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Research Article

Boundary Value Problems for First-Order Impulsive Functional q-Integrodifference Equations

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We discuss the existence and uniqueness of solutions for a first-order boundary value problem for impulsive functional q_k-integrodifference equations. The main results are obtained with the aid of some classical fixed point theorems. Illustrative examples are also presented.

1. Introduction

In this paper, we study the boundary value problem for impulsive functional q_k-integro-difference equation of the following the form:

$$\begin{gathered} D_{q_k} x(t) = f\left(t, x(t), x(\theta(t)), \left(S_{q_k} x\right)(t)\right), \\ t \in J := [0, T], \quad t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ \alpha x(0) = \beta x(T) + \sum_{i=0}^{m} \gamma_i \int_{t_i}^{t_{i+1}} x(s)\, d_{q_i} s, \end{gathered} \tag{1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times \mathbb{R}^3 \to \mathbb{R}$, $\theta : J \to J$,

$$\left(S_{q_k} x\right)(t) = \int_{t_k}^{t} \phi(t, s, x(s))\, d_{q_k} s, \quad k = 0, 1, 2, \ldots, m, \tag{2}$$

$\phi : J^2 \times \mathbb{R} \to [0, \infty)$ is a continuous function, $I_k \in C(\mathbb{R}, \mathbb{R})$, $\Delta x(t_k) = x(t_k^+) - x(t_k)$ for $k = 1, 2, \ldots, m$, $x(t_k^+) = \lim_{h \to 0} x(t_k + h)$, α, β, γ_i, $i = 0, 1, \ldots, m$ are real constants, and $0 < q_k < 1$ for $k = 0, 1, 2, \ldots, m$.

The notions of q_k-derivative and q_k-integral on finite intervals were introduced recently by the authors in [1]. Their basic properties were studied and applications existence and uniqueness results were proved for initial value problems for first- and second-order impulsive q_k-difference equations. In this paper, we continue the study on this new subject by considering the boundary value problem (1).

The book by Kac and Cheung [2] covers many of the fundamental aspects of the quantum calculus. In recent years, the topic of q-calculus has attracted the attention of several researchers and a variety of new results can be found in the papers [3–15] and the references cited therein. On the other hand, for some monographs on the impulsive differential equations we refer to [16–18].

The rest of this paper is organized as follows. In Section 2, we recall the notions of q_k-derivative and q_k-integral on finite intervals and present a preliminary result which will be used in this paper. In Section 3, we will consider the existence results for problem (1) while in Section 4, we will give examples to illustrate our main results.

2. Preliminaries

In this section, we recall the notions of q_k-derivative and q_k-integral on finite intervals. For a fixed $k \in \mathbb{N} \cup \{0\}$ let $J_k := [t_k, t_{k+1}] \subset \mathbb{R}$ be an interval and let $0 < q_k < 1$ be a constant. We define q_k-derivative of a function $f : J_k \to \mathbb{R}$ at a point $t \in J_k$ as follows.

Definition 1. Assume $f : J_k \to \mathbb{R}$ is a continuous function and let $t \in J_k$. Then the expression

$$D_{q_k} f(t) = \frac{f(t) - f(q_k t + (1 - q_k) t_k)}{(1 - q_k)(t - t_k)}, \quad t \neq t_k,$$
$$D_{q_k} f(t_k) = \lim_{t \to t_k} D_{q_k} f(t) \tag{3}$$

is called the q_k-derivative of function f at t.

We say that f is q_k-differentiable on J_k provided that $D_{q_k} f(t)$ exists for all $t \in J_k$. Note that if $t_k = 0$ and $q_k = q$ in (3), then $D_{q_k} f = D_q f$, where D_q is the well-known q-derivative of the function $f(t)$ defined by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1 - q)t}. \tag{4}$$

In addition, we should define the higher q_k-derivative of functions.

Definition 2. Let $f : J_k \to \mathbb{R}$ be a continuous function; we call the second-order q_k-derivative $D^2_{q_k} f$ provided that $D_{q_k} f$ is q_k-differentiable on J_k with $D^2_{q_k} f = D_{q_k}(D_{q_k} f) : J_k \to \mathbb{R}$. Similarly, we define higher order q_k-derivative $D^n_{q_k} : J_k \to \mathbb{R}$.

The q_k-integral is defined as follows.

Definition 3. Assume $f : J_k \to \mathbb{R}$ is a continuous function. Then the q_k-integral is defined by

$$\int_{t_k}^{t} f(s)\, d_{q_k} s = (1 - q_k)(t - t_k) \sum_{n=0}^{\infty} q_k^n f(q_k^n t + (1 - q_k^n) t_k) \tag{5}$$

for $t \in J_k$. Moreover, if $a \in (t_k, t)$, then the definite q_k-integral is defined by

$$\begin{aligned}\int_{a}^{t} f(s)\, d_{q_k} s &= \int_{t_k}^{t} f(s)\, d_{q_k} s - \int_{t_k}^{a} f(s)\, d_{q_k} s \\ &= (1 - q_k)(t - t_k) \sum_{n=0}^{\infty} q_k^n f(q_k^n t + (1 - q_k^n) t_k) \\ &\quad - (1 - q_k)(a - t_k) \sum_{n=0}^{\infty} q_k^n f(q_k^n a + (1 - q_k^n) t_k).\end{aligned} \tag{6}$$

Note that if $t_k = 0$ and $q_k = q$, then (5) reduces to q-integral of a function $f(t)$, defined by

$$\int_{0}^{t} f(s)\, d_q s = (1 - q)t \sum_{n=0}^{\infty} q^n f(q^n t) \quad \text{for } t \in [0, \infty). \tag{7}$$

For the basic properties of q_k-derivative and q_k-integral we refer to [1].

Let $J = [0, T]$, $J_0 = [t_0, t_1]$, and $J_k = (t_k, t_{k+1}]$ for $k = 1, 2, \ldots, m$. Let $PC(J, \mathbb{R}) = \{x : J \to \mathbb{R} : x(t)$ be continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $k = 1, 2, \ldots, m\}$. $PC(J, \mathbb{R})$ is a Banach space with the norm $\|x\|_{PC} = \sup\{|x(t)|; t \in J\}$.

