

## Research Article

# Universal Projective Synchronization of Two Different Hyperchaotic Systems with Unknown Parameters

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Universal projective synchronization (UPS) of two chaotic systems is defined. Based on the Lyapunov stability theory, an adaptive control method is derived such that UPS of two different hyperchaotic systems with unknown parameters is realized, which is up to a scaling function matrix and three kinds of reference systems, respectively. Numerical simulations are used to verify the effectiveness of the scheme.

## 1. Introduction

Chaotic synchronization has gained extensively and intensively attention in recent decades since Pecora and Carroll's pioneering and meaningful work [1]. It has applications in vast areas of physics, chemistry, biology, and others particularly in secure communication. Various synchronization modes are proposed, such as complete synchronization [1], phase synchronization [2], projective synchronization [3], and antisynchronization [4]. Projective synchronization (PS) [5] as a generalization of complete synchronization (CS) and antisynchronization (AS), has attracted much attention for the scaling factor in which the state vectors become proportional.

The projective synchronization is firstly applied in a class of systems with partial linearity. Yan and Li [6] applied this notion in a general class of chaotic systems including nonpartially-linear systems in 2005: this synchronization mode was called generalized projective synchronization (GPS). In [7], the modified projective synchronization (MPS) is presented to synchronize two identical systems up to a scaling constant matrix. In 2007, Chen and Li [8] gave a new synchronization mode—function projective synchronization (FPS) in which scaling function matrix substitutes the scaling constant matrix. Tang et al. [9] presented FPS in another familiar way by using a function instead of the scaling factor

of GPS. Du et al. [10] applied Tang's definition to synchronize different chaotic systems with uncertain parameters. A more general definition called modified function projective synchronization (MFPS) was given by Du et al. [11] in 1999 and was used in secure communication. In recent years, many efforts are concentrated on MFPS [12–14].

From the development, we know that the scaling factor is expanded from constant to constant matrix and from function to matrix function.

In this paper, we will introduce a new type of synchronization called universal projective synchronization (UPS) which concerns the displacement. Adaptive synchronization is an effective method for dealing with unknown parameters [10, 15]. Zhang et al. [16] proposed a new hyperchaotic system. UPS of hyperchaotic  $Lü$  system [17] and the new one with unknown parameters will be investigated by using adaptive control method. Three kinds of reference systems are chosen for the UPS. The introduction of UPS will expand the scope of synchronization.

The rest of this paper is organized as follows. In Section 2, the definition of UPS is given. In Section 3, the UPS of hyperchaotic  $Lü$  system and the new one is studied. Adaptive control schemes are proposed. Simulation results verify the effectiveness of the schemes. The conclusion is given in Section 4.

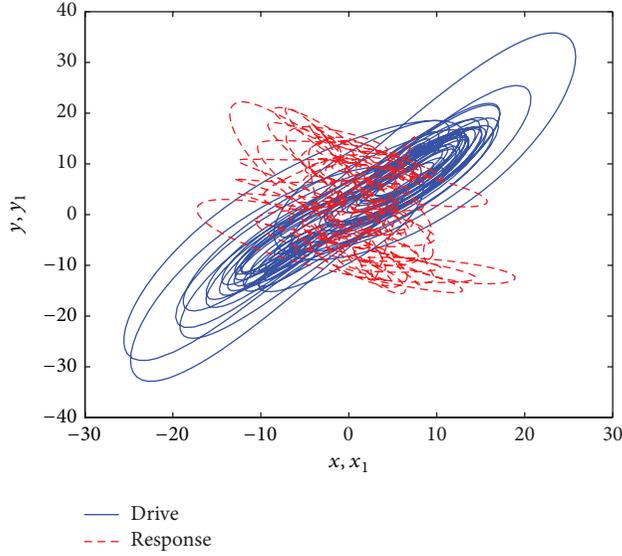


FIGURE 1: The trajectories projection on  $x - y(x_1 - y_1)$  plane of the drive system and the response system.

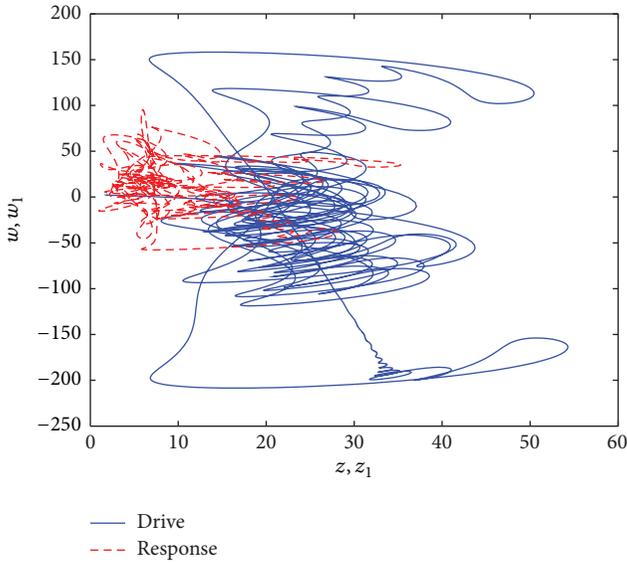


FIGURE 2: The trajectories projection on  $z - w(z_1 - w_1)$  plane of the drive system and the response system.

## 2. The Definition of UPS

The drive system and the response system are defined as follows:

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}), \quad (1)$$

$$\dot{\mathbf{y}} = \mathbf{g}(t, \mathbf{y}) + \mathbf{u}(t, \mathbf{x}, \mathbf{y}), \quad (2)$$

where  $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$  and  $\mathbf{y} = (y_1, y_2, \dots, y_n)^T \in \mathbb{R}^n$  are the state vectors;  $\mathbf{f}, \mathbf{g} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are continuous nonlinear vector function;  $\mathbf{u}(t, \mathbf{x}, \mathbf{y}) \in \mathbb{R}^n$  is the controller to

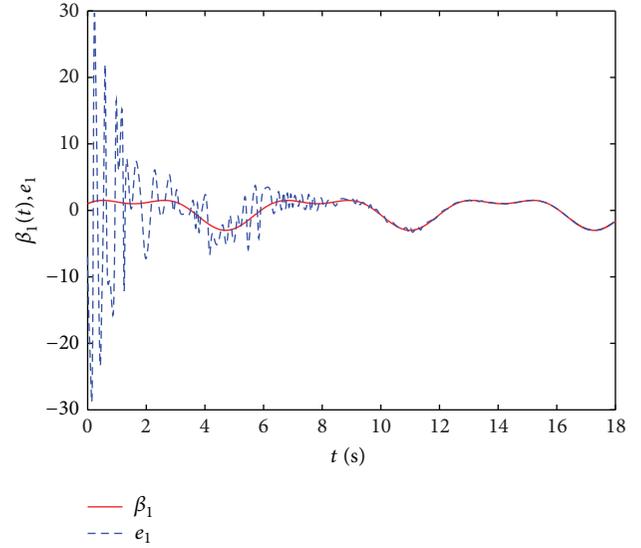


FIGURE 3: The time evolution of the error  $e_1$  and the function  $\beta_1(t)$ .

