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Spanning Trees with Few Peripheral Branch Vertices

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Abstract. Let T be a tree, a vertex of degree one is a *leaf* of T and a vertex of degree at least three is a *branch vertex* of T. The set of leaves of T is denoted by $L(T)$ and the set of branch vertices of T is denoted by $B(T)$. For two distinct vertices u, v of T, let $P_T[u,v]$ denote the unique path in T connecting u and v. Let T be a tree with $B(T) \neq \emptyset$, for each vertex $x \in L(T)$, set $y_x \in B(T)$ such that $(V(P_T[x,y_x]) \setminus \{y_x\}) \cap B(T) = \emptyset$. We delete $V(P_T[x,y_x]) \setminus \{y_x\}$ from T for all $x \in L(T)$. The resulting graph is a subtree of T and is denoted by $\text{R\_Stem}(T)$. It is called the *reducible stem* of T. A leaf of $\text{R\_Stem}(T)$ is called a *peripheral branch vertex* of T. In this paper, we give some sharp sufficient conditions on the independence number and the degree sum for a graph G to have a spanning tree with few peripheral branch vertices.

1. Introduction

In this paper, we only consider finite simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex $v \in V(G)$, we use $N_G(v)$ and $\deg_G(v)$ (or $N(v)$ and $\deg(v)$ if there is no ambiguity) to denote the set of neighbors of v and the degree of v in G, respectively. For any $X \subseteq V(G)$, we denote by $|X|$ the cardinality of X. Sometimes, we use $|G|$ (and G) to denote $|V(G)|$ (and $V(G)$ respectively). We define $N_G(X) = \bigcup_{x \in X} N_G(x)$ and $\deg_G(X) = \sum_{x \in X} \deg_G(x)$. We use $G - X$ to denote the graph obtained from G by deleting the vertices in X together with their incident edges. We define $G - uv$ to be the graph obtained from G by deleting the edge $uv \in E(G)$, and $G + uv$ to be the graph obtained from G by adding an edge uv between two non-adjacent vertices u and v of G. For two vertices u and v of G, the distance between u and v in G is denoted by $d_G(u,v)$. We use K_n to denote the complete graph on n vertices. We write $A := B$ to rename B as A.

For an integer $m \geq 2$, let $\alpha^m(G)$ denote the number defined by

$$\alpha^m(G) = \max\{|S| : S \subseteq V(G), d_G(x,y) \geq m \text{ for all distinct vertices } x, y \in S\}.$$

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For an integer $p \geq 2$, we define

$$\sigma_p^m(G) = \min\{\deg_G(S) : S \subseteq V(G), |S| = p, d_G(x,y) \geq m \text{ for all distinct vertices } x, y \in S\}.$$

For convenience, we define $\sigma_p^m(G) = +\infty$ if $\alpha^m(G) < p$. We note that, $\alpha^2(G)$ is often written $\alpha(G)$, which is the independence number of G, and $\sigma_p^2(G)$ is often written $\sigma_p(G)$, which is the minimum degree sum of p independent vertices.

Let T be a tree. A vertex of degree one is a *leaf* of T and a vertex of degree at least three is a *branch vertex* of T. There are several well-known conditions (such as independence number conditions and degree sum conditions) ensuring that a graph G contains a spanning tree with a bounded number of leaves or branch vertices (see [1, 12, 14, 16]). Win [16] obtained a sufficient condition related to the independence number for l-connected graphs, which confirms a conjecture of Las Vergnas [11]. Broersma and Tuinstra [1] gave a degree sum condition for a connected graph to contain a spanning tree with at most k leaves.

Theorem 1.1. (see Win [16]) *Let $l \geq 1$ and $k \geq 2$ be integers and let G be an l-connected graph. If $\alpha(G) \leq k + l - 1$, then G has a spanning tree with at most k leaves.*

Theorem 1.2. (see Broerma and Tuinstra [1]) *Let G be a connected graph and let $k \geq 2$ be an integer. If $\sigma_2(G) \geq |G| - k + 1$, then G has a spanning tree with at most k leaves.*

The set of leaves of T is denoted by $L(T)$ and the set of branch vertices of T is denoted by $B(T)$. The subtree $T - L(T)$ of T is called the *stem* of T and is denoted by $\text{Stem}(T)$. Then, many researchers studied spanning trees in connected graphs whose stems have a bounded number of leaves or branch vertices (see [7, 8, 15, 17] for more details). We introduce here some results on spanning trees whose stems have a few leaves or branch vertices.

