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Research Article

Almost Surely Exponential Stability of Numerical Solutions for Stochastic Pantograph Equations

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Our effort is to develop a criterion on almost surely exponential stability of numerical solution to stochastic pantograph differential equations, with the help of the discrete semimartingale convergence theorem and the technique used in stable analysis of the exact solution. We will prove that the Euler-Maruyama (EM) method can preserve almost surely exponential stability of stochastic pantograph differential equations under the linear growth conditions. And the backward EM method can reproduce almost surely exponential stability for highly nonlinear stochastic pantograph differential equations. A highly nonlinear example is provided to illustrate the main theory.

1. Introduction

Stochastic unbounded delay systems play an important role in a variety of application areas, including biology, epidemiology, mechanic, economics, and finance. The systems provide powerful models, such as infinite delay Kolmogorov-type systems in mathematic biology [1–4], stochastic neural networks [5–8], and stochastic pantograph equations in science and engineering. The pantograph equation which is a very special unbounded delay equation was used by Ockendon and Tayler [9] in 1971 to study how the electric current is collected by the pantograph of an electric locomotive, from where it gets its name. Such systems have received an increasing attention (see e.g., [3, 10–13]).

Unfortunately, most of stochastic differential equations cannot be solved explicitly. Especially, explicit solutions can rarely be obtained for nonlinear stochastic pantograph equations, so numerical methods have recently received more and more attention (see [6, 8, 13–15]). Most of research efforts have been devoted to the convergence and mean-square stability of various numerical methods for the linear delay systems [14, 16–19]. Recently, several authors were devoted to the convergence in probability of the Euler-Maruyama (EM) method for the nonlinear delay systems. For example, Mao [20] and Milošević [21] developed the convergence in probability of the EM approximate solution for nonlinear SDDE and neutral SDDE under the Khasminskii-type conditions, respectively. Zhou et al. [22, 23] established the convergence in probability of the EM approximation for neutral stochastic functional differential equation under the polynomial growth conditions.

Stability theory of numerical solution is one key problem in numerical analysis. Compared with the convergence, the study on stability of numerical methods for delay systems is relatively scarce due to infinite time-delay that is often the source of instability. Research efforts have been devoted to various stabilities of numerical methods for SDEs. For example, Higham et al. [15, 24–26] investigated stability of numerical methods for SDEs. Pang et al. [27] showed that the EM discretization can capture almost surely and moment exponential stability for all sufficiently small time-step under appropriate conditions for linear hybrid SDE. Mao et al. [28] showed that the backward EM method can reproduce almost surely exponential stability of nonlinear hybrid SDE. Mao and Szpruch [29] developed almost surely asymptotic properties of implicit numerical methods for nonlinear SDEs, via a stochastic version of LaSalle principle. However, the stability of numerical method for highly nonlinear stochastic delay system is less studied, due to time-delay that is often the source of instability (see [2, 30, 31]), which is the main topic of the present paper.

The stability of numerical method is inspired by Wu et al.'s paper [32], in which they first studied that the backward EM method can reproduce almost surely exponential stability of nonlinear stochastic differential delay system

$$\mathrm{d}x(t) = f(x(t), x(t-\tau))\,\mathrm{d}t + g(x(t), x(t-\tau))\,\mathrm{d}w(t), \tag{1}$$

by using of the discrete semi-martingale convergence theorem, under the following conditions

$$2x^T f(x,0) \le -\lambda_1 |x|^2, \tag{2}$$

$$|f(x,y) - f(x,0)|^2 \le \lambda_2 |y|^2, \tag{3}$$

$$|g(x,y)|^2 \le \lambda_3 |x|^2 + \lambda_4 |y|^2. \tag{4}$$

Wu et al.'s work is a very important contribution to numerical SDDE theory. Certainly, we also see that the theory imposes the one-sided linear growth on the drift coefficient (i.e., (3)) and the linear growth on the diffusion coefficient (i.e., (4)), which are rather strong so that many highly nonlinear SDDEs are excluded. Recently, Zhou et al. [33] studied exponential stability of stochastic functional differential equation under polynomial growth condition. To the best of author's knowledge, there is no work on stability of numerical method for stochastic pantograph differential equations. In the paper, our effort is to develop a new criterion on almost surely exponential stability of numerical solution to stochastic pantograph differential equations, with the help of the discrete semimartingale convergence theorem and the technique used in stability analysis of the exact solution. We will prove that the Euler-Maruyama method can preserve the almost surely exponential stability of stochastic pantograph differential equations under the linear growth conditions. The backward EM method can preserve almost surely exponential stability for highly nonlinear stochastic pantograph differential equations under sufficiently small step size.

2. The Global Solution

Throughout this paper, unless otherwise specified, let $|x|$ be the Euclidean norm in $x \in \mathbb{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}(A^T A)}$, while its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$, satisfying the usual conditions (i.e., it is increasing and right continuous and $\mathcal{F}_0$ contains all P-null sets). Let $w(t) = (w_1(t), w_2(t), \ldots, w_m(t))$ be m-dimensional Brownian motion.

Consider an n-dimensional stochastic pantograph equation

$$\mathrm{d}x(t) = f(x(t), x(qt))\,\mathrm{d}t + g(x(t), x(qt))\,\mathrm{d}w(t), \quad 0 < q < 1 \tag{5}$$

on $t \ge 0$ with the initial data $x_0 \in R^n$); $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are locally Lipschitz continuous.

(H1) The Local Lipschitz Condition. For each integer $R \ge 1$, there exists a positive constant K_R such that

$$\begin{aligned} &|f(x_1, y_1) - f(x_2, y_2)| \vee |g(x_1, y_1) - g(x_2, y_2)| \\ &\qquad \le K_R(|x_1 - x_2| + |y_1 - y_2|) \end{aligned} \tag{6}$$

for all $x_k, y_k \in \mathbb{R}^n$ with $|x_k| \vee |y_k| \le R$ $(k = 1, 2)$.