We now consider the following linear case:

$$\begin{aligned}D_{q_k} x(t) &= h(t), \quad t \in [0, T], \ t \neq t_k, \\ \Delta x(t_k) &= I_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\ \alpha x(0) &= \beta x(T) + \sum_{i=0}^{m} \gamma_i \int_{t_i}^{t_{i+1}} x(s)\, d_{q_i} s,\end{aligned} \tag{8}$$

where $h : J \to \mathbb{R}$.

Lemma 4. *Let $\alpha \neq \beta + \sum_{i=0}^{m} \gamma_i (t_{i+1} - t_i)$. The unique solution of problem (8) is given by*

$$\begin{aligned}x(t) = {} & \frac{\beta}{\Omega} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}} s + \frac{\beta}{\Omega} \sum_{k=1}^{m} I_k(x(t_k)) \\ & + \frac{1}{\Omega} \sum_{i=0}^{m} \gamma_i \int_{t_i}^{t_{i+1}} \int_{t_i}^{u} h(s)\, d_{q_i} s\, d_{q_i} u \\ & + \frac{1}{\Omega} \sum_{i=1}^{m} \sum_{k=1}^{i} \gamma_i (t_{i+1} - t_i) \int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}} s \\ & + \frac{1}{\Omega} \sum_{i=1}^{m} \sum_{k=1}^{i} \gamma_i (t_{i+1} - t_i) I_k(x(t_k)) \\ & + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}} s + I_k(x(t_k)) \right) \\ & + \int_{t_k}^{t} h(s)\, d_{q_k} s,\end{aligned} \tag{9}$$

with $\sum_{i=a}^{b}(\cdot) = 0$ for $a > b$, where

$$\Omega = \frac{1}{\alpha - \beta - \sum_{i=0}^{m} \gamma_i (t_{i+1} - t_i)}. \tag{10}$$

Proof. For $t \in J_0$, q_0-integrating (8), it follows

$$x(t) = x_0 + \int_0^t h(s)\, d_{q_0} s, \tag{11}$$

which leads to

$$x(t_1) = x_0 + \int_0^{t_1} h(s)\, d_{q_0} s. \tag{12}$$

For $t \in J_1$, taking q_1-integral to (8), we have

$$x(t) = x(t_1^+) + \int_{t_1}^{t} h(s)\, d_{q_1} s. \tag{13}$$

Since $x(t_1^+) = x(t_1) + I_1(x(t_1))$, then we have

$$x(t) = x_0 + \int_0^{t_1} h(s)\, d_{q_0} s + \int_{t_1}^{t} h(s)\, d_{q_1} s + I_1(x(t_1)). \tag{14}$$

Again q_2-integrating (8) from t_2 to t, where $t \in J_2$, then

$$\begin{aligned} x(t) &= x(t_2^+) + \int_{t_2}^{t} h(s)\, d_{q_2}s \\ &= x_0 + \int_0^{t_1} h(s)\, d_{q_0}s + \int_{t_1}^{t_2} h(s)\, d_{q_1}s \\ &\quad + \int_{t_2}^{t} h(s)\, d_{q_2}s + I_1(x(t_1)) + I_2(x(t_2)). \end{aligned} \tag{15}$$

Repeating the above process, for $t \in J$, we obtain

$$\begin{aligned} x(t) &= x_0 + \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}}s + I_k(x(t_k)) \right) \\ &\quad + \int_{t_k}^{t} h(s)\, d_{q_k}s. \end{aligned} \tag{16}$$

In particular, for $t = T$, we have

$$\begin{aligned} x(T) &= x_0 + \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}}s + I_k(x(t_k)) \right) \\ &\quad + \int_{t_m}^{T} h(s)\, d_{q_m}s. \end{aligned} \tag{17}$$

Further, q_i-integrating (16) from t_i to t_{i+1}, it follows

$$\begin{aligned} &\int_{t_i}^{t_{i+1}} x(u)\, d_{q_i}u \\ &\quad = x_0(t_{i+1} - t_i) \\ &\qquad + \sum_{k=1}^{i} \left(\int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}}s + I_k(x(t_k)) \right)(t_{i+1} - t_i) \\ &\qquad + \int_{t_i}^{t_{i+1}} \int_{t_i}^{u} h(s)\, d_{q_i}s\, d_{q_i}u. \end{aligned} \tag{18}$$

Applying the boundary condition of (8), one has

$$\begin{aligned} \alpha x_0 &= \beta x_0 + \beta \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}}s + I_k(x(t_k)) \right) \\ &\quad + \beta \int_{t_m}^{T} h(s)\, d_{q_m}s + x_0 \sum_{i=0}^{m} \gamma_i (t_{i+1} - t_i) \\ &\quad + \sum_{i=0}^{m} \gamma_i \sum_{k=1}^{i} \left(\int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}}s + I_k(x(t_k)) \right)(t_{i+1} - t_i) \\ &\quad + \sum_{i=0}^{m} \gamma_i \int_{t_i}^{t_{i+1}} \int_{t_i}^{u} h(s)\, d_{q_i}s\, d_{q_i}u \\ &= \beta x_0 + \beta \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}}s + \beta \sum_{k=1}^{m} I_k(x(t_k)) \\ &\quad + \beta \int_{t_m}^{T} h(s)\, d_{q_m}s + x_0 \sum_{i=0}^{m} \gamma_i (t_{i+1} - t_i) \\ &\quad + \sum_{i=0}^{m} \sum_{k=1}^{i} \gamma_i (t_{i+1} - t_i) \int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}}s \\ &\quad + \sum_{i=0}^{m} \sum_{k=1}^{i} \gamma_i (t_{i+1} - t_i) I_k(x(t_k)) \\ &\quad + \sum_{i=0}^{m} \gamma_i \int_{t_i}^{t_{i+1}} \int_{t_i}^{u} h(s)\, d_{q_i}s\, d_{q_i}u. \end{aligned} \tag{19}$$

