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Research Article

On Connectivity of Fatou Components concerning a Family of Rational Maps

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I. N. Baker established the existence of Fatou component with any given finite connectivity by the method of quasi-conformal surgery. M. Shishikura suggested giving an explicit rational map which has a Fatou component with finite connectivity greater than 2. In this paper, considering a family of rational maps $R(z, t)$ that A. F. Beardon proposed, we prove that $R(z, t)$ has Fatou components with connectivities 3 and 5 for any $t \in (0, 1/12]$. Furthermore, there exists $t \in (0, 1/12]$ such that $R(z, t)$ has Fatou components with connectivity nine.

1. Introduction and Main Results

By Sullivan's theorem [1], each Fatou component of a rational map is eventually periodic. Moreover, for periodic Fatou components, there are only four possibilities: attracting basin, parabolic basin, Siegel disk, and Herman ring. Attracting basins and parabolic basins are either simply connected or infinitely connected, a Siegel disk is simply connected, and a Herman ring is doubly connected. However, for nonperiodic Fatou components, the corresponding connectivity may be bigger than two.

For any given $n \in \mathbb{Z}^+$, Baker et al. [2] proved that there exists a rational map R which has a Fatou component with connectivity n by the method of quasiconformal surgery. M. Shishikura suggested giving an explicit example such that it has a Fatou component with finite connectivity greater than two. Beardon [3] investigated the family of rational maps as follows:

$$R(z, t) = \frac{z^2 \left(1 + t^{12} z^3\right)}{(1 - tz)^3 \left(1 - t^4 z\right)}, \quad \left(z \in \overline{\mathbb{C}},\ t \in \mathbb{R}\right). \quad (1)$$

He proved the following result.

Theorem A. *For sufficiently small $t > 0$, there exists a Fatou component D of $R(z, t)$ with connectivity three or four.*

At the same time, he claimed that one may be able to compute the connectivity of D by further discussion. Qiao and Gao [4] verified that D has connectivity three for $t \in (0, 10^{-4})$. Moreover, for any given positive integer n, two different families of rational maps were constructed such that one of them has a Fatou component with connectivity n (see [4, 5]). However, the degree of rational maps satisfies with those conditions are increased as the number n increases. As the first step to study the problem of connectivity number of Fatou components in rational maps space with fixed degree, we just investigate the connectivity of any other Fatou component of $R(z, t)$ as the real parameter t varies. In fact, we have the following results.

Theorem 1. *Suppose that $R(z, t)$ is defined as in (1); then we have the following.*

(1) *For any $t \in (0, 1/12]$, there exist two Fatou components D_t and $\widetilde{D_t}$ of $R(z, t)$ with connectivities three and five, respectively. Moreover, $R : \widetilde{D_t} \to D_t$ is an unbranched covering.*

(2) *There exists* $t \in (0, 1/12]$ *such that* $R(z,t)$ *has one Fatou component with connectivity nine.*

Remark 2. In order to draw the graphs of Julia sets and Fatou components of such rational maps in complex plane, we consider its conformal conjugate. Let $g(z) = 1/z$, and put $\widetilde{R}(z,t) = g \circ R \circ g^{-1}(z,t) = (z(z-t)^3(z-t^4))/(z^3+t^{12})$. It is easy to see that ∞ is a superattracting fixed point of $\widetilde{R}(z,t)$ for any $t \in (0, 1/12]$. By Figure 1, $R(z,t)$ has Fatou components with connectivities three, five, and nine for some $t \in (0, 1/12]$ since $R(z,t)$ and $\widetilde{R}(z,t)$ have the same dynamical properties. Furthermore, by Figure 2, we know that $R(z, 0.004355)$ has Fatou components with connectivities eight and fourteen, and we conjecture that for any large integral $N \in \mathbb{N}$, there exists $t_0 \in (0, 1/12)$ such that $R(z,t_0)$ has a Fatou component with its connectivity bigger than N.

2. Preliminary Lemmas

For the fundamental concepts and classical results of iteration theory of rational maps, see [2, 3, 6, 7]. In order to prove Theorem 1, we need the following four lemmas. Except for Lemma 3, the others are certain modifications of results which have been verified in [3].

Lemma 3 (see [8, Proposition 2.5]). *Let f be a rational map of a degree larger than one, and let $\{U_i\}_{i=1}^{p}$ be its one (super)attractive or parabolic cycle of periodic Fatou components. If one of U_i $(i = 1, \ldots, p)$ is not simply connected, then $\cup_{i=1}^{p} U_i$ contains at least two different critical values for critical points in itself.*

In what follows, $t \in (0, 1/12]$ in $R(z,t)$. It is easy to see that $z = 0$ (resp., $z = \infty$) is a superattracting (resp., repelling) fixed point of $R(z,t)$. Let D_{00} be the Fatou component that contains $z = 0$.