(H2) The Polynomial Growth Conditions. For all $x \in \mathbb{R}^n$, there exist positive constants $\alpha, \beta, \gamma, a, \bar{a}, \tilde{a}, b, \bar{b}, \tilde{b}$ such that

$$\begin{aligned} &\langle x(s), f(x(s), x(qs)) \rangle \\ &\quad \le -a|x(s)|^{\alpha+2} + \bar{a}\left(|x(qs)|^{\beta+2} + |x(s)|^{\beta+2}\right) - \tilde{a}|x(s)|^2, \end{aligned} \tag{7}$$

$$|g(x(s), x(qs))|^2 \le b|x(s)|^{\gamma+2} + \bar{b}|x(qs)|^{\gamma+2} + \tilde{b}|x(s)|^2. \tag{8}$$

Lemma 1 (see [1]). *Let $\alpha, b > 0$, $\kappa(x) \in C(\mathbb{R}_+; \mathbb{R})$; if $\kappa(x) = o(|x|^\alpha)$ $(|x| \to \infty)$, then*

$$\sup_{x\in\mathbb{R}_+} [\kappa(x) - b|x|^\alpha] < \infty. \tag{9}$$

Theorem 2. *Let (H1) and (H2) hold with $\alpha \ge \beta \vee \gamma$, $2\tilde{a} > \tilde{b}$, $2a > 2\bar{a}(1 + 1/q) + b + \bar{b}/q$. Then for any initial data x_0, there almost surely exists a unique global solution $x(t)$ to (5) on $t \ge 0$.*

Proof. Under (H1), applying the standing truncation technique to (5) for any initial data x_0, there exists a unique maximal local strong solution $0 < t < \nu_e$, where ν_e is the explosion time. To show this solution is global, we only need to show that $\nu_e = \infty$ a.s. Let $k_0 > 0$ be sufficiently large such that $k_0 > |x_0|$. For each integer $k \ge k_0$, define the stopping time

$$\nu_k = \inf\{t \in [0, \nu_e) : |x(t)| \ge k\}, \quad k \in N, \tag{10}$$

where throughout this paper, we set $\inf \emptyset = \infty$ (as usual, $\emptyset$ = the empty set). By the definition of the stopping time ν_k, it is obvious that ν_k is an increasing function with k, so $\nu_k \to \nu_\infty \le \nu_e$ $(k \to \infty)$ a.s. If we can show that $\nu_\infty = \infty$ a.s., then $\nu_e = \infty$ a.s. which implies that $x(t)$ is global. In other words, we only prove that $P(\nu_k \le t) \to 0$ $(k \to \infty, t > 0)$. Define $V(x) = |x|^2$, by $P(\nu_k \le t)V(x(\nu_k)) \le EV(x(t \wedge \nu_k))$; we only need to prove that $EV(x(t \wedge \nu_k)) < +\infty$ according to $V(x(\nu_k)) = |x(\nu_k)|^2 = k^2 \to \infty$. For $\varepsilon > 0$, by the Itô formula, we have

$$\begin{aligned} V(x(t \wedge \nu_k)) = V(x(0)) + \int_0^{t\wedge\nu_k} \Big(&2x^T f(x(s), x(qs)) \\ &+ |g(x(s), x(qs))|^2\Big)\,\mathrm{d}s \\ &+ M(t), \end{aligned} \tag{11}$$

where $M(t) = \int_0^{t\wedge\nu_k} \mathrm{e}^{\varepsilon s} V_x(x(s)) g(x(s), x(qs)) \mathrm{d}w(s)\mathrm{d}s$ is a real-valued continuous local martingale with $M(0) = 0$. By (7) and (8), we may also estimate

$$\begin{aligned}
&2x^T f(x(s), x(qs)) + |g(x(s), x(qs))|^2 \\
&\quad\le -2a|x(s)|^{\alpha+2} + 2\bar{a}\left(|x(qs)|^{\beta+2} + |x(s)|^{\beta+2}\right) - 2\tilde{a}|x(s)|^2 \\
&\qquad + b|x(s)|^{\gamma+2} + \bar{b}|x(qs)|^{\gamma+2} + \tilde{b}|x(s)|^2 \\
&\quad\le \frac{2\bar{a}}{q}\left(q|x(qs)|^{\beta+2} - |x(s)|^{\beta+2}\right) \\
&\qquad + \frac{\bar{b}}{q}\left(q|x(qs)|^{\gamma+2} - |x(s)|^{\gamma+2}\right) - 2a|x(s)|^{\alpha+2} \\
&\qquad - \left(2\tilde{a} - \tilde{b}\right)|x(s)|^2 + 2\bar{a}\left(1 + \frac{1}{q}\right)|x(s)|^{\beta+2} \\
&\qquad + \left(b + \frac{\bar{b}}{q}\right)|x(s)|^{\gamma+2}.
\end{aligned} \tag{12}$$

Denoted by

$$\begin{aligned}
I(x(s)) &= \left(2\tilde{a} - \tilde{b}\right)|x(s)|^2 + 2a|x(s)|^{\alpha+2} \\
&\quad - 2\bar{a}\left(1 + \frac{1}{q}\right)|x(s)|^{\beta+2} - \left(b + \frac{\bar{b}}{q}\right)|x(s)|^{\gamma+2}.
\end{aligned} \tag{13}$$

Noting that $2\tilde{a} > \tilde{b}$, $2a > 2\bar{a}(1 + 1/q) + b + \bar{b}/q$, $\alpha \ge \beta \vee \gamma$, by Lemma 1, there exists a positive c_0 such that $I(x(s)) \ge c_0|x(s)|^2$. Substitution for this and (12) into (11) yields

$$\begin{aligned}
V(x(t\wedge\nu_k)) &\le V(x(0)) + M(t) \\
&\quad + \frac{2\bar{a}}{q}\int_0^{t\wedge\nu_k}\left(q|x(qs)|^{\beta+2} - |x(s)|^{\beta+2}\right)\mathrm{d}s \\
&\quad + \frac{\bar{b}}{q}\int_0^{t\wedge\nu_k}\left(q|x(qs)|^{\gamma+2} - |x(s)|^{\gamma+2}\right)\mathrm{d}s \\
&\quad - c_0\int_0^{t\wedge\nu_k}|x(s)|^2\mathrm{d}s.
\end{aligned} \tag{14}$$

Making use of the property of the integral, we may estimate

$$\begin{aligned}
&\int_0^t \mathrm{e}^{\varepsilon s}\left(q|x(qs)|^{\beta+2} - |x(s)|^{\beta+2}\right)\mathrm{d}s \\
&\quad = \int_0^t \left(q|x(qs)|^{\beta+2} - |x(s)|^{\beta+2}\right)\mathrm{d}s \\
&\quad \le \int_0^{qt}|x(s)|^{\beta+2}\mathrm{d}s - \int_0^t|x(s)|^{\beta+2}\mathrm{d}s \\
&\quad = -\int_{qt}^t|x(s)|^{\beta+2}\mathrm{d}s.
\end{aligned} \tag{15}$$

Similarly,

$$\int_0^t\left(q|x(qs)|^{\gamma+2} - |x(s)|^{\gamma+2}\right)\mathrm{d}s \le -\int_{qt}^t|x(s)|^{\gamma+2}\mathrm{d}s. \tag{16}$$

Substituting (15) and (16) into (14) and taking expectation yield

$$\begin{aligned}
\mathbb{E}V(x(t\wedge\nu_k)) &\le \mathbb{E}\|\xi\|^2 - \frac{2\bar{a}}{q}\int_{qt}^{t\wedge\nu_k}|x(s)|^{\beta+2}\mathrm{d}s \\
&\quad - \frac{\bar{b}}{q}\int_{q(t\wedge\nu_k)}^{t\wedge\nu_k}|x(s)|^{\gamma+2}\mathrm{d}s \\
&\quad - c_0\int_0^{t\wedge\nu_k}|x(s)|^2\mathrm{d}s \le \mathbb{E}\|\xi\|^2,
\end{aligned} \tag{17}$$

which implies that there exists almost surely a unique global solution. The proof is complete. □

For stability, we need to impose a stronger condition on the coefficients as follows.

