2\Gamma_n}{\partial x_3^2} = -\epsilon_n \cdot \Gamma_n \quad \text{or} \quad \frac{\partial^2\gamma_n}{\partial x_2^2} + \frac{\partial^2\gamma_n}{\partial x_3^2} = -\epsilon_n \cdot \gamma_n. \tag{4.1}$$

Substituting (4.1) into the wave equation (3.20), we infer that

$$-k^2 - \frac{\lambda+2\mu}{\rho} \cdot \left[\frac{-n^2\cdot\pi^2}{l^2} - \epsilon_n\right] = 0 \quad \text{or} \quad k = \pm\sqrt{\frac{\lambda+2\mu}{\rho}} \cdot \sqrt{\frac{n^2\cdot\pi^2}{l^2} + \epsilon_n}.$$

Usually ϵ_n lies in a limited range. Generating $\Gamma_n(t,x_2,x_3) = e^{i\cdot k\cdot t} \cdot \gamma_n(x_2,x_3)$ with large ϵ_n requires high energy. Thus we expect, by assuming ϵ_n relatively small, that the vibration frequencies of Class I ($*$-exact deformations) should locate around (1.1):

$$\frac{|k|}{2\pi} \approx \sqrt{\frac{\lambda+2\mu}{\rho}} \cdot \frac{n}{2l} \text{ Hz} \quad \text{(Class I)}$$

where n is a positive integer.

Similarly we may deduce from Proposition 3.7 that some vibration frequencies of Class II ($*$-closed deformations) should appear around (1.2):

$$\sqrt{\frac{\mu}{\rho}} \cdot \frac{n}{2l} \text{ Hz} \quad \text{(Class II)}.$$

On the other hand, we may infer from Proposition 3.8 that some other vibration frequencies of Class II ($*$-closed deformations) may appear at

$$\sqrt{\frac{\mu}{\rho}} \cdot \frac{\sqrt{\epsilon}}{2\pi} \text{ Hz} \quad \text{(Class II)}$$

randomly.

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