

Research Article

On Hesitant Fuzzy Reducible Weighted Bonferroni Mean and Its Generalized Form for Multicriteria Aggregation

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Due to convenience and powerfulness in dealing with vagueness and uncertainty of real situation, hesitant fuzzy set has received more and more attention and has been a hot research topic recently. To differently process and effectively aggregate hesitant fuzzy information and capture their interrelationship, in this paper, we propose the hesitant fuzzy reducible weighted Bonferroni mean (HFRWBM) and present its four prominent characteristics, namely, reductibility, monotonicity, boundedness, and idempotency. Then, we further investigate its generalized form, that is, the generalized hesitant fuzzy reducible weighted Bonferroni mean (GHFRWBM). Based on the discussion of model parameters, some special cases of the HFRWBM and GHFRWBM are studied in detail. In addition, to deal with the situation that multicriteria have connections in hesitant fuzzy information aggregation, a three-step aggregation approach has been proposed on the basis of the HFRWBM and GHFRWBM. In the end, we apply the proposed aggregation operators to multicriteria aggregation and give an example to illustrate our results.

1. Introduction

In order to better understand the vagueness and uncertainty of the real world and thus be able to explain it, the fuzzy set (FS) theory has been extended and generalized to many other forms, such as interval-valued fuzzy set [1], type-2 fuzzy set [2, 3], fuzzy multiset [4], intuitionistic fuzzy set [5], interval-valued intuitionistic fuzzy set [6], and hesitant fuzzy set [7]. Due to convenience and powerfulness in dealing with vagueness and uncertainty of real situation, hesitant fuzzy set has received more and more attention from researchers and obtained some significant research results [8–12]. To aggregate the hesitant fuzzy information, a lot of aggregation operators have been developed and investigated, such as the hesitant fuzzy averaging aggregation (HFA) operator, the hesitant fuzzy geometric aggregation (HFGA) operator, the hesitant fuzzy ordered weighted averaging (HFOWA) operator, the hesitant fuzzy ordered weighted geometric (HFOWG) operator, and the hesitant fuzzy hybrid aggregation (HFHA) operator [13].

The Bonferroni mean (BM) was introduced by Bonferroni [14] six decades ago but has been a hot research topic recently for its important role in the information aggregation,

which can provide for the aggregation lying between the max and min operators and logical “oring” and “anding” operators. A prominent characteristic of BM is that it not only considers the importance of each criterion but also reflects the interrelationship of the individual criterion [15]. The extended BMs include the generalized Bonferroni mean [16], the intuitionistic fuzzy Bonferroni mean [17], the interval-valued intuitionistic fuzzy Bonferroni mean [18], the revised Bonferroni mean [19], the intuitionistic fuzzy geometric Bonferroni mean [12], the hesitant fuzzy geometric Bonferroni mean, and the hesitant fuzzy Choquet geometric Bonferroni mean [20].

In reality, more differences are required to be considered in the multicriteria aggregation process due to different functions and influences among arguments in practical applications, especially, in hesitant fuzzy environment. To overcome this limitation, some weighted BMs have been developed, such as the intuitionistic fuzzy weighted Bonferroni mean [17], the interval-valued intuitionistic fuzzy weighted Bonferroni mean [18], the intuitionistic fuzzy normalized weighted Bonferroni mean [12], and the revised generalized weighted Bonferroni mean [19]. To suit the hesitant fuzzy environment, Zhou and He [12] proposed the weighted hesitant fuzzy

Bonferroni mean, and Zhu et al. [20] defined the weighted hesitant fuzzy geometric Bonferroni mean. However, a problem arises; that is, the HFBM, HFGBM, and HFCGBM cannot be obtained, respectively, when all the weights of the aggregated arguments are the same, which means the above weighted hesitant fuzzy BMs have not the reducibility that is a basic property among the weighted operators.

With the analysis above, in this paper, we propose the hesitant fuzzy reducible weighted Bonferroni mean and its generalized form and organize this paper as follows. Some basic concepts and operations are reviewed in Section 2. Sections 3 and 4, respectively, propose the hesitant fuzzy reducible weighted Bonferroni mean (HFRWBM) and the generalized hesitant fuzzy reducible weighted Bonferroni mean (GHFRWBM) and study their desirable properties. In Section 5, based on HFRWBM and GHFRWBM, a three-step aggregation approach has been proposed to deal with the hesitant fuzzy multicriteria aggregation, and a practical example is provided to demonstrate their application. The paper ends in Section 6 with concluding remarks.

2. Preliminaries

The Bonferroni mean (BM) and the generalized Bonferroni mean (GBM) were originally introduced by Bonferroni [14] and Zhou and He [16]. As two extensions of the arithmetic average, they are two very useful multicriteria aggregation operators, which consider the interrelationships among arguments. Now we introduce the concepts of the BM and GBM as follows.

Definition 1 (see [14]). Let $p, q \geq 0$, and a_i ($i = 1, 2, \dots, n$) be a collection of nonnegative numbers. Then, the Bonferroni mean (BM) is defined as

$$BM^{p,q}(a_1, a_2, \dots, a_n) = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^n a_i^p a_j^q \right)^{1/(p+q)}. \quad (1)$$

Definition 2 (see [16]). Let $p, q, r \geq 0$, and a_i ($i = 1, 2, \dots, n$) be a collection of nonnegative numbers. Then, the generalized Bonferroni mean (GBM) is defined as

$$GBM^{p,q,r}(a_1, a_2, \dots, a_n) = \left(\frac{1}{n(n-1)(n-2)} \sum_{\substack{i,j,k=1 \\ i \neq j \neq k}}^n a_i^p a_j^q a_k^r \right)^{1/(p+q+r)}. \quad (2)$$

It is obvious that the GBM could reduce to the BM if $r = 0$, and the GBM can represent the interrelationship of any three criteria. However, it is noted that the above BM and GBM can only deal with the situation that the arguments are represented by real number and ignore the weight vector of the aggregated arguments. To deal with this issue, Xu and Yager [17] extended BM to the fuzzy environment and proposed the following intuitionistic fuzzy Bonferroni mean

(IFBM) and intuitionistic fuzzy weighted Bonferroni mean (IFWBM).