Since $T = t_{m+1}$ and $\sum_{i=a}^{b}(\cdot) = 0$ for $a > b$, we have

$$\begin{aligned} &x_0 \left(\alpha - \beta - \sum_{i=0}^{m} \gamma_i (t_{i+1} - t_i) \right) \\ &\quad = \beta \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}}s + \beta \sum_{k=1}^{m} I_k(x(t_k)) \\ &\qquad + \sum_{i=1}^{m} \sum_{k=1}^{i} \gamma_i (t_{i+1} - t_i) \int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}}s \\ &\qquad + \sum_{i=1}^{m} \sum_{k=1}^{i} \gamma_i (t_{i+1} - t_i) I_k(x(t_k)) \\ &\qquad + \sum_{i=0}^{m} \gamma_i \int_{t_i}^{t_{i+1}} \int_{t_i}^{u} h(s)\, d_{q_i}s\, d_{q_i}u. \end{aligned} \tag{20}$$

Therefore,

$$\begin{aligned} x_0 &= \frac{\beta}{\Omega} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}}s + \frac{\beta}{\Omega} \sum_{k=1}^{m} I_k(x(t_k)) \\ &\quad + \frac{1}{\Omega} \sum_{i=0}^{m} \gamma_i \int_{t_i}^{t_{i+1}} \int_{t_i}^{u} h(s)\, d_{q_i}s\, d_{q_i}u \\ &\quad + \frac{1}{\Omega} \sum_{i=1}^{m} \sum_{k=1}^{i} \gamma_i (t_{i+1} - t_i) \int_{t_{k-1}}^{t_k} h(s)\, d_{q_{k-1}}s \\ &\quad + \frac{1}{\Omega} \sum_{i=1}^{m} \sum_{k=1}^{i} \gamma_i (t_{i+1} - t_i) I_k(x(t_k)). \end{aligned} \tag{21}$$

Substituting the constant x_0 into (16), we obtain (9) as requested. □

3. Main Results

In view of Lemma 4, we define an operator $\mathscr{K}:\mathrm{PC}(J,\mathbb{R})\to\mathrm{PC}(J,\mathbb{R})$ by

$$\begin{aligned}(\mathscr{K}x)(t) &= \frac{\beta}{\Omega}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k} f\left(s,x(s),x(\theta(s)),\left(S_{q_{k-1}}x\right)(s)\right)d_{q_{k-1}}s \\ &+ \frac{\beta}{\Omega}\sum_{k=1}^{m} I_k\left(x\left(t_k\right)\right) + \frac{1}{\Omega}\sum_{i=0}^{m}\gamma_i \\ &\times \int_{t_i}^{t_{i+1}}\int_{t_i}^{u} f\left(s,x(s),x(\theta(s)),\left(S_{q_i}x\right)(s)\right)d_{q_i}s\,d_{q_i}u \\ &+ \frac{1}{\Omega}\sum_{i=1}^{m}\sum_{k=1}^{i}\gamma_i\left(t_{i+1}-t_i\right) \\ &\times \int_{t_{k-1}}^{t_k} f\left(s,x(s),x(\theta(s)),\left(S_{q_{k-1}}x\right)(s)\right)d_{q_{k-1}}s \\ &+ \frac{1}{\Omega}\sum_{i=1}^{m}\sum_{k=1}^{i}\gamma_i\left(t_{i+1}-t_i\right)I_k\left(x\left(t_k\right)\right) \\ &+ \int_{t_k}^{t} f\left(s,x(s),x(\theta(s)),\left(S_{q_k}x\right)(s)\right)d_{q_k}s \\ &+ \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} f\left(s,x(s),x(\theta(s)),\left(S_{q_{k-1}}x\right)(s)\right)d_{q_{k-1}}s\right. \\ &\qquad \left. + I_k\left(x\left(t_k\right)\right)\right).\end{aligned} \tag{22}$$

It should be noticed that problem (1) has solutions if and only if the operator $\mathscr{K}$ has fixed points.

Our first result is an existence and uniqueness result for the impulsive boundary value problem (1) by using Banach's contraction mapping principle.

Further, for convenience we set

$$\begin{aligned}\Lambda_1 &= \frac{|\beta|+|\Omega|}{|\Omega|}\sum_{k=1}^{m+1}\left[\left(L_1+L_2\right)\left(t_k-t_{k-1}\right) + \frac{\phi_0 L_3\left(t_k-t_{k-1}\right)^2}{1+q_{k-1}}\right] \\ &+ \frac{1}{|\Omega|}\sum_{i=0}^{m}\left|\gamma_i\right|\left[\left(L_1+L_2\right)\frac{\left(t_{i+1}-t_i\right)^2}{1+q_i} + \frac{\phi_0 L_3\left(t_{i+1}-t_i\right)^3}{1+q_i+q_i^2}\right] \\ &+ \frac{1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left|\gamma_i\right|\left(t_{i+1}-t_i\right)\left[\left(L_1+L_2\right)\left(t_k-t_{k-1}\right) + \frac{\phi_0 L_3\left(t_k-t_{k-1}\right)^2}{1+q_{k-1}}\right] \\ &+ \frac{L_4}{|\Omega|}\sum_{i=1}^{m} i\left|\gamma_i\right|\left(t_{i+1}-t_i\right) + \frac{m\left(|\beta|+|\Omega|\right)L_4}{|\Omega|},\end{aligned} \tag{23}$$

$$\begin{aligned}\Lambda_2 &= \frac{|\beta|+|\Omega|}{|\Omega|}M_1\sum_{k=1}^{m+1}\left(t_k-t_{k-1}\right) + \frac{M_1}{|\Omega|}\sum_{i=0}^{m}\left|\gamma_i\right|\frac{\left(t_{i+1}-t_i\right)^2}{1+q_i} \\ &+ \frac{M_1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left|\gamma_i\right|\left(t_{i+1}-t_i\right)\left(t_k-t_{k-1}\right) \\ &+ \frac{M_2}{|\Omega|}\sum_{i=1}^{m} i\left|\gamma_i\right|\left(t_{i+1}-t_i\right) + \frac{m\left(|\beta|+|\Omega|\right)M_2}{|\Omega|}.\end{aligned} \tag{24}$$

Theorem 5. *Assume the following.*

(H_1) *The function* $\phi: J^2\times\mathbb{R}\to\mathbb{R}$ *is continuous and there exists a constant* $\phi_0>0$ *such that*

$$\left|\phi(t,s,y)-\phi(t,s,z)\right| \le \phi_0\left|y-z\right|, \tag{25}$$

for each $t,s\in J$ *and* $y,z\in\mathbb{R}$.