Lemma 4. *The nonzero critical points of $R(z,t)$ lie outside the circle $\{z : |z| = 3\}$.*

Proof. $R(z,t)$ has exactly eight critical points, that is, $0, 1/t, 1/t$ and ξ_j $(j = 1,2,3,4,5)$, which are the solutions of the equation $R'(z,t) = 0$. Obviously, $z = 1/t$ lies outside the circle $\{|z| = 3\}$. By a calculation, each ξ_j $(j = 1,2,3,4,5)$ satisfies the following equation:

$$2 + tz \quad H(z,t) = 0, \tag{2}$$

where $H(z,t) = t^{16}z^4 - 2t^{12}(1+2t^3)z^3 + 5t^{11}z^2 - 2t^4z + 1 - t^3$. Putting

$$M = \sup\left\{|H(z,t)| : |z| \le 3, t \in \left(0, \frac{1}{12}\right]\right\}, \tag{3}$$

we can deduce that

$$M \le 81t^{16} + 54t^{12}\left(1+2t^3\right) + 45t^{11} + 6t^4 + t^3 + 1 \le 2. \tag{4}$$

By (2), the nonzero critical points of $R(z,t)$ lie outside the circle $\{z : |z| = 3\}$. $\square$

Lemma 5. *D_{00} is simply connected and $\{z : |z| \le 1/2\} \subset D_{00} \subset \{z : |z| \le 3/2\}$.*

Proof. For $z \in K = \{z : 0 < |z| \le 3/2\}$, we have

$$\begin{aligned}
\left|\frac{R(z,t)}{z^2}\right| &\le \frac{1 + t^{12}|z|^3}{(1 - t|z|)^3(1 - t^4|z|)} \\
&\le \frac{1 + (3/2)^3 t^{12}}{(1 - (3/2)t)^3(1 - (3/2)t^4)} < \frac{3}{2}, \\
\left|\frac{R(z,t)}{z^2}\right| &\ge \frac{1 - t^{12}|z|^3}{(1 + t|z|)^3(1 + t^4|z|)} \\
&\ge \frac{1 - (3/2)^3 t^{12}}{(1 + (3/2)t)^3(1 + (3/2)t^4)} > \frac{2}{3}.
\end{aligned} \tag{5}$$

If $|z| \le 1/2$, then $|R(z)| \le (3/2)|z|^2 \le (3/4)|z|$; we can deduce that $\{z : |z| \le 1/2\} \subset D_{00}$.

Suppose that D_{00} meets the circle $\{z : |z| = 3/2\}$; take a point $\omega \in D_{00} \cap \{z : |z| = 3/2\}$ and join ω to the origin by a curve $\sigma \subset D_{00}$. It is easy to see that $R^n \to 0$ uniformly on σ; then there exists a unique positive integer k, such that $R^k(\sigma)$ meets the circle $\{z : |z| = 3/2\}$, but $R^n(\sigma)$ does not meet $\{z : |z| = 3/2\}$ for $n > k$. Let ζ be a point where $R^k(\sigma)$ meets $\{z : |z| = 3/2\}$; we have

$$|\zeta| > |R(\zeta)| > \frac{2}{3}|\zeta|^2 = |\zeta|. \tag{6}$$

It is a contradiction and thus $D_{00} \subset \{z : |z| \le 3/2\}$. Obviously, D_{00} contains only one critical point $z = 0$ by Lemma 4; then D_{00} is simply connected by Lemma 3. $\square$

Lemma 6. *$R^{-1}(D_{00})$ only consists of two Fatou components, that is, D_{00} and D_{01}, which contains a triply connected domain Ω. Here, $\Omega = \{z : 1/2t^4 - 1/t \le |z| \le 3/2t^4 + 1/t, |z - 1/t^4| \ge 1/2t^4\}$.*

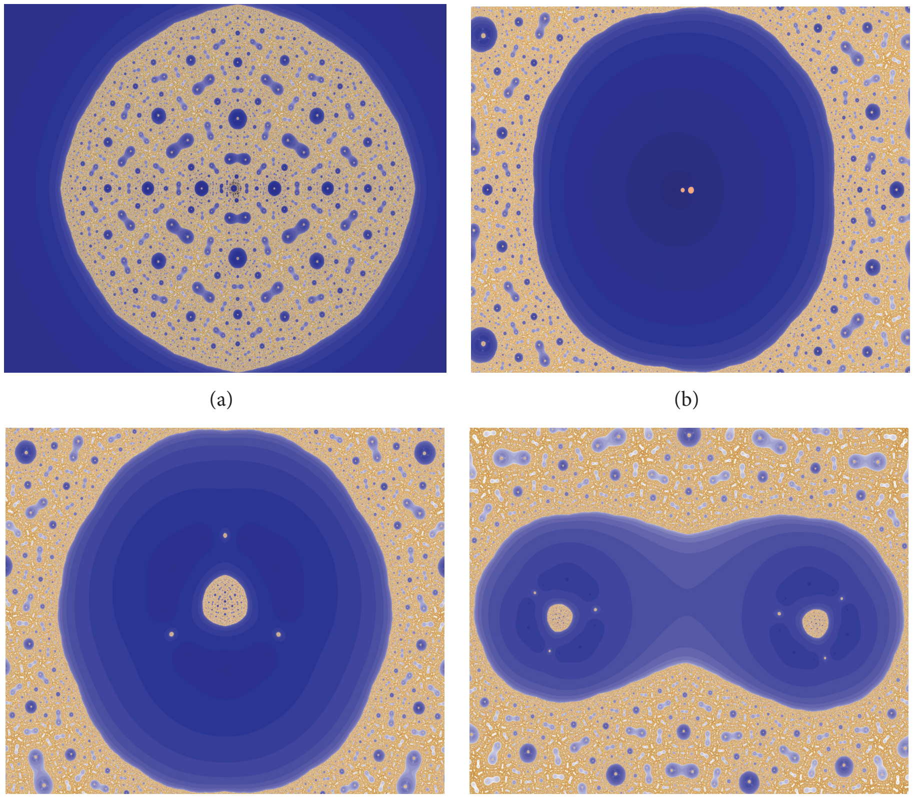
Proof. Take $z \in \Omega$; by a simple calculation, we have

