

Invariant Subsets and Homological Properties of Orlicz Modules over Group Algebras

Rüya Üster and Serap Öztop*

Abstract. Let G be a locally compact group with left Haar measure. We study the closed convex left invariant subsets of $L^\Phi(G)$ and characterize affine mappings from the space of nonnegative functions in $L^1(G)$ of norm 1 into $L^\Phi(G)$ spaces. We apply the results to the study of the multipliers of $L^\Phi(G)$. We also investigate the homological properties of $L^\Phi(G)$ as a Banach left $L^1(G)$ -module such as projectivity, injectivity and flatness.

1. Introduction

An Orlicz space $L^\Phi(G)$ is a type of function space generalizing the L^p spaces. Besides the L^p spaces, a variety of function spaces that arise in analysis are contained naturally in Orlicz spaces. For example, $L \log L$, which is a Banach space related to Hardy-Littlewood maximal functions, and certain Sobolev spaces are contained in Orlicz spaces as subspaces. Linear properties of Orlicz spaces have been studied thoroughly (see [21], for example). Recently, Orlicz spaces and weighted Orlicz spaces over a locally compact group G are considered as Banach algebras and certain cohomological properties of them are also investigated (see [1, 16–19]). However, little attention has been paid to their possible module properties (see [17], for example).

Our goal in this paper is to study possible module structures on Orlicz spaces over the group algebra of a locally compact group G . It is well-known that $L^p(G)$ is a left or right Banach $L^1(G)$ -module (see [10, Theorem 32.16]). In the context of Banach modules, Lau [13] studied the closed convex invariant subsets of $L^p(G)$ ($1 \leq p < \infty$); in particular, he proved that the only compact convex left or right translation-invariant subset of $L^p(G)$ is the origin. We obtain similar results for the $L^\Phi(G)$ spaces. Therefore we extend some standard results for $L^p(G)$ to $L^\Phi(G)$ spaces. Specifically, we observe that the idea of Lau can be applied to investigate the multipliers of $L^\Phi(G)$ spaces. Consequently we also obtain some insight into the multipliers problem for $L^\Phi(G)$ spaces.

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*Corresponding author.

On the other hand, Dales and Polyakov in [6] gave a detailed study of the homological properties of modules over the group algebra of a locally compact group. By following their work, we study some homological properties of Orlicz spaces considered as $L^1(G)$ -modules such as projectivity, injectivity and flatness.

The structure of the paper is as follows. First we introduce essential definitions and basic information that will be used in this paper. In Section 3, we study the closed convex left-invariant subsets of $L^\Phi(G)$. We prove that if G is locally compact but noncompact, then each closed convex left-invariant subset C of $L^\Phi(G)$ must contain the origin. We also prove that if C is compact, then C contains only the origin. In Section 4, we study affine continuous mappings on subsets of Orlicz spaces. In Section 5, we apply our results to extend some classical theorems concerning multipliers for the $L^p(G)$ spaces ($1 \leq p \leq \infty$) to multipliers for the $L^\Phi(G)$ spaces. In particular, we extend the characterization of Brainerd and Edwards [4] from $L^1(G)$ to $L^\Phi(G)$. Finally in Section 6, we study the projectivity of Orlicz modules over the group algebra of a locally compact group G and give necessary and sufficient conditions for projectivity. We also have results on the injectivity and flatness of Orlicz modules over the group algebra of a locally compact group.

2. Preliminaries

In this section, we provide an overview of some basic definitions and state some technical results that will be crucial in the rest of this paper. In this paper, G denotes a locally compact group with a fixed left Haar measure μ .

First we recall some facts concerning Young functions and Orlicz spaces. Our main reference is [21]. A nonzero function $\Phi: [0, \infty) \rightarrow [0, \infty]$ is called a Young function if Φ is convex, $\Phi(0) = 0$ and $\lim_{x \rightarrow \infty} \Phi(x) = +\infty$. For a Young function Φ , its complementary function Ψ is given by

$$\Psi(y) = \sup\{xy - \Phi(x) : x \geq 0\}, \quad y \geq 0.$$

It is easy to check that Ψ is also a Young function. Also, if Ψ is the complementary function of Φ , then Φ is the complementary function of Ψ , and (Φ, Ψ) is called a complementary pair. We have the Young inequality

$$xy \leq \Phi(x) + \Psi(y), \quad x, y \geq 0$$

for complementary functions Φ and Ψ . By definition, a Young function can have the value ∞ at a point, and hence be discontinuous at such a point. However, we always consider a pair of complementary Young functions (Φ, Ψ) with both Φ and Ψ continuous and strictly increasing. In particular, they attain positive values on $(0, \infty)$. Note that even though Φ is continuous, it may happen that Ψ is not continuous.