Definition 3 (see [17]). Let $p, q, r \geq 0$, and α_i ($i = 1, 2, \dots, n$) be a collection of intuitionistic fuzzy values. The intuitionistic fuzzy Bonferroni mean (IFBM) and the intuitionistic fuzzy weighted Bonferroni mean (IFWBM) are, respectively, defined as

$$IFBM^{p,q}(\alpha_1, \alpha_2, \dots, \alpha_n) = \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1, i \neq j}^n (\alpha_i^p \otimes \alpha_j^q) \right)^{1/(p+q)}, \quad (3)$$

$$IFWBM^{p,q}(\alpha_1, \alpha_2, \dots, \alpha_n) = \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1, i \neq j}^n ((w_i \alpha_i^p) \otimes (w_j \alpha_j^q)) \right)^{1/(p+q)}. \quad (4)$$

Recently, Torra and Narukawa [7] originally proposed the hesitant fuzzy set which is a more general fuzzy set and permits the membership having a set of possible values.

Definition 4 (see [13]). Let X be a fixed set; a hesitant fuzzy set (HFS) on X is in terms of a function that when applied to X returns a subset of $[0, 1]$.

To be easily understood, Xia and Xu [13] expressed the HFS by a mathematical symbol:

$$E = \{ \langle x, h_E(x) \rangle \mid x \in X \}, \quad (5)$$

where $h_E(x)$ is a set of some values in $[0, 1]$ and called the hesitant fuzzy element (HFE), denoting the possible membership degrees of the element x to the set E .

Definition 5 (see [13]). For a HFE h , $s(h) = \sum_{\gamma \in h} \gamma / l(h)$ is called the score function of h , where $l(h)$ is the number of values in h and γ is element of the hesitant fuzzy set h . For two HFEs, h_1 and h_2 , if $s(h_1) > s(h_2)$, then $h_1 > h_2$; if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

There are some operational laws about any three HFEs; h , h_1 , and h_2 are as below:

- (1) $h^\lambda = \bigcup_{\gamma \in h} \{\gamma^\lambda\}$, $\lambda > 0$,
- (2) $\lambda h = \bigcup_{\gamma \in h} \{1 - (1 - \gamma)^\lambda\}$, $\lambda > 0$,
- (3) $h_1 \otimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 \gamma_2\}$,
- (4) $h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2\}$,
- (5) $\lambda(h_1 \oplus h_2) = \lambda h_1 \oplus \lambda h_2$,
- (6) $(h_1 \otimes h_2)^\lambda = h_1^\lambda \otimes h_2^\lambda$.

By referencing to the IFBM and IFWBM, the following hesitant fuzzy Bonferroni mean (HFBM) and the hesitant fuzzy weighted Bonferroni mean (HFWBM) could be defined:

$$\begin{aligned} & \text{HFBM}^{p,q}(h_1, h_2, \dots, h_n) \\ &= \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1, i \neq j}^n (h_i^p \otimes h_j^q) \right)^{1/(p+q)}, \end{aligned} \tag{6}$$

$$\begin{aligned} & \text{HFWBM}^{p,q}(h_1, h_2, \dots, h_n) \\ &= \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1, i \neq j}^n ((w_i h_i^p) \otimes (w_j h_j^q)) \right)^{1/(p+q)}. \end{aligned}$$

According to the classic arithmetic averaging (AA) and weighted averaging (WA) operators, the intuitionistic fuzzy arithmetic averaging (IFAA) and intuitionistic fuzzy weighted averaging (IFWA) operators, and the hesitant fuzzy arithmetic averaging (HFAA) and hesitant fuzzy weighted averaging (HFWA), it is found that the following *reducibility* is the basic property among these operators; that is, if $w_i = 1/n, i = 1, 2, \dots, n$, then AA = WA, IFAA = IFWA, and HFAA = HFWA.

It is obvious that the above HFBM and HFWBM are not equal when $w_i = 1/n$ and have not the *reducibility*. To deal with this issue, Xia et al. [19] proposed the generalized weighted Bonferroni mean (GWBM) and the corresponding generalized hesitant fuzzy weighted Bonferroni mean (GHFWBM) based on the GWBM, which are described as follows:

$$\begin{aligned} & \text{GWBM}^{p,q,r}(a_1, a_2, \dots, a_n) \\ &= \left(\sum_{i,j,k=1}^n w_i w_j w_k a_i^p a_j^q a_k^r \right)^{1/(p+q+r)}, \end{aligned} \tag{7}$$

$$\begin{aligned} & \text{GHFWBM}^{p,q,r}(h_1, h_2, \dots, h_n) \\ &= \left(\bigoplus_{i,j,k=1}^n w_i w_j w_k h_i^p h_j^q h_k^r \right)^{1/(p+q+r)}. \end{aligned}$$

However, another question arises; that is, the above GWBM and GHFWBM just consider the whole correlation-ship between the criterion and all criteria and cannot reflect the interrelationship between the individual criterion and other criteria which is the main advantage of the BM [15]. To overcome this drawback and introduce the reducibility, we propose the hesitant fuzzy reducible weighted Bonferroni mean (HFRWBM) and its generalized form called generalized hesitant fuzzy reducible weighted Bonferroni mean (GHFRWBM) in next sections.

3. Hesitant Fuzzy Reducible Weighted Bonferroni Mean (HFRWBM)

In multicriteria aggregation, the performance of an alternative under a criterion may be presented by several possible

values. To aggregate all the possible values of an alternative under the criteria and reflect the interrelationship between the individual criterion and other criteria, we give a new hesitant fuzzy weighted Bonferroni mean with the reducibility, which is defined as follows.

Definition 6. Let $p, q \geq 0$, and $h_i (i = 1, 2, \dots, n)$ be a collection of HFEs with the weight vector $w = (w_1, w_2, \dots, w_n)$ such that $w_i \geq 0, w_i \neq 1$, and $\sum_{i=1}^n w_i = 1$. If

$$\begin{aligned} & \text{HFRWBM}_w^{p,q}(h_1, h_2, \dots, h_n) \\ &= \left(\bigoplus_{i,j=1, i \neq j}^n \frac{w_i w_j}{1 - w_i} (h_i^p \otimes h_j^q) \right)^{1/(p+q)}, \end{aligned} \tag{8}$$

then HFRWBM^{p,q} is called a hesitant fuzzy reducible weighted Bonferroni mean (HFRWBM).