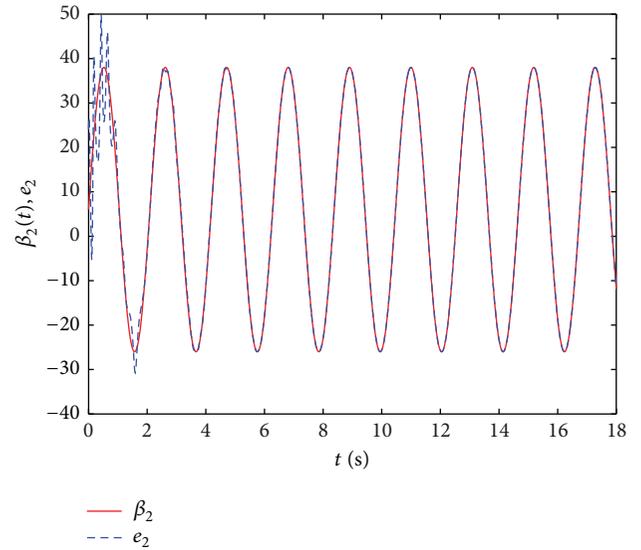


FIGURE 4: The time evolution of the error  $e_2$  and the function  $\beta_2(t)$ .

be designed. If there is no controller  $\mathbf{u}(t, \mathbf{x}, \mathbf{y})$ , (1) and (2) are chaotic systems. We define the state error vector

$$\mathbf{e} = (e_1, e_2, \dots, e_n)^T \triangleq \mathbf{x} - \mathbf{A}(t)\mathbf{y}, \quad (3)$$

where  $\mathbf{A}(t) = \text{diag}(\alpha_1(t), \alpha_2(t), \dots, \alpha_n(t))$ ;  $\alpha_i(t)$  ( $i = 1, 2, 3, \dots, n$ ) is continuous differentiable function with boundness, and  $\alpha_i(t) \neq 0$  for all  $t$ . Obviously, the error system is

$$\dot{\mathbf{e}} = \mathbf{f}(t, \mathbf{x}) - \dot{\mathbf{A}}(t)\mathbf{y} - \mathbf{A}(t)(\mathbf{g}(t, \mathbf{y}) + \mathbf{u}(t, \mathbf{x}, \mathbf{y})). \quad (4)$$

Suppose there exists a reference system

$$\dot{\mathbf{z}} = \mathbf{h}(t, \mathbf{z}), \quad (5)$$

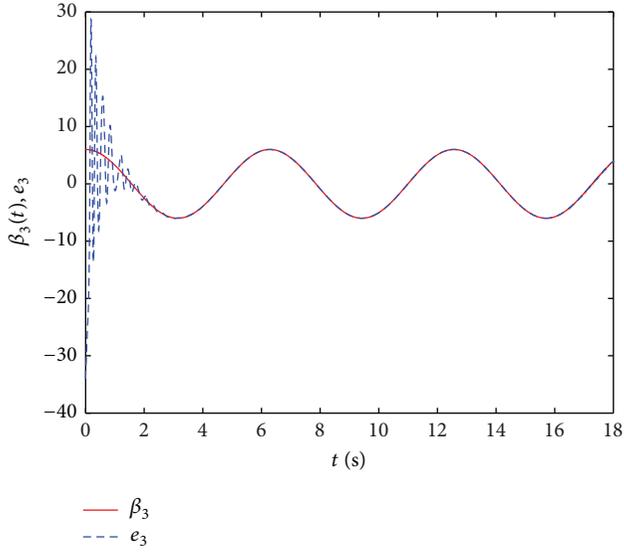


FIGURE 5: The time evolution of the error  $e_3$  and the function  $\beta_3(t)$ .

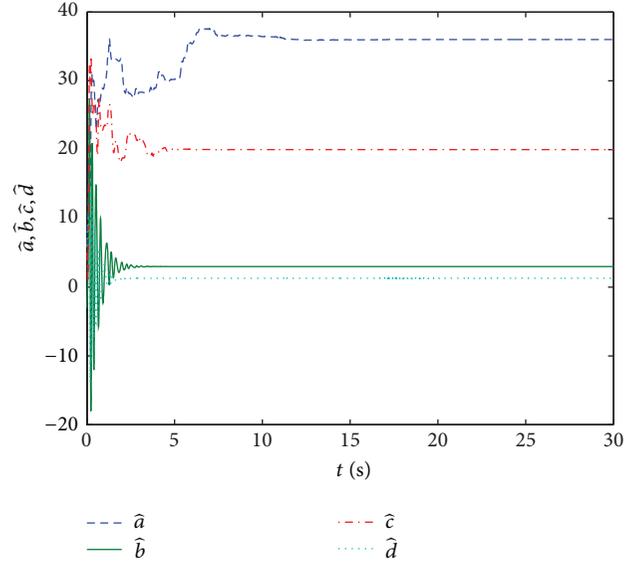


FIGURE 7: The time evolution of the estimated values  $\hat{a}$ ,  $\hat{b}$ ,  $\hat{c}$ , and  $\hat{d}$ .

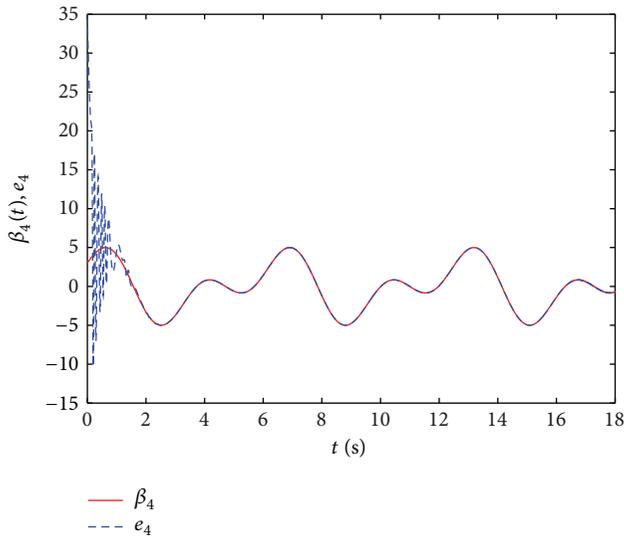


FIGURE 6: The time evolution of the error  $e_4$  and the function  $\beta_4(t)$ .

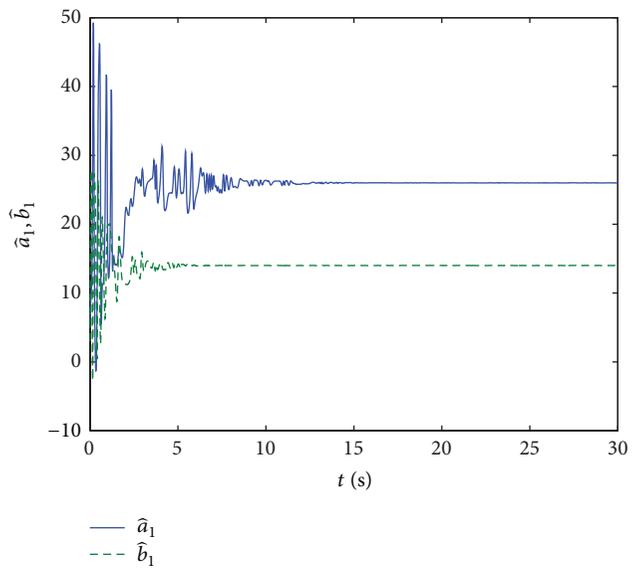


FIGURE 8: The time evolution of the estimated values  $\hat{a}_1$  and  $\hat{b}_1$ .

where  $\mathbf{z} = (z_1, z_2, \dots, z_n)^T \in R^n$ ,  $\mathbf{h} : R^n \rightarrow R^n$  is a continuous differentiable function. Suppose the reference system is an attractor.