Let us recall some facts concerning Orlicz spaces. Suppose that G is a locally compact group with a fixed Haar measure $d\mu$ and (Φ, Ψ) is a complementary Young pair. We define

$$\mathcal{L}^\Phi(G) = \left\{ f: G \rightarrow \mathbb{C} : f \text{ is measurable and } \int_G \Phi(|f(x)|) d\mu(x) < \infty \right\}.$$

Since $\mathcal{L}^\Phi(G)$ is not always a linear space, we define the Orlicz space $L^\Phi(G)$ to be

$$L^\Phi(G) = \left\{ f: G \rightarrow \mathbb{C} : \int_G \Phi(\alpha|f(x)|) d\mu(x) < \infty \text{ for some } \alpha > 0 \right\},$$

where f indicates a μ -equivalence class of measurable functions with respect to the Haar measure $d\mu$. Then the Orlicz space is a Banach space under the Orlicz norm $\|\cdot\|_\Phi$ defined for $f \in L^\Phi(G)$ by

$$\|f\|_\Phi = \sup \left\{ \int_G |f(x)\nu(x)| d\mu(x) : \int_G \Psi(|\nu(x)|) d\mu(x) \leq 1 \right\},$$

where Ψ is the complementary function to Φ . One can also define the Luxemburg norm $\|\cdot\|_\Phi^o$ on $L^\Phi(G)$ by

$$\|f\|_\Phi^o = \inf \left\{ k > 0 : \int_G \Phi\left(\frac{|f(x)|}{k}\right) d\mu(x) \leq 1 \right\}.$$

It is known that these two norms are equivalent; that is,

$$\|\cdot\|_\Phi^o \leq \|\cdot\|_\Phi \leq 2\|\cdot\|_\Phi^o$$

and

$$\|f\|_\Phi \leq 1 \quad \text{if and only if} \quad \int_G \Phi(|f(x)|) d\mu(x) \leq 1.$$

Let us recall that a step function takes finitely many values on measurable sets of G . We define $M^\Phi(G)$ as the closure of the linear space of all step functions in $L^\Phi(G)$. Then $M^\Phi(G)$ is a Banach space and contains $C_c(G)$, the space of all continuous functions on G with compact support, as a dense subspace [21, Proposition 3.4.3]. Moreover, $M^\Phi(G)^*$, the dual of $M^\Phi(G)$, can be identified with $L^\Psi(G)$ in a natural way [21, Theorem 4.1.6]. Another useful characterization of $M^\Phi(G)$ is that $f \in M^\Phi(G)$ if and only if for every $\alpha > 0$, $\alpha f \in \mathcal{L}^\Phi(G)$ [21, Definition 3.4.2 and Proposition 3.4.3].

A Young function Φ satisfies the Δ_2 -condition if there exists a constant $K > 0$ such that

$$\Phi(2x) \leq K\Phi(x), \quad x \geq 0.$$

In this case, we write $\Phi \in \Delta_2$. If $\Phi \in \Delta_2$, then it follows that $L^\Phi(G) = M^\Phi(G)$ so that $L^\Phi(G)^* = L^\Psi(G)$ [21, Corollary 3.4.5]. If in addition $\Psi \in \Delta_2$, then the Orlicz space $L^\Phi(G)$ is a reflexive Banach space.

Definition 2.1. Let Φ_1 and Φ_2 be two Young functions. Then Φ_1 is stronger than Φ_2 , $\Phi_1 \succ \Phi_2$ (or $\Phi_2 \prec \Phi_1$) in symbols, if

$$\Phi_2(x) \leq \Phi_1(ax), \quad x \geq 0$$

for some $a > 0$. Further we say that Φ_1 and Φ_2 are strongly equivalent if $\Phi_1 \succ \Phi_2$ and $\Phi_2 \succ \Phi_1$. Then we write $\Phi_1 \approx \Phi_2$.

We have said that Orlicz spaces are generalization of Lebesgue spaces. If we take $\Phi(x) = x^p/p$, $1 < p < \infty$, then the complementary Young function is $\Psi(y) = y^q/q$, with $1/p + 1/q = 1$, and the space $L^\Phi(G)$ becomes the Lebesgue space $L^p(G)$ and the norm $\|\cdot\|_\Phi$ is equivalent to the classical norm $\|\cdot\|_p$.

Our results can be applied to various Young functions (see [3, Example 8.3], [15, Proposition 2.11] and [21, p. 15]) and concrete groups G (see [8]); we point out a few below.