$$\begin{aligned}
t|z| - 1 &\ge \frac{1}{2t^3} - 2 > 800, \\
\frac{1}{2} - t^3 &< t^4|z| < \frac{3}{2} + t^3, \\
\left|1 - t^4 z\right| &\ge \frac{1}{2}.
\end{aligned} \tag{7}$$

It is easy to see that $1/(t|z| - 1) < 1.002/t|z|$; we have

$$\begin{aligned}
\left|\frac{z^2\left(1 + t^{12}z^3\right)}{(1 - tz)^3}\right| &\le \frac{|z|^2\left(1 + t^{12}|z|^3\right)}{(t|z| - 1)^3} \\
&< (1.002)^3\frac{|z|^2\left(1 + t^{12}|z|^3\right)}{(t|z|)^3} < 3t.
\end{aligned} \tag{8}$$

Moreover, $R(\Omega) \subset \{z : |z| \le 6t\} \subset \{z : |z| \le 1/2\}$. By Lemma 5, we have $R(\Omega) \subset D_{00}$ and $\Omega \subsetneq D_{00}$, and there exists



(a) (b) (c) (d)

Figure 1: The Julia set $J(\widetilde{R}(z, 1/20))$ and three Fatou components of $\widetilde{R}(z, 1/20)$ with connectivities 3, 5, and 9, respectively.

at least one Fatou component $D_{01}(\neq D_{00})$ of $R^{-1}(D_{00})$. Noting that there are three (resp., two) zeros of $R(z,t)$ in Ω (resp., D_{00}), this implies that $R^{-1}(D_{00}) = D_{00} \cup D_{01}$ and $D_{01} \supset \Omega$. □

3. Proofs

Let $D \subset \mathbb{C}$ be a bounded Fatou component of $R(z,t)$. We denote the connectivity of D by $n(D)$ and the unbounded component of $\overline{\mathbb{C}} \setminus D$ by $\mathrm{Out}(D)$. Let $\mathrm{Int}(D) = \overline{\mathbb{C}} \setminus \mathrm{Out}(D)$ and $d|_D := \deg(R : D \to R(D))$. Moreover, we say that a component of $R^{-n}(D)$ $(n \in \mathbb{N})$ is a component $\widetilde{D}$ such that $R^n(\widetilde{D}) = D$. We say that D surrounds a point $z(\in \mathbb{C})$ (or a domain $U(\subset \mathbb{C})$) if it satisfies $z \in \mathrm{Int}(D)$ (or $U \subset \mathrm{Int}(D)$) and denote by $D \circlearrowleft z$ (or $D \circlearrowleft U$). Denote the number of zeros and poles of $R(z,t)$ in the interior of Jordan curve $\gamma \subset \mathbb{C}$ by $N(R,\gamma)$ and $P(R,\gamma)$. In order to prove Theorem 1, we need the following propositions. Considering the connectivity of D_{01} in Lemma 6, we have the following result.

Proposition 7. *D_{01} is a triply-connected domain.*

Proof. By a simple calculation, we have

$$R'(z,t) = \frac{zQ(z,t)}{(1-tz)^4(1-t^4z)^2}. \tag{9}$$

Here

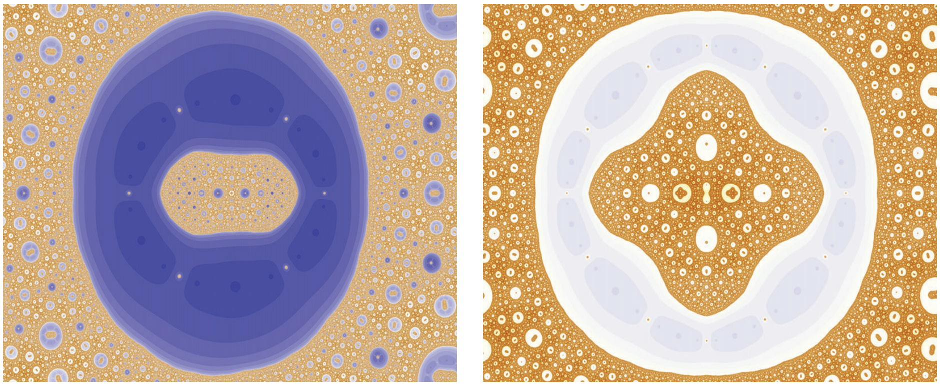
$$\begin{aligned} Q(z,t) = {} & t^{17}z^5 - 2t^{13}\left(1+2t^3\right)z^4 + 5t^{12}z^3 \\ & - 2t^5z^2 + t\left(1-t^3\right)z + 2. \end{aligned} \tag{10}$$