(H_2) *The function* $f: J\times\mathbb{R}^3\to\mathbb{R}$ *is continuous and there exist constants* $L_1,L_2,L_3>0$ *such that*

$$\begin{aligned}&\left|f\left(t,y_1,y_2,y_3\right)-f\left(t,z_1,z_2,z_3\right)\right| \\ &\quad \le L_1\left|y_1-z_1\right| + L_2\left|y_2-z_2\right| + L_3\left|y_3-z_3\right|,\end{aligned} \tag{26}$$

for each $t\in J$ *and* $y_i,z_i,\in\mathbb{R}$, $i=1,2,3$.

(H_3) *The functions* $I_k:\mathbb{R}\to\mathbb{R}$ *are continuous and there exists a constant* $L_4>0$ *such that*

$$\left|I_k(y)-I_k(z)\right| \le L_4\left|y-z\right|, \tag{27}$$

for each $y,z\in\mathbb{R}$, $k=1,2,\ldots,m$.

If

$$\Lambda_1 \le \delta < 1, \tag{28}$$

where Λ_1 *is defined by* (23), *then the boundary value problem* (1) *has a unique solution on* J.

Proof. Firstly, we transform the boundary value problem (1) into a fixed point problem, $x=\mathscr{K}x$, where the operator $\mathscr{K}$ is defined by (22). Using the Banach contraction principle, we will show that $\mathscr{K}$ has a fixed point which is the unique solution of the boundary value problem (1).

Let M_1 and M_2 be nonnegative constants such that $\sup_{t\in J}|f(t,0,0,0)| = M_1$ and $\sup\{|I_k(0)| : k=1,2,\ldots,m\} = M_2$. By choosing a positive constant r as

$$r \ge \frac{\Lambda_2}{1-\varepsilon}, \tag{29}$$

where $\delta \le \varepsilon < 1$ and Λ_2 defined by (24), we will show that $\mathscr{K}B_r \subset B_r$, where a suitable ball B_r is defined by $B_r = \{x \in \mathrm{PC}(J,\mathbb{R}) : \|x\| \le r\}$. For $x \in B_r$, we have

$$
\begin{aligned}
&|\mathscr{K}x(t)| \\
&\quad \le \frac{|\beta|}{|\Omega|}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\left|f\left(s,x(s),x(\theta(s)),\left(S_{q_{k-1}}x\right)(s)\right)\right| d_{q_{k-1}}s \\
&\qquad + \frac{|\beta|}{|\Omega|}\sum_{k=1}^{m}\left|I_k(x(t_k))\right| + \frac{1}{|\Omega|}\sum_{i=0}^{m}|\gamma_i| \\
&\qquad \times \int_{t_i}^{t_{i+1}}\int_{t_i}^{u}\left|f\left(s,x(s),x(\theta(s)),\left(S_{q_i}x\right)(s)\right)\right| d_{q_i}s\, d_{q_i}u \\
&\qquad + \frac{1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|(t_{i+1}-t_i) \\
&\qquad \times \int_{t_{k-1}}^{t_k}\left|f\left(s,x(s),x(\theta(s)),\left(S_{q_{k-1}}x\right)(s)\right)\right| d_{q_{k-1}}s \\
&\qquad + \frac{1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|(t_{i+1}-t_i)\left|I_k(x(t_k))\right| \\
&\qquad + \int_{t_k}^{t}\left|f\left(s,x(s),x(\theta(s)),\left(S_{q_k}x\right)(s)\right)\right| d_{q_k}s \\
&\qquad + \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\left|f\left(s,x(s),x(\theta(s)),\left(S_{q_{k-1}}x\right)(s)\right)\right| d_{q_{k-1}}s\right. \\
&\qquad\qquad \left. + \left|I_k(x(t_k))\right|\right) \\
&\quad \le \frac{|\beta|+|\Omega|}{|\Omega|} \\
&\qquad \times \sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\left|f\left(s,x(s),x(\theta(s)),\left(S_{q_{k-1}}x\right)(s)\right)\right| d_{q_{k-1}}s \\
&\qquad + \frac{1}{|\Omega|}\sum_{i=0}^{m}|\gamma_i| \\
&\qquad \times \int_{t_i}^{t_{i+1}}\int_{t_i}^{u}\left|f\left(s,x(s),x(\theta(s)),\left(S_{q_i}x\right)(s)\right)\right| d_{q_i}s\, d_{q_i}u \\
&\qquad + \frac{1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|(t_{i+1}-t_i) \\
&\qquad \times \int_{t_{k-1}}^{t_k}\left|f\left(s,x(s),x(\theta(s)),\left(S_{q_{k-1}}x\right)(s)\right)\right| d_{q_{k-1}}s \\
&\qquad + \frac{1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|(t_{i+1}-t_i)\left|I_k(x(t_k))\right| \\
&\qquad + \frac{|\beta|+|\Omega|}{|\Omega|}\sum_{k=1}^{m}\left|I_k(x(t_k))\right| \\
&\quad \le \frac{|\beta|+|\Omega|}{|\Omega|}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\left(\left|f\left(s,x(s),x(\theta(s)),\left(S_{q_{k-1}}x\right)(s)\right)\right.\right. \\
&\qquad\qquad \left.\left. - f(s,0,0,0)\right| + \left|f(s,0,0,0)\right|\right) d_{q_{k-1}}s \\
&\qquad + \frac{1}{|\Omega|}\sum_{i=0}^{m}|\gamma_i| \\
&\qquad \times \int_{t_i}^{t_{i+1}}\int_{t_i}^{u}\left(\left|f\left(s,x(s),x(\theta(s)),\left(S_{q_i}x\right)(s)\right)\right.\right. \\
&\qquad\qquad \left.\left. - f(s,0,0,0)\right| + \left|f(s,0,0,0)\right|\right) d_{q_i}s\, d_{q_i}u \\
&\qquad + \frac{1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|(t_{i+1}-t_i) \\
&\qquad \times \int_{t_{k-1}}^{t_k}\left(\left|f\left(s,x(s),x(\theta(s)),\left(S_{q_{k-1}}x\right)(s)\right)\right.\right. \\
&\qquad\qquad \left.\left. - f(s,0,0,0)\right| + \left|f(s,0,0,0)\right|\right) d_{q_{k-1}}s \\
&\qquad + \frac{1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|(t_{i+1}-t_i)\left(\left|I_k(x(t_k)) - I_k(0)\right| + \left|I_k(0)\right|\right) \\
&\qquad + \frac{|\beta|+|\Omega|}{|\Omega|}\sum_{k=1}^{m}\left(\left|I_k(x(t_k)) - I_k(0)\right| + \left|I_k(0)\right|\right) \\
&\quad \le \frac{|\beta|+|\Omega|}{|\Omega|} \\
&\qquad \times \sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\left[r\left(L_1+L_2+\phi_0L_3\int_{t_{k-1}}^{s} d_{q_{k-1}}u\right)+M_1\right] d_{q_{k-1}}s \\
&\qquad + \frac{1}{|\Omega|}\sum_{i=0}^{m}|\gamma_i| \\
&\qquad \times \int_{t_i}^{t_{i+1}}\int_{t_i}^{u}\left[r\left(L_1+L_2+\phi_0L_3\int_{t_i}^{s} d_{q_i}v\right)+M_1\right] d_{q_i}s\, d_{q_i}u \\
&\qquad + \frac{1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|(t_{i+1}-t_i) \\
&\qquad \times \int_{t_{k-1}}^{t_k}\left[r\left(L_1+L_2+\phi_0L_3\int_{t_{k-1}}^{s} d_{q_{k-1}}u\right)+M_1\right] d_{q_{k-1}}s \\
&\qquad + \frac{1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|(t_{i+1}-t_i)(rL_4+M_2) \\
&\qquad + \frac{|\beta|+|\Omega|}{|\Omega|}\sum_{k=1}^{m}(rL_4+M_2) \\
&\quad = r\Lambda_1 + \Lambda_2 \le r,
\end{aligned} \tag{30}
$$