- (1) If $\Phi(x) = x \ln(1+x)$, then $\Psi(x) \approx \cosh x - 1$.
- (2) If $\Phi(x) = \cosh x - 1$, then $\Psi(x) \approx x \ln(1+x)$.
- (3) If $\Phi(x) = e^x - x - 1$, then $\Psi(x) = (1+x) \ln(1+x) - x$.
- (4) If $\Phi(x) = x \log x$ ($0 \leq x < \infty$), then the Orlicz space L^Φ is the Zygmund space $L \log L$.
- (5) If $\Phi(x) = e^{x^\beta} - 1$, $x \geq 0$ with $\beta > 0$, then the Orlicz space L^Φ is the Zygmund space $\exp L^\beta$.
- (6) The group $SU(2)$ is compact. If f is an integrable function on $SU(2)$, then its integral with respect to the Haar measure is

$$\int_{SU(2)} f(x) d\mu(x) = \frac{1}{2\pi^2} \int_0^\pi \int_0^\pi \int_0^{2\pi} f \circ \Phi(\theta, \varphi, \psi) \sin^2 \theta \sin \varphi d\theta d\varphi d\psi,$$

where Φ denotes the map $(\theta, \varphi, \psi) \mapsto x = (x_1, x_2, x_3, x_4)$ in which $x_1 = \cos \theta$, $x_2 = \sin \theta \cos \varphi$, $x_3 = \sin \theta \sin \varphi \cos \psi$, $x_4 = \sin \theta \sin \varphi \sin \psi$.

- (7) The group $GL(n, \mathbb{R})$ is noncompact. The Haar measure on it is $|\det A|^{-n} dA$, where dA is the Lebesgue measure on $M(n, \mathbb{R})$.

As usual, $C_b(G)$ is the space of bounded continuous complex-valued functions on G , $C_0(G)$ is the subspace of $C_b(G)$ consisting of all those functions that vanish at infinity, and $M(G)$ denotes the set of all complex regular Borel measures on G which is a Banach space with respect to the norm defined by

$$\|\mu\| = |\mu|(G) < \infty, \quad \mu \in M(G).$$

The convolution of $\mu \in M(G)$ and $f \in L^\Phi(G)$ is defined by

$$(\mu * f)(x) = \int_G f(y^{-1}x) d\mu(y).$$

For each $a \in G$, let δ_a denote the measure in $M(G)$ such that $\delta_a(A) = 1_A(a)$, where 1_A is the characteristic function of a subset A of G ,

$$(\delta_a * f)(x) = f(a^{-1}x) = L_{a^{-1}}f(x).$$

Given any function f on G , the function \tilde{f} on G will be defined by $\tilde{f}(x) = f(x^{-1})$ for each $x \in G$.

Definition 2.2. Let X and Y be normed spaces and C, D convex subsets of X, Y respectively. A map $f: C \rightarrow D$ is called affine if

$$f(\alpha x + (1 - \alpha)y) = \alpha f(x) + (1 - \alpha)f(y)$$

for all $x, y \in C$ and $\alpha \in [0, 1]$.

Throughout this paper we are mainly interested in Orlicz spaces $L^\Phi(G)$ with the Δ_2 -condition on its Young function Φ .

3. Invariant subsets

In this section we generalize the results of [13] concerning the characterization of closed convex invariant subsets of the $L^p(G)$ to the Orlicz spaces of a locally compact group G . Our results are motivated by the corresponding results for $L^p(G)$. The proof is somewhat similar to that of the $L^p(G)$ case, but new ideas are needed to fully adapt it to the Orlicz spaces case. We note that $L^\Phi(G)$ is an essential Banach $L^1(G)$ -module with respect to the convolution [17, Lemma 3.2], it is also an $M(G)$ -module. Throughout this paper we are mainly interested in the left invariant subsets of $L^\Phi(G)$, but similar results obtain for the right invariant subsets of $L^\Phi(G)$. Let us give the definition of the left invariant subsets of $L^\Phi(G)$.

Definition 3.1. A subset C of the space $L^\Phi(G)$ is called left invariant if $L_x f \in C$ for each $f \in C$ and $x \in G$.

Note that if $f \in L^\Phi(G)$, $x \in G$, then $L_x f \in L^\Phi(G)$ and $\|L_x f\|_\Phi = \|f\|_\Phi$ (see [17, p. 10]).

Let K be a subset of $L^\Phi(G)$, then $\text{co } K$ will denote the convex hull of K . Besides the norm topology on $L^\Phi(G)$, we will consider the weak-topology $\omega = \sigma(L^\Phi(G), L^\Phi(G)^*)$, where $L^\Phi(G)^* = L^\Psi(G)$ and (Φ, Ψ) is a complementary pair. The weak topology on $L^\Phi(G)$ will be considered only occasionally, so unless otherwise specified, we will refer

to the topological properties of sets and functions in $L^\Phi(G)$ with respect to the norm topology.

Further, the following subsets of $M(G)$ will be crucial in our investigation:

- (1) $P(G) = \{\mu \in M(G) : \|\mu\| = 1 \text{ and } \mu \geq 0\}$,
- (2) $P_1(G) = \{h \in L^1(G) : \|h\|_1 = 1 \text{ and } h \geq 0\}$,
- (3) $E(G) = \{\delta_a : a \in G\}$.

One can consider that $P_1(G) = P(G) \cap L^1(G)$.

The following definition and lemma appear in [13, Lemma 3.1].