Note that

$$\begin{aligned} Q\left(-\frac{2}{t}+5t^2, t\right) = {} & -t^3 + 35t^6 - 122t^9 + 524t^{12} - 910t^{15} \\ & - 1775t^{18} + 7750t^{21} - 8750t^{24} \\ & + 3125t^{27} < 0, \\ Q\left(-\frac{2}{t}+6t^2, t\right) = {} & 42t^6 - 144t^9 + 648t^{12} - 1560t^{15} \\ & - 1800t^{18} + 12960t^{21} - 18144t^{24} \\ & + 7776t^{27} > 0, \end{aligned} \tag{11}$$

we can deduce that there exists a point $c_0 \in (-2/t+5t^2, -2/t+6t^2)$ such that $Q(c_0,t) = 0$ (c_0 is a critical point of $R(z,t)$). Note that $(-2/t+5t^2, -2/t+6t^2) \subset (-2/t, -1/t)$ and

$$R\left(-\frac{x}{t}, t\right) = \frac{x^2\left(1-t^9x^9\right)}{t^2(1+x)^3\left(1+t^3x\right)}, \tag{12}$$



(a) (b)

FIGURE 2: Two Fatou components of $\widetilde{R}(z, 0.004355)$ with connectivities 8 and 14, respectively.

and for any $x \in [1,2]$, we have $|R(-x/t,t)| > 3/2$; thus $|R(c_0,t)| > 3/2$. By Lemma 5, it follows that $c_0 \notin D_{01}$. Moreover, three critical points $0, 1/t$ (twice) lie outside of D_{01}. Hence, D_{01} contains at most four critical points. By Lemma 6, since the triply-connected domain $\Omega \subset D_{01}$ and $1/t^4 \in J(R(z,t)) \cap \{|z-1/t^4| < 1/2t^4\}$, then $\chi(D_1) \le -1$. Applying the Riemann-Hurwitz formula to the threefold covering map $R : D_1 \to D_{00}$, we have

$$\chi(D_1) + \sum (k_j - 1) = 3\chi(D_{00}) = 3, \tag{13}$$

so $\sum(k_j-1) \ge 4$. It follows that $\sum(k_j-1) = 4$ and $\chi(D_1) = -1$, and thus $n(D_{01}) = 3$. □

By Lemma 5, $\overline{D_{00}}$ is bounded. Since ∞ is a repelling fixed point, then each component in the preimage of D_{00} is bounded. In fact, we have the following result.

Proposition 8. *Each Fatou component of $R(z,t)$ is bounded.*

Proof. By Proposition 7, D_{01} contains four critical points and they tend to $z = 0$ under $R^n(z,t)$ $(n \to \infty)$. Note that $R(1/t,t) = \infty$; then the dynamics of $R(z,t)$ are decided by the forward orbit of the critical point c_0 in Proposition 7. If $\lim_{n\to\infty} R^n(c_0) \ne 0$, there exists at most one cycle of periodic components which is distinct from D_{00} by Sullivan's theorem. Assume that this cycle exists, denoted by $U_1, \ldots, U_p$; then

$$F(R(z,t)) = \bigcup_{n=0}^{\infty} R^{-n}(D_{00}) \cup \left(\bigcup_{n=0}^{\infty} \bigcup_{i=1}^{p} R^{-n}(U_i) \right). \tag{14}$$

Otherwise, $F(R(z,t)) = \cup_{n=0}^{\infty} R^{-n}(D_{00})$.

Below we will prove that each component of $F(R(z,t))$ is bounded. From the above analysis, let D be any component of $F(R(z,t))$; there exists $n \in \mathbb{N}$ such that $R^n(D) = D_{00}$ or $R^n(D) = U_1$ (if it exists). In order to show that D is bounded, we need to prove that $\overline{U_1}$ is bounded. Suppose that $U_1, \ldots, U_p$ exist, and note that $\cup_{n=0}^{+\infty} R^n(c_0) \subset \mathbb{R} \cup \{\infty\}$; we have $U_1, \ldots, U_p$ which are (super)attracting or parabolic components by the Sullivan theorem, and thus $c_0 \in \cup_{i=1}^{p} U_i$. Without loss of generality, $c_0 \in U_1$. By Lemma 3, U_i $(i = 1, \ldots, p)$ is simply connected; then $\overline{U_1}$ is contained in the bounded component of $\mathbb{C}$. □

Remark 9. By Proposition 8, $z = \infty$ and its preimages are buried points; here a point z_0 in Julia set is called buried point if it is not on the boundary of any Fatou component.

By Lemma 6 and Propositions 7 and 8, let M_1 be the component of $\overline{\mathbb{C}} \setminus D_{01}$ which contains $1/t^4$. Clearly, $M_1 \ne \{1/t^4\}$, and in the following, we denote $M = \mathrm{Int}(D_{01}) \setminus (D_{01} \cup M_1)$; then $1/t \in M$. Below we consider the connectivity of $R^{-1}(D_{01})$.