Then, we can transform the HFRWBM in to the interrelationship HFRWBM form as follows:

$$\begin{aligned} & \text{HFRWBM}_w^{p,q}(h_1, h_2, \dots, h_n) \\ &= \left(\bigoplus_{i=1}^n w_i h_i^p \left(\bigoplus_{j=1, i \neq j}^n \frac{w_j}{1 - w_i} h_j^q \right) \right)^{1/(p+q)}. \end{aligned} \tag{9}$$

We see that the term $\bigoplus_{j=1, i \neq j}^n (w_j / (1 - w_i)) h_j^q$ is the weighted power average satisfaction of all criteria except H_i , and $\bigoplus_{j=1, i \neq j}^n (w_j / (1 - w_i)) = 1$. We denote the term as v_i^q . Thus

$$\begin{aligned} & \text{HFRWBM}_w^{p,q}(h_1, h_2, \dots, h_n) \\ &= \left(\bigoplus_{i=1}^n w_i (h_i^p \otimes v_i^q) \right)^{1/(p+q)}. \end{aligned} \tag{10}$$

Here then v_i^q is the weighted power average satisfaction to all criteria except H_i , and HFRWBM^{p,q} represents the weighted interrelationship between the individual criterion h_i and other criteria v_i which is similar to the BM.

Based on the operational law of HFRWBM, we further derive the following results.

Theorem 7. Let $p, q \geq 0$, and $h_i (i = 1, 2, \dots, n)$ be a collection of HFEs with the weight vector $w = (w_1, w_2, \dots, w_n)$ such that $w_i \geq 0, w_i \neq 1$ and $\sum_{i=1}^n w_i = 1$; then the aggregation value by using the HFRWBM is a HFE, and

$$\begin{aligned} & \text{HFRWBM}_w^{p,q}(h_1, h_2, \dots, h_n) \\ &= \left(\bigoplus_{i,j=1, i \neq j}^n \kappa_{ij} \right)^{1/(p+q)} \\ &= \bigcup_{\kappa_{i,j} \in \kappa_{ij, i \neq j}} \left\{ \left(1 - \prod_{i,j=1, i \neq j}^n (1 - \kappa_{i,j}) \right)^{1/(p+q)} \right\}, \end{aligned} \tag{11}$$

where $\kappa_{ij} = (w_i w_j / (1 - w_i)) (h_i^p \otimes h_j^q)$ can be considered as “bonding satisfaction” factor used as a calculation unit, capturing the connection between h_i and h_j .

Proof. By the operational laws (1)–(6) of HFEs, we obtain

$$\begin{aligned}\kappa_{ij} &= \frac{w_i w_j}{1 - w_i} (h_i^p \otimes h_j^q) \\ &= \frac{w_i w_j}{1 - w_i} \bigcup_{\substack{r_i \in h_i \\ r_j \in h_j}} (r_i^p r_j^q) \\ &= \bigcup_{\substack{r_i \in h_i \\ r_j \in h_j}} \left(1 - (1 - r_i^p r_j^q)^{w_i w_j / (1 - w_i)} \right),\end{aligned}\quad (12)$$

which is also a HFE; then (8) can be written as

$$\text{HFRWBM}_w^{p,q}(h_1, h_2, \dots, h_n) = \left(\bigoplus_{i,j=1, i \neq j}^n \kappa_{ij} \right)^{1/(p+q)}.\quad (13)$$

Furthermore, from the operational laws of HFEs, we have

$$\left(\bigoplus_{i,j=1, i \neq j}^n \kappa_{ij} \right) = \bigcup_{k_{i,j} \in \kappa_{i,j}} \left\{ 1 - \prod_{i,j=1, i \neq j}^n (1 - k_{i,j}) \right\}.\quad (14)$$

Therefore,

$$\begin{aligned}\text{HFRWBM}_w^{p,q}(h_1, h_2, \dots, h_n) &= \left(\bigoplus_{i,j=1, i \neq j}^n \kappa_{ij} \right)^{1/(p+q)} \\ &= \bigcup_{k_{i,j} \in \kappa_{i,j}} \left\{ \left(1 - \prod_{i,j=1, i \neq j}^n (1 - k_{i,j}) \right)^{1/(p+q)} \right\}.\end{aligned}\quad (15)$$

Then the proof is completed. \square

From (8), we can see that the HFRWBM can fully represent the interrelationship between the individual criterion and other criteria by two types of conjunction calculations, that is, \oplus and \otimes . Moreover, after the original data being operated by the conjunction calculations, their values have been changed. In multicriteria aggregation problems, the HFRWBM can result in the advantage of the BM considering the individual criterion and other criteria in aggregation process, which can take much more hesitant fuzzy information into account.

The κ_{ij} can be considered as “bonding satisfaction” factor, which is the basic calculation unit of the HFRWBM. According to (8), we can derive the reducibility of HFRWBM and other useful properties as follows.

Property 1 (reducibility). Let $p, q \geq 0$, and h_i ($i = 1, 2, \dots, n$) be a collection of HFEs with the weight vector $w = (1/n, 1/n, \dots, 1/n)$; then

$$\text{HFRWBM}_w^{p,q}(h_1, h_2, \dots, h_n) = \text{HFBM}^{p,q}(h_1, h_2, \dots, h_n).\quad (16)$$