Theorem 1.3. (see Tsugaki and Zhang [15]) *Let G be a connected graph and let $k \geq 2$ be an integer. If $\sigma_3(G) \geq |G| - 2k + 1$, then G has a spanning tree whose stem has at most k leaves.*

Theorem 1.4. (see Kano and Yan [7]) *Let G be a connected graph and let $k \geq 2$ be an integer. If either $\alpha^4(G) \leq k$ or $\sigma_{k+1}(G) \geq |G| - k - 1$, then G has a spanning tree whose stem has at most k leaves.*

Theorem 1.5. (see Kano and Yan [8]) *Let G be a connected graph. If $\sigma_4^4(G) \geq |G| - 5$, then G has a spanning tree whose stem is a spider.*

Theorem 1.6. (see Yan [17]) *Let G be a connected graph and k be a non-negative integer. If one of the following conditions holds, then G has a spanning tree whose stem has at most k branch vertices.*

(a) $\alpha^4(G) \leq k + 2$,

(b) $\sigma^4_{k+3}(G) \geq |G| - 2k - 3$.

On the other hand, for a positive integer $t \geq 3$, a graph G is said to be a $K_{1,t}$-free graph if it contains no $K_{1,t}$ as an induced subgraph. If $t = 3$, a $K_{1,3}$-free graph is also called a claw-free graph. Many independence number conditions and degree sum conditions ensuring that a $K_{1,t}$-free graph G contains a spanning tree which (or whose stem) has a bounded number of leaves or branch vertices have been derived (see [2, 3, 5, 6, 9, 10, 13]).

In this paper, we would like to introduce a new concept on spanning tree problem. For two distinct vertices u and v of T, let $P_T[u, v]$ denote the unique path in T connecting u and v. Let T be a tree with $B(T) \neq \emptyset$. For every $x \in L(T)$, set $y_x \in B(T)$ such that $(V(P_T[x, y_x]) \setminus \{y_x\}) \cap B(T) = \emptyset$. We delete $V(P_T[x, y_x]) \setminus \{y_x\}$ from T for all $x \in L(T)$. The resulting graph is denoted by $\mathrm{R\_Stem}(T)$. It is called the *reducible stem* of T. The path that connects x to y_x but does not contain y_x, is called a *leaf-branch path of T incident to x* and denoted by B_x. Let $B = \bigcup_{x \in L(T)} V(B_x)$, then $\mathrm{R\_Stem}(T) = T - B$ (see Figure 1.1 for an example of T and $\mathrm{R\_Stem}(T)$).

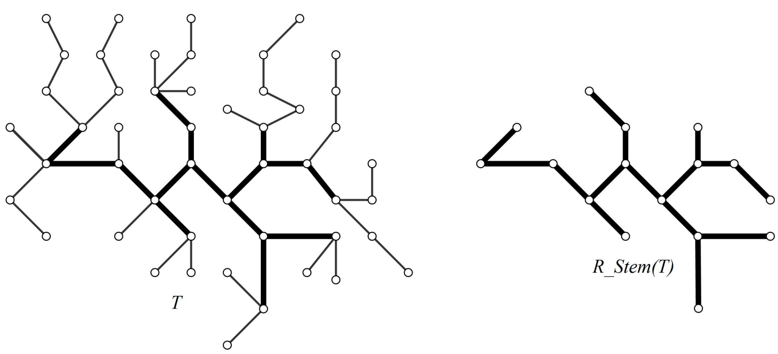

Figure 1.1: Tree T and $\mathrm{R\_Stem}(T)$.

A leaf of $\mathrm{R\_Stem}(T)$ is also called a peripheral branch vertex of T (see [12]). We denote by $P(B(T))$ the peripheral branch vertex set of T. Then $P(B(T)) = L(\mathrm{R\_Stem}(T))$.

We would like to study sufficient conditions for a graph to have a spanning tree T with few peripheral branch vertices, i.e., $\mathrm{R\_Stem}(T)$ has a few leaves. In particular, we state the following theorem.

Theorem 1.7. *Let G be a connected graph and let $k \geq 2$ be an integer. If one of the following conditions holds, then G has a spanning tree with at most k peripheral branch vertices.*

(i) $\alpha(G) \leq 2k+2$,

(ii) $\sigma^4_{k+1}(G) \geq \lfloor \frac{|G|-k}{2} \rfloor$.

Here, the notation $\lfloor r \rfloor$ stands for the biggest integer that does not exceed the real number r.