(H3) The Polynomial Growth Conditions. For all $x \in \mathbb{R}^n$, there exist positive constants α, β, γ, a, $\bar{a}$, $\tilde{a}$, b, $\bar{b}$, $\tilde{b}$, and ε with $\varepsilon \le 2\tilde{a} - \tilde{b}$ such that

$$\begin{aligned}
&\langle x(s), f(x(s), x(qs))\rangle \\
&\quad \le -a|x(s)|^{\alpha+2} + \bar{a}\left(\mathrm{e}^{-(1-q)\varepsilon s}|x(qs)|^{\beta+2} + |x(s)|^{\beta+2}\right) \\
&\qquad - \tilde{a}|x(s)|^2,
\end{aligned} \tag{18}$$

$$\begin{aligned}
&|g(x(s), x(qs))|^2 \\
&\quad \le b|x(s)|^{\gamma+2} + \bar{b}\mathrm{e}^{-(1-q)\varepsilon s}|x(qs)|^{\gamma+2} + \tilde{b}|x(s)|^2.
\end{aligned} \tag{19}$$

Theorem 3. *Let (H1) and (H3) hold with* $\alpha \ge \beta \vee \gamma$, $2\tilde{a} > \tilde{b}$, $2a > 2\bar{a}(1 + 1/q) + b + \bar{b}/q$. *Then for any initial data* x_0, *the solution* $x(t)$ *is pth moment and almost surely exponentially stable; that is,*

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \le -\frac{\varepsilon}{2}, \tag{20}$$

where $\varepsilon \le 2\tilde{a} - \tilde{b}$.

Proof. Clearly, (H3) implies that (H2) by letting $\varepsilon = 0$, so there exists a unique global solution. Define $V(x) = |x|^2$, for any $\varepsilon > 0$; by the Itô formula, we have

$$\begin{aligned}
\mathrm{e}^{\varepsilon t}V(x(t)) &= V(x(0)) + \int_0^t \mathrm{e}^{\varepsilon s}\left(LV(x(s), x(qs))\right. \\
&\quad \left.+\varepsilon V(x(s))\right)\mathrm{d}s + \overline{M}(t),
\end{aligned} \tag{21}$$

where $\overline{M}(t) = \int_0^t e^{\varepsilon s} V_x(x(s)) g(x(s), x(qs)) dw(s) ds$ is a real-valued continuous local martingale with $M(0) = 0$. By (18) and (19), we may also estimate

$$\begin{aligned} & LV(x(s), x(qs)) + \varepsilon V(x(s)) \\ & \quad \leq -2a|x(s)|^{\alpha+2} + 2\overline{a}\left(e^{-(1-q)\varepsilon s}|x(qs)|^{\beta+2} + |x(s)|^{\beta+2}\right) \\ & \qquad - 2\tilde{a}|x(s)|^2 + b|x(s)|^{\gamma+2} + \overline{b}e^{-(1-q)\varepsilon s}|x(qs)|^{\gamma+2} \\ & \qquad + \tilde{b}|x(s)|^2 + \varepsilon|x(s)|^2 \\ & \quad \leq \frac{2\overline{a}}{q}\left(qe^{-(1-q)\varepsilon s}|x(qs)|^{\beta+2} - |x(s)|^{\beta+2}\right) \\ & \qquad + \frac{\overline{b}}{q}\left(qe^{-(1-q)\varepsilon s}|x(qs)|^{\gamma+2} - |x(s)|^{\gamma+2}\right) - 2a|x(s)|^{\alpha+2} \\ & \qquad - \left(2\tilde{a} - \tilde{b} - \varepsilon\right)|x(s)|^2 + 2\overline{a}\left(1 + \frac{1}{q}\right)|x(s)|^{\beta+2} \\ & \qquad + \left(b + \frac{\overline{b}}{q}\right)|x(s)|^{\gamma+2}. \end{aligned} \tag{22}$$

Denoted by

$$\begin{aligned} I(x(s)) = {} & \left(2\tilde{a} - \tilde{b} - \varepsilon\right)|x(s)|^2 + 2a|x(s)|^{\alpha+2} \\ & - 2\overline{a}\left(1 + \frac{1}{q}\right)|x(s)|^{\beta+2} - \left(b + \frac{\overline{b}}{q}\right)|x(s)|^{\gamma+2}. \end{aligned} \tag{23}$$

Noting that $2\tilde{a} > \tilde{b}$, $2a > 2\overline{a}(1 + 1/q) + b + \overline{b}/q$, $\varepsilon < 2\tilde{a} - \tilde{b}$, $\alpha \geq \beta \vee \gamma$, by Lemma 1, there exists a positive $\overline{c}_0$ such that $I(x(s)) \geq \overline{c}_0|x(s)|^2$. Substitution for this and (22) into (21) yields

$$\begin{aligned} e^{\varepsilon t}V(x(t)) \leq {} & V(x(0)) + \overline{M}(t) \\ & + \frac{2\overline{a}}{q}\int_0^t e^{\varepsilon s}\left(qe^{-(1-q)\varepsilon s}|x(qs)|^{\beta+2} - |x(s)|^{\beta+2}\right) ds \\ & + \frac{\overline{b}}{q}\int_0^t e^{\varepsilon s}\left(qe^{-(1-q)\varepsilon s}|x(qs)|^{\gamma+2} - |x(s)|^{\gamma+2}\right) ds \\ & - c_0\int_0^t e^{\varepsilon s}|x(s)|^2 ds. \end{aligned} \tag{24}$$

Making use of the property of the integral, we may estimate

$$\begin{aligned} & \int_0^t e^{\varepsilon s}\left(qe^{-(1-q)\varepsilon s}|x(qs)|^{\beta+2} - |x(s)|^{\beta+2}\right) ds \\ & \quad = \int_0^t \left(qe^{\varepsilon qs}|x(qs)|^{\beta+2} - e^{\varepsilon s}|x(s)|^{\beta+2}\right) ds \\ & \quad \leq \int_0^{qt} e^{\varepsilon s}|x(s)|^{\beta+2} ds - \int_0^t e^{\varepsilon s}|x(s)|^{\beta+2} ds \\ & \quad = -\int_{qt}^t e^{\varepsilon s}|x(s)|^{\beta+2} ds. \end{aligned} \tag{25}$$