Proof. Since $w_i = 1/n$ ($i = 1, 2, \dots, n$), then, by Definition 6, we have

$$\begin{aligned}\text{HFRWBM}_w^{p,q}(h_1, h_2, \dots, h_n) &= \left(\frac{1}{n(n-1)} \bigoplus_{i,j=1, i \neq j}^n (h_i^p \otimes h_j^q) \right)^{1/(p+q)} \\ &= \text{HFBM}^{p,q}(h_1, h_2, \dots, h_n),\end{aligned}\quad (17)$$

which complete the proof of the property. \square

Property 2 (monotonicity). Let $h_\alpha = \{h_{\alpha 1}, h_{\alpha 2}, \dots, h_{\alpha n}\}$ and $h_\beta = \{h_{\beta 1}, h_{\beta 2}, \dots, h_{\beta n}\}$ be two collections of HFEs, if, for any $\gamma_{\alpha i} \in h_{\alpha i}$ and $\gamma_{\beta i} \in h_{\beta i}$, we have $\gamma_{\alpha i} \leq \gamma_{\beta i}$ for all i ($i = 1, 2, \dots, n$), and thus

$$\begin{aligned}\text{HFRWBM}_w^{p,q}(h_{\alpha 1}, h_{\alpha 2}, \dots, h_{\alpha n}) &\leq \text{HFRWBM}_w^{p,q}(h_{\beta 1}, h_{\beta 2}, \dots, h_{\beta n}).\end{aligned}\quad (18)$$

Proof. Using the terms above and (11), we have

$$\begin{aligned}\left(1 - \prod_{i,j=1, i \neq j}^n (1 - k_{\alpha i,j}) \right)^{1/(p+q)} &\leq \left(1 - \prod_{i,j=1, i \neq j}^n (1 - k_{\beta i,j}) \right)^{1/(p+q)}.\end{aligned}\quad (19)$$

Then,

$$\begin{aligned}\left(\bigoplus_{i,j=1, i \neq j}^n \kappa_{\alpha i,j} \right)^{1/(p+q)} &= \bigcup_{k_{\alpha i,j} \in \kappa_{\alpha i,j}} \left\{ \left(1 - \prod_{i,j=1, i \neq j}^n (1 - k_{\alpha i,j}) \right)^{1/(p+q)} \right\} \\ &\leq \left(\bigoplus_{i,j=1, i \neq j}^n \kappa_{\beta i,j} \right)^{1/(p+q)} \\ &= \bigcup_{k_{\beta i,j} \in \kappa_{\beta i,j}} \left\{ \left(1 - \prod_{i,j=1, i \neq j}^n (1 - k_{\beta i,j}) \right)^{1/(p+q)} \right\},\end{aligned}\quad (20)$$

which complete the proof of the property. \square

Property 3 (boundedness). Let $p, q \geq 0$, and h_i ($i = 1, 2, \dots, n$) be a collection of HFEs with the weight vector $w = (w_1, w_2, \dots, w_n)$ such that $w_i \geq 0$, $w_i \neq 1$, and $\sum_{i=1}^n w_i = 1$; we have

$$h^- \leq \text{HFRWBM}_w^{p,q}(h_1, h_2, \dots, h_n) \leq h^+, \quad (21)$$

where h^- and h^+ are lower and upper bounds, respectively [8], with $h^- = \min_i \{\gamma_i \mid \gamma_i \in h_i\}$ and $h^+ = \max_i \{\gamma_i \mid \gamma_i \in h_i\}$.

Proof. Since $h^- \leq \gamma_i \leq h^+$, for all i , then

$$\begin{aligned} (h^-)^{p+q} &= (h^-)^p \otimes (h^-)^q \leq h_i^p \otimes h_j^q \\ &\leq (h^+)^p \otimes (h^+)^q = (h^+)^{p+q}, \\ &\left((h^-)^{p+q} \bigoplus_{i,j=1,i \neq j}^n \frac{w_i w_j}{1-w_i} \right)^{1/(p+q)} \\ &\leq \left(\bigoplus_{i,j=1,i \neq j}^n \frac{w_i w_j}{1-w_i} (h_i^p \otimes h_j^q) \right)^{1/(p+q)} \\ &\leq \left((h^+)^{p+q} \bigoplus_{i,j=1,i \neq j}^n \frac{w_i w_j}{1-w_i} \right)^{1/(p+q)}. \end{aligned} \tag{22}$$

Therefore,

$$\begin{aligned} h^- &\leq \text{HFRWBM}_w^{p,q}(h_1, h_2, \dots, h_n) \\ &= \left(\bigoplus_{i,j=1,i \neq j}^n \frac{w_i w_j}{1-w_i} (h_i^p \otimes h_j^q) \right)^{1/(p+q)} \leq h^+, \end{aligned} \tag{23}$$

which complete the proof of the property. \square

Property 4 (idempotency). Let $p, q \geq 0$, and h_i ($i = 1, 2, \dots, n$) be a collection of HFEs with the weight vector $w = (w_1, w_2, \dots, w_n)$ such that $w_i \geq 0$, $w_i \neq 1$, and $\sum_{i=1}^n w_i = 1$. If all h_i are equal; that is, $h_i = h$ for all i , then

$$\text{HFRWBM}_w^{p,q}(h_1, h_2, \dots, h_n) = h. \tag{24}$$

Proof. Since $h_i = h$ ($i = 1, 2, \dots, n$), then

$$\begin{aligned} &\text{HFRWBM}_w^{p,q}(h_1, h_2, \dots, h_n) \\ &= \left(\bigoplus_{i,j=1,i \neq j}^n \frac{w_i w_j}{1-w_i} (h_i^p \otimes h_j^q) \right)^{1/(p+q)} \\ &= \left(\bigoplus_{i,j=1,i \neq j}^n \frac{w_i w_j}{1-w_i} h^{p+q} \right)^{1/(p+q)} \\ &= \left(h^{p+q} \bigoplus_{i,j=1,i \neq j}^n \frac{w_i w_j}{1-w_i} \right)^{1/(p+q)} = (h^{p+q})^{1/(p+q)} = h, \end{aligned} \tag{25}$$

which complete the proof of the property. \square

Now, if we change the parameters p and q of the HFRWBM, then we can get some special cases as follows.