Proposition 10. *$R^{-1}(D_{01})$ consists of three Fatou components with connectivities 5, 3, and 3, respectively.*

Proof. We claim that $R^{-1}(D_{01})$ consists of three Fatou components. On one hand, since $R(\mathrm{Out}(D_{01})) = R(M_1) = \mathrm{Out}(D_{00})$ and $D_{01} \subset \mathrm{Out}(D_{00})$, there exists at least one component of $R^{-1}(D_{01})$ in $\mathrm{Out}(D_{01})$ and M_1, respectively. On the other hand, since $R(z,t)$ is monotone increasing from 0 to $+\infty$ for $z \in (0, 1/t)$ and $D_{01} \supset \Omega$ by Lemma 6, there exists a unique component D_2 of $R^{-1}(D_{01})$ with $D_2 \cap (0, 1/t) \ne \emptyset$ and $D_2 \subset M$. We claim that

$$D_2 \circlearrowleft \frac{1}{t}. \tag{15}$$

Assume that (15) is true (in what follows, we will return to the proof of this fact later in the proof), and by the definition of interior at the beginning of this section, $1/t \in \mathrm{Int}(D_2)$. Since $1/t$ is a critical point with multiplities 2 and $1/t^4$ $, \infty \notin \mathrm{Int}(D_2)$, then $R(z,t)$ is a 3-fold map from $\mathrm{Int}(D_2)$ to some neighborhood U of ∞. Furthermore, we can easily deduce that D_2 is the unique component of $R^{-1}(D_{01})$ in $\mathrm{Int}(D_2)$ (otherwise, $R(z,t) : \mathrm{Int}(D_2) \to U$ is at least a 4-fold map; it is a contradiction). Hence, $d|_{D_2} = 3$ owing to $R(z,t) : D_2 \to D_{01}$ which is a proper map. Obviously, we have $d|_{R(z,t)} = 5$, so the number of components of $R^{-1}(D_{01})$ in $\mathrm{Out}(D_{01})$ and M_1 is exactly one, respectively, denoted by D_1 and D_3. Clearly, D_1, D_2, and D_3 are mutually disjoint preimage components of $R^{-1}(D_{01})$ and $d|_{D_1} = d|_{D_3} = 1$.

Below we prove (15). In fact, if D_2 does not surround $1/t$, we distinguish two cases to discuss and get a contradiction.

(i) If D_2 does not surround D_{00}, it is easy to see that $R(\mathrm{Int}(D_2)) = \mathrm{Int}(D_{01})$ and $D_{00} \subset \mathrm{Int}(D_{01})$, and since D_2 has no pole, then $R^{-1}(D_{00}) \cap \mathrm{Int}(D_2) \neq \emptyset$, but it is a contradiction to Lemma 6.

(ii) If $D_2 \circlearrowleft D_{00}$, note that $R(z,t) < 0$ for $z \in (1/t, 1/t^4)$ and $\lim_{z\to 1/t+} R(z,t) = -\infty$, and we can deduce that there exists at least one component (denoted by $\widehat{D_2}$) of $R^{-1}(D_{01})$ with $\widehat{D_2} \cap (1/t, 1/t^4) \neq \emptyset$. Obviously, $\widehat{D_2} \subset \mathrm{Out}(D_{01})$ and $\widehat{D_2} \cap D_2 = \emptyset$ since D_2 does not surround $1/t$. If $\widehat{D_2} \circlearrowleft D_{00}$, then $\widehat{D_2} \circlearrowleft 1/t$, and by a similar discussion as used in the case of D_2, it is easy to see that $d|_{\widehat{D_2}} = 3$, but it is a contradiction to the fact that both M and M_1 contain some connected components of $R^{-1}(D_{01})$. If $\widehat{D_2}$ does not surround D_{00}, we also get a contradiction by a similar discussion of case (i). Hence, we get $D_2 \circlearrowleft 1/t$.

Next we will acquire the connectivity of D_i $(i = 1,2,3)$. Obviously, $c_0 \notin D_1 \cup D_3$, and by the Riemann-Hurwitz formula, $n(D_1) = n(D_3) = 3$. Furthermore, we claim that the "free" critical point c_0 in Proposition 7 is not contained in D_2, and thus $n(D_2) = 5$. In order to prove that $c_0 \notin D_2$, we turn to show the stronger result as follows:

$$\overline{D_2} \cap \mathbb{R}^- = \emptyset. \tag{16}$$

Otherwise, assume that $x \in \overline{D_2} \cap \mathbb{R}^-$. Note that t is a real parameter and $D_2 \cap \mathbb{R}^+ \neq \emptyset$; then D_2 is symmetric with respect to real axis $\mathbb{R}$. We choose a Jordan curve γ in the interior of D_2 such that γ is very close to $\partial\mathrm{Out}(D_2)$ and symmetric with respect to $\mathbb{R}$, and we take a point z_0 in $\gamma \cap \{z \mid z \in \mathbb{C}, \mathrm{Im}\, z \geq 0\}$ such that z_0 is one of the nearest points from x; denote the arc of γ from $\overline{z_0}$ to z_0 in counterclockwise direction by Γ. Moreover, we can also choose a Jordan arc η between x and z_0 in $\gamma \cap \{z \mid z \in \mathbb{C}, \mathrm{Im}\, z \geq 0\}$ such that $\eta \cap \Gamma = \{z_0\}$. Set $\tilde{\eta} = \{z \mid \overline{z} \in \eta\}$ and take $\tilde{\Gamma} = \Gamma \cup \eta \cup \tilde{\eta}$; then $\tilde{\Gamma}$ is a Jordan curve in $\overline{D_2}$ and $\tilde{\Gamma} \cap \partial D_2 \subset \{x\}$. Since $N(R, \tilde{\Gamma}) = 2$, $P(R, \tilde{\Gamma}) = 3$, and $\arg R(z,t)$ changes by -2π as z goes around $\tilde{\Gamma}$ by argument principle, but $\arg R(z,t)$ changes by -6π since $d|_{D_2} = 3$; it is a contradiction. Hence, we get (16). □