Similarly,

$$\begin{aligned} & \int_0^t e^{\varepsilon s}\left(qe^{-(1-q)\varepsilon s}|x(qs)|^{\gamma+2} - |x(s)|^{\gamma+2}\right) ds \\ & \quad \leq -\int_{qt}^t e^{\varepsilon s}|x(s)|^{\gamma+2} ds. \end{aligned} \tag{26}$$

Substituting for (25) and (26) into (24) yields

$$\begin{aligned} e^{\varepsilon t}V(x(t)) \leq {} & V(x(0)) + M(t) - \frac{2\overline{a}}{q}\int_{qt}^t e^{\varepsilon s}|x(s)|^{\beta+2} ds \\ & - \frac{\overline{b}}{q}\int_{qt}^t e^{\varepsilon s}|x(s)|^{\gamma+2} ds - \overline{c}_0\int_0^t e^{\varepsilon s}|x(s)|^2 ds. \end{aligned} \tag{27}$$

Applying the nonnegative semimartingale convergence theorem (see [33]) in (27), we have

$$\limsup_{t \to \infty} e^{\varepsilon t}V(x(t)) < \infty \text{ a.s.} \tag{28}$$

That is, there is a finite positive random variable C_0 such that $\sup_{0 \leq t < \infty} e^{\varepsilon t}V(x(t)) \leq C_0$ a.s. This implies $\limsup_{t \to \infty}(1/t)\log|x(t)| \leq -\varepsilon/2$ a.s. The proof is complete. □

3. Stability of the Numerical Solution for Linear SPDE

In the section, we will establish almost surely exponential stability of EM method numerical solution under the following linear growth conditions.

(H4) The Linear Growth Conditions. For any $x \in \mathbb{R}^n$, there exist positive constants $a, \overline{a}, c, \overline{c}, b, \overline{b}$ such that

$$\langle x(s), f(x(s), x(qs))\rangle \leq -a|x(s)|^2 + \overline{a}e^{-(1-q)\varepsilon s}|x(qs)|^2, \tag{29}$$

$$\left|f(x(s), x(qs))\right|^2 \leq c|x(s)|^2 + \overline{c}e^{-(1-q)\varepsilon s}|x(qs)|^2, \tag{30}$$

$$\left|g(x(s), x(qs))\right|^2 \leq b|x(s)|^2 + \overline{b}e^{-(1-q)\varepsilon s}|x(qs)|^2. \tag{31}$$

Clearly, let $\alpha = \beta = \gamma = \tilde{a} = \tilde{b} = 0$ in (H2) and (H3), which implies (H4). Similar to Theorems 2 and 3, we may obtain the following result.

Theorem 4. *Let (H1) and (H4) hold with* $2a > (2\overline{a} + \overline{b} + \overline{c})/q + b + c$. *Then for any initial data* x_0, *there almost surely exists a unique global solution* $x(t)$ *to* (5) *on* $t \geq 0$. *Moreover, for*

$\varepsilon \le 2a - b - c - (2\bar{a} + \bar{b} + \bar{c})/q$, the solution is almost surely exponentially stable; that is,

$$\limsup_{t\to\infty} \frac{1}{t} \log |x(t)| \le -\frac{\varepsilon}{2} \quad a.s. \tag{32}$$

Now we define the Euler-Maruyama approximate solution for (5). Given a step size $\Delta \in (0,1)$, compute the discrete approximations $X_k \approx x_0$, $t_k = k\Delta$ by setting $X_0 = x_0$ and performing

$$X_{k+1} = X_k + f\left(X_k, X_{[qk]}\right)\Delta + g\left(X_k, X_{[qk]}\right)\Delta w_k, \tag{33}$$

where the increments $\Delta w_k = w(t_{k+1}) - w(t_k)$, $k = 1, \ldots,$ are independent $N(0,\Delta)$-distributed Gaussian random variables $\mathscr{F}_{t_k}$-measurable at the mesh-point t_k.

The following discrete semimartingale theorem plays an important role in the section.

Lemma 5 (see [32]). *Let $\{A_i\}$ and $\{U_i\}$ be two nonnegative random variables such that both A_i and U_i are $\mathscr{F}_{i-1}$-measurable for $i = 1, 2, \ldots$ with $A_0 = U_0 = 0$ a.s. Let $\{M_i\}$ be a real-valued local martingale with $M_0 = 0$ a.s. Let ζ be a nonnegative $\mathscr{F}_0$-measurable random variable. Assume that $\{X_i\}$ is a nonnegative semimartingale with the Doob-Mayer decomposition $X_i = \zeta + A_i - U_i + M_i$. If $\lim_{t\to\infty} A_i < \infty$ a.s., then for almost all $\omega \in \Omega \lim_{t\to\infty} X_i < \infty$, $\lim_{t\to\infty} U_i < \infty$. That is, all of the three processes X_i and U_i converge to finite random variables.*

Theorem 6. *Let (H1) and (H4) hold with $2a > (2\bar{a} + m\bar{b} + \bar{c})/q + mb + c$. Then for $\varepsilon \le 2a - mb - c - (2\bar{a} + m\bar{b} + \bar{c})/q$, there exists a sufficiently small $\Delta^* \in (0,1)$ such that the approximate solution $\{X_k\}$ defined by (33) satisfies*

$$\limsup_{k\to\infty} \frac{1}{k\Delta} \log \left|X_k\right| \le -\frac{\varepsilon}{2} \quad a.s. \tag{34}$$

Proof. By (33) and (H4), we may compute

$$\begin{aligned}
\left|X_{k+1}\right|^2 &= \Big\langle X_k + f\left(X_k, X_{[qk]}\right)\Delta + g\left(X_k, X_{[qk]}\right)\Delta w_k, \\
&\qquad X_k + f\left(X_k, X_{[qk]}\right)\Delta + g\left(X_k, X_{[qk]}\right)\Delta w_k \Big\rangle \\
&= \left|X_k\right|^2 + 2\left\langle X_k, f\left(X_k, X_{[qk]}\right)\right\rangle \Delta \\
&\quad + \left|f(X_k, X_{[qk]})\right|^2 \Delta^2 + \left|g(X_k, X_{[qk]})\Delta w_k\right|^2 \\
&\quad + 2\left\langle X_k + f\left(X_k, X_{[qk]}\right)\Delta, g\left(X_k, X_{[qk]}\right)\Delta w_k \right\rangle \\
&= \left|X_k\right|^2 + 2\langle X_k, f\left(X_k, X_{[qk]}\right)\rangle\Delta \\
&\quad + \left(\left|f(X_k, X_{[qk]})\right|^2\Delta + \left|g(X_k, X_{[qk]})\right|^2\right)\Delta \\
&\quad + \left|g(X_k, X_{[qk]})\right|^2 \left(\left|\Delta w_k\right|^2 - m\Delta\right) \\
&\quad + 2\langle X_k + f\left(X_k, X_{[qk]}\right)\Delta, g\left(X_k, X_{[qk]}\right)\Delta w_k \rangle \\
&= \left(1 - 2a\Delta + c\Delta^2 + mb\Delta\right)\left|X_k\right|^2 \\
&\quad + \left(2\bar{a} + \bar{c}\Delta + m\bar{b}\right) \mathrm{e}^{-(1-q)\varepsilon k\Delta}\left|X_{[qk]}\right|^2 \Delta \\
&\quad + \left|g(X_k, X_{[qk]})\right|^2 \left(\left|\Delta w_k\right|^2 - m\Delta\right) \\
&\quad + 2\left\langle X_k + f\left(X_k, X_{[qk]}\right)\Delta, g\left(X_k, X_{[qk]}\right)\Delta w_k \right\rangle \\
&= \left(1 - 2a\Delta + c\Delta^2 + mb\Delta\right)\left|X_k\right|^2 \\
&\quad + \left(2\bar{a} + \bar{c}\Delta + m\bar{b}\right) \mathrm{e}^{-(1-q)\varepsilon k\Delta}\left|X_{[qk]}\right|^2 \Delta + s_k,
\end{aligned} \tag{35}$$