Case 1. If $p = 0$ and $q = 0$, based on (8), we have

$$\text{HFRWBM}_w^{0,0}(h_1, h_2, \dots, h_n) = \bigoplus_{i=1}^n w_i = 1. \tag{26}$$

Case 2. If $p = 1$ and $q = 0$, based on (8), we get

$$\begin{aligned} &\text{HFRWBM}_w^{1,0}(h_1, h_2, \dots, h_n) \\ &= \left(\bigoplus_{i,j=1,i \neq j}^n \frac{w_i w_j}{1-w_i} (h_i^1 \otimes h_j^0) \right)^{1/1} \\ &= \bigoplus_i^n w_i h_i = \text{HFWA}(h_1, h_2, \dots, h_n). \end{aligned} \tag{27}$$

Case 3. If $p = 1$ and $q = 1$, then (8) is transformed as

$$\begin{aligned} &\text{HFRWBM}_w^{1,1}(h_1, h_2, \dots, h_n) \\ &= \left(\bigoplus_{i,j=1,i \neq j}^n \frac{w_i w_j}{1-w_i} (h_i \otimes h_j) \right)^{1/2}, \end{aligned} \tag{28}$$

which we call a hesitant fuzzy reducible square weighted Bonferroni mean (HFRSWBM).

Case 4. If $p = 2$ and $q = 0$, then (8) is transformed as

$$\text{HFRWBM}_w^{2,0}(h_1, h_2, \dots, h_n) = \left(\bigoplus_{i=1}^n w_i h_i^2 \right)^{1/2}, \tag{29}$$

which we call a hesitant fuzzy Euclidean distance weighted Bonferroni mean (HFEDWBM).

Based on the above four special cases, we find that (1) Case 1 presents orthonormality of the HFRWBM operator which is an important property of the weighted aggregation operator; (2) Case 2 proves the HFWA operator is just a special case of the new hesitant fuzzy weighted aggregation operator; (3) Case 3 proposes the simplest interrelated HFRWBM operator; (4) Case 4 shows that the new HFRWBM operator could be used to compute the Euclidean distance between two HFEs. Generally, the HFRWBM^{1,1} is recommended to use in hesitant fuzzy multicriteria aggregation.

4. Generalized Hesitant Fuzzy Reducible Weighted Bonferroni Mean (GHFRWBM)

In this section, by considering the correlation of any three aggregated arguments instead of any two, we further extend the HFRWBM to a generalized form, that is, the generalized hesitant fuzzy reducible weighted Bonferroni mean (GHFRWBM), based on the HFRWBM operator.

Definition 8. Let $p, q, r \geq 0$, and h_i ($i = 1, 2, \dots, n$) be a collection of HFEs with the weight vector $w = (w_1, w_2, \dots, w_n)$ such that $w_i \geq 0, w_i \neq 1$, and $\sum_{i=1}^n w_i = 1$. If

$$\begin{aligned} & \text{GHFRWBM}_w^{p,q,r}(h_1, h_2, \dots, h_n) \\ &= \left(\bigoplus_{i,j,t=1, i \neq j \neq t}^n \frac{w_i w_j w_t}{(1-w_i)(1-w_i-w_j)} \right. \\ & \quad \left. \times (h_i^p \otimes h_j^q \otimes h_t^r) \right)^{1/(p+q+r)}, \end{aligned} \tag{30}$$

then $\text{GHFRWBM}_w^{p,q,r}$ is called a generalized hesitant fuzzy reducible weighted Bonferroni mean.

Then, we can transform the GHFRWBM in to two inter-relationship forms as follows:

$$\begin{aligned} & \text{GHFRWBM}_w^{p,q,r}(h_1, h_2, \dots, h_n) \\ &= \left(\bigoplus_{i=1}^n w_i h_i^p \bigoplus_{j=1, i \neq j}^n n_j h_j^q \bigoplus_{t=1, t \neq i \neq j}^n m_t h_t^r \right)^{1/(p+q+r)}, \end{aligned} \tag{31}$$

$$\begin{aligned} & \text{GHFRWBM}_w^{p,q,r}(h_1, h_2, \dots, h_n) \\ &= \left(\bigoplus_{i=1}^n w_i h_i^p \left(\bigoplus_{j,t=1, i \neq j \neq t}^n u_{j,t} (h_j^q \otimes h_t^r) \right) \right)^{1/(p+q+r)}, \end{aligned} \tag{32}$$

where $n_j = w_j/(1-w_i)$, $m_t = w_t/(1-w_i-w_j)$, and $u_{j,t} = w_j w_t / (1-w_i)(1-w_i-w_j)$.

From (31), we see that the term $\bigoplus_{j=1, i \neq j}^n n_j h_j^q$ is the weighted power average satisfaction to all criteria except i , with $\sum_{j=1, j \neq i}^n n_j = 1$. The term $\bigoplus_{t=1, t \neq i \neq j}^n m_t h_t^r$ is the weighted power average satisfaction of all criteria except i and j , with $\sum_{t=1, t \neq j \neq i}^n m_t = 1$. Then, the new operator represents the interrelationship between any three aggregated arguments, which is similar to the GBM. From (32), we see that the term $\bigoplus_{j,t=1, i \neq j \neq t}^n u_{j,t} (h_j^q \otimes h_t^r)$ is the weighted power average satisfaction to any two criteria i with $\sum_{j,t=1, j \neq t \neq i}^n u_{j,t} = 1$, which represents another interrelationship form between any three aggregated arguments.

Moreover, based on the operational law of HFEs, HFRWBM, and GHFRWBM, we further derive the following results.

Theorem 9. Let $p, q, r \geq 0$, and h_i ($i = 1, 2, \dots, n$) be a collection of HFEs with the weight vector $w = (w_1, w_2, \dots, w_n)$

such that $w_i \geq 0, w_i \neq 1$, and $\sum_{i=1}^n w_i = 1$; then the aggregation value by using the GHFRWBM is a HFE, and

$$\begin{aligned} & \text{GHFRWBM}_w^{p,q,r}(h_1, h_2, \dots, h_n) \\ &= \left(\bigoplus_{\substack{i,j,t=1 \\ i \neq j \neq t}}^n \kappa_{ijt} \right)^{1/(p+q+r)} \\ &= \bigcup_{k_{i,j,t} \in \kappa_{ijt, i \neq j \neq t}} \left\{ \left(1 - \prod_{\substack{i,j,t=1 \\ i \neq j \neq t}}^n (1 - k_{i,j,t}) \right)^{1/(p+q+r)} \right\}, \end{aligned} \tag{33}$$

where $n_j = w_j/(1-w_i)$, $m_t = w_t/(1-w_i-w_j)$, and $\kappa_{ijt} = w_i n_j m_t (h_i^p \otimes h_j^q \otimes h_t^r)$. κ_{ijt} can be considered a “bonding satisfaction” factor, capturing the connection among h_i, h_j , and h_t .