Definition 3.2. Let ν denote the separated locally convex topology on $M(G)$ determined by the family of seminorms $Q = \{p_f : f \in C_b(G)\}$, such that

$$|p_f(\mu)| = |\langle \mu, f \rangle| = \left| \int_G f d\mu \right|$$

for each $\mu \in M(G)$. Then in the ν -topology, $\mu_\alpha \rightarrow \mu$ means that $\int_G f d\mu_\alpha \rightarrow \int_G f d\mu$. We denote the weak topology of $M(G)$ by $\sigma(M(G), L^\infty(G))$, while the weak* topology by $\sigma(M(G), C_0(G))$. In this way, the ν -topology is $\sigma(M(G), C_b(G))$. Since $C_0(G) \subseteq C_b(G) \subseteq L^\infty(G)$, we have $\sigma(M(G), C_0(G)) \leq \sigma(M(G), C_b(G)) \leq \sigma(M(G), L^\infty(G))$.

Lemma 3.3. *For a locally compact group G , we have*

$$P(G) = \overline{P_1(G)}^\nu = \overline{\text{co } E(G)}^\nu.$$

As a consequence of the Hahn-Banach separation theorem, let us note that a convex set in a Banach space which is closed in the norm topology is also closed in the weak topology.

The following lemma and theorem can be proven by adapting arguments similar to the one presented in [13, Lemma 3.2] and [13, Theorem 4.1(a)]. Therefore we omit the proofs.

Lemma 3.4. *Let G be a locally compact group.*

- (1) *For each $f \in L^\Phi(G)$, the mapping $\mu \mapsto \mu * f$ from $M(G)$ into $L^\Phi(G)$ is continuous when $M(G)$ has the weak* topology and $L^\Phi(G)$ has the weak topology.*
- (2) *For each $f \in L^1(G)$, the mapping $h \mapsto f * h$ from $L^\Phi(G)$ into $L^\Phi(G)$ is continuous when $L^\Phi(G)$ has the weak topology.*

Theorem 3.5. *Let G be a locally compact group and C a closed convex subset of $L^\Phi(G)$. Then C is left invariant if and only if $h * C \subseteq C$ for each $h \in P_1(G)$.*

We observe the following consequence.

Corollary 3.6. *Let I be a closed linear subspace of $L^\Phi(G)$. Then $L^1(G) * I \subseteq I$ if and only if I is a left invariant subspace of $L^\Phi(G)$.*

Note that the set $\overline{\text{co}\{L_x f : x \in G\}}$ is a closed, convex and left invariant subset of $L^\Phi(G)$. Hence, we have the following consequence.

Corollary 3.7. *Let $f \in L^\Phi(G)$. Then*

$$\overline{\text{co}\{L_x f : x \in G\}} = \overline{\{\varphi * f : \varphi \in P_1(G)\}}.$$

Theorem 3.8. *Let G be a locally compact group. Then G is noncompact if and only if each closed, convex, left invariant nonempty subset of $L^\Phi(G)$ contains the origin.*

Proof. Assume that G is noncompact, and take C a closed, convex, left invariant nonempty subset of $L^\Phi(G)$. Since G is noncompact for each compact subset H of G , there exists an element $x_H \in G \setminus H$. It follows that the net $\{\delta_{x_H}\}_H$ weak* converges to zero in $M(G)$. Using Lemma 3.4 we conclude that the net $\{\delta_{x_H} * f\}_H$ converges to zero in the weak topology of $L^\Phi(G)$. The set C is closed and convex, so weakly closed. Since $\{\delta_{x_H} * f\}_H \subseteq C$, then its weak limit is in C , that is, C contains the origin.

Conversely suppose G is compact. The set $C = \{1_G\}$ is a closed, convex, left invariant subset of $L^\Phi(G)$ but does not contain the origin. \square

Theorem 3.9. *Let G be a locally compact group. Then G is noncompact if and only if each compact, convex, left invariant nonempty subset of $L^\Phi(G)$ is the origin.*

Proof. Let G be noncompact, C a compact, convex, left invariant nonempty subset of $L^\Phi(G)$ and f arbitrary in C . Let $h \in L^\Psi(G)$, where Ψ is the complementary Young pair of Φ , and consider the function $k(s) = \langle L_s f, h \rangle$ for $s \in G$. Then $k = f * \tilde{h} \in C_0(G)$. Furthermore, k is a bounded, continuous, almost periodic function on G . In order to justify that k is almost periodic, i.e., $f * \tilde{h} \in AP(G)$, we notice that, for each $x \in G$,

$$\begin{aligned} L_x(f * \tilde{h})(y) &= (f * \tilde{h})(xy) = \int_G f(z) \tilde{h}(z^{-1}xy) d\mu(z) = \int_G f(xz) \tilde{h}(z^{-1}y) d\mu(z) \\ &= \int_G L_x f(z) \tilde{h}(z^{-1}y) d\mu(z) = (L_x f * \tilde{h})(y), \quad y \in G. \end{aligned}$$