where $s_k = |g(X_k, X_{[qk]})|^2(|\Delta w_k|^2 - m\Delta) + 2\langle X_k + f(X_k, X_{[qk]})\Delta, g(X_k, X_{[qk]})\Delta w_k\rangle$. By (35), it is easy to obtain

$$\begin{aligned}
&\mathrm{e}^{\varepsilon(k+1)\Delta}\left|X_{k+1}\right|^2 - \mathrm{e}^{\varepsilon k\Delta}\left|X_k\right|^2 \\
&\quad = \left(1 - 2a\Delta + c\Delta^2 + mb\Delta - \mathrm{e}^{-\varepsilon\Delta}\right)\mathrm{e}^{\varepsilon(k+1)\Delta}\left|X_k\right|^2 \\
&\qquad + \left(2\bar{a} + \bar{c}\Delta + m\bar{b}\right)\mathrm{e}^{\varepsilon(kq+1)\Delta}\left|X_{[qk]}\right|^2\Delta + \mathrm{e}^{\varepsilon(k+1)\Delta}s_k.
\end{aligned} \tag{36}$$

With the help of recursive method, it is not difficult to get

$$\begin{aligned}
&\mathrm{e}^{\varepsilon k\Delta}\left|X_k\right|^2 \\
&\quad = \left|X_0\right|^2 + \left(1 - 2a\Delta + c\Delta^2 + mb\Delta - \mathrm{e}^{-\varepsilon\Delta}\right) \\
&\qquad \times \sum_{i=0}^{k-1} \mathrm{e}^{\varepsilon(i+1)\Delta}\left|X_i\right|^2 + \left(2\bar{a} + \bar{c}\Delta + m\bar{b}\right) \\
&\qquad \times \sum_{i=0}^{k-1} \mathrm{e}^{\varepsilon(i+1)\Delta}\left[\mathrm{e}^{-(1-q)\varepsilon i\Delta}\left|X_{[qi]}\right|^2 - \mathrm{e}^{\varepsilon\Delta}\left|X_i\right|^2\right]\Delta \\
&\qquad + \left(2\bar{a} + \bar{c}\Delta + m\bar{b}\right)\mathrm{e}^{\varepsilon\Delta}\sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(i+1)\Delta}\left|X_i\right|^2\Delta + \sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(i+1)\Delta}s_i \\
&\quad = \left|X_0\right|^2 - \Bigg[2a - c\Delta - mb - \frac{1 - \mathrm{e}^{-\varepsilon\Delta}}{\Delta} \\
&\qquad\qquad - \left(2\bar{a} + \bar{c}\Delta + m\bar{b}\right)\mathrm{e}^{\varepsilon\Delta}\Bigg] \\
&\qquad \times \sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(i+1)\Delta}\left|X_i\right|^2\Delta \\
&\qquad + \sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(i+1)\Delta}\left(\mathrm{e}^{-(1-q)\varepsilon i\Delta}\left|X_{[qi]}\right|^2 - \mathrm{e}^{\varepsilon\Delta}\left|X_i\right|^2\right)\Delta \\
&\qquad + \sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(i+1)\Delta}s_i.
\end{aligned} \tag{37}$$

Obviously, $\sum_{i=0}^{k-1} \mathrm{e}^{\varepsilon(i+1)\Delta} s_i$ is a martingale. Let $[qi] = j$; then $j \le qi < j+1$, so $qi - 1 < j \le qi$. If $0 \le i \le k-1$, then $-1 < j \le q(k-1) \le [qk] + 1 - q \le [qk] + 1$. This implies that

$$\begin{aligned}
&\sum_{i=0}^{k-1} \mathrm{e}^{\varepsilon(i+1)\Delta} \left[\mathrm{e}^{-(1-q)\varepsilon i\Delta} \left| X_{[qi]} \right|^2 - \mathrm{e}^{\varepsilon\Delta} \left| X_i \right|^2 \right] \Delta \\
&\quad = \sum_{i=0}^{k-1} \mathrm{e}^{\varepsilon(qi+1)\Delta} \left| X_{[qi]} \right|^2 - \sum_{i=0}^{k-1} \mathrm{e}^{\varepsilon(i+2)\Delta} \left| X_i \right|^2 \\
&\quad \le \sum_{i=0}^{[qk]+1} \mathrm{e}^{\varepsilon(i+2)\Delta} \left| X_i \right|^2 - \sum_{i=0}^{k-1} \mathrm{e}^{\varepsilon(i+2)\Delta} \left| X(i\Delta) \right|^{\alpha+2} \\
&\quad \le - \sum_{i=[qk]+2}^{k-1} \mathrm{e}^{\varepsilon(i+2)\Delta} \left| X_i \right|^2 .
\end{aligned} \tag{38}$$

Denote by

$$f(\Delta) = 2a - c\Delta - mb + \frac{\mathrm{e}^{-\varepsilon\Delta} - 1}{\Delta} - \mathrm{e}^{\varepsilon\Delta} \left(2\overline{a} + \overline{c}\Delta + m\overline{b} \right). \tag{39}$$

By the Taylor series, we have

$$\begin{gathered}
\mathrm{e}^{-\varepsilon\Delta} = 1 - \varepsilon\Delta + \frac{(\varepsilon\Delta)^2}{2} - \frac{(\varepsilon\Delta)^3}{3!} + o(\varepsilon\Delta) > 1 - \varepsilon\Delta, \\
\frac{\mathrm{e}^{-\varepsilon\Delta} - 1}{\Delta} > -\varepsilon.
\end{gathered} \tag{40}$$