Proof. By the operational laws (1)–(6) of HFEs and (12), we get

$$\begin{aligned} \kappa_{ijt} &= w_i n_j m_t (h_i^p \otimes h_j^q \otimes h_t^r) \\ &= \bigcup_{\substack{r_i \in h_i \\ r_j \in h_j \\ r_t \in h_t}} (1 - (1 - r_i^p r_j^q r_t^r)^{w_i n_j m_t}), \end{aligned} \tag{34}$$

which is also a HFE; then (8) can be written as

$$\begin{aligned} & \text{GHFRWBM}_w^{p,q,r}(h_1, h_2, \dots, h_n) \\ &= \left(\bigoplus_{i,j,t=1, i \neq j \neq t}^n \kappa_{ijt} \right)^{1/(p+q+r)}. \end{aligned} \tag{35}$$

Furthermore, from the operational law of HFEs, we have

$$\begin{aligned} & \left(\bigoplus_{i,j,t=1, i \neq j \neq t}^n \kappa_{ijt} \right)^{1/(p+q+r)} \\ &= \bigcup_{k_{i,j,t} \in \kappa_{ijt, i \neq j \neq t}} \left\{ 1 - \prod_{i,j,t=1, i \neq j \neq t}^n (1 - k_{i,j,t}) \right\}. \end{aligned} \tag{36}$$

Therefore,

$$\begin{aligned} & \text{GHFRWBM}_w^{p,q,r}(h_1, h_2, \dots, h_n) \\ &= \left(\bigoplus_{\substack{i,j,t=1 \\ i \neq j \neq t}}^n \kappa_{ijt} \right)^{1/(p+q+r)} \\ &= \bigcup_{k_{i,j,t} \in \kappa_{ijt, i \neq j \neq t}} \left\{ \left(1 - \prod_{\substack{i,j,t=1 \\ i \neq j \neq t}}^n (1 - k_{i,j,t}) \right)^{1/(p+q+r)} \right\}. \end{aligned} \tag{37}$$

Then the proof is completed. \square

According to (30), we can derive the reducibility, monotonicity, boundedness, and idempotency of the GHFRWBM operator as follows.

Property 5 (reducibility). Let $p, q, r \geq 0$, and h_i ($i = 1, 2, \dots, n$) be a collection of HFEs with the weight vector $w = (1/n, 1/n, \dots, 1/n)$; then

$$\begin{aligned} \text{GHFRWBM}_w^{p,q,r}(h_1, h_2, \dots, h_n) \\ = \text{GHFBM}_w^{p,q,r}(h_1, h_2, \dots, h_n). \end{aligned} \tag{38}$$

Proof. Since $w_i = 1/n$ ($i = 1, 2, \dots, n$), then by Definition 8, we have

$$\begin{aligned} \text{GHFRWBM}_w^{p,q,r}(h_1, h_2, \dots, h_n) \\ = \left(\frac{1}{n(n-1)(n-2)} \right. \\ \left. \times \bigoplus_{i,j,t=1, i \neq j \neq t}^n (h_i^p \otimes h_j^q \otimes h_t^r) \right)^{1/(p+q+r)} \\ = \text{GHFBM}_w^{p,q,r}(h_1, h_2, \dots, h_n), \end{aligned} \tag{39}$$

which complete the proof of the property. \square

Property 6 (monotonicity). Let $h_\alpha = \{h_{\alpha 1}, h_{\alpha 2}, \dots, h_{\alpha n}\}$ and $h_\beta = \{h_{\beta 1}, h_{\beta 2}, \dots, h_{\beta n}\}$ be two collections of HFEs, if, for any $\gamma_{\alpha i} \in h_{\alpha i}$ and $\gamma_{\beta i} \in h_{\beta i}$, we have $\gamma_{\alpha i} \leq \gamma_{\beta i}$ for all i ($i = 1, 2, \dots, n$), and thus

$$\begin{aligned} \text{GHFRWBM}_w^{p,q,r}(h_{\alpha 1}, h_{\alpha 2}, \dots, h_{\alpha n}) \\ \leq \text{GHFRWBM}_w^{p,q,r}(h_{\beta 1}, h_{\beta 2}, \dots, h_{\beta n}). \end{aligned} \tag{40}$$

Proof. Using the terms above and (33), we have

$$\begin{aligned} \left(1 - \prod_{\substack{i,j,t=1 \\ i \neq j \neq t}}^n (1 - k_{\alpha_{i,j,t}}) \right)^{1/(p+q+r)} \\ \leq \left(1 - \prod_{\substack{i,j,t=1 \\ i \neq j \neq t}}^n (1 - k_{\beta_{i,j,t}}) \right)^{1/(p+q+r)}. \end{aligned} \tag{41}$$

Then,

$$\begin{aligned} \left(\bigoplus_{i,j,t=1, i \neq j \neq t}^n k_{\alpha_{i,j,t}} \right)^{1/(p+q+r)} \\ = \bigcup_{k_{\alpha_{i,j,t}} \in K_{\alpha_{i,j,t}, i \neq j \neq t}} \left\{ \left(1 - \prod_{\substack{i,j,t=1 \\ i \neq j \neq t}}^n (1 - k_{\alpha_{i,j,t}}) \right)^{1/(p+q+r)} \right\} \\ \leq \left(\bigoplus_{i,j,t=1, i \neq j \neq t}^n k_{\beta_{i,j,t}} \right)^{1/(p+q+r)} \\ = \bigcup_{k_{\beta_{i,j,t}} \in K_{\beta_{i,j,t}, i \neq j \neq t}} \left\{ \left(1 - \prod_{\substack{i,j,t=1 \\ i \neq j \neq t}}^n (1 - k_{\beta_{i,j,t}}) \right)^{1/(p+q+r)} \right\}, \end{aligned} \tag{42}$$

which complete the proof of the property. \square

Property 7 (boundedness). Let $p, q, r \geq 0$, and h_i ($i = 1, 2, \dots, n$) be a collection of HFEs with the weight vector $w = (w_1, w_2, \dots, w_n)$ such that $w_i \geq 0, w_i \neq 1$, and $\sum_{i=1}^n w_i = 1$; we have

$$h^- \leq \text{GHFRWBM}_w^{p,q,r}(h_1, h_2, \dots, h_n) \leq h^+, \tag{43}$$

where h^- and h^+ are lower and upper bounds, respectively [8], with $h^- = \min_i \{\gamma_i \mid \gamma_i \in h_i\}$ and $h^+ = \max_i \{\gamma_i \mid \gamma_i \in h_i\}$.