By a similar argument as the one used in (15), we can deduce that $D_3 \circlearrowleft 1/t^4$ and $D_1 \circlearrowleft D_{01}$. In fact, it is decided by the "similarity" of the Julia sets $J(R(z,t))$. By the definition of M_1 and Lemma 6, for any $x_0 \in M_1 \cap \mathbb{R}$, $x_0 > 1/2t^4$. Let $x_1 \in \mathbb{R}^-$ be the largest point of ∂D_{01} and let $x_2 \in \mathbb{R}^+$ be the largest point of ∂D_{00}; by Lemma 5, $R(x_1,t) = x_2 < 2$. Furthermore, by Lemma 6 and (16), the unique "free" critical point c_0 in Proposition 7 satisfies $c_0 \in (x_1, 0)$. Note that $R(z,t)$ is monotone increasing in (x_1, c_0) and monotone decreasing in $(c_0, 0)$; by a calculation, we can easily deduce that $|R(c_0,t)| < 1/t^2$ for any $x \in (-1/t^4, 0)$. It is easy to see that $R(z,t)$ is monotone increasing in $(0, 1/t)$ from 0 to $+\infty$; then the equation $R(z,t) = x_0$ has only one real root in $(-1/t^4, 1/t^4)$ since $R(z,t) < 0$ for $z \in (1/t, 1/t^4)$. Since

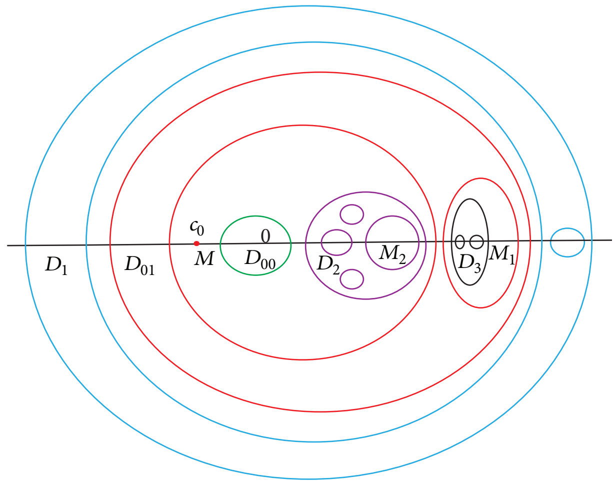


Figure 3: Schematic diagram of $\cup_{i=0}^{2} R^{-i}(D_{00})$.

$D_2 \subset M$ and $M \cap \mathbb{R} \subset (-1/t^4, 1/t^4)$, we can deduce that the number of bounded components of $\overline{\mathbb{C}} \setminus D_2$ which intersects with $\mathbb{R}$ is two (see Figure 3).

For any component D in the preimage of D_i $(i = 1,2,3)$, it is easy to see that $n(D) \geq 3$ by the Riemann-Hurwitz formula. Furthermore, the connectivity $n(D)$ is decided by the number of critical points in D and local degree $d|_D$. It is easy to see that there exists at most one Fatou component which contains the free critical point c_0; the following Proposition 13 shows that even if there is no critical point in D, its local degree may be larger than one. Therefore, we cannot give a complete description of connectivity of this family of rational maps. In order to get Proposition 13, we first consider the number of poles in $\mathrm{Int}(D)$.

Proposition 11. *No preimage component of D_2 or D_3 surrounds 0, $1/t$ or $1/t^4$.*

Proof. We argue by contradiction and induction. Let D be a preimage component of D_2 or D_3; we distinguish the following three cases to discuss.

(i) Suppose that $D \circlearrowleft 0$; it is easy to see that $D \circlearrowleft D_{00}$ since $0 \in D_{00}$ and $D \neq D_{00}$.

(ii) Suppose that $D \circlearrowleft 1/t$; then $D \subset M_2$ or $D \subset \mathrm{Out}(D_2)$. If $D \subset M_2$, since $R(z,t) > 0$ for any $z \in D \cap (0, 1/t)$ $(\neq \emptyset)$ and $R(z,t) < 0$ for any $z \in D \cap (1/t, 1/t^4)$ $(\neq \emptyset)$, $R(D) \circlearrowleft D_{00}$ by symmetry. If $D \subset \mathrm{Out}(D_2)$, then $D \circlearrowleft D_{00}$ or there exist two points $z_1 \in (0, 1/t) \cap D$ and $z_2 \in (1/t, 1/t^4) \cap D$. Since $R(z_1,t) > 0$ and $R(z_2,t) < 0$, then $R(D)$ surrounds D_{00} by symmetry.

(iii) Suppose that $D \circlearrowleft 1/t^4$; then either D or $R(D)$ surrounds D_{00} by the similar proof of (i) and (ii).