Therefore $f(\Delta) > 2a - c\Delta - mb - \varepsilon - \mathrm{e}^{\varepsilon\Delta}(2\overline{a} + \overline{c}\Delta + m\overline{b})$. Noting that $2a > (2\overline{a} + m\overline{b} + \overline{c})/q + mb + c, q \in (0,1)$, for any given ε, choose a sufficiently small Δ^* $(\Delta^* < \overline{\Delta})$ such that for all $\Delta < \Delta^*$,

$$2a - c\Delta - mb - \varepsilon - \mathrm{e}^{\varepsilon\Delta} \left(2\overline{a} + \overline{c}\Delta + m\overline{b} \right) > 0. \tag{41}$$

Substituting for (38) and (41) into (37), Lemma 5 implies that there exists a positive constant C_0 such that $\mathrm{e}^{\varepsilon k\Delta} |X_k|^2 \le C_0$. That is, $\limsup_{k\to\infty} (1/k\Delta) \log |X_k| \le -\varepsilon/2$ a.s. The proof is complete. □

4. Stability of Numerical Solution for Highly Nonlinear SPDE

In the section, we will prove that the backward EM method can preserve the almost surely exponential stability of the true solution of highly nonlinear stochastic pantograph differential equation.

(H5) The Polynomial Growth Conditions. For any $x \in \mathbb{R}^d$, there exist positive constants $\alpha, a, \overline{a}, \widetilde{a}, b, \overline{b}, \widetilde{b}$ such that

$$\begin{aligned}
&\langle x(s), f(x(s), x(qs)) \rangle \\
&\quad \le -a|x(s)|^{\alpha+2} + \overline{a}\mathrm{e}^{-(1-q)\varepsilon s} |x(qs)|^{\alpha+2} - \widetilde{a}|x(s)|^2,
\end{aligned} \tag{42}$$

$$\begin{aligned}
&|g(x(s), x(qs))|^2 \\
&\quad \le b|x(s)|^{\alpha+2} + \overline{b}\mathrm{e}^{-(1-q)\varepsilon s} |x(qs)|^{\alpha+2} + \widetilde{b}|x(s)|^2.
\end{aligned} \tag{43}$$

By Theorems 2 and 3, (5) has a unique global solution and the solution is almost surely exponentially stable.

Given a step size $\Delta \in (0,1)$, and for $t \in [0,T]$, $M\Delta = T$ for some positive integer M. Let $t_k = k\Delta$ $(k \ge 0)$, $[t/\Delta]$ be the integer part of t/Δ. Define the backward Euler-Maruyama method as follows:

$$X_{k+1} = X_k + f\left(X_{k+1}, X_{[q(k+1)]} \right) \Delta + g\left(X_k, X_{[qk]} \right) \Delta w(t_k). \tag{44}$$

To guarantee that this method is well defined, we impose the following one-sided Lipschitz condition on the drift coefficient $f(x,y)$.

(H6) One-Sided Lipschitz Condition. There exists a positive constant $\lambda > 0$ such that for any $x_i \in \mathbb{R}^n$ $(i = 1, 2)$

$$\langle x_1 - x_2, f(x_1, y) - f(x_2, y) \rangle \le \lambda |x_1 - x_2|^2. \tag{45}$$

Applying a fixed point theorem one can prove that (44) has a unique solution X_{k+1}; given X_k if $\lambda\Delta < 1$, then the backward EM scheme (44) is well defined (see, e.g., [28]). From now on we always assume that $\Delta < \lambda^{-1}$.

Theorem 7. *Let (H1), (H5) and (H6) hold with $2\widetilde{a} > m\widetilde{b}$, $2a > mb + (2\overline{a} + m\overline{b})/q$; there exists a sufficiently small $\Delta^* \in (0,1)$ such that the approximate solution $\{X_k\}$ defined by (44) satisfies*

$$\limsup_{k\to\infty} \frac{1}{k\Delta} \log |X_k| \le -\frac{\varepsilon}{2} \quad a.s., \tag{46}$$

where $\varepsilon < 2\widetilde{a} - m\widetilde{b}$.

Proof. By (44) and (H5), we may compute

$$\begin{aligned}
|X_{k+1}|^2 &= \left\langle X_{k+1}, X_k + f\left(X_{k+1}, X_{[q(k+1)]} \right) \Delta \right. \\
&\qquad \left. + g\left(X_k, X_{[qk]} \right) \Delta w_k \right\rangle \\
&= \left\langle X_{k+1}, f\left(X_{k+1}, X_{[q(k+1)]} \right) \right\rangle \Delta \\
&\quad + \left\langle X_{k+1}, X_k + g\left(X_k, X_{[qk]} \right) \Delta w_k \right\rangle \\
&\le -a|X_{k+1}|^{\alpha+2}\Delta + \overline{a}\mathrm{e}^{-\varepsilon(1-q)(k+1)\Delta} \left| X_{[q(k+1)]} \right|^{\alpha+2} \Delta \\
&\quad - \widetilde{a}|X_{k+1}|^2\Delta + \frac{1}{2}|X_{k+1}|^2 + \frac{1}{2}|X_k|^2 \\
&\quad + \frac{1}{2}\left| g\left(X_k, X_{[qk]} \right) \right|^2 |\Delta w_k|^2 \\
&\quad + \left\langle X_k, g\left(X_k, X_{[qk]} \right) \right\rangle \Delta w_k
\end{aligned}$$

$$\begin{aligned}
&= \left(\frac{1}{2} - \tilde{a}\Delta\right)\left|X_{k+1}\right|^2 + \bar{a}\mathrm{e}^{-\varepsilon(1-q)(k+1)\Delta}\left|X_{[q(k+1)]}\right|^{\alpha+2}\Delta \\
&\quad - a\left|X_{k+1}\right|^{\alpha+2}\Delta + \frac{1}{2}\left|X_k\right|^2 \\
&\quad + \frac{1}{2}\left(b\left|X_k\right|^{\alpha+2} + \bar{b}\mathrm{e}^{-\varepsilon(1-q)k\Delta}\left|X_{[qk]}\right|^{\alpha+2} + \tilde{b}\left|X_k\right|^2\right) \\
&\quad \times \left|\Delta w_k\right|^2 + \langle X_k, g\left(X_k, X_{[qk]}\right)\rangle \Delta w_k.
\end{aligned} \tag{47}$$