*Definition 1.* For the scaling diagonal matrix function  $\mathbf{A}(t)$ , it is said that the drive system (1) and the response system (2) are universal projective synchronization (UPS) in the sense of the system (5), if the error system (4) and the reference system (5) are completely synchronized; that is,

$$\lim_{t \rightarrow \infty} \|\mathbf{e} - \mathbf{z}\| = 0. \quad (6)$$

Now we give another dual definition of UPS. We define the state error vector

$$\mathbf{e}^* = (e_1^*, e_2^*, \dots, e_n^*)^T \triangleq \mathbf{A}(t) \mathbf{x} - \mathbf{y}; \quad (7)$$

then the error system is

$$\dot{\mathbf{e}}^* = \dot{\mathbf{A}}(t) \mathbf{x} + \mathbf{A}(t) \mathbf{f}(t, \mathbf{x}) - \mathbf{g}(t, \mathbf{y}) - \mathbf{u}(t, \mathbf{x}, \mathbf{y}). \quad (8)$$

*Definition 2.* For the scaling diagonal matrix function  $\mathbf{A}(t)$ , it is said that the drive system (1) and the response system (2) are universal projective synchronization (UPS) in the sense of the system (5), if there exists a scaling diagonal matrix function  $\mathbf{A}(t)$  such that the error system (8) and the reference system (5) are completely synchronized; that is,

$$\lim_{t \rightarrow \infty} \|\mathbf{e}^* - \mathbf{z}\| = 0. \quad (9)$$

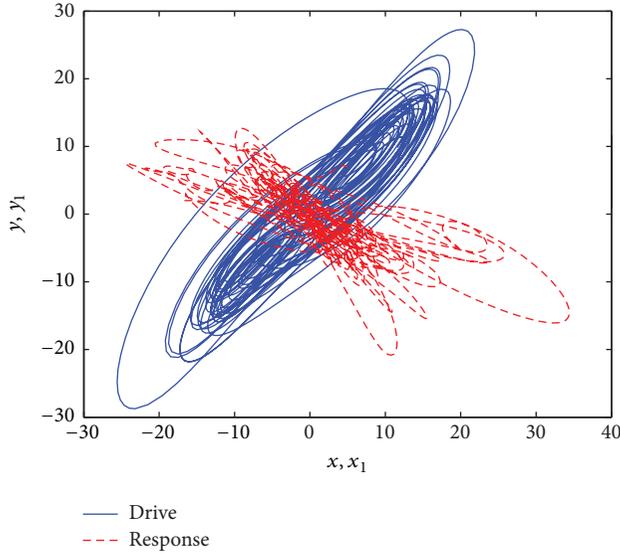


FIGURE 9: The trajectories projection on  $x - y(x_1 - y_1)$  plane of the drive system and the response system.

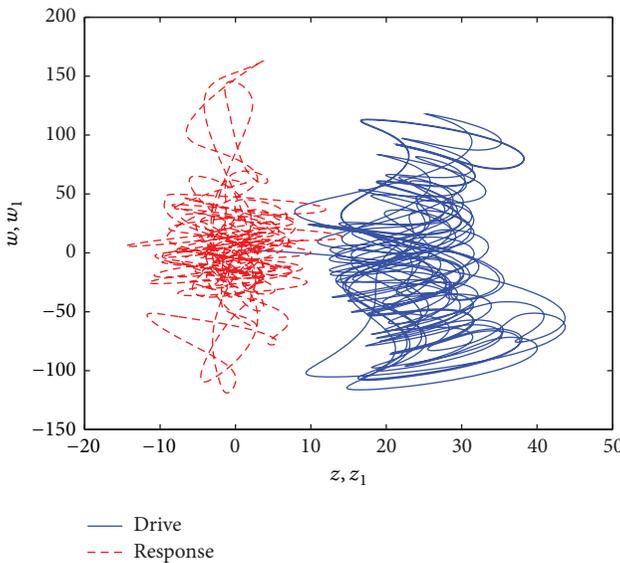


FIGURE 10: The trajectories projection on  $z - w(z_1 - w_1)$  plane of the drive system and the response system.

*Remark 3.* It is easy to see that UPS degenerated to MFPS if the reference system denotes constant vector  $\mathbf{0}$ ; that is, the initial value and state change rate of system (5) are all  $\mathbf{0}$ .

*Remark 4.* The reference system (5) expresses a constant vector  $\xi$ , if its initial value is  $\xi$  and the state change rate is  $\mathbf{0}$ . UPS is in the sense of constant vector  $\xi$ ; that is, there exists a displacement  $\xi$  between the drive system and response system. System (5) can be other attractors, such as periodic function, quasiperiod function, chaos, and hyperchaos.

Thereinafter UPS is always in the sense of Definition 1.

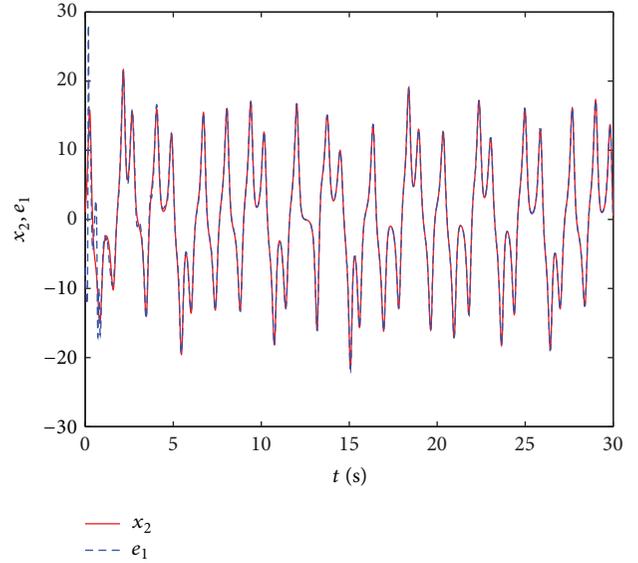


FIGURE 11: The time evolution of the error  $e_1$  and the state  $x_2$ .

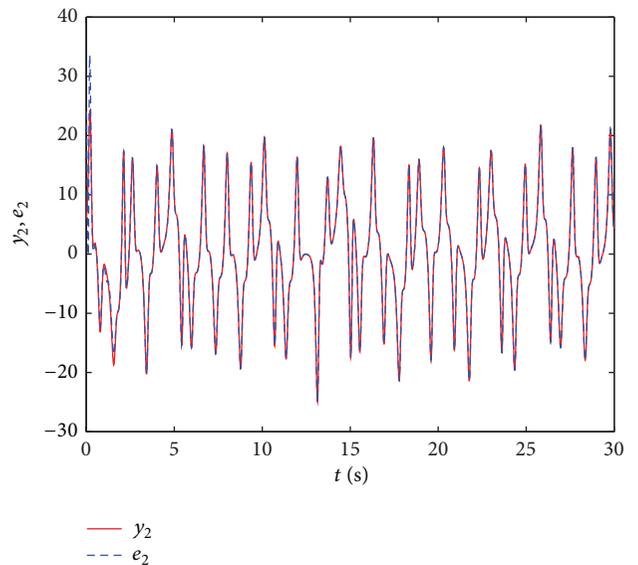


FIGURE 12: The time evolution of the error  $e_2$  and the state  $y_2$ .

### 3. UPS of Hyperchaotic Systems with Uncertain Parameters

We take hyperchaotic Lü system [17] as the drive system, which is described by

$$\begin{aligned}
 \dot{x} &= a(y - x) + w, \\
 \dot{y} &= -xz + cy, \\
 \dot{z} &= xy - bz, \\
 \dot{w} &= xz + dw,
 \end{aligned} \tag{10}$$

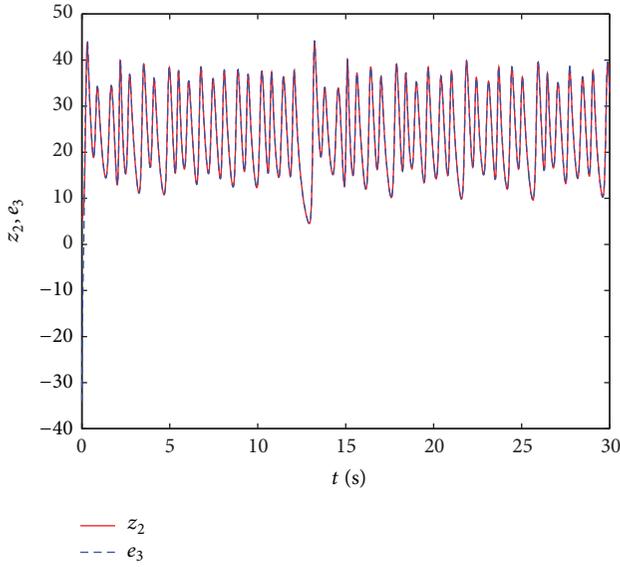


FIGURE 13: The time evolution of the error  $e_3$  and the state  $z_2$ .