To end this section, we give an example to show that our main results are sharp. Let $k \geq 2$ and $m \geq 1$ be integers, and let $D_1, D_2, \ldots, D_{k+1}$ and $H_1, H_2, \ldots, H_{k+1}$ be $2k+2$ disjoint copies of the complete graph K_m of order m. Let $w, x_1, x_2, \ldots, x_{k+1}$ be $k+2$ vertices not contained in $V(D_1) \cup V(D_2) \cup \cdots \cup V(D_{k+1}) \cup V(H_1) \cup V(H_2) \cup \cdots \cup V(H_{k+1})$. Join w to all vertices of $\{x_1, x_2, \ldots, x_{k+1}\}$ and join x_i to all the vertices in $V(D_i) \cup V(H_i)$ for every $1 \leq i \leq k+1$. Let G denote the resulting graph (see Figure 1.2). Then $\alpha(G) = 2k+3$.

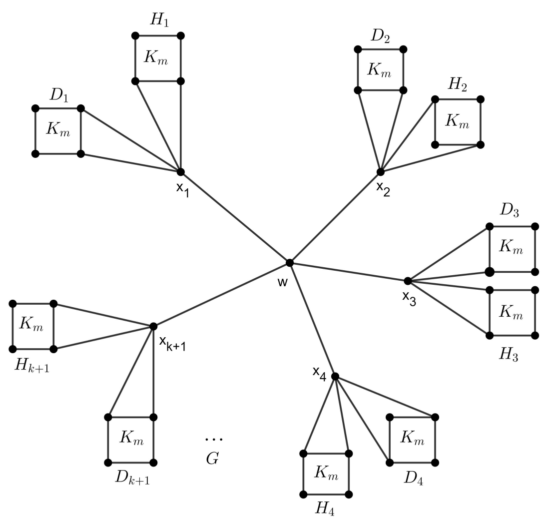

Figure 1.2: Graph G.

Moreover, let S be a subset of $V(G)$ such that $|S| = k+1$ and $d_G(x,y) \geq 4$ for all distinct vertices $x, y \in S$, then $S \cap (V(D_i) \cup V(H_i)) \neq \emptyset$ for every $1 \leq i \leq k+1$. Therefore, for every $1 \leq i \leq k+1$, take $y_i \in V(D_i) \cup V(H_i)$. We then obtain

$$\sigma^4_{k+1}(G) = \sum_{i=1}^{k+1} \deg_G(y_i) = (k+1)m = \left\lfloor \frac{|G|-k}{2} \right\rfloor - 1.$$

But G has no spanning tree with at most k peripheral branch vertices. Then, our main results are sharp.

Since $\sigma_{k+1}(G) \le \sigma^4_{k+1}(G)$, we have a corollary of Theorem 1.7 as follows.

Corollary 1.8. *Let G be a connected graph and let $k \ge 2$ be an integer. If $\sigma_{k+1}(G) \ge \lfloor \frac{|G|-k}{2} \rfloor$, then G has a spanning tree with at most k peripheral branch vertices.*

We also note that in the above example, if $m \le k+1$ then $\sigma_{k+1}(G) = \sigma^4_{k+1}(G) = \lfloor \frac{|G|-k}{2} \rfloor - 1$. So, the condition $\sigma_{k+1}(G) \ge \lfloor \frac{|G|-k}{2} \rfloor$ of Corollary 1.8 is tight.

2. Proof of the main result

Let T be a tree. For two distinct vertices u and v of T, we always define the *orientation* of $P_T[u,v]$ to be from u to v. If $v \in V(P)$, then v^+ and v^- denote the successor and predecessor of v on P if they exist, respectively. For any $X \subseteq V(G)$, set $(N(X) \cap P_T[u,v])^- = \{x^- \mid x \in V(P_T[u,v]) \setminus \{u\}$ and $x \in N(X)\}$ and $(N(X) \cap P_T[u,v])^+ = \{x^+ \mid x \in V(P_T[u,v]) \setminus \{v\}$ and $x \in N(X)\}$. For an integer $t \ge 1$, we let $N_t(X) = \{x \in V(G) \mid |N(x) \cap X| = t\}$. We refer to [4] for terminology and notation not defined here.

Proof of Theorem 1.7. Suppose, to the contrary, each spanning tree of G contains at least $k+1$ peripheral branch vertices. Let $\mathcal{T} = \{T : T$ is a subgraph of G and T is a tree$\}$, and let $\mathcal{T}_{k+1} = \{T : T \in \mathcal{T}$ and $|P(B(T))| = k+1\}$. Choose a maximal tree T in $\mathcal{T}_{k+1}$ (a tree T in $\mathcal{T}_{k+1}$ such that $|V(T)|$ is maximum) which satisfies the following two conditions:

(C1) $|\,\mathrm{R\_Stem}(T)|$ is as small as possible,

(C2) $|L(T)|$ is as small as possible subject to (C1).