That is,

$$\begin{aligned}
&(1 + 2\tilde{a}\Delta)\left|X_{k+1}\right|^2 \\
&= \left(1 + m\tilde{b}\Delta\right)\left|X_k\right|^2 + 2\bar{a}\mathrm{e}^{-\varepsilon(1-q)(k+1)\Delta}\left|X_{[q(k+1)]}\right|^{\alpha+2}\Delta \\
&\quad - 2a\left|X_{k+1}\right|^{\alpha+2}\Delta + \left(mb\left|X_k\right|^{\alpha+2} + m\bar{b}\mathrm{e}^{-\varepsilon(1-q)k\Delta}\left|X_{[qk]}\right|^{\alpha+2}\right) \\
&\quad \times \Delta + s_k^{\Delta},
\end{aligned} \tag{48}$$

where

$$\begin{aligned}
s_k^{\Delta} &= \left(b\left|X_k\right|^{\alpha+2} + \bar{b}\mathrm{e}^{-\varepsilon(1-q)k\Delta}\left|X_{[qk]}\right|^{\alpha+2} + \tilde{b}\left|X_k\right|^2\right) \\
&\quad \times \left(\left|\Delta w_k\right|^2 - m\Delta\right) + 2\left\langle X_k, g\left(X_k, X_{[qk]}\right)\right\rangle \Delta w_k.
\end{aligned} \tag{49}$$

According to (48), we may obtain

$$\begin{aligned}
&(1 + 2\tilde{a}\Delta)\left[\mathrm{e}^{\varepsilon(k+1)\Delta}\left|X_{k+1}\right|^2 - \mathrm{e}^{\varepsilon k\Delta}\left|X_k\right|^2\right] \\
&\quad = \left[1 + m\tilde{b}\Delta - (1 + 2\tilde{a}\Delta)\,\mathrm{e}^{-\varepsilon\Delta}\right]\mathrm{e}^{\varepsilon(k+1)\Delta}\left|X_k\right|^2 \\
&\qquad + 2\bar{a}\mathrm{e}^{\varepsilon q(k+1)\Delta}\left|X_{[q(k+1)]}\right|^{\alpha+2}\Delta - 2a\mathrm{e}^{\varepsilon(k+1)\Delta}\left|X_{k+1}\right|^{\alpha+2}\Delta \\
&\qquad + mb\mathrm{e}^{\varepsilon(k+1)\Delta}\left|X_k\right|^{\alpha+2}\Delta + m\bar{b}\mathrm{e}^{-\varepsilon(qk+1)\Delta}\left|X_{[qk]}\right|^{\alpha+2}\Delta \\
&\qquad + \mathrm{e}^{\varepsilon(k+1)\Delta}s_k^{\Delta}.
\end{aligned} \tag{50}$$

With the help of recursive method, compute

$$\begin{aligned}
&(1 + 2\tilde{a}\Delta)\,\mathrm{e}^{\varepsilon k\Delta}\left|X_k\right|^2 \\
&\quad = (1 + 2\tilde{a}\Delta)\left|X_0\right|^2 + \left[1 + m\tilde{b}\Delta - (1 + 2\tilde{a}\Delta)\,\mathrm{e}^{-\varepsilon\Delta}\right] \\
&\qquad \times \sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(i+1)\Delta}\left|X_i\right|^2 - 2a\sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(i+1)\Delta}\left|X_{i+1}\right|^{\alpha+2}\Delta \\
&\qquad + 2\bar{a}\sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon q(i+1)\Delta}\left|X_{[q(i+1)]}\right|^{\alpha+2}\Delta \\
&\qquad + mb\sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(i+1)\Delta}\left|X_i\right|^{\alpha+2}\Delta \\
&\qquad + m\bar{b}\sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(qi+1)\Delta}\left|X_{[qi]}\right|^{\alpha+2}\Delta + \sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(i+1)\Delta}s_i^{\Delta} \\
&\quad = (1 + 2\tilde{a}\Delta)\left|X_0\right|^2 - \left[1 + 2\tilde{a}\Delta - \left(1 + \overline{m}\tilde{b}\Delta\right)\mathrm{e}^{\varepsilon\Delta}\right] \\
&\qquad \times \sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon i\Delta}\left|X_i\right|^2 - 2a\sum_{i=1}^{k}\mathrm{e}^{\varepsilon i\Delta}\left|X_i\right|^{\alpha+2}\Delta \\
&\qquad + mb\sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(i+1)\Delta}\left|X_i\right|^{\alpha+2}\Delta + 2\bar{a}\sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(i+1)\Delta}\left|X_i\right|^{\alpha+2}\Delta \\
&\qquad + m\bar{b}\sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(i+2)\Delta}\left|X_i\right|^{\alpha+2}\Delta \\
&\qquad + 2\bar{a}\sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(i+1)\Delta}\left(\mathrm{e}^{-\varepsilon(1-q)(i+1)\Delta}\left|X_{[q(i+1)]}\right|^{\alpha+2}\right. \\
&\qquad\qquad \left.-\left|X_i\right|^{\alpha+2}\right)\Delta \\
&\qquad + m\bar{b}\sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(i+1)\Delta}\left(\mathrm{e}^{-\varepsilon(1-q)i\Delta}\left|X_{[qi]}\right|^{\alpha+2} - \mathrm{e}^{\varepsilon\Delta}\left|X_i\right|^{\alpha+2}\right)\Delta \\
&\qquad + S_k,
\end{aligned} \tag{51}$$

where $S_k = \sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(i+1)\Delta}s_k^{\Delta}$. Obviously, S_k is a martingale. Let $[qi] = j$; then $j \le qi < j+1$, so $qi - 1 < j \le qi$. If $0 \le i \le k-1$; then $-1 < j \le q(k-1) \le [qk] + 1 - q \le [qk] + 1$. This implies

$$\begin{aligned}
&\sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(i+1)\Delta}\left(\mathrm{e}^{-\varepsilon(1-q)i\Delta}\left|X_{[qi]}\right|^{\alpha+2} - \mathrm{e}^{\varepsilon\Delta}\left|X_i\right|^{\alpha+2}\right) \\
&\quad = \sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(1+qi)\Delta}\left|X_{[qi]}\right|^{\alpha+2} - \sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(i+2)\Delta}\left|X_i\right|^{\alpha+2} \\
&\quad = \sum_{i=0}^{[qk]+1}\mathrm{e}^{\varepsilon(i+2)\Delta}\left|X_i\right|^{\alpha+2} - \sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon(i+2)\Delta}\left|X_i\right|^{\alpha+2} \\
&\quad \le -\sum_{i=[qk]+2}^{k-1}\mathrm{e}^{\varepsilon(i+2)\Delta}\left|X_i\right|^{\alpha+2}.
\end{aligned} \tag{52}$$

Similarly,

$$\begin{aligned}
&\sum_{i=0}^{k-1}\mathrm{e}^{-\varepsilon(i+1)\Delta}\left[\mathrm{e}^{\varepsilon(1-q)(i+1)\Delta}\left|X_{[q(i+1)]})\right|^{\alpha+2} - \left|X_i\right|^{\alpha+2}\right) \\
&\quad \le -\sum_{i=[qk]+2}^{k-1}\mathrm{e}^{\varepsilon(i+1)\Delta}\left|X_i\right|^{\alpha+2}.
\end{aligned} \tag{53}$$