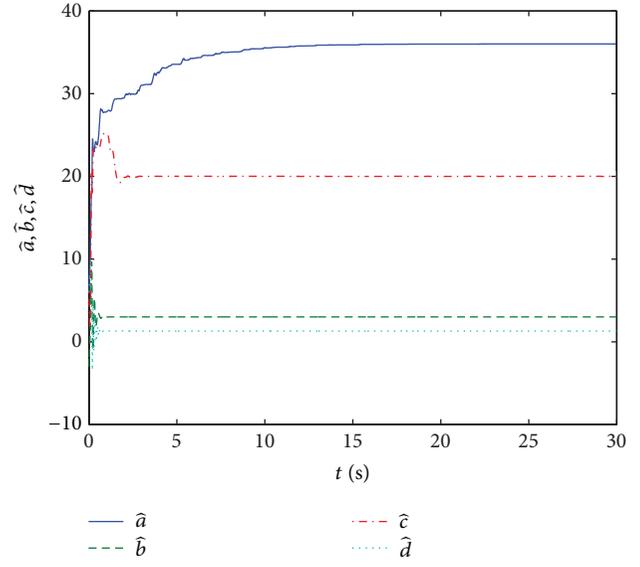


FIGURE 15: The time evolution of the estimated values  $\hat{a}$ ,  $\hat{b}$ ,  $\hat{c}$ , and  $\hat{d}$ .

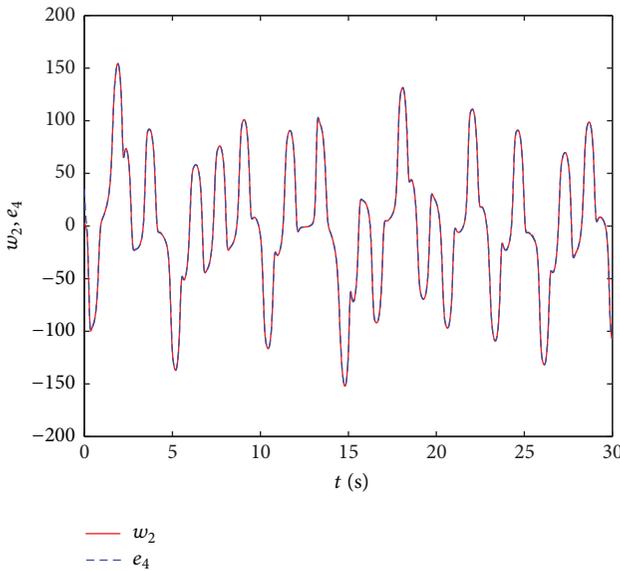


FIGURE 14: The time evolution of the error  $e_4$  and the state  $w_2$ .

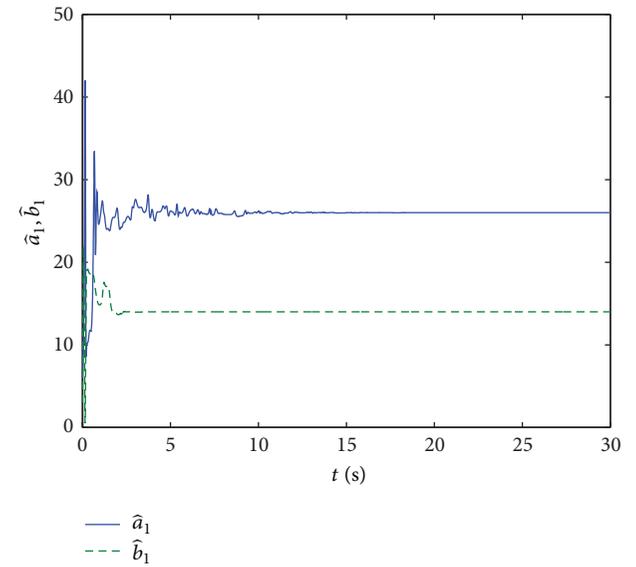


FIGURE 16: The time evolution of the estimated values  $\hat{a}_1$  and  $\hat{b}_1$ .

where  $x, y, z, w$  are state variables and  $a, b, c, d$  are uncertain parameters, which need to be estimated. The new hyperchaotic system presented by Zhang et al. [16], as the response system, is given by

$$\begin{aligned}
 \dot{x}_1 &= a_1(y_1 - x_1) + u_1, \\
 \dot{y}_1 &= b_1 y_1 - x_1 z_1 + w_1 + u_2, \\
 \dot{z}_1 &= x_1 y_1 - 3.36 z_1 + u_3, \\
 \dot{w}_1 &= -x_1 z_1 + w_1 + u_4,
 \end{aligned} \tag{11}$$

where  $x_1, y_1, z_1, w_1$  are state variables,  $a_1, b_1$  are uncertain parameters, which need to be estimated, and  $u_1, u_2, u_3, u_4$  are the nonlinear control laws. Suppose the scaling matrix  $A(t) = \text{diag}(\alpha_1(t), \alpha_2(t), \alpha_3(t), \alpha_4(t))$ . Obviously the errors  $e_1 = x - \alpha_1(t)x_1, e_2 = y - \alpha_2(t)y_1, e_3 = z - \alpha_3(t)z_1$ , and  $e_4 = w - \alpha_4(t)w_1$ . The estimations of  $a, b, c, d, a_1, b_1$  are  $\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{a}_1, \hat{b}_1$ , respectively.  $\tilde{a} = \hat{a} - a, \tilde{b} = \hat{b} - b, \tilde{c} = \hat{c} - c, \tilde{d} = \hat{d} - d, \tilde{a}_1 = \hat{a}_1 - a_1$ , and  $\tilde{b}_1 = \hat{b}_1 - b_1$  are the corresponding parameter estimate errors.

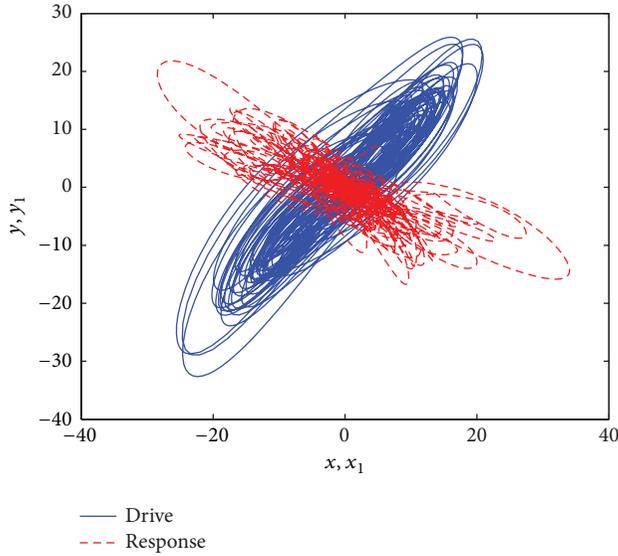


FIGURE 17: The trajectories projection on  $x - y(x_1 - y_1)$  plane of the drive system and the response system.

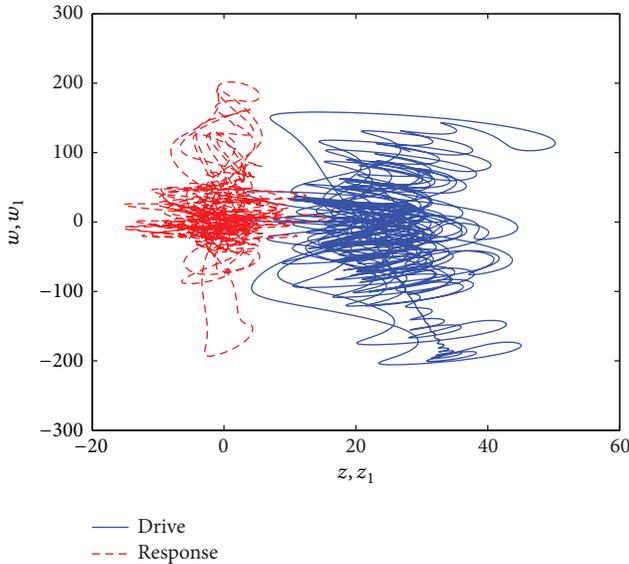


FIGURE 18: The trajectories projection on  $z - w(z_1 - w_1)$  plane of the drive system and the response system.

Now we discuss the UPS with three kinds of reference systems.