Claim 2.1. There does not exist a tree S in G such that $V(S) = V(T)$ and $|P(B(S))| \le k$.

Proof. Suppose, to the contrary, there exists a tree S in G such that $V(S) = V(T)$ and $|P(B(S))| \le k$. Since $|P(B(S))| \le k$, S is not a spanning tree of G. Then there exists $u \in V(G) - V(S)$ such that u is adjacent to a vertex $v \in S$. Let S_1 be a tree obtained from S by adding the edge uv. Then S_1 is a tree in G such that $|V(S_1)| = |V(T)| + 1$ and $|P(B(S_1))| \le k+1$.

If $|P(B(S_1))| = k+1$, then S_1 contradicts the maximality of T (since $|V(S_1)| = |V(S)| + 1 = |V(T)| + 1 > |V(T)|$). So we may assume that $|P(B(S_1))| \le k$. By repeating this process, we can recursively construct a set of trees $\{S_i \mid i \ge 1\}$ in G such that S_i satisfies that $|P(B(S_i))| \le k$ and $|V(S_{i+1})| = |V(S_i)| + 1$ for each $i \ge 1$. Since G has no spanning tree with at most k peripheral branch vertices and $|V(G)|$ is finite, the process must terminate after a finite number of steps, i.e., there exists some $h \ge 1$ such that S_{h+1} is a tree in G with $|P(B(S_{h+1}))| = k+1$. But this contradicts the maximality of T. So the claim holds. □

Set $P(B(T)) = \{x_1, x_2, \dots, x_{k+1}\}$. By the definition of peripheral branch vertex, we have the following claim.

Claim 2.2. For every $i \in \{1, 2, \dots, k+1\}$, there exist at least two leaf-branch paths of T which are incident to x_i.

Now we will prove the following two claims to show that $\alpha(G) \geq 2k+3$.

Claim 2.3. For each $i \in \{1, 2, \dots, k+1\}$, there exist $y_i, z_i \in L(T)$ such that B_{y_i}, B_{z_i} are incident to x_i and $N_G(y_i) \cap (V(\text{R\_Stem}(T)) - \{x_i\}) = \emptyset$ and $N_G(z_i) \cap (V(\text{R\_Stem}(T)) - \{x_i\}) = \emptyset$.

Proof. Let $\{a_{ij}\}_{j=1}^m$ be the subset of $L(T)$ such that $B_{a_{ij}}$ is incident to x_i. By Claim 2.2, we obtain $m \geq 2$.

Suppose that there are more than $m-2$ vertices in $\{a_{ij}\}_{j=1}^m$ satisfying

$$N_G(a_{ij}) \cap (V(\text{R\_Stem}(T)) - \{x_i\}) \neq \emptyset.$$

Without loss of generality, we may assume that $N_G(a_{ij}) \cap (V(\text{R\_Stem}(T)) - \{x_i\}) \neq \emptyset$ for all $j = 2, \dots, m$. Set $b_{ij} \in N_G(a_{ij}) \cap (V(\text{R\_Stem}(T)) - \{x_i\})$ and $v_{ij} \in N_T(x_i) \cap V(P_T[a_{ij}, x_i])$ for all $j \in \{2, \dots, m\}$. Consider the tree

$$T' := T + \{a_{ij}b_{ij}\}_{j=2}^m - \{x_i v_{ij}\}_{j=2}^m.$$

Then T' satisfies $|V(T')| = |V(T)|$, $|P(B(T'))| \leq |P(B(T))|$ and $|\text{R\_Stem}(T')| < |\text{R\_Stem}(T)|$, where x_i is not in $V(\text{R\_Stem}(T'))$. This contradicts either Claim 2.1 or Condition (C1). Therefore, Claim 2.3 holds. □

Set $U = \{y_i, z_i\}_{i=1}^{k+1}$. By the maximality of T we have $N_G(U) \subseteq V(T)$.

Claim 2.4. U is an independent set in G.