As $\{L_x f : x \in G\}$ is contained in the compact set C , there is a net $\{x_\alpha\}_\alpha \subseteq G$ such that $\{L_{x_\alpha} f\}_\alpha$ converges in the normed space $L^\Phi(G)$. Since $h \in L^\Psi(G)$,

$$\|L_{x_\alpha} f * \tilde{h} - L_{x_\beta} f * \tilde{h}\|_\infty \leq \|L_{x_\alpha} f - L_{x_\beta} f\|_\Phi \|h\|_\Psi \rightarrow 0,$$

so $\{L_{x_\alpha} f * \tilde{h}\}_\alpha$ converges in $(C_b(G), \|\cdot\|_\infty)$. Since G is a locally compact noncompact group, the set $AP(G) \cap C_0(G)$ reduces to zero, so $k \equiv 0$. Hence $f * \tilde{h} = 0$ for each

$h \in L^\Psi(G)$. Now we want to show that $f = 0$. Let U_α be a decreasing set of compact neighborhoods of e and let $h_\alpha = 1_{U_\alpha}/|U_\alpha|$; then $\|f * \widetilde{h_\alpha} - f\|_\Phi \rightarrow 0$. Since $h_\alpha \in L^\Psi(G)$ for each α , it follows that $f = 0$. Hence the set C contains only the origin.

Conversely suppose that G is compact. The set $C = \{1_G\}$ is a compact, convex and left invariant subset of $L^\Phi(G)$ but does not contain the origin. \square

Let us remind that more information about almost periodic functions on locally compact groups can be found in [14] for example.

4. Affine mappings commuting with translations

Our main aim is to determine the affine mappings and characterize the compact multipliers of Orlicz spaces. Lau [13] studied the affine mappings T from a left invariant closed convex subset A of $L^q(G)$ into a closed convex left (right) invariant subset of $L^p(G)$ commuting with left (right) translations. We shall study affine continuous mappings T on subsets of Orlicz spaces.

The proof of the next theorem is similar to that of [13, Theorem 5.1]; we therefore omit it.

Theorem 4.1. *Let Φ and Ψ be Young functions and let B and C be closed left invariant convex subsets of $L^\Psi(G)$ and $L^\Phi(G)$, respectively. If $T: B \rightarrow C$ is continuous and affine, then the following are equivalent.*

- (1) *T commutes with left translations (i.e., $T(L_x f) = L_x T f$ whenever $x \in G$ and $f \in B$).*
- (2) *$T(h * f) = h * T(f)$ for each $h \in P_1(G)$ and $f \in B$.*

We have the following consequence.

Corollary 4.2. *If $T: L^\Psi(G) \rightarrow L^\Phi(G)$ is a linear bounded operator, then the following are equivalent.*

- (1) *$T(L_x f) = (L_x T) f$ whenever $x \in G$ and $f \in L^\Psi(G)$.*
- (2) *$T(h * f) = h * T(f)$ for each $h \in L^1(G)$ and $f \in L^\Psi(G)$.*

Theorem 4.3. *Let G be a locally compact noncompact group. Let C be a nonempty closed convex left invariant subset of $L^1(G)$, and B a nonempty weakly compact closed convex left invariant subset of $L^\Phi(G)$. If $T: B \rightarrow C$ is a continuous affine map and $T(L_x f) = L_x T f$ for each $f \in B$, then $T = 0$.*

Proof. Since B and C are closed and convex, they are weakly closed. Since T is affine and continuous, T is also weakly continuous when B and C have their respective weak topologies. As B is weakly compact, convex and T is weakly continuous, $T(B)$ is a weakly compact convex left invariant subset of $L^1(G)$. Hence by [13, Theorem 4.8], $T(B) = \{0\}$. \square

Theorem 4.4. *Let G be a locally compact group, C be a weakly compact, closed, bounded, left invariant subset of $L^\Phi(G)$, and let $T: P_1(G) \rightarrow C$ be a continuous affine map. Then the following are equivalent.*

- (1) *T commutes with all left translations.*
- (2) *There exists an $f \in C$ such that $T(h) = h * f$ for $h \in P_1(G)$.*

Proof. Let $x \in G$ and suppose that $T(L_x h) = L_x T(h)$ whenever $h \in P_1(G)$. By Theorem 4.1, we have $T(k * h) = k * T(h)$ for $k, h \in P_1(G)$. Suppose that $\{U_\alpha\} \subseteq P_1(G)$ is a bounded approximate identity for $L^1(G)$. Since $T(U_\alpha) \in C$ is bounded and C is weakly compact, there is an $f \in C$ such that $\{T(U_\alpha)\}_\alpha$ converges to f in the weak topology. Then

$$T(h) = \lim_\alpha T(h * U_\alpha) = \lim_\alpha h * T(U_\alpha) = h * f.$$

Conversely let $x \in G$ and suppose that there is $f \in C$ such that $T(h) = h * f$ for all $h \in P_1(G)$. Then

$$L_x T(h) = L_x(h * f) = \delta_{x^{-1}} * (h * f) = (\delta_{x^{-1}} * h) * f = L_x h * f = T(L_x h).$$