**3.1. UPS in the Sense of Vector Function.** Take the vector function  $\beta(t) = (\beta_1(t), \beta_2(t), \beta_3(t), \beta_4(t))^T$  as the reference system. Let  $\tilde{\mathbf{e}} = (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4)^T \triangleq (e_1 - \beta_1, e_2 - \beta_2, e_3 - \beta_3, e_4 - \beta_4)^T$ .

**Theorem 5.** For the given scaling matrix  $\mathbf{A}(t)$  and the vector function  $\beta(t)$ , the UPS between drive system (10) and response

system (11) with unknown parameters will occur by the control  $\mathbf{u} = (u_1, u_2, u_3, u_4)$  and parameter update law as follows:

$$\begin{aligned} u_1 &= \frac{1}{\alpha_1(t)} \left( \hat{a}(y-x) + w - \dot{\alpha}_1(t)x_1 \right. \\ &\quad \left. - \alpha_1(t)\hat{a}_1(y_1 - x_1) - \dot{\beta}_1(t) + k_1\tilde{e}_1 \right), \\ u_2 &= \frac{1}{\alpha_2(t)} \left( -xz + \hat{c}y - \dot{\alpha}_2(t)y_1 + \alpha_2(t)x_1z_1 \right. \\ &\quad \left. - \alpha_2(t)\hat{b}_1y_1 - \alpha_2(t)w_1 - \dot{\beta}_2(t) + k_2\tilde{e}_2 \right), \\ u_3 &= \frac{1}{\alpha_3(t)} \left( xy - \hat{b}z - \dot{\alpha}_3(t)z_1 - \alpha_3(t)x_1y_1 \right. \\ &\quad \left. + 3.36\alpha_3(t)z_1 - \dot{\beta}_3(t) + k_3\tilde{e}_3 \right), \\ u_4 &= \frac{1}{\alpha_4(t)} \left( xz + \hat{d}w - \dot{\alpha}_4(t)w_1 + \alpha_4(t)x_1z_1 \right. \\ &\quad \left. - \alpha_4(t)w_1 - \dot{\beta}_4(t) + k_4\tilde{e}_4 \right), \end{aligned} \quad (12)$$

$$\dot{\hat{a}} = (y-x)\tilde{e}_1,$$

$$\dot{\hat{b}} = -z\tilde{e}_3,$$

$$\dot{\hat{c}} = y\tilde{e}_2,$$

$$\dot{\hat{d}} = w\tilde{e}_4,$$

$$\dot{\hat{a}}_1 = -\alpha_1(t)(y_1 - x_1)\tilde{e}_1,$$

$$\dot{\hat{b}}_1 = -\alpha_2(t)y_1\tilde{e}_2,$$

(13)

where  $k_1, k_2, k_3, k_4$  are positive control gains.

*Proof.* Choose the following Lyapunov function:

$$V = \frac{1}{2} \left( \tilde{\mathbf{e}}^T \tilde{\mathbf{e}} + \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2 + \tilde{a}_1^2 + \tilde{b}_1^2 \right). \quad (14)$$

Then the derivative of  $V$  is given by

$$\begin{aligned} \dot{V} &= \frac{1}{2} \left( 2\tilde{\mathbf{e}}^T \dot{\tilde{\mathbf{e}}} + 2\tilde{a}\dot{\tilde{a}} + 2\tilde{b}\dot{\tilde{b}} + 2\tilde{c}\dot{\tilde{c}} + 2\tilde{d}\dot{\tilde{d}} \right. \\ &\quad \left. + 2\tilde{a}_1\dot{\tilde{a}}_1 + 2\tilde{b}_1\dot{\tilde{b}}_1 \right) \\ &= \tilde{\mathbf{e}}^T \dot{\tilde{\mathbf{e}}} + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} + \tilde{d}\dot{\tilde{d}} + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{b}_1\dot{\tilde{b}}_1 \\ &= \tilde{e}_1 \left( a(y-x) + w - \dot{\alpha}_1x_1 - \alpha_1a_1(y_1 - x_1) \right. \\ &\quad \left. - \alpha_1u_1 - \dot{\beta}_1 \right) \\ &\quad + \tilde{e}_2 \left( -xz + cy - \dot{\alpha}_2y_1 - \alpha_2(b_1y_1 - x_1z_1 + w_1) \right. \\ &\quad \left. - \alpha_2u_2 - \dot{\beta}_2 \right) \\ &\quad + \tilde{e}_3 \left( xy - bz - \dot{\alpha}_3z_1 - \alpha_3(x_1y_1 - c_1z_1) - \alpha_3u_3 - \dot{\beta}_3 \right) \\ &\quad + \tilde{e}_4 \left( xz + dw - \dot{\alpha}_4w_1 - \alpha_4(-x_1z_1 + w_1) - \alpha_4u_4 - \dot{\beta}_4 \right) \\ &\quad + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} + \tilde{d}\dot{\tilde{d}} + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{b}_1\dot{\tilde{b}}_1. \end{aligned} \quad (15)$$

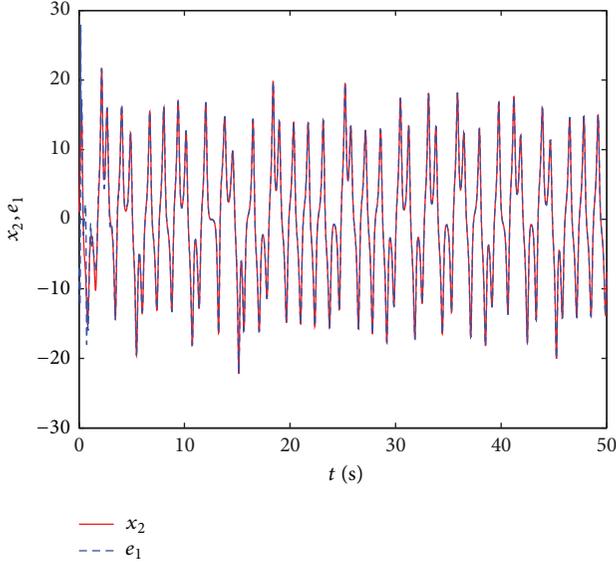


FIGURE 19: The time evolution of the error  $e_1$  and the state  $x_2$ .

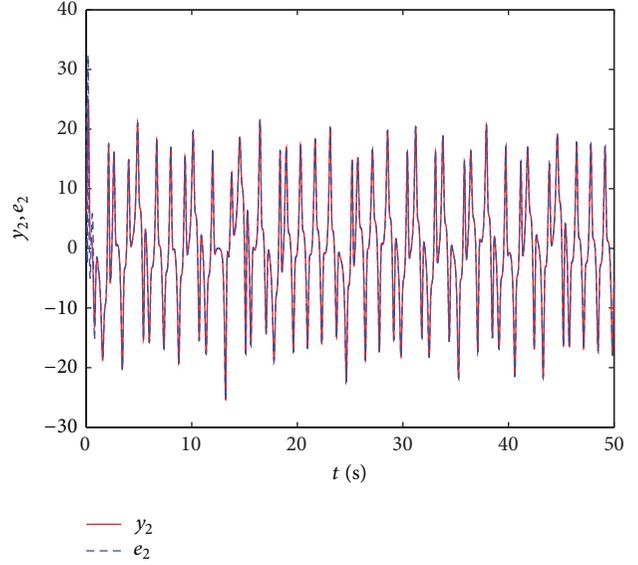


FIGURE 20: The time evolution of the error  $e_2$  and the state  $y_2$ .