Proof. Suppose that there exist two vertices $u, v \in U$ such that $uv \in E(G)$. Without loss of generality, we may assume that $v = y_i$ for some $i \in \{1, 2, \dots, k+1\}$. Set $v_i \in N_T(x_i) \cap V(B_{y_i})$. Consider the tree $T' := T + uy_i - v_i x_i$. Then $V(T') = V(T)$ and $|P(B(T'))| \leq |P(B(T))|$. If $\deg_T(x_i) = 3$ then x_i is not a branch vertex of T'. Hence $|\text{R\_Stem}(T')| < |\text{R\_Stem}(T)|$, this contradicts either Claim 2.1 or Condition (C1). Otherwise, we have $|P(B(T'))| = |P(B(T))|$, $|\text{R\_Stem}(T')| = |\text{R\_Stem}(T)|$ and $|L(T')| < |L(T)|$, where either T' has only one new leaf and y_i, u are not leaves of T' or y_i is still a leaf of T' but T' has no new leaf and u is not a leaf of T'. This contradicts Condition (C2). The proof of Claim 2.4 is completed. □

Since $k \geq 2$, then $|L(\text{R\_Stem}(T))| = |P(B(T))| \geq 3$. Hence, we have $|B(\text{R\_Stem}(T))| \geq 1$. Let u be a vertex in $B(\text{R\_Stem}(T))$. By Claims 2.3 and 2.4, we conclude that $U \cup \{u\}$

is an independent set in G. This implies that $\alpha(G) \geq 2k+3$. As either $\alpha(G) \leq 2k+2$, or $\sigma_{k+1}^4(G) \geq \lfloor \frac{|G|-k}{2} \rfloor$, we conclude that $\sigma_{k+1}^4(G) \geq \lfloor \frac{|G|-k}{2} \rfloor$.

Claim 2.5. For every $i, j \in \{1, 2, \ldots, k+1\}$ where $i \neq j$, $N_G(y_i) \cap V(B_{y_j}) = \emptyset$ and $N_G(y_i) \cap V(B_{z_j}) = \emptyset$.

Proof. By the same role of y_j and z_j, we only need to prove $N_G(y_i) \cap V(B_{y_j}) = \emptyset$. Suppose the assertion of the claim is false. Then there exists a vertex $x \in N_G(y_i) \cap V(B_{y_j})$. Set $T' := T + xy_i$. Then T' is a subgraph of G including a unique cycle C, which contains both x_i and x_j.

Since $k \geq 2$, then $|L(\text{R\_Stem}(T))| = |P(B(T))| \geq 3$. Hence, we obtain $|B(\text{R\_Stem}(T))| \geq 1$. Then there exists a branch vertex of R_Stem(T) contained in C. Let e be an edge incident to such a vertex in C and R_Stem(T). By removing the edge e from T' we obtain a tree T'' of G satisfying $V(T'') = V(T)$ and $|P(B(T''))| \leq k$, the reason is that either R_Stem(T'') has only one new leaf and x_i, x_j are not leaves of R_Stem(T'') or x_i (or x_j) is still a leaf of R_Stem(T'') but R_Stem(T'') has no new leaf and x_j (or x_i respectively) is not a leaf of R_Stem(T''). This is a contradiction with Claim 2.1. So Claim 2.5 is proved. □

Claim 2.6. For every $1 \leq i < j \leq k+1$, $d_G(y_i, y_j) \geq 4$ and $d_G(z_i, z_j) \geq 4$.

Proof. We first prove that $d_G(y_i, y_j) \geq 4$. Let $P[y_i, y_j]$ be a shortest path connecting y_i and y_j in G. Assume that all vertices of $P[y_i, y_j]$ are contained in $(V(G) - V(\text{R\_Stem}(T))) \cup \{x_i, x_j\}$.

Let $t_i \in B_{y_i} \cup \{x_i\}$, $t_j \in B_{y_j} \cup \{x_j\}$ such that $t_i, t_j \in P[y_i, y_j]$ and

$$P_{P[y_i,y_j]}[t_i, t_j] \cap B_{y_i} = \{t_i\}, \quad P_{P[y_i,y_j]}[t_i, t_j] \cap B_{y_j} = \{t_j\}.$$

Set $P[t_i, t_j] := P_{P[y_i,y_j]}[t_i, t_j]$. For every vertex $p \in L(T)$ such that $B_p \cap P[t_i, t_j] \neq \emptyset$. Let $v_p \in B(T)$ such that $(V(P_T[p, v_p]) \setminus \{v_p\}) \cap B(T) = \emptyset$. Let $v_p^- \in V(B_p) \cap N_T(v_p)$. Remove all the edges $v_p v_p^-$ of T and add $P[t_i, t_j]$. Then the resulting subgraph T' of G includes a unique cycle C, which contains the vertices x_i and x_j. Since $k \geq 2$, then $|L(\text{R\_Stem}(T))| = |P(B(T))| \geq 3$. Hence, we obtain $|B(\text{R\_Stem}(T))| \geq 1$. Then, there exists a branch vertex u of R_Stem(T) contained in C. Let e be an edge in C which is incident to u. Denote by T'' the tree obtained from T' by removing the edge e (see Figure 2.1). Then $V(T) \subseteq V(T') = V(T'')$ and $|P(B(T''))| \leq k$, where either R_Stem(T'') has only one new leaf and x_i, x_j are not leaves of R_Stem(T'') or x_i (or x_j) is still a leaf of R_Stem(T'') but R_Stem(T'') has no new leaf and x_j (or x_i respectively) is not a leaf of R_Stem(T''). This contradicts either the maximality of T or Claim 2.1. Therefore,