In all, if D surrounds any of 0, $1/t$ or $1/t^4$, then either D or $R(D)$ surrounds D_{00}. Note that if $R(D)$ surrounds D_{00}, we have $D \subsetneq R^{-1}(D_2)$ and $D \subsetneq R^{-1}(D_3)$ since $D_2 \cap \mathbb{R}^- = \emptyset$ and $D_3 \subset M_1$. To get the conclusion in this proposition, it suffices to prove that no preimage component of D_2 or D_3 surrounds D_{00}.

Below we prove that no preimage component of D_2 surrounds D_{00} by induction.

Let D be a component of $R^{-1}(D_2)$. Obviously, we have $D \cap D_{00} = \emptyset$. Suppose that $D \circlearrowleft D_{00}$; we will get contradictions by discussion.

(iv) Suppose that $D \subset \mathrm{Out}(D_{01})$, and since $(-\infty, -1/t^4) \cap D \neq \emptyset$, we get $R(D) \cap \mathbb{R}^{-1} \neq \emptyset$ which contradicts with $D_2 \cap \mathbb{R}^{-1} = \emptyset$.

(v) Suppose that $D \subset M$, and choosing a Jordan curve γ in D such that $\gamma \circlearrowleft D_{00}$, it is easy to know that $N(R, \gamma) = 2$ and $P(R, \gamma) = 0$ or 3. Since $R(D) = D_2$ and D_2 does not surround D_{00}, the contradiction can be deduced by argument principle.

Hence, any component D of $R^{-1}(D_2)$ cannot surround D_{00}. Assume that any component of $R^{-n}(D_2)$ cannot surround D_{00}. Again, let D be a component of $R^{-(n+1)}(D_2)$. Suppose that $D \circlearrowleft D_{00}$; we still distinguish two cases to discuss.

(vi) Suppose that $D \subset \mathrm{Out}(D_{01})$, and since $R(z,t) < 0$ for any $z \in D \cap (-\infty, -1/t^4)$ $(\neq \emptyset)$ and $R(z,t) > 0$ for any $z \in D \cap (1/t^4, +\infty)$ $(\neq \emptyset)$, then $R(D)$ surrounds D_{00} by symmetry which contradicts with the assumption.

(vii) Suppose that $D \subset M$, and since $R(D)$ is a component of $R^{-n}(D_2)$ and $R(D) \neq D$, then $R(D)$ cannot surround D_{00} by assumption. By a similar analysis as used in the case (v), we also deduce a contradiction.

Therefore, we get that no preimage component of D_2 surrounds D_{00}. By a similar discussion as the above used in D_2, any preimage component of D_3 cannot surround D_{00}. □

However, the conclusion in Proposition 11 cannot fit for D_1. For simplicity, the symbol $(1 \times n)$ $(n \in \mathbb{N})$ is $\underbrace{1 \cdots 1}_{n}$ and $D_{(1\times 0)i} = D_i$ for $i = 1, 2, 3$.

Proposition 12. *Given that $n \in [1, +\infty)$, only one component of $R^{-n}(D_1)$ surrounds 0, $1/t$, and $1/t^4$. Moreover, only one component of $R^{-n}(D_1)$ surrounds $1/t$ but does not surround 0 and $1/t^4$. In addition, only one component of $R^{-n}(D_1)$ surrounds $1/t^4$ but does not surround 0 and $1/t$.*

Proof. By induction and a similar discussion as the one used in Proposition 10, it is easy to get the following conclusion. For any integer $n \in \mathbb{N}$, $R^{-1}(D_{(1\times n)})$ consists of three components: one is contained in $\mathrm{Out}(D_{(1\times n)})$, denoted by $D_{(1\times(n+1))}$; one is contained in M_2, denoted by $D_{(1\times n)2}$; one is contained in M_1, denoted by $D_{(1\times n)3}$. Moreover, $D_{(1\times(n+1))} \circlearrowleft D_{(1\times n)}$ and $D_{(1\times n)i} \circlearrowleft D_{(1\times(n+1))i}$ $(i = 2, 3)$. By a similar discussion as used in Proposition 11, we can deduce that any preimage component of $D_{(1\times m)2}$ or $D_{(1\times m)3}$ $(m = 1, \ldots, n-1)$ cannot surround 0 or $1/t$. Since $R^n(D_{(1\times n)i}) = D_i$ $(i = 1, 2, 3)$, then $D_{(1\times(n+1))}$, $D_{(1\times n)2}$ and $D_{(1\times n)3}$ are satisfied with conditions of this proposition in turn as follows:

$$D_{00} \longleftarrow D_{01} \longleftarrow \left\{ \begin{matrix} D_2 \\ D_1 \\ D_3 \end{matrix} \right. \longleftarrow \left\{ \begin{matrix} D_{12} \\ D_{11} \\ D_{13} \end{matrix} \right. \longleftarrow \left\{ \begin{matrix} D_{112} \\ D_{111} \\ D_{113} \end{matrix} \right. \longleftarrow \cdots . \tag{17}$$

□

By Proposition 10, we can deduce that $n(D_{(1\times n)1}) = n(D_{(1\times n)3}) = 3$, $n(D_{(1\times n)2}) = 5$ for any $n \in [1, +\infty)$. However, for the other components in the preimage of D_i $(i = 1, 2, 3)$, we have the following result.