With the controls (12) and the parameter update laws (13), we have

$$\dot{V} = -k_1 \tilde{e}_1^2 - k_2 \tilde{e}_2^2 - k_3 \tilde{e}_3^2 - k_4 \tilde{e}_4^2. \quad (16)$$

Because  $\dot{V}$  is negative simidefnite,  $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4 \in L_\infty$ ,  $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{a}_1, \tilde{b}_1 \in L_\infty$ . From the definition of  $\tilde{\mathbf{e}}$ , we know  $\dot{\tilde{e}}_1, \dot{\tilde{e}}_2, \dot{\tilde{e}}_3, \dot{\tilde{e}}_4 \in L_\infty$ . From  $\dot{V} = -k_1 \tilde{e}_1^2 - k_2 \tilde{e}_2^2 - k_3 \tilde{e}_3^2 - k_4 \tilde{e}_4^2 = -\tilde{\mathbf{e}}^T K \tilde{\mathbf{e}}$  ( $K = \text{diag}(k_1, k_2, k_3, k_4)$ ), we have

$$\begin{aligned} \int_0^t \lambda_{\min}(K) \|\tilde{\mathbf{e}}\|^2 dt &\leq \int_0^t \tilde{\mathbf{e}}^T K \tilde{\mathbf{e}} dt \\ &\leq \int_0^t -\dot{V} dt = V(0) - V(t) \leq V(0), \end{aligned} \quad (17)$$

where  $\lambda_{\min}(K)$  is the minimum eigenvalue of  $K$ , so  $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4 \in L_\infty$ ; according to Barbalat's Lemma we have

$$\lim_{t \rightarrow \infty} \|\tilde{\mathbf{e}}(t)\| = 0. \quad (18)$$

The UPS of system (10) and system (11) in the sense of vector function  $\boldsymbol{\beta}(t)$  is achieved.

Now we illustrate the effectiveness of the method. Suppose the vector function  $\boldsymbol{\beta}(t) = (2 \sin t + \cos 2t, 32 \sin(3t) + 6, 6 \cos t, 2.7 \sin(2t) + 3 \cos t)^T$ . The "unknown" parameters are  $a = 36, b = 3, c = 20, d = 1.3, a_1 = 26$ , and  $b_1 = 14$ . Control gain is  $(k_1, k_2, k_3, k_4) = (2, 3, 4, 5)$ . The scaling matrix is  $\mathbf{A}(t) = \text{diag}(\sin 2t + 2, -\cos 3t - 3, \sin t + \cos t + 3, 2 \sin t - 3)$ . The initial values of the drive system and response system are  $(3, -4, 2, 2)$  and  $(5, 7, 9, 11)$ , respectively. The parameter estimate initial values are  $\hat{a}(0) = 6, \hat{b}(0) = -3, \hat{c}(0) = 6, \hat{d}(0) = 6, \hat{a}_1(0) = 6$ , and  $\hat{b}_1(0) = 3$ . Simulation results are as Figures 1, 2, 3, 4, 5, 6, 7, and 8.  $\square$

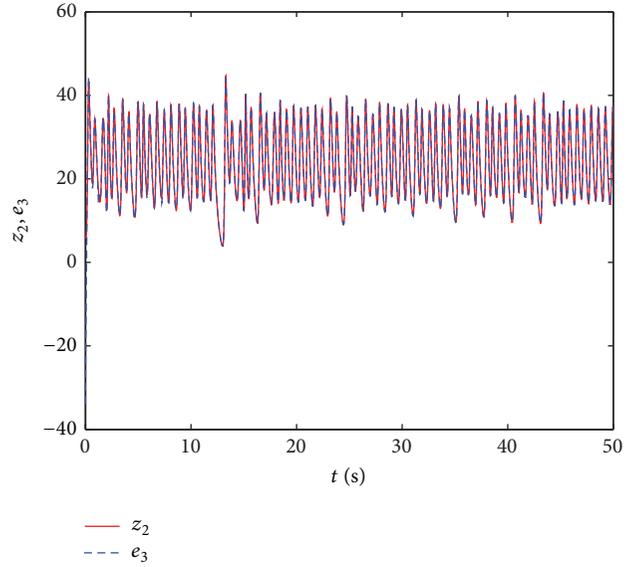


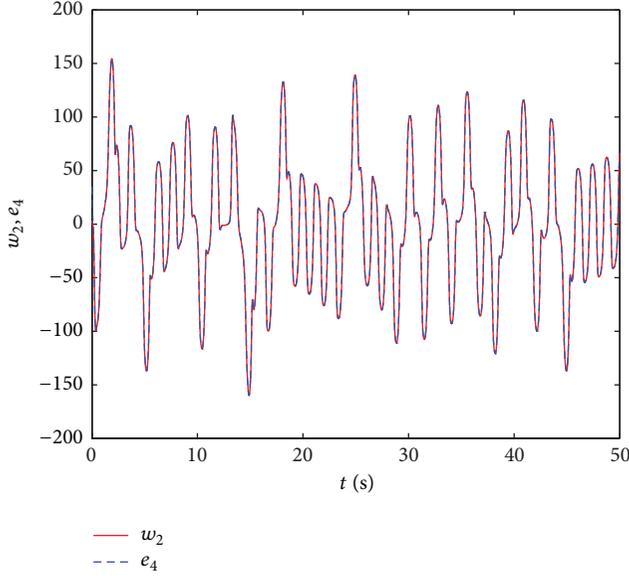
FIGURE 21: The time evolution of the error  $e_3$  and the state  $z_2$ .

3.2. *UPS in the Sense of a Hyperchaotic System.* We take hyperchaotic Wang system [18] as the reference system, which is given by

$$\begin{aligned} \dot{x}_2 &= a_2(-x_2 + y_2) + w_2, \\ \dot{y}_2 &= c_2 x_2 - y_2 - x_2 z_2, \\ \dot{z}_2 &= x_2 y_2 - b_2 z_2, \\ \dot{w}_2 &= -y_2 z_2 + d_2 w_2, \end{aligned} \quad (19)$$

where  $x_2, y_2, z_2, w_2$  are state variables and  $a_2 = 10, b_2 = 8/3, c_2 = 28$ , and  $d_2 = -1$  are parameters.

Let  $\tilde{\mathbf{e}} = (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4)^T \triangleq (e_1 - x_2, e_2 - y_2, e_3 - z_2, e_4 - w_2)^T$ .

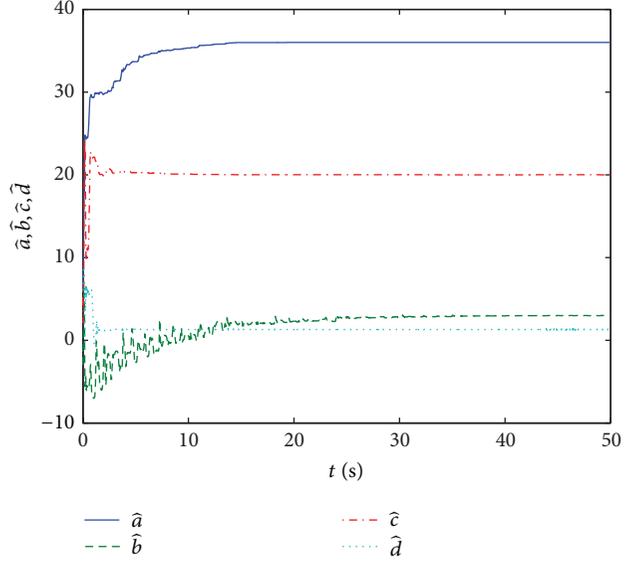
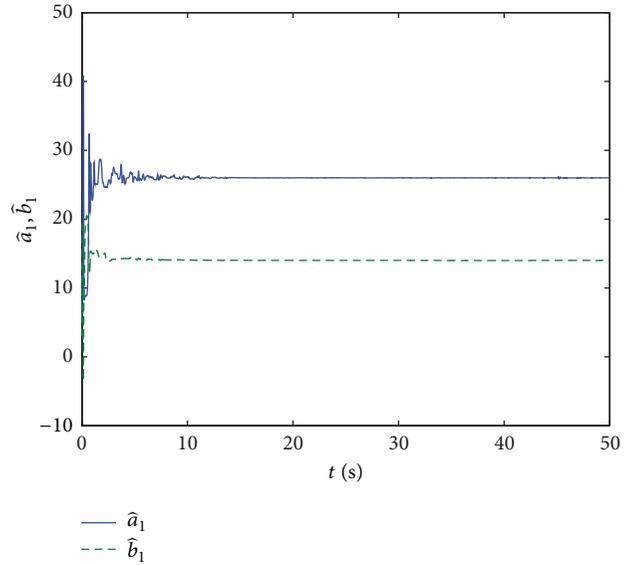
FIGURE 22: The time evolution of the error  $e_4$  and the state  $w_2$ .