Proof. Since $h^- \leq \gamma_i \leq h^+$, for all i , then

$$\begin{aligned} (h^-)^{p+q+r} &= (h^-)^p \otimes (h^-)^q \otimes (h^-)^r \\ &\leq h_i^p \otimes h_j^q \otimes h_t^r \leq (h^+)^p \otimes (h^+)^q \otimes (h^+)^r \\ &= (h^+)^{p+q+r}, \end{aligned} \tag{44}$$

$$\begin{aligned} h^- &\leq \left(\bigoplus_{i,j,t=1, i \neq j \neq t}^n \frac{w_i w_j w_t}{(1-w_i)(1-w_i-w_j)} \right. \\ &\quad \left. \times (h_i^p \otimes h_j^q \otimes h_t^r) \right)^{1/(p+q+r)} \leq h^+. \end{aligned}$$

Therefore,

$$h^- \leq \text{GHFRWBM}_w^{p,q,r}(h_1, h_2, \dots, h_n) \leq h^+, \tag{45}$$

which complete the proof of the property. \square

Property 8 (idempotency). Let $p, q, r \geq 0$, and h_i ($i = 1, 2, \dots, n$) be a collection of HFEs with the weight vector $w = (w_1, w_2, \dots, w_n)$ such that $w_i \geq 0, w_i \neq 1$, and $\sum_{i=1}^n w_i = 1$. If all h_i are equal; that is, $h_i = h$ for all i , then

$$\text{GHFRWBM}_w^{p,q,r}(h_1, h_2, \dots, h_n) = h. \tag{46}$$

Proof. Since $h_i = h$ ($i = 1, 2, \dots, n$), then

$$\begin{aligned} & \text{GHFRWBM}_w^{p,q,r}(h_1, h_2, \dots, h_n) \\ &= \left(\bigoplus_{\substack{i,j,t=1 \\ i \neq j \neq t}}^n \frac{w_i w_j w_t}{(1-w_i)(1-w_i-w_j)} (h)^{p+q+r} \right)^{1/(p+q+r)} \\ &= (h^{p+q})^{1/(p+q)} = h, \end{aligned} \tag{47}$$

which complete the proof of the property. □

Now, if we change the parameters p, q , and r of the HFRWBM, then we can get some special cases as follows.

Case 1. If $r = 0$, based on (30), we have

$$\begin{aligned} & \text{GHFRWBM}_w^{p,q,r}(h_1, h_2, \dots, h_n) \\ &= \text{HFRWBM}_w^{p,q}(h_1, h_2, \dots, h_n). \end{aligned} \tag{48}$$

Case 2. If $p = 0, q = 0$, and $r = 0$, based on (30), we get

$$\text{GHFRWBM}_w^{0,0,0}(h_1, h_2, \dots, h_n) = \bigoplus_{i=1}^n w_i = 1. \tag{49}$$

Case 3. If $p = 1, q = 0$, and $r = 0$, based on (30), we have

$$\begin{aligned} & \text{HFRWBM}_w^{1,0,0}(h_1, h_2, \dots, h_n) = \bigoplus_i^n w_i h_i \\ &= \text{HFWA}(h_1, h_2, \dots, h_n). \end{aligned} \tag{50}$$

Case 4. If $p = 1, q = 1$, and $r = 1$, then (30) is transformed as

$$\begin{aligned} & \text{GHFRWBM}_w^{1,1,1}(h_1, h_2, \dots, h_n) \\ &= \left(\bigoplus_{\substack{i,j,t=1, i \neq j \neq t}}^n \frac{w_i w_j w_t}{(1-w_i)(1-w_i-w_j)} \right. \\ & \quad \left. \times (h_i \otimes h_j \otimes h_t) \right)^{1/3}, \end{aligned} \tag{51}$$

which we call a generalized hesitant fuzzy reducible triple weighted BM (GHFRTWBM).

Case 5. If $p = 2, q = 0$, and $r = 0$, then (30) is transformed as

$$\text{GHFRWBM}_w^{2,0,0}(h_1, h_2, \dots, h_n) = \left(\bigoplus_{i=1}^n w_i h_i^2 \right)^{1/2}, \tag{52}$$

which we call a generalized hesitant fuzzy Euclidean distance weighted BM (GHFREWBM).

Based on the above five cases, we find that (1) Case 1 proves that the GHFRWBM is a generalized form of the

HFRWBM and reduces to the HFRWBM when $r = 0$; (2) Case 2 presents orthonormality of the GHFRWBM operator; (3) Case 3 indicates the HFWA operator is a special case of the GHFRWBM; (4) Case 4 proposes the simplest interrelated GHFRWBM; (5) Case 5 shows that the GHFRWBM could be used to compute the Euclidean distance between two HFEs.

5. An Approach to Hesitant Fuzzy Multicriteria Aggregation

Based on Definitions 6 and 8 and Theorems 7 and 9, below we develop a three-step approach for multicriteria aggregation under hesitant fuzzy environment.