which yields $\mathscr{K}B_r \subset B_r$.

For any $x, y \in \mathrm{PC}(J, \mathbb{R})$ and for each $t \in J$, we have

$$\begin{aligned}
&\left|\mathcal{K}x(t) - \mathcal{K}y(t)\right| \\
&\leq \frac{|\beta| + |\Omega|}{|\Omega|} \\
&\quad \times \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} \left| f\left(s, x(s), x(\theta(s)), \left(S_{q_{k-1}}x\right)(s)\right) \right. \\
&\qquad \left. - f\left(s, y(s), y(\theta(s)), \left(S_{q_{k-1}}y\right)(s)\right) \right| d_{q_{k-1}}s \\
&\quad + \frac{1}{|\Omega|} \sum_{i=0}^{m} |\gamma_i| \\
&\quad \times \int_{t_i}^{t_{i+1}} \int_{t_i}^{u} \left| f\left(s, x(s), x(\theta(s)), \left(S_{q_i}x\right)(s)\right) \right. \\
&\qquad \left. - f\left(s, y(s), y(\theta(s)), \left(S_{q_i}y\right)(s)\right) \right| d_{q_i}s \, d_{q_i}u \\
&\quad + \frac{1}{|\Omega|} \sum_{i=1}^{m} \sum_{k=1}^{i} |\gamma_i| (t_{i+1} - t_i) \\
&\quad \times \int_{t_{k-1}}^{t_k} \left| f\left(s, x(s), x(\theta(s)), \left(S_{q_{k-1}}x\right)(s)\right) \right. \\
&\qquad \left. - f\left(s, y(s), y(\theta(s)), \left(S_{q_{k-1}}y\right)(s)\right) \right| d_{q_{k-1}}s \\
&\quad + \frac{1}{|\Omega|} \sum_{i=1}^{m} \sum_{k=1}^{i} |\gamma_i| (t_{i+1} - t_i) \left| I_k(x(t_k)) - I_k(y(t_k)) \right| \\
&\quad + \frac{|\beta| + |\Omega|}{|\Omega|} \sum_{k=1}^{m} \left| I_k(x(t_k)) - I_k(y(t_k)) \right| \\
&\leq \frac{|\beta| + |\Omega|}{|\Omega|} \|x - y\| \\
&\quad \times \sum_{k=1}^{m+1} \left[(L_1 + L_2)(t_k - t_{k-1}) + \frac{\phi_0 L_3 (t_k - t_{k-1})^2}{1 + q_{k-1}} \right] \\
&\quad + \frac{\|x - y\|}{|\Omega|} \sum_{i=0}^{m} |\gamma_i| \left[(L_1 + L_2) \frac{(t_{i+1} - t_i)^2}{1 + q_i} \right. \\
&\qquad \left. + \frac{\phi_0 L_3 (t_{i+1} - t_i)^3}{1 + q_i + q_i^2} \right] \\
&\quad + \frac{\|x - y\|}{|\Omega|} \sum_{i=1}^{m} \sum_{k=1}^{i} |\gamma_i| (t_{i+1} - t_i) \left[(L_1 + L_2)(t_k - t_{k-1}) \right. \\
&\qquad \left. + \frac{\phi_0 L_3 (t_k - t_{k-1})^2}{1 + q_{k-1}} \right] \\
&\quad + \frac{L_4 \|x - y\|}{|\Omega|} \sum_{i=1}^{m} i |\gamma_i| (t_{i+1} - t_i) \\
&\quad + \frac{m(|\beta| + |\Omega|) L_4}{|\Omega|} \|x - y\| \\
&= \Lambda_1 \|x - y\|,
\end{aligned} \tag{31}$$

which implies that $\|\mathcal{K}x - \mathcal{K}y\| \leq \Lambda_1 \|x - y\|$. Since $\Lambda_1 < 1$, $\mathcal{K}$ is a contraction. Therefore, by Banach's contraction mapping principle, we conclude that $\mathcal{K}$ has a fixed point which is the unique solution of problem (1). □

The second existence result is based on Krasnoselskii's fixed point theorem.