**Theorem 6.** For the given scaling matrix  $\mathbf{A}(t)$  and the reference system which is hyperchaotic Wang system, the UPS between drive system (10) and response system (11) with unknown parameters will occur by the control  $\mathbf{u} = (u_1, u_2, u_3, u_4)$  and parameter update law as follows:

$$\begin{aligned}
 u_1 &= \frac{1}{\alpha_1(t)} (\hat{a}(y-x) + w - \dot{\alpha}_1(t)x_1 - \alpha_1(t)\hat{a}_1(y_1 - x_1) \\
 &\quad - (a_2(-x_2 + y_2) + w_2) + k_1\tilde{e}_1), \\
 u_2 &= \frac{1}{\alpha_2(t)} (-xz + \hat{c}y - \dot{\alpha}_2(t)y_1 + \alpha_2(t)x_1z_1 - \alpha_2(t)\hat{b}_1y_1 \\
 &\quad - \alpha_2(t)w_1 - (c_2x_2 - y_2 - x_2z_2) + k_2\tilde{e}_2), \\
 u_3 &= \frac{1}{\alpha_3(t)} (xy - \hat{b}z - \dot{\alpha}_3(t)z_1 - \alpha_3(t)x_1y_1 + 3.36\alpha_3(t)z_1 \\
 &\quad - (x_2y_2 - b_2z_2) + k_3\tilde{e}_3), \\
 u_4 &= \frac{1}{\alpha_4(t)} (xz + \hat{d}w - \dot{\alpha}_4(t)w_1 + \alpha_4(t)x_1z_1 - \alpha_4(t)w_1 \\
 &\quad - (-\gamma_2z_2 + d_2w_2) + k_4\tilde{e}_4),
 \end{aligned} \tag{20}$$

$$\begin{aligned}
 \dot{\hat{a}} &= (y-x)\tilde{e}_1, \\
 \dot{\hat{b}} &= -z\tilde{e}_3, \\
 \dot{\hat{c}} &= y\tilde{e}_2, \\
 \dot{\hat{d}} &= w\tilde{e}_4,
 \end{aligned} \tag{21}$$

$$\begin{aligned}
 \dot{\hat{a}}_1 &= -\alpha_1(t)(y_1 - x_1)\tilde{e}_1, \\
 \dot{\hat{b}}_1 &= -\alpha_2(t)y_1\tilde{e}_2.
 \end{aligned}$$

FIGURE 23: The time evolution of the estimated values  $\hat{a}, \hat{b}, \hat{c}, \hat{d}$ .FIGURE 24: The time evolution of the estimated values  $\hat{a}_1, \hat{b}_1$ .

The proof is similar to Theorem 5 (omitted!).

Now we illustrate the effectiveness of this method. Suppose the initial value of hyperchaotic Wang is  $(1, 6, 6, 3)$ . The "unknown" parameters are  $a = 36, b = 3, c = 20, d = 1.3, a_1 = 26$ , and  $b_1 = 14$ . Control gain is  $(k_1, k_2, k_3, k_4) = (12, 13, 14, 15)$ . The scaling matrix is  $\mathbf{A}(t) = \text{diag}(\sin 2t + 2, -\cos 3t - 3, \sin t + \cos t + 3, 2 \sin t - 3)$ . The initial values of the drive system and response system are  $(3, -4, 2, 2)$  and  $(5, 7, 9, 11)$ , respectively. The parameter estimate initial values are  $\hat{a}(0) = 6, \hat{b}(0) = -3, \hat{c}(0) = 6, \hat{d}(0) = 6, \hat{a}_1(0) = 6$ , and  $\hat{b}_1(0) = 3$ . Simulation results are as Figures 9, 10, 11, 12, 13, 14, 15, and 16.

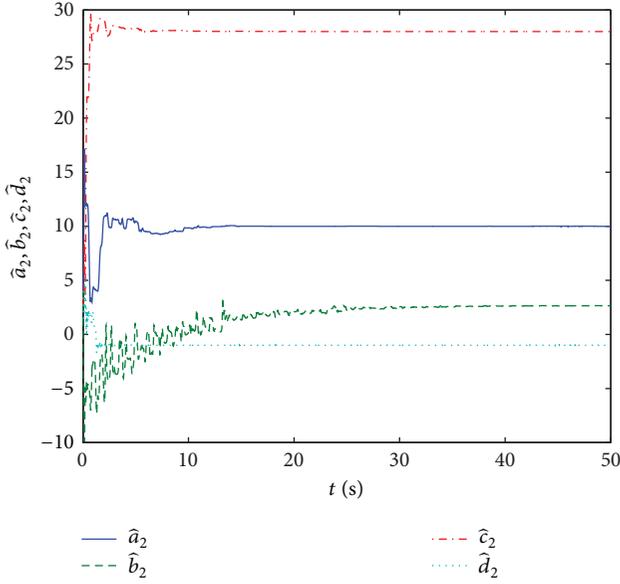


FIGURE 25: The time evolution of the estimated values  $\hat{a}_2, \hat{b}_2, \hat{c}_2, \hat{d}_2$ .

**3.3. UPS in the Sense of a Hyperchaotic Wang System with Unknown Parameters.** We still use hyperchaotic Wang system as the reference system, but the parameters  $a_2, b_2, c_2, d_3$  are uncertain. Suppose the estimated values of  $a_2, b_2, c_2, d_3$  are  $\hat{a}_2, \hat{b}_2, \hat{c}_2, \hat{d}_2$ , respectively; the parameter errors are  $(\tilde{a}_2, \tilde{b}_2, \tilde{c}_2, \tilde{d}_2) = (\hat{a}_2 - a_2, \hat{b}_2 - b_2, \hat{c}_2 - c_2, \hat{d}_2 - d_2)$ . Let  $\tilde{\mathbf{e}} = (\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4)^T \triangleq (e_1 - x_2, e_2 - y_2, e_3 - z_2, e_4 - w_2)^T$ .

**Theorem 7.** For the given scaling matrix  $\mathbf{A}(t)$  and the reference system which is hyperchaotic Wang system with unknown parameters, the UPS between drive system (10) and response system (11) with unknown parameters will occur by the control  $\mathbf{u} = (u_1, u_2, u_3, u_4)$  and parameter update law as follows:

$$\begin{aligned}
 u_1 &= \frac{1}{\alpha_1(t)} (\hat{a}(y-x) + w - \dot{\alpha}_1(t) x_1 - \alpha_1(t) \hat{a}_1 (y_1 - x_1) \\
 &\quad - \hat{a}_2 (-x_2 + y_2) - w_2 + k_1 \tilde{e}_1), \\
 u_2 &= \frac{1}{\alpha_2(t)} (-xz + \hat{c}y - \dot{\alpha}_2(t) y_1 + \alpha_2(t) x_1 z_1 - \alpha_2(t) \hat{b}_1 y_1 \\
 &\quad - \alpha_2(t) w_1 - \hat{c}_2 x_2 + y_2 + x_2 z_2 + k_2 \tilde{e}_2), \\
 u_3 &= \frac{1}{\alpha_3(t)} (xy - \hat{b}z - \dot{\alpha}_3(t) z_1 - \alpha_3(t) x_1 y_1 + 3.36 \alpha_3(t) z_1 \\
 &\quad - x_2 y_2 + \hat{b}_2 z_2 + k_3 \tilde{e}_3), \\
 u_4 &= \frac{1}{\alpha_4(t)} (xz + \hat{d}w - \dot{\alpha}_4(t) w_1 + \alpha_4(t) x_1 z_1 - \alpha_4(t) w_1 \\
 &\quad + y_2 z_2 - \hat{d}_2 w_2 + k_4 \tilde{e}_4),
 \end{aligned} \tag{22}$$

$$\begin{aligned}
 \dot{\hat{a}} &= (y-x) \tilde{e}_1, \\
 \dot{\hat{b}} &= -z \tilde{e}_3, \\
 \dot{\hat{c}} &= y \tilde{e}_2, \\
 \dot{\hat{d}} &= w \tilde{e}_4,
 \end{aligned} \tag{23}$$

$$\begin{aligned}
 \dot{\hat{a}}_1 &= -\alpha_1(t) (y_1 - x_1) \tilde{e}_1, \\
 \dot{\hat{b}}_1 &= -\alpha_2(t) y_1 \tilde{e}_2, \\
 \dot{\hat{a}}_2 &= -(-x_2 + y_2) \tilde{e}_1, \\
 \dot{\hat{b}}_2 &= z_2 \tilde{e}_3, \\
 \dot{\hat{c}}_2 &= -x_2 \tilde{e}_2, \\
 \dot{\hat{d}}_2 &= -w_2 \tilde{e}_4.
 \end{aligned} \tag{24}$$