This completes the proof. \square

5. Compact multipliers

A multiplier is an important concept for Banach spaces. We want to study the problem of multipliers on L^Φ spaces. We apply our results in Section 3 to extend some classical theorems concerning multipliers for the L^p spaces, $1 \leq p < \infty$, to multipliers for L^Φ spaces of a locally compact group. For $p = 1$ and G is a locally compact group, the space of multipliers is precisely $M(G)$. This is a famous result of Wendel [24]. For a locally compact noncompact group, Sakai [23] proved that a compact multiplier on $L^1(G)$ commuting with left (right) translations is identically zero. On the other hand Akemann [2] proved that if G is a compact group, then $M(L^1(G), L^1(G))$ consists only of compact operators, where $M(L^1(G), L^1(G))$ denotes the set of all bounded linear operators on $L^1(G)$ that commute with the left (right) translations.

Our main purpose is to obtain some results on the compact multipliers problem for Orlicz spaces. We characterize the compact multipliers of $L^\Phi(G)$ by taking $L^\Phi(G)$ as an

$L^1(G)$ -module. We are able to extend these results to compact multipliers from $L^1(G)$ to $L^\Phi(G)$.

Let us start with the definition of the left multipliers of $L^\Phi(G)$.

Definition 5.1. A bounded linear operator T from $L^\Psi(G)$ to $L^\Phi(G)$ is called a left multiplier for the pair $(L^\Psi(G), L^\Phi(G))$ if $T(L_x f) = L_x(Tf)$ for all $f \in L^\Psi(G)$ and $x \in G$. We denote the set of left multipliers for the pair $(L^\Psi(G), L^\Phi(G))$ by $M(L^\Psi(G), L^\Phi(G))$.

For the following theorem we assume that both Φ and Ψ are Young functions satisfying the Δ_2 -condition.

Theorem 5.2. *Let G be a locally compact group. For a bounded linear operator T from $L^1(G)$ to $L^\Phi(G)$, the following are equivalent.*

- (1) $T \in M(L^1(G), L^\Phi(G))$.
- (2) *There exists an $f \in L^\Phi(G)$ such that $T(h) = h * f$ for $h \in L^1(G)$.*

Proof. Suppose first that $T \in M(L^1(G), L^\Phi(G))$. Let $\{u_\alpha\}_\alpha$ be an approximate identity for $L^1(G)$ such that $\|u_\alpha\|_1 = 1$ for all α . Then for all $h \in L^1(G)$ we have $\|h * u_\alpha - h\|_1 \rightarrow 0$. It follows that $\{T(h * u_\alpha)\}_\alpha$ converges to $T(h)$ in the norm topology for all $h \in L^1(G)$. On the other hand, the net $\{T(u_\alpha)\}_\alpha$ is bounded in the reflexive space $L^\Phi(G)$, so passing to a subnet if necessary, we may assume that $\{T(u_\alpha)\}_\alpha$ converges in the weak topology to some $f \in L^\Phi(G)$. Then using Corollary 4.2 and Lemma 3.4, we have

$$T(h) = \lim_{\alpha} T(h * u_\alpha) = \lim_{\alpha} h * T(u_\alpha) = h * f.$$

The proof of the converse part is similar to the proof of the corresponding part of Theorem 4.4. □

Theorem 5.3. *Let G be a compact group. Every multiplier T from $L^1(G)$ into $L^\Phi(G)$ is compact.*

Proof. Let $T \in M(L^1(G), L^\Phi(G))$. By Theorem 5.2 there exists $f \in L^\Phi(G)$ such that $T(h) = h * f$, for all $h \in L^1(G)$. We will prove that T is a compact operator, i.e., T takes the unit ball of $L^1(G)$ into a relatively compact set of $L^\Phi(G)$. As G is compact and the mapping $L_x f$ is continuous, the set $\overline{\text{co}\{L_x f : x \in G\}}^{\|\cdot\|_\Phi} \subseteq L^\Phi(G)$ is compact, so using Corollary 3.7 it follows that $\{\varphi * f : \varphi \in P_1(G)\}$ is relatively compact in $L^\Phi(G)$. If $h \in L^1(G)$ and $\|h\|_1 \leq 1$, then $h = (h_1 - h_2) + i(h_3 - h_4)$, where $h_j > 0$ and $\|h_j\|_1 < 1$, $j = 1, 2, 3, 4$. It follows that $T(h_j)$ is in $C = \{\lambda \varphi * f : \lambda \in [0, 1]\}$. Since C is relatively compact, it follows that $T(h)$ lies in the relatively compact set $(C - C) + i(C - C)$. Hence T is a compact operator. □

Theorem 5.4. *For a locally compact noncompact group G , the only compact multiplier T from $L^\Psi(G)$ into $L^\Phi(G)$ is zero.*

Proof. Let T be a compact operator and $TL_x = L_xT$. Then, the closure of the set $\{T(f) : \|f\|_\Psi \leq 1\}$ is a compact, convex, left invariant, nonempty subset of $L^\Phi(G)$, so by Theorem 3.9, it reduces to zero. Hence $T = 0$. \square

The following consequence is apparent from Theorem 4.3.