Substituting for (52) and (53) into (51) yields

$$\begin{aligned}
&(1 + 2\tilde{a}\Delta)\,\mathrm{e}^{\varepsilon k\Delta}\left|X_k\right|^2 \\
&\quad = (1 + 2\tilde{a}\Delta)\left|X_0\right|^2 - \left[1 + 2\tilde{a}\Delta - \left(1 + m\tilde{b}\Delta\right)\mathrm{e}^{\varepsilon\Delta}\right] \\
&\qquad \times \sum_{i=0}^{k-1}\mathrm{e}^{\varepsilon i\Delta}\left|X_i\right|^2 + \left(-2a + mb\mathrm{e}^{\varepsilon\Delta} + 2\bar{a}\mathrm{e}^{\varepsilon\Delta} + m\bar{b}\mathrm{e}^{2\varepsilon\Delta}\right)
\end{aligned}$$

$$\begin{aligned}
&\times \sum_{i=1}^{k-1} \mathrm{e}^{\varepsilon i\Delta}\left|X_i\right|^{\alpha+2}\Delta - 2a\mathrm{e}^{\varepsilon k\Delta}\left|X_k\right|^{\alpha+2}\Delta \\
&+ \left(2\overline{a} + mb + m\overline{b}\mathrm{e}^{\varepsilon\Delta}\right)\mathrm{e}^{\varepsilon\Delta}\left|X_0\right|^{\alpha+2}\Delta \\
&- 2\overline{a} \sum_{i=[qk]+2}^{k-1} \mathrm{e}^{\varepsilon(i+1)\Delta}\left|X_i\right|^{\alpha+2}\Delta \\
&- m\overline{b} \sum_{i=[qk]+2}^{k-1} \mathrm{e}^{\varepsilon(i+2)\Delta}\left|X_i\right|^{\alpha+2}\Delta + S_k.
\end{aligned} \tag{54}$$

Denote by $f(\Delta) = 1 + 2\tilde{a}\Delta - (1 + m\tilde{b}\Delta)\mathrm{e}^{\varepsilon\Delta}$. Differentiating with respect to Δ yields

$$\begin{aligned}
f'(\Delta) &= 2\tilde{a} - m\tilde{b}\mathrm{e}^{\varepsilon\Delta} - \left(1 + m\tilde{b}\Delta\right)\varepsilon\mathrm{e}^{\varepsilon\Delta}, \\
f''(\Delta) &= -2m\tilde{b}\varepsilon\mathrm{e}^{\varepsilon\Delta} - \left(1 + m\tilde{b}\Delta\right)\varepsilon^2\mathrm{e}^{\varepsilon\Delta}.
\end{aligned} \tag{55}$$

Clearly, $f'(0) = 2\tilde{a} - m\tilde{b} - \varepsilon > 0$, $f''(0) < 0$; then there exists a $\overline{\Delta} > 0$ such that $f'(\overline{\Delta}) = 0$. $f(\Delta)$ is an increase function for $\Delta < \overline{\Delta}$ and noting that $f(0) = 0$ therefore there exists a sufficiently small Δ^* ($\Delta^* < \overline{\Delta}$) such that for all $\Delta < \Delta^*$,

$$1 + 2\tilde{a}\Delta - \left(1 + m\tilde{b}\Delta\right)\mathrm{e}^{\varepsilon\Delta} > 0. \tag{56}$$

On the other hand, since $2\tilde{a} > (2\overline{a} + m\overline{b})/q + b$, $q \in (0,1)$, then $2a > 2\overline{a} + m\overline{b} + m\tilde{b}$, and there exists sufficiently small $\Delta < \Delta^*$ such that

$$2a - 2\overline{a}\mathrm{e}^{\varepsilon\Delta} - m\left(b + \overline{b}\mathrm{e}^{\varepsilon\Delta}\right)\mathrm{e}^{\varepsilon\Delta} > 0. \tag{57}$$

Substituting for (56) and (57) into (54), Lemma 5 implies that there exists a positive constant C_0 such that

$$(1 + 2a\Delta)\,\mathrm{e}^{\varepsilon k\Delta}\left|X_k\right|^2 \le C_0. \tag{58}$$

That is,

$$\limsup_{k\to\infty} \frac{1}{k\Delta}\log\left|X_k\right| \le -\frac{\varepsilon}{2} \quad \text{a.s.} \tag{59}$$

The proof is complete. □

Example 8. Consider the following nonlinear scalar SPDE

$$\begin{aligned}
\mathrm{d}x(t) &= \left[-0.5x(t) - 5x^5(t) + 3\mathrm{e}^{-0.25\varepsilon t}x^5(qt)\right]\mathrm{d}t \\
&\quad + x^3(t)\,\mathrm{d}w(t),
\end{aligned} \tag{60}$$

where $w(t)$ is scalar Brownian motion. Define $f(x, y, t) = -0.5x - 5x^5 + 3\mathrm{e}^{-0.25\varepsilon t}y^5$, $g(x, y, t) = x^3$. Compute

$$\begin{aligned}
&f(x_1, y, t) - f(x_2, y, t) \\
&\quad \le -0.5(x_1 - x_2) - 5\left(x_1^5 - x_2^5\right) \\
&\quad \le -0.5(x_1 - x_2)\left[1 + 10\left(x_1^4 + x_1^3x_2 + x_2^2x_2^2 \right.\right. \\
&\qquad\qquad \left.\left. + x_1x_2^3 + x_2^4\right)\right].
\end{aligned} \tag{61}$$

Noting that $a^2 + b^2 \ge (a+b)^2/2$, compute

$$\begin{aligned}
&x_1^4 + x_1^3x_2 + x_2^2x_2^2 + x_1x_2^3 + x_2^4 \\
&\quad \ge \frac{\left(x_1^2 + x_2^2\right)^2}{2} + x_1x_2\left(x_1^2 + x_2^2\right) + \left(x_1x_2\right)^2 \\
&\quad \ge \left[\frac{x_1^2 + x_2^2}{2} + x_1x_2\right]^2,
\end{aligned} \tag{62}$$

which implies

$$\left\langle x_1 - x_2, f(x_1, y, t) - f(x_2, y, t)\right\rangle \le -0.5(x_1 - x_2)^2. \tag{63}$$

That implies that $f(x, y, t)$ satisfies the one-sided growth condition. By using the inequality $a^pb^q \le (p/(p+q))a^{p+q} + (q/(p+q))b^{p+q}$, it is easy to compute

$$\begin{aligned}
\left\langle x, f(x, y, t)\right\rangle &\le -0.5x^2 - 5x^6 + 3\mathrm{e}^{-0.25\varepsilon t}y^5x \\
&\le -0.5x^2 - 4.5x^6 + 2.5y^6
\end{aligned} \tag{64}$$

and $|g(x(t), y(t))|^2 \le x^6$. By Theorems 2 and 7, (60) has unique global solution and the solution is almost surely exponentially stable.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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