Proposition 13. *Given that $n \in \mathbb{N}$, let D be a component of preimage of $D_{(1\times n)2}$ or $D_{(1\times n)3}$, and if $c_0 \in \mathrm{Out}(D)$, then $d|_D = 1$ or $d|_D = 2$.*

Proof. By Propositions 11 and 12, $\mathrm{Int}(D)$ contains no pole of $R(z,t)$, and thus $R(\mathrm{Int}(D)) = \mathrm{Int}R(D)$ and R is a proper map in $\mathrm{Int}(D)$. We distinguish three cases to discuss the local degree of D as follows.

(1) If $c_0 \in \mathrm{Out}(D)$, then $d|_{\mathrm{Int}(D)} = 1$ by the Riemann-Hurwitz formula. Obviously, $d|_D = 1$.

(2) If $c_0 \in D$, then $d|_D \geq 2$. By the Riemann-Hurwitz formula, $d|_{\mathrm{Int}(D)} = 2$, so $d|_D = 2$.

(3) If $c_0 \in \mathrm{Int}(D) \setminus D$, then $d|_{\mathrm{Int}(D)} = 2$ by the Riemann-Hurwitz formula. Furthermore, we can deduce that $d|_D = 2$. Otherwise, $d|_D = 1$. Note that $\mathrm{Int}(D)$ has no poles of $R(z,t)$, and for any bounded components of $\overline{\mathbb{C}} \setminus R(D)$, the forward components of it in $\mathrm{Int}(D)$ are corresponding to all bounded components of $\overline{\mathbb{C}} \setminus D$. Hence, $R^{-1}(R(D)) \cap \mathrm{Int}(D) = D$; it is a contradiction to $d|_{\mathrm{Int}(D)} = 2$.

□

Proof of Theorem 1. (1) It is an immediate result of Propositions 7 and 10.

(2) We show that $R(z, 1/20)$ has a Fatou component with connectivity 9.

Put $t_0 = 1/20$, and by Proposition 7, the "free" critical point c_0 satisfies $c_0 \in (-2/t_0 + 5t_0^2, -2/t_0 + 6t_0^2)$. Set $R(x) = R(x, t_0)$, $r_1(x) = x^2$, $r_2(x) = 1 + t_0^{12}x^3$, $r_3(x) = (1 - t_0 x)^3$, $r_4(x) = 1 - t_0^4 x$ $(x \in \mathbb{R})$; then $R(x) = (r_1(x) r_2(x))/(r_3(x) r_4(x))$. Put $I_1 = [-2/t_0 + 5t_0^2, -2/t_0 + 6t_0^2]$, and for any $x_1 \in I_1$, we have

$$\begin{aligned} &\frac{r_1\left(-2/t_0 + 6t_0^2\right) r_2\left(-2/t_0 + 5t_0^2\right)}{r_3\left(-2/t_0 + 5t_0^2\right) r_4\left(-2/t_0 + 5t_0^2\right)} \\ &\quad \leq R\left(x_1\right) \leq \frac{r_1\left(-2/t_0 + 5t_0^2\right) r_2\left(-2/t_0 + 6t_0^2\right)}{r_3\left(-2/t_0 + 6t_0^2\right) r_4\left(-2/t_0 + 6t_0^2\right)}. \end{aligned} \tag{18}$$

Furthermore, we can deduce that

$$R\left(x_1\right) \in [59.237, 59.252] := I_2, \quad \forall x_1 \in I_1. \tag{19}$$

If necessary, we enlarge or reduce the length of corresponding interval, by a similar discussion as the one used in inequality 5, as follows:

$$R\left(x_2\right) \in [-465.125, -464.357] := I_3, \quad \forall x_2 \in I_2;$$

$$R\left(x_3\right) \in [15.065, 15.188] := I_4, \quad \forall x_3 \in I_3;$$

$$R\left(x_4\right) \in [15108, 16564] := I_5, \quad \forall x_4 \in I_4;$$

$$R(x_5) \in [-0.714, -0.445] := I_6, \quad \forall x_5 \in I_5;$$
$$R(x_6) \in [0.178, 0.478] := I_7, \quad \forall x_6 \in I_6. \tag{20}$$

For example,

$$\frac{r_1(59.252)\, r_2(59.252)}{r_3(59.237)\, r_4(59.252)} \le R(x_2) \le \frac{r_1(59.237)\, r_2(59.237)}{r_3(59.252)\, r_4(59.237)}. \tag{21}$$

By Lemmas 5 and 6, $I_7 \cup I_6 \subset D_{00}$ and $I_5 \subset D_{01}$. Note that $I_4 = [15.065, 15.188]$; then $I_4 \subset D_2$. Let D be the Fatou component that contains c_0, and since $R(c_0) > 0$, then $R(D) \cap \mathbb{R}^- = \emptyset$ by Proposition 11, and thus $c_0 \in \mathrm{Out}(R(D))$. As $R(z, t_0) = -1/(t_0)^4$ has an approximate root $22.9777 (> 1/t_0)$, we have $R(D) \subset \mathrm{Out}(D_2)$. By Proposition 12, we can deduce that $R^2(D)$ cannot surround D. By Proposition 13, $d|_D = 2$ and $d_{R^n(D)} = 1$ $(n = 1, 2)$, and thus $n(R(D)) = n(R^2(D)) = 5$, $n(D) = 9$ by the Riemann-Hurwitz formula. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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