$P[y_i, y_j] \cap (\text{R\_Stem}(T) - \{x_i, x_j\}) \neq \emptyset$. Set $v \in P[y_i, y_j] \cap (\text{R\_Stem}(T) - \{x_i, x_j\})$. Hence, by combining with Claim 2.3, we obtain

$$d_G(y_i, y_j) = d_{P[y_i,y_j]}(y_i, y_j) \geq d_{P[y_i,y_j]}(y_i, v) + d_{P[y_i,y_j]}(v, y_j) \geq 2 + 2 = 4.$$

Now, using the same arguments, we also obtain that $d_G(z_i, z_j) \geq 4$. This completes the

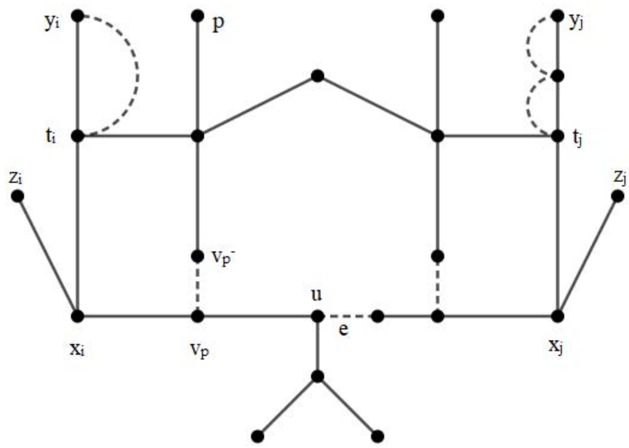

Figure 2.1: Tree T''.

proof of Claim 2.6. □

Claim 2.7. If $p \in L(T) - U$, then $\sum_{u \in U} |N_G(u) \cap B_p| \leq |B_p| - 1$.

Proof. Set $v_p \in B(T)$ such that $(V(P_T[p, v_p]) \setminus \{v_p\}) \cap B(T) = \emptyset$. Let $V(B_p) \cap N_T(v_p) = \{v_p^-\}$. Then we consider $B_p = P_T[p, v_p^-]$.

Subclaim 2.7.1. For every $i \in \{1, 2, \dots, k+1\}$, if $x \in N_G(y_i) \cap B_p$ then $x^- \notin N_G(U - \{y_i\}) \cap B_p$.

Suppose that there exists $x^- \in N_G(z) \cap B_p$ with $z \in U - \{y_i\}$. Let $T' := T + \{xy_i, x^- z\} - \{xx^-, v_p v_p^-\}$. Then T' is a tree in G satisfying $V(T') = V(T)$, $|P(B(T'))| = |P(B(T))|$, $|\text{R\_Stem}(T')| = |\text{R\_Stem}(T)|$ and $|L(T')| < |L(T)|$, where y_i, z are not leaves of T' (see Figure 2.2). Hence this contradicts Condition (C2).

Subclaim 2.7.2. If $x \in B_p$, then x is adjacent to at most 2 vertices in U.

Indeed, we can prove a stronger statement that if $x \in N_G(y_i) \cap B_p$ then $x \notin N_G(y_j) \cap B_p$ and $x \notin N_G(z_j) \cap B_p$ for all $1 \leq i, j \leq k+1$, $i \neq j$. Suppose, to the contrary, there exist i and j, with $1 \leq i, j \leq k+1$, $i \neq j$, such that $x \in N_G(y_i) \cap B_p$ and $x \in N_G(w)$, where $w = y_j$ or $w = z_j$. Without loss of generality, we assume that $w = y_j$. Set $T' := T + \{xy_i, xy_j\} - \{v_p v_p^-\}$. Then T' is a subgraph of G that includes a unique cycle C, which contains two vertices x_i and x_j. Since $k \geq 2$, then $|L(\text{R\_Stem}(T))| = |P(B(T))| \geq 3$.