Lemma 6 ((Krasnoselskii's fixed point theorem) [19]). *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be the operators such that (a) $Ax + By \in M$ whenever $x, y \in M$; (b) A is compact and continuous; (c) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.*

We use the following notations:

$$\begin{aligned}
\Lambda_3 = {} & \frac{|\beta| + |\Omega|}{|\Omega|} \sum_{k=1}^{m+1} (t_k - t_{k-1}) + \frac{1}{|\Omega|} \sum_{i=0}^{m} |\gamma_i| \frac{(t_{i+1} - t_i)^2}{1 + q_i} \\
& + \frac{1}{|\Omega|} \sum_{i=1}^{m} \sum_{k=1}^{i} |\gamma_i| (t_{i+1} - t_i)(t_k - t_{k-1}),
\end{aligned} \tag{32}$$

$$\Lambda_4 = \frac{m|\beta| N}{|\Omega|} + \frac{N}{|\Omega|} \sum_{i=1}^{m} \sum_{k=1}^{i} |\gamma_i| (t_{i+1} - t_i) + mN. \tag{33}$$

Theorem 7. *Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that (H_3) holds. In addition, we suppose the following:*

(H_4) $|f(t, z_1, z_2, z_3)| \leq \mu(t)$, $\forall (t, z_1, z_2, z_3) \in J \times \mathbb{R}^3$, and $\mu \in C(J, \mathbb{R}^+)$,

(H_5) there exists a constant $N > 0$ such that $|I_k(x)| \leq N$ for all $x \in \mathbb{R}$, for $k = 1, 2, \ldots, m$.

Then the impulsive functional q_k-integrodifference boundary value problem (1) has at least one solution on J provided that

$$\left(\frac{|\beta| + |\Omega|}{|\Omega|} \right) m L_4 + \frac{L_4}{|\Omega|} \sum_{i=1}^{m} i |\gamma_i| (t_{i+1} - t_i) < 1. \tag{34}$$

Proof. Let $\sup_{t \in J} |\mu(t)| = \|\mu\|$. By choosing a suitable ball $B_R = \{x \in \mathrm{PC}(J, \mathbb{R}) : \|x\| \leq R\}$, where

$$R \geq \|\mu\| \Lambda_3 + \Lambda_4, \tag{35}$$

and Λ_3, Λ_4 are defined by (32) and (33), respectively; we define the operators $\mathscr{A}_1$ and $\mathscr{A}_2$ on B_R by

$$\begin{aligned}
&(\mathscr{A}_1 x)(t) \\
&= \frac{\beta}{\Omega}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k} f\left(s, x(s), x(\theta(s)), \left(S_{q_{k-1}}x\right)(s)\right) d_{q_{k-1}}s \\
&\quad + \frac{1}{\Omega}\sum_{i=0}^{m}\gamma_i \\
&\quad \times \int_{t_i}^{t_{i+1}}\int_{t_i}^{u} f\left(s, x(s), x(\theta(s)), \left(S_{q_i}x\right)(s)\right) d_{q_i}s\, d_{q_i}u \\
&\quad + \frac{1}{\Omega}\sum_{i=1}^{m}\sum_{k=1}^{i}\gamma_i(t_{i+1}-t_i) \\
&\quad \times \int_{t_{k-1}}^{t_k} f\left(s, x(s), x(\theta(s)), \left(S_{q_{k-1}}x\right)(s)\right) d_{q_{k-1}}s \\
&\quad + \sum_{0<t_k<t}\int_{t_{k-1}}^{t_k} f\left(s, x(s), x(\theta(s)), \left(S_{q_{k-1}}x\right)(s)\right) d_{q_{k-1}}s \\
&\quad + \int_{t_k}^{t} f\left(s, x(s), x(\theta(s)), \left(S_{q_k}x\right)(s)\right) d_{q_k}s, \\
&(\mathscr{A}_2 x)(t) \\
&= \frac{\beta}{\Omega}\sum_{k=1}^{m} I_k(x(t_k)) + \frac{1}{\Omega}\sum_{i=1}^{m}\sum_{k=1}^{i}\gamma_i(t_{i+1}-t_i) I_k(x(t_k)) \\
&\quad + \sum_{0<t_k<t} I_k(x(t_k)).
\end{aligned} \tag{36}$$

For any $x, y \in B_R$, we have

$$\begin{aligned}
&\left\|\mathscr{A}_1 x + \mathscr{A}_2 y\right\| \\
&\le \|\mu\|\left[\frac{|\beta|}{|\Omega|}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k} d_{q_{k-1}}s + \frac{1}{|\Omega|}\sum_{i=0}^{m}|\gamma_i|\int_{t_i}^{t_{i+1}}\int_{t_i}^{u} d_{q_i}s\, d_{q_i}u\right. \\
&\qquad + \frac{1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|(t_{i+1}-t_i)\int_{t_{k-1}}^{t_k} d_{q_{k-1}}s \\
&\qquad \left. + \sum_{k=1}^{m}\int_{t_{k-1}}^{t_k} d_{q_{k-1}}s + \int_{t_m}^{T} d_{q_k}s\right] + \frac{m|\beta|N}{|\Omega|} \\
&\quad + \frac{N}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|(t_{i+1}-t_i) + mN \\
&= \|\mu\|\left[\frac{|\beta|+|\Omega|}{|\Omega|}\sum_{k=1}^{m+1}(t_k - t_{k-1}) + \frac{1}{|\Omega|}\sum_{i=0}^{m}|\gamma_i|\frac{(t_{i+1}-t_i)^2}{1+q_i}\right. \\
&\qquad \left. + \frac{1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|(t_{i+1}-t_i)(t_k - t_{k-1})\right] \\
&\quad + \frac{m|\beta|N}{|\Omega|} + \frac{N}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|(t_{i+1}-t_i) + mN \\
&\le R.
\end{aligned} \tag{37}$$

This implies that $\mathscr{A}_1 x + \mathscr{A}_2 y \in B_R$.