Theorem 5.5. *Let G be a locally compact noncompact group. Let A be a weakly compact closed left invariant subspace of $L^\Phi(G)$. If T is a bounded linear operator from A into $L^1(G)$ commuting with left translations, then T is zero.*

6. Projectivity, injectivity and flatness

Homological properties of $L^p(G)$ as an $L^1(G)$ -module have been studied by Dales and Polyakov [6]. In this section we characterize some homological properties of $L^\Phi(G)$ considered as $L^1(G)$ -modules, such as projectivity, injectivity and flatness.

First we recall some homological properties of a Banach algebra. Let A be a Banach algebra and E be a Banach left A -module. We denote by $A\text{-mod}$ and $\text{mod-}A$ the categories of Banach left A -modules and of Banach right A -modules, respectively. These classes of modules are defined in [5, 0.3.2] and [9, Definition 2.6.1]. Suppose that $E \in A\text{-mod}$. Set

$$AE = \text{span}\{ax : a \in A, x \in E\}.$$

Then E is called essential if $\overline{AE} = E$, where the overbar denotes closure. In the case A has a bounded approximate identity $\{e_\alpha\}_\alpha$, E is essential if and only if $\lim_\alpha e_\alpha x = x$ for all $x \in E$ (see [7, Corollary 15.3]).

We denote by $B(E, F)$ the Banach space of all bounded linear operators from E to F . We call a $T \in B(E, F)$ admissible if there exists $S \in B(F, E)$ with $T \circ S = \text{id}_F$. If $E, F \in A\text{-mod}$, then ${}_AB(E, F)$ denotes the closed linear subspace of $B(E, F)$ consisting of all left A -module morphisms. An operator $T \in {}_AB(E, F)$ is called a retraction if there exists $S \in {}_AB(F, E)$ with $T \circ S = \text{id}_F$. Similar definitions apply when $E, F \in \text{mod-}A$.

Definition 6.1. Let A be a Banach algebra, and let $P \in A\text{-mod}$. Then P is called a projective A -module if for each $E, F \in A\text{-mod}$, for each admissible epimorphism $T \in {}_AB(E, F)$ and for each $S \in {}_AB(P, F)$, there exists $R \in {}_AB(P, E)$ such that $T \circ R = S$.

Definition 6.2. Let A be a Banach algebra and $J \in A\text{-mod}$. Then J is called injective if for each $E, F \in A\text{-mod}$, each admissible monomorphism $T \in {}_AB(E, F)$ and each $S \in {}_AB(E, J)$, there exists $R \in {}_AB(F, J)$ such that $R \circ T = S$.

Definition 6.3. Let A be a Banach algebra, and let $E \in A\text{-mod}$. Then E is flat if the dual E^* of E is injective in $\text{mod-}A$.

It is known that $L^1(G)$ has the approximation property by [22, Example 4.5] and $\widehat{\otimes}$ denotes the projective tensor product of any Banach spaces.

We investigate the homological properties of $L^\Phi(G)$ induced by the structure of the underlying group. We will use [6, Proposition 1.2(i)] for the following theorem.

Theorem 6.4. Let G be a locally compact group and Φ be a Young function. Then $L^\Phi(G)$ is a projective $L^1(G)$ -module if and only if G is compact.

Proof. First suppose that G is compact. It is known that if G is compact, then every essential Banach left $L^1(G)$ -module is projective [20, 1.2]. Therefore $L^\Phi(G)$ is a projective $L^1(G)$ -module.

For the converse, suppose on the contrary that G is not compact. Let V and W be compact symmetric neighborhoods of the identity element e_G such that $V^2 \subseteq W$. Let $f(x) = \frac{\chi_V * \chi_V(x)}{\mu(V)} = \frac{1}{\mu(V)} \int_G \chi_V(y) \chi_V(y^{-1}x) d\mu(y)$ for $x \in G$. Then $f \in C_c(G)$, $f(G) \in [0, 1]$, with $f(e_G) = 1$, $\text{supp}(f) \subseteq V$, $\|f\|_\infty = 1$ and $\|f\|_1 \leq 1$. It is clear that $f * f \neq 0$ and $\text{supp}(f * f) \subseteq W$. Since $L^1(G)$ has the approximation property by [22, Example 4.5] and also $L^\Phi(G)$ is essential by [17, Lemma 3.2], then by [6, Proposition 1.2] there exists a $T \in {}_{L^1(G)}B(L^\Phi(G), L^1(G))$ such that $T(f * f) \neq 0$.

Set $\eta = \|f * T(f)\|_1/2$ and fix $k \in \mathbb{N}$. Choose a function $g \in C_c(G)$ with support K_1 such that $\|T(f) - g\|_1 < 1/k$ and $\|f * (T(f) - g)\|_1 < \eta$. It follows that $\|f * g\|_1 > \eta$.