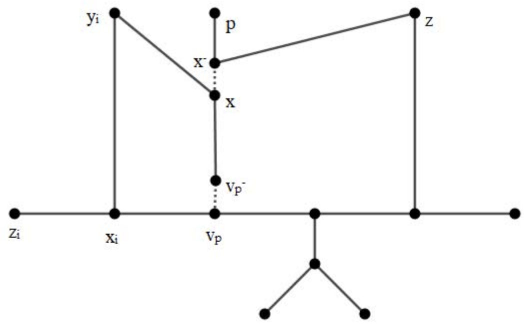

Figure 2.2: Tree T'.

Hence, we obtain $|B(\text{R\_Stem}(T))| \geq 1$. Then, there exists a branch vertex of R_Stem(T) contained in C. Let e be an edge which is incident to such a vertex in C. By removing the edge e we obtain a tree T'' of G (see Figure 2.3).

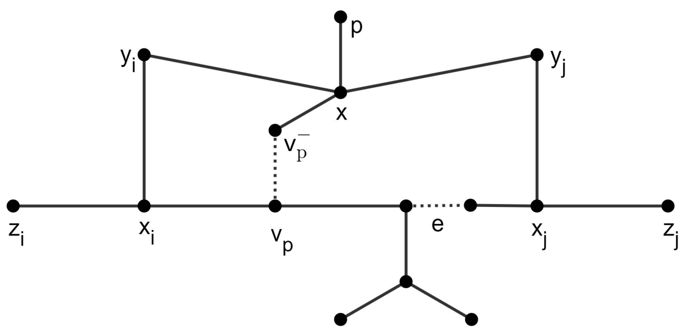

Figure 2.3: Tree T''.

Then $V(T'') = V(T)$ and $|P(B(T''))| \leq k$, where x_i and x_j are not leaves of R_Stem(T''). This contradicts either the maximality of T or Claim 2.1. Therefore, we have $|U \cap N_G(x)| \leq 2$. The proof of Subclaim 2.7.2 is completed.

Subclaim 2.7.3. $p \notin N_G(U)$ and $v_p^- \notin N_G(U)$.

Suppose, to the contrary, $z \in N_G(y_i)$ for some $z \in \{p, v_p^-\}$ and $y_i \in U$. Consider the tree $T' := T + y_i z - v_p v_p^-$. Then T' is a tree in G satisfying $V(T') = V(T)$, $|P(B(T'))| = |P(B(T))|$, $|\text{R\_Stem}(T')| = |\text{R\_Stem}(T)|$ and $|L(T')| < |L(T)|$. This contradicts Condition (C2). Therefore, Subclaim 2.7.3 holds.

Now, by Subclaims 2.7.1–2.7.3 we conclude that $\{p\}$, $N_G(y_i) \cap B_p$, $(N_G(U - \{y_i\}) \cap B_p)^+$ and $(N_2(U) - N(y_i)) \cap B_p$ are pairwise disjoint subsets in B_p for every $1 \leq i \leq k+1$. Recall that $N_3(U) \cap B_p = \emptyset$ by Subclaim 2.7.2. Then by combining with Subclaim 2.7.3

we obtain

$$\sum_{u\in U}|N_G(u)\cap B_p| = |N_G(y_i)\cap B_p| + |N_G(U-\{y_i\})\cap B_p| + |(N_2(U)-N(y_i))\cap B_p|$$
$$= |N_G(y_i)\cap B_p| + |(N_G(U-\{y_i\})\cap B_p)^+| + |(N_2(U)-N(y_i))\cap B_p|$$
$$\leq |B_p|-1.$$

Claim 2.7 is proved. □

Claim 2.8. For every $1 \leq i \leq k+1$, $\sum_{u\in U}|N_G(u)\cap B_{y_i}| \leq |B_{y_i}|-1$ and $\sum_{u\in U}|N_G(u)\cap B_{z_i}| \leq |B_{z_i}|-1$.

Proof. By the same role of y_i and z_i, we only need to prove $\sum_{u\in U}|N_G(u)\cap B_{y_i}| \leq |B_{y_i}|-1$. Set $V(B_{y_i})\cap N_T(x_i) = \{x_i^-\}$. Now we consider $B_{y_i} = P_T[y_i, x_i^-]$.

By Claim 2.5, we obtain the following.

Subclaim 2.8.1. $N_G(U)\cap B_{y_i} = N_G(\{y_i, z_i\})\cap B_{y_i}$.

Subclaim 2.8.2. If $x \in N_G(y_i)\cap B_{y_i}$ then $x^- \notin N_G(z_i)\cap B_{y_i}$.