To show that $\mathscr{A}_2$ is a contraction, for $x, y \in \mathrm{PC}(J,\mathbb{R})$, we have

$$\begin{aligned}
&\left\|\mathscr{A}_2 x - \mathscr{A}_2 y\right\| \\
&\le \frac{|\beta|}{|\Omega|}\sum_{k=1}^{m}\left|I_k(x(t_k)) - I_k(y(t_k))\right| \\
&\quad + \frac{1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|(t_{i+1}-t_i)\left|I_k(x(t_k)) - I_k(y(t_k))\right| \\
&\quad + \sum_{k=1}^{m}\left|I_k(x(t_k)) - I_k(y(t_k))\right| \\
&\le \left[\left(\frac{|\beta|+|\Omega|}{|\Omega|}\right)mL_4 + \frac{L_4}{|\Omega|}\sum_{i=1}^{m} i|\gamma_i|(t_{i+1}-t_i)\right]\|x-y\|.
\end{aligned} \tag{38}$$

From (34), it follows that $\mathscr{A}_2$ is a contraction.

Next, the continuity of f implies that operator $\mathscr{A}_1$ is continuous. Further, $\mathscr{A}_1$ is uniformly bounded on B_R by

$$\left\|\mathscr{A}_1 x\right\| \le \|\mu\|\Lambda_3. \tag{39}$$

Now we will prove the compactness of $\mathscr{A}_1$. Setting $\sup_{(t,z_1,z_2,z_3)\in J\times B_R^3}|f(t,z_1,z_2,z_3)| = f^* < \infty$, then for each $\tau_1, \tau_2 \in (t_l, t_{l+1})$ for some $l \in \{0,1,\ldots,m\}$ with $\tau_2 > \tau_1$, we have

$$\begin{aligned}
&\left|(\mathscr{A}_1 x)(\tau_2) - (\mathscr{A}_1 x)(\tau_1)\right| \\
&\quad = \left|\int_{t_l}^{\tau_2} f\left(s, x(s), x(\theta(s)), \left(S_{q_l}x\right)(s)\right) d_{q_l}s\right. \\
&\qquad \left. - \int_{t_l}^{\tau_1} f\left(s, x(s), x(\theta(s)), \left(S_{q_l}x\right)(s)\right) d_{q_l}s\right| \\
&\quad \le |\tau_2 - \tau_1| f^*.
\end{aligned} \tag{40}$$

As $\tau_1 \to \tau_2$, the right hand side above tends to zero independently on x. Therefore, the operator $\mathscr{A}_1$ is equicontinuous. Since $\mathscr{A}_1$ maps bounded subsets into relatively compact subsets, it follows that $\mathscr{A}_1$ is relative compact on B_R. Hence, by the Arzelá-Ascoli theorem, $\mathscr{A}_1$ is compact on B_R. Thus, all the assumptions of Lemma 6 are satisfied. Hence, by the conclusion of Lemma 6, the impulsive functional q_k-integrodifference boundary value problem (1) has at least one solution on J. □

Our third existence result is based on Leray-Schauder degree theory. Before proving the result, we set

$$\begin{aligned}\Lambda_5 = {} & \frac{|\beta| + |\Omega|}{|\Omega|}\sum_{k=1}^{m+1}\left[\xi_1\left(t_k - t_{k-1}\right) + \xi_2\xi_3\frac{\left(t_k - t_{k-1}\right)^2}{1 + q_{k-1}}\right] \\ & + \frac{1}{|\Omega|}\sum_{i=0}^{m}|\gamma_i|\left[\xi_1\frac{\left(t_{i+1} - t_i\right)^2}{1 + q_i} + \xi_2\xi_3\frac{\left(t_{i+1} - t_i\right)^3}{1 + q_i + q_i^2}\right] \\ & + \frac{1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|\left(t_{i+1} - t_i\right) \\ & \times\left[\xi_1\left(t_k - t_{k-1}\right) + \xi_2\xi_3\frac{\left(t_k - t_{k-1}\right)^2}{1 + q_{k-1}}\right] \\ & + \frac{1}{|\Omega|}\sum_{i=1}^{m} i\,|\gamma_i|\left(t_{i+1} - t_i\right)\xi_4 + \frac{m\left(|\beta| + |\Omega|\right)\xi_4}{|\Omega|},\end{aligned} \tag{41}$$

$$\begin{aligned}\Lambda_6 = {} & \frac{|\beta| + |\Omega|}{|\Omega|}\sum_{k=1}^{m+1}\left[\xi_2 Q_2\frac{\left(t_k - t_{k-1}\right)^2}{1 + q_{k-1}} + Q_1\left(t_k - t_{k-1}\right)\right] \\ & + \frac{1}{|\Omega|}\sum_{i=0}^{m}|\gamma_i|\left[\xi_2 Q_2\frac{\left(t_{i+1} - t_i\right)^3}{1 + q_i + q_i^2} + Q_1\frac{\left(t_{i+1} - t_i\right)^2}{1 + q_i}\right] \\ & + \frac{1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|\left(t_{i+1} - t_i\right) \\ & \times\left[\xi_2 Q_2\frac{\left(t_k - t_{k-1}\right)^2}{1 + q_{k-1}} + Q_1\left(t_k - t_{k-1}\right)\right] \\ & + \frac{L_4}{|\Omega|}\sum_{i=1}^{m} i\,|\gamma_i|\left(t_{i+1} - t_i\right)Q_3 + \frac{m\left(|\beta| + |\Omega|\right)Q_3}{|\Omega|}.\end{aligned} \tag{42}$$

Theorem 8. *Assume that $f : J\times\mathbb{R}^3 \to \mathbb{R}$ and $\phi : J^2\times\mathbb{R} \to \mathbb{R}$ are continuous functions. In addition we suppose the following.*

(H_6) *There exist constants $\xi_1, \xi_2 > 0$ and $Q_1 \geq 0$ such that*

$$\begin{aligned}\left|f\left(t, z_1, z_2, z_3\right)\right| \leq \xi_1\left|z_1\right| + \xi_2\left|z_3\right| + Q_1 \\ \forall\left(t, z_1, z_2, z_3\right) \in J\times\mathbb{R}^3.\end{aligned} \tag{43}$$