Since G is compact, there exist $s_1, s_2, \dots, s_k \in G$ such that the sets $s_j(V \cup K)^2$ are pairwise disjoint for $j = 1, 2, \dots, k$. Indeed, set $s_1 = e$ and consider the set $s_1(V \cup K)^2(V \cup K)^{-2}$ which is a compact subset of G . Since G is not compact, there exists $s_2 \in G \setminus s_1(V \cup K)^2(V \cup K)^{-2}$. This way one can choose inductively each $s_j \in G \setminus \bigcup_{i=1}^{j-1} s_i(V \cup K)^2(V \cup K)^{-2}$.

Set $f_j = s_j * f$ and $\lambda = \sum_{j=1}^k f_j * f$. Since s_1V, \dots, s_kV are pairwise disjoint, we have $\|\sum_{j=1}^k f_j\|_\infty = 1$ and $\|\sum_{j=1}^k f_j * g\|_1 = k\|f * g\|_1$. Then

$$\|T(\lambda)\|_1 = \left\| \sum_{j=1}^k f_j * T(f) \right\|_1 \geq \left\| \sum_{j=1}^k f_j * g \right\|_1 - 1 = k\|f * g\|_1 - 1 > k\eta - 1.$$

On the other hand,

$$\begin{aligned} \left\| \sum_{j=1}^k f_j * f \right\|_\Phi &= \sup \left\{ \int_G \left| \sum_{j=1}^k f_j * f(x) h(x) \right| d\mu(x) : \int_G \Psi(|h(x)|) d\mu(x) \leq 1 \right\} \\ &\leq \sup \left\{ \int_G \sum_{j=1}^k |f_j * f(x) h(x)| d\mu(x) : \int_G \Psi(|h(x)|) d\mu(x) \leq 1 \right\} \end{aligned}$$

$$\begin{aligned}
&= \sup \left\{ \sum_{j=1}^k \int_G |f * f(x) h(s_j x)| d\mu(x) : \int_G \Psi(|h(x)|) d\mu(x) \leq 1 \right\} \\
&= k \|f * f\|_{\Phi}.
\end{aligned}$$

We obtain $\|\sum_{j=1}^k f_j * f\|_{\Phi} \leq k \|f * f\|_{\Phi}$. Since $f \in C_c(G)$, it follows that $f * f \in L^{\Phi}(G)$ and $\|\lambda\|_{\Phi} \leq k \|f * f\|_{\Phi}$.

Thus

$$k\eta - 1 \leq \|T(\lambda)\|_1 \leq \|T\| \|\lambda\|_{\Phi} \leq k \|T\| \|f * f\|_{\Phi}.$$

Consequently,

$$k\eta \leq k \|T\| \|f * f\|_{\Phi} + 1.$$

This holds for each $k \in \mathbb{N}$, which is the required contradiction. Therefore G must be compact, as claimed. \square

Remark 6.5. For the “if” part we can also use the result of Racher [20, Proposition 1.4] related to the projectivity of Banach modules. It is known that Orlicz space is separable if and only if G is separable (see [21, Theorem 3.5.1]).

Observation 6.6. Let G be a locally compact group, (Φ, Ψ) complementary pair and $\Psi \in \Delta_2$. If $L^{\Phi}(G)$ is a projective $L^1(G)$ -module with G being separable (second countable) then G is compact.

Now we seek to determine when the module $L^{\Phi}(G)$ is injective and flat. Let us recall that if $E \in A\text{-mod}$, then $E^* \in \text{mod-}A$ is the dual module of E with the module operation specified by the formula

$$\langle x, \lambda a \rangle = \langle ax, \lambda \rangle, \quad a \in A, x \in E, \lambda \in E^*.$$

Similarly $E^* \in A\text{-mod}$ when $E \in \text{mod-}A$ and in this case the module operation in E^* is specified by

$$\langle x, a\lambda \rangle = \langle xa, \lambda \rangle, \quad a \in A, x \in E, \lambda \in E^*.$$

We know that every module (left or right) over an amenable algebra is flat and every dual module is injective [9, VII.2.29]. Moreover, Johnson proved that the Banach algebra $L^1(G)$ is amenable if and only if the group G is amenable [11, Theorem 2.5]. If we combine these results we have the following observations.

Remark 6.7. Let Φ be a Young function. If G is an amenable group, then

- (1) $L^{\Phi}(G)$ is a flat Banach left $L^1(G)$ -module,
- (2) $L^{\Psi}(G)$, the dual module of $L^{\Phi}(G)$, is an injective Banach right $L^1(G)$ -module,
- (3) if $\Phi, \Psi \in \Delta_2$, then since $L^{\Phi}(G)$ is a dual module, $L^{\Phi}(G)$ is both an injective and flat Banach left $L^1(G)$ -module.

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Rüya Üster and Serap Öztop

Department of Mathematics, Faculty of Science, İstanbul University, İstanbul, Turkey

E-mail address: ruya.uster@istanbul.edu.tr, oztops@istanbul.edu.tr