Suppose that there exists $x \in N_G(y_i)\cap B_{y_i}$ such that $x^- \in N_G(z_i)\cap B_{y_i}$. Consider the tree $T' := T + \{xy_i, z_ix^-\} - \{xx^-, x_i^-x_i\}$. Then $V(T') = V(T)$ and $|P(B(T'))| \leq |P(B(T))|$. If $\deg_T(x_i) = 3$ then x_i is not a branch vertex of T'. Hence $|\text{R\_Stem}(T')| < |\text{R\_Stem}(T)|$, this contradicts either Claim 2.1 or Condition (C1). Otherwise, we have $|P(B(T'))| = |P(B(T))|$, $|\text{R\_Stem}(T')| = |\text{R\_Stem}(T)|$ and $|L(T')| < |L(T)|$, where y_i and z_i are not leaves of T'. This is a contradiction with Condition (C2). Therefore, Subclaim 2.8.2 holds.

Subclaim 2.8.3. $x_i^- \notin N_G(z_i)$.

Suppose, to the contrary, $x_i^-z_i \in E(G)$. Consider the tree $T' := T + x_i^-z_i - x_ix_i^-$. Then T' is a tree in G satisfying $V(T') = V(T)$, $|P(B(T'))| = |P(B(T))|$, $|\text{R\_Stem}(T')| = |\text{R\_Stem}(T)|$ and $|L(T')| < |L(T)|$, where z_i is not a leaf of T'. This contradicts Condition (C2). Therefore, Subclaim 2.8.3 holds.

By Subclaims 2.8.1–2.8.3, we conclude that $\{y_i\}$, $N_G(y_i)\cap B_{y_i}$ and $(N_G(z_i)\cap B_{y_i})^+$ are pairwise disjoint subsets in B_{y_i}. Combining with Subclaim 2.8.1, we have

$$\sum_{u\in U}|N_G(u)\cap B_{y_i}| = |N_G(y_i)\cap B_{y_i}| + |N_G(z_i)\cap B_{y_i}|$$
$$= |N_G(y_i)\cap B_{y_i}| + |(N_G(z_i)\cap B_{y_i})^+| \leq |B_{y_i}|-1.$$

This completes the proof of Claim 2.8. □

By Claims 2.3, 2.7 and 2.8, we obtain that

$$\deg_G(U) = \sum_{i=1}^{k+1}(\deg_G(y_i)+\deg_G(z_i))$$

$$\begin{aligned}
&\le \sum_{i=1}^{k+1}(|B_{y_i}|-1) + \sum_{i=1}^{k+1}(|B_{z_i}|-1) + \sum_{p\in L(T)-U}(|B_p|-1) + 2(k+1)\\
&= |G| - |\,\mathrm{R_Stem}(T)| - |L(T)-U|\\
&\le |G| - |\,\mathrm{R_Stem}(T)|.
\end{aligned}$$

On the other hand, since $k \ge 2$, then $|L(\mathrm{R\_Stem}(T))| = |P(B(T))| = k+1 \ge 3$. Hence, we obtain $|B(\mathrm{R\_Stem}(T))| \ge 1$. So we have $|\,\mathrm{R\_Stem}(T)| \ge k+2$. Hence

$$\begin{aligned}
&\sum_{i=1}^{k+1}\deg_G(y_i) + \sum_{i=1}^{k+1}\deg_G(z_i) \le |G| - k - 2\\
\Longrightarrow\quad &\min\left\{\sum_{i=1}^{k+1}\deg_G(y_i), \sum_{i=1}^{k+1}\deg_G(z_i)\right\} \le \left\lfloor \frac{|G|-k-2}{2} \right\rfloor.
\end{aligned}$$

Combining with Claim 2.6, we obtain

$$\sigma^4_{k+1}(G) \le \min\left\{\sum_{i=1}^{k+1}\deg_G(y_i), \sum_{i=1}^{k+1}\deg_G(z_i)\right\} \le \left\lfloor \frac{|G|-k}{2} \right\rfloor - 1.$$

Thus, G does not satisfy either the condition $\alpha(G) \le 2k+2$, or the condition $\sigma^4_{k+1}(G) \ge \lfloor \frac{|G|-k}{2} \rfloor$, a contradiction. Therefore, G has a spanning tree with at most k peripheral branch vertices if either $\alpha(G) \le 2k+2$, or $\sigma^4_{k+1}(G) \ge \lfloor \frac{|G|-k}{2} \rfloor$.

The proof of Theorem 1.7 is completed. □

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