Step 1. For a hesitant fuzzy multicriteria aggregation problem, let $Y = \{y_1, y_2, \dots, y_m\}$ be a set of m alternative, and let $C = \{c_1, c_2, \dots, c_n\}$ be a set of n criteria, whose weight vector is $w = (w_1, w_2, \dots, w_n)^T$, such that $w_j \geq 0$ and $\sum_{j=1}^n w_j = 1$, where w_j denotes the important degree of the criteria c_j . The decision makers provide all possible values so that the alternative y_i satisfies the criteria c_j represented by a HFE $h_{ij} = \cup_{r_{ij} \in h_{ij}} \{r_{ij}\}$ and all h_{ij} construct a hesitant fuzzy multicriteria aggregation matrix $H_{ij} = (h_{ij})_{m \times n}$ with $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

Step 2. Utilize the HFRWBM or GHFRWBM (here, we recommend to take square or triple operator, respectively; i.e., $p = q = 1$ or $p = q = r = 1$) to aggregate all the performance values h_{ij} of the i th line and get the overall performance value h_i corresponding to the alternative y_i ; that is, $h_i = \text{HFRWBM}_w^{1,1}(h_{i1}, h_{i2}, \dots, h_{in})$ or $h_i = \text{GHFRWBM}_w^{1,1,1}(h_{i1}, h_{i2}, \dots, h_{in})$.

Step 3. Compute the scores $s(h_i)$ of h_i and rank all the alternatives y_i ($i = 1, 2, \dots, n$) according to $s(h_i)$ in descending order and choose the optimal alternative.

In the following, we apply the HFRWBM and GHFRWBM to a multicriteria aggregation problem under the hesitant fuzzy environment.

Example. Let us consider a company which intends to rent a house for its local office. Three alternatives y_i ($i = 1, 2, 3$) are available, and the decision makers consider three criteria to decide which house to choose: c_1 (price), c_2 (location), and c_3 (transportation). The weight vector of three criteria is $w = (0.4, 0.3, 0.3)^T$ [21]. Assume that the characteristics of the alternatives y_i with respect to the criteria c_j ($j = 1, 2, 3$) are represented by HFEs $h_{ij} = \cup_{r_{ij} \in h_{ij}} \{r_{ij}\}$, where r_{ij} indicates the degree to which the alternative y_i satisfies the criterion c_j . All h_{ij} are contained in a hesitant fuzzy multicriteria aggregation matrix $H_{ij} = (h_{ij})_{3 \times 3}$ (see Table 1).

In Table 1, all criteria c_j are transformed to the benefit type criteria and do not need normalization. We first utilize the HFRWBM (here, we take $p = q = 1$, that is, the HFRSWBM) to aggregate all the performance values h_{ij} of the i th line and get the overperformance value h_i corresponding

TABLE 1: The hesitant fuzzy multicriteria aggregation matrix.

Alternatives	Criteria		
	c_1	c_2	c_3
y_1	(0.68, 0.77)	(0.45, 0.55, 0.60)	(0.82, 0.95)
y_2	(0.72, 0.83, 0.91)	(0.78)	(0.15, 0.27, 0.35)
y_3	(0.21, 0.33)	(0.82)	(0.67, 0.75)

to the alternative y_i ; then we calculate the scores of all the alternatives next:

$$s(h_1) = 0.5051, \quad s(h_2) = 0.3892, \quad s(h_3) = 0.4453. \quad (53)$$

Therefore, $y_1 > y_3 > y_2$, and the alternative y_1 is the optimal alternative.

Now, we use the GHFRWBM operator (here, we take $p = q = r = 1$, i.e., the GHFRSWBM) to aggregate the hesitant fuzzy data in Table 1 and obtain the scores of three alternatives as follows:

$$s(h_1) = 0.2078, \quad s(h_2) = 0.1754, \quad s(h_3) = 0.1321. \quad (54)$$

Therefore, we give the ranking of y_i as $y_1 > y_2 > y_3$, and the alternative y_1 is still the optimal alternative.

Based on the aforementioned numerical results, there are three phenomena that could be found.

In the first, the whole ranking of the alternatives has changed, despite the fact that the optimal decision has been unchanged. That is because the HFRWBM operator pays more attention to some arguments, whose performances are too high or too low, while the GHFRWBM operator focuses on the whole arguments instead. In other words, the HFRWBM operator reflects the interrelationship between the individual criterion and other criteria, but the GHFRWBM operator presents the interrelationship of any three criteria.

In the second, the aggregated scores by the GHFRWBM are smaller than the corresponding aggregated scores by the HFRWBM. That is because the GHFRWBM operator is a three-layer multiplication calculation and the HFRWBM operator is a two-layer multiplication calculation, which could be seen from (8) and (30). Therefore, we will get the smaller aggregated scores and the lower identification when the GHFRWBM operator is used to aggregate any four or more criteria at once. Just for this, we propose the GHFRWBM operator to use a three-layer multiplication calculation, that is, $\text{GHFRWBM}^{p,q,r}$.

In the last, the aggregated hesitant fuzzy values derived by the HFRWBM or GHFRWBM operators depend on the choice of the parameters p , q , and r . In general, the bigger parameters are, the more calculation effort needed. Then, we recommend taking $p = q = r = 1$ and corresponding to the HFRSWBM and GHFRWBM, respectively, which is not only an intuitive and simple method but also represents an interrelationship of three individual arguments.

6. Concluding Remarks

The Bonferroni mean (BM) operator is a traditional mean type aggregation operator bounded by the max and min operators, which can capture the expressed interrelationship of the individual arguments. In this paper, we have extended the weighted BM to accommodate hesitant fuzzy environment and proposed the hesitant fuzzy reducible weighted Bonferroni mean (HFRWBM) and defined the generalized form of this new operator. Next, we have proved and discussed the desirable characteristics of the HFRWBM and GHFRWBM operators which include reducibility, monotonicity, boundedness, and idempotency. Some special cases have been given to represent these desirable characteristics and compare the different HFRWBMs and GHFRWBMs with changed parameters p , q , and r . It is worth noting that the aggregated hesitant fuzzy values derived by the HFRWBM or GHFRWBM operator depend on the choice of the above parameters, and these parameters are not robust.

In the end, to deal with the situation that the criteria have connections in hesitant fuzzy multicriteria aggregation, a three-step aggregation approach has been proposed on the basis of the HFRWBM and GHFRWBM hesitant fuzzy aggregation operators. Then, an example has been provided to illustrate our results.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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