*Proof.* Choose the following Lyapunov function:

$$\begin{aligned}
 V &= \frac{1}{2} (\tilde{\mathbf{e}}^T \tilde{\mathbf{e}} + \tilde{a}^2 + \tilde{b}^2 + \tilde{c}^2 + \tilde{d}^2 + \tilde{a}_1^2 \\
 &\quad + \tilde{b}_1^2 + \tilde{a}_2^2 + \tilde{b}_2^2 + \tilde{c}_2^2 + \tilde{d}_2^2).
 \end{aligned} \tag{25}$$

Then the derivative of  $V$  is given by

$$\begin{aligned}
 \dot{V} &= \frac{1}{2} (2\tilde{\mathbf{e}}^T \dot{\tilde{\mathbf{e}}} + 2\tilde{a}\dot{\tilde{a}} + 2\tilde{b}\dot{\tilde{b}} + 2\tilde{c}\dot{\tilde{c}} + 2\tilde{d}\dot{\tilde{d}} + 2\tilde{a}_1\dot{\tilde{a}}_1 + 2\tilde{b}_1\dot{\tilde{b}}_1 \\
 &\quad + 2\tilde{a}_2\dot{\tilde{a}}_2 + 2\tilde{b}_2\dot{\tilde{b}}_2 + 2\tilde{c}_2\dot{\tilde{c}}_2 + 2\tilde{d}_2\dot{\tilde{d}}_2) \\
 &= \tilde{\mathbf{e}}^T \dot{\tilde{\mathbf{e}}} + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} + \tilde{d}\dot{\tilde{d}} + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{b}_1\dot{\tilde{b}}_1 \\
 &\quad + \tilde{a}_2\dot{\tilde{a}}_2 + \tilde{b}_2\dot{\tilde{b}}_2 + \tilde{c}_2\dot{\tilde{c}}_2 + \tilde{d}_2\dot{\tilde{d}}_2 \\
 &= \tilde{e}_1 (a(y-x) + w - \dot{\alpha}_1 x_1 - \alpha_1 a_1 (y_1 - x_1) \\
 &\quad - \alpha_1 u_1 - a_2 (-x_2 + y_2) - w_2) \\
 &\quad + \tilde{e}_2 (-xz + cy - \dot{\alpha}_2 y_1 - \alpha_2 (b_1 y_1 - x_1 z_1 + w_1) \\
 &\quad - \alpha_2 u_2 - c_2 x_2 + y_2 + x_2 z_2) \\
 &\quad + \tilde{e}_3 (xy - bz - \dot{\alpha}_3 z_1 - \alpha_3 (x_1 y_1 - c_1 z_1) \\
 &\quad - \alpha_3 u_3 - x_2 y_2 + b_2 z_2) \\
 &\quad + \tilde{e}_4 (xz + dw - \dot{\alpha}_4 w_1 - \alpha_4 (-x_1 z_1 + w_1) \\
 &\quad - \alpha_4 u_4 + y_2 z_2 - d_2 w_2) \\
 &\quad + \tilde{a}\dot{\tilde{a}} + \tilde{b}\dot{\tilde{b}} + \tilde{c}\dot{\tilde{c}} + \tilde{d}\dot{\tilde{d}} + \tilde{a}_1\dot{\tilde{a}}_1 + \tilde{b}_1\dot{\tilde{b}}_1 \\
 &\quad + \tilde{a}_2\dot{\tilde{a}}_2 + \tilde{b}_2\dot{\tilde{b}}_2 + \tilde{c}_2\dot{\tilde{c}}_2 + \tilde{d}_2\dot{\tilde{d}}_2.
 \end{aligned} \tag{26}$$

With the controls (22) and the parameter update laws (23) and (24), we have

$$\dot{V} = -k_1 \tilde{e}_1^2 - k_2 \tilde{e}_2^2 - k_3 \tilde{e}_3^2 - k_4 \tilde{e}_4^2. \tag{27}$$

Because  $\dot{V}$  is negative semidefinite,  $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4 \in L_\infty$ ,  $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2, \tilde{c}_2, \tilde{d}_2 \in L_\infty$ . From the definition of  $\tilde{\mathbf{e}}$ , we know  $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4 \in L_\infty$ . From  $\dot{V} = -k_1\tilde{e}_1^2 - k_2\tilde{e}_2^2 - k_3\tilde{e}_3^2 - k_4\tilde{e}_4^2 = -\tilde{\mathbf{e}}^T K \tilde{\mathbf{e}}$  ( $K = \text{diag}(k_1, k_2, k_3, k_4)$ ), we have

$$\begin{aligned} \int_0^t \lambda_{\min}(K) \|\tilde{\mathbf{e}}\|^2 dt &\leq \int_0^t \tilde{\mathbf{e}}^T K \tilde{\mathbf{e}} dt \\ &\leq \int_0^t -\dot{V} dt = V(0) - V(t) \leq V(0), \end{aligned} \quad (28)$$

where  $\lambda_{\min}(K)$  is the minimum eigenvalue of  $K$ , so  $\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \tilde{e}_4 \in L_\infty$ ; according to Barbalat's Lemma we have

$$\lim_{t \rightarrow \infty} \|\tilde{\mathbf{e}}(t)\| = 0. \quad (29)$$

The UPS of system (10) and system (11) in the sense of hyperchaotic Wang system (19) with uncertain parameters is achieved.

Now we illustrate the effectiveness of this method. Suppose the initial value of hyperchaotic Wang system is (1, 6, 6, 3). The "unknown" parameters are  $a = 36$ ,  $b = 3$ ,  $c = 20$ ,  $d = 1.3$ ,  $a_1 = 26$ ,  $b_1 = 14$ ,  $a_2 = 10$ ,  $b_2 = 8/3$ ,  $c_2 = 28$ , and  $d_2 = -1$ . Control gain is  $(k_1, k_2, k_3, k_4) = (12, 13, 14, 15)$ . The scaling matrix is  $\mathbf{A}(t) = \text{diag}(\sin 2t + 2, -\cos 3t - 3, \sin t + \cos t + 3, 2 \sin t - 3)$ . The initial values of the drive system and the response system are (3, -4, 2, 2) and (5, 7, 9, 11), respectively. The parameter estimate initial values are  $\hat{a}(0) = 6$ ,  $\hat{b}(0) = -3$ ,  $\hat{c}(0) = 6$ ,  $\hat{d}(0) = 6$ ,  $\hat{a}_1(0) = 6$ ,  $\hat{b}_1(0) = 3$ ,  $\hat{a}_2(0) = 4$ ,  $\hat{b}_2(0) = 4$ ,  $\hat{c}_2(0) = 4$ , and  $\hat{d}_2(0) = 4$ . Simulation results are as Figures 17, 18, 19, 20, 21, 22, 23, 24, and 25.

From the above analysis, we find that, in the form, the adaptive laws equations (13), (21), and (23) corresponding three kinds of reference systems are the same.  $\square$

## 4. Conclusion

In this paper, we have defined universal projective synchronization. For different hyperchaotic systems with uncertain parameters, we have given adaptive control method such that UPS has occurred and the parameters have been determined. Three kinds of reference systems have been concerned and we find that, in the form, the adaptive laws are the same. Numerical simulations have verified the effectiveness of the scheme.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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