(H_7) *There exist constants $\xi_3 > 0$ and $Q_2 \geq 0$ such that*

$$\left|\phi\left(t, s, z\right)\right| \leq \xi_3|z| + Q_2 \quad \forall\left(t, s, z\right) \in J^2\times\mathbb{R}. \tag{44}$$

(H_8) *There exist constants $\xi_4 > 0$ and $Q_3 \geq 0$ such that*

$$\left|I_k\left(z\right)\right| \leq \xi_4|z| + Q_3 \quad \forall z \in \mathbb{R},\ k = 1, 2, \ldots, m. \tag{45}$$

If

$$\Lambda_5 < 1, \tag{46}$$

where Λ_5 is given by (41)*, then the impulsive functional q_k-integrodifference boundary value problem* (1) *has at least one solution on J.*

Proof. We define an operator $\mathscr{K} : \mathrm{PC}(J, \mathbb{R}) \to \mathrm{PC}(J, \mathbb{R})$ as in (22) and consider the fixed point problem:

$$x = \mathscr{K}x. \tag{47}$$

We are going to prove that there exists a fixed point $x \in \mathrm{PC}(J, \mathbb{R})$ satisfying (47). It is sufficient to show that $\mathscr{K} : \overline{B}_\rho \to \mathrm{PC}(J, \mathbb{R})$ satisfies

$$x \neq \lambda\mathscr{K}x, \quad \forall x \in \partial B_\rho,\ \forall\lambda \in [0, 1], \tag{48}$$

where $B_\rho = \{x \in \mathrm{PC}(J, \mathbb{R}) : \max_{t\in J}|x(t)| < \rho, \rho > 0\}$. We define

$$H\left(\lambda, x\right) = \lambda\mathscr{K}x, \quad x \in \mathrm{PC}\left(J, \mathbb{R}\right),\ \lambda \in [0, 1]. \tag{49}$$

It is easy to see that the operator $\mathscr{K}$ is continuous, uniformly bounded, and equicontinuous. Then, by the Arzelá-Ascoli Theorem, a continuous map h_λ defined by $h_\lambda(x) = x - H(\lambda, x) = x - \lambda\mathscr{K}x$ is completely continuous. If (48) is true, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

$$\begin{aligned}& \deg\left(h_\lambda, B_\rho, 0\right) \\ & \quad = \deg\left(I - \lambda\mathscr{K}, B_\rho, 0\right) = \deg\left(h_1, B_\rho, 0\right) \\ & \quad = \deg\left(h_0, B_\rho, 0\right) = \deg\left(I, B_\rho, 0\right) = 1 \neq 0, \quad 0 \in B_\rho,\end{aligned} \tag{50}$$

where I denotes the identity operator. By the nonzero property of Leray-Schauder degree, $h_1(x) = x - \mathscr{K}x = 0$ for at least one $x \in B_\rho$. In order to prove (48), we assume that $x = \lambda\mathscr{K}x$ for some $\lambda \in [0, 1]$. Then

$$\begin{aligned}& \left|\mathscr{K}x\left(t\right)\right| \\ & \leq \sup_{t\in J}\left\{\frac{\beta}{\Omega}\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k} f\left(s, x\left(s\right), x\left(\theta\left(s\right)\right), \left(S_{q_{k-1}}x\right)\left(s\right)\right)d_{q_{k-1}}s\right. \\ & \qquad + \frac{\beta}{\Omega}\sum_{k=1}^{m} I_k\left(x\left(t_k\right)\right) + \frac{1}{\Omega}\sum_{i=0}^{m}\gamma_i \\ & \qquad \times\int_{t_i}^{t_{i+1}}\int_{t_i}^{u} f\left(s, x\left(s\right), x\left(\theta\left(s\right)\right), \left(S_{q_i}x\right)\left(s\right)\right)d_{q_i}s\, d_{q_i}u \\ & \qquad + \frac{1}{\Omega}\sum_{i=1}^{m}\sum_{k=1}^{i}\gamma_i\left(t_{i+1} - t_i\right) \\ & \qquad \times\int_{t_{k-1}}^{t_k} f\left(s, x\left(s\right), x\left(\theta\left(s\right)\right), \left(S_{q_{k-1}}x\right)\left(s\right)\right)d_{q_{k-1}}s\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Omega}\sum_{i=1}^{m}\sum_{k=1}^{i}\gamma_i\left(t_{i+1}-t_i\right)I_k\left(x\left(t_k\right)\right)\\
&+\int_{t_k}^{t} f\left(s,x\left(s\right),x\left(\theta\left(s\right)\right),\left(S_{q_k}x\right)\left(s\right)\right)d_{q_k}s\\
&+\sum_{0<t_k<t}\left(\left(\int_{t_{k-1}}^{t_k} f\left(s,x\left(s\right),x\left(\theta\left(s\right)\right),\left(S_{q_{k-1}}x\right)\left(s\right)\right)d_{q_{k-1}}s\right.\right.\\
&\qquad\left.\left.+I_k\left(x\left(t_k\right)\right)\right)\right\}\\
&\leq \frac{|\beta|+|\Omega|}{|\Omega|}\\
&\quad\times\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\left|f\left(s,x\left(s\right),x\left(\theta\left(s\right)\right),\left(S_{q_{k-1}}x\right)\left(s\right)\right)\right|d_{q_{k-1}}s\\
&\quad+\frac{1}{|\Omega|}\sum_{i=0}^{m}|\gamma_i|\\
&\quad\times\int_{t_i}^{t_{i+1}}\int_{t_i}^{u}\left|f\left(s,x\left(s\right),x\left(\theta\left(s\right)\right),\left(S_{q_i}x\right)\left(s\right)\right)\right|d_{q_i}s d_{q_i}u\\
&\quad+\frac{1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|\left(t_{i+1}-t_i\right)\\
&\quad\times\int_{t_{k-1}}^{t_k}\left|f\left(s,x\left(s\right),x\left(\theta\left(s\right)\right),\left(S_{q_{k-1}}x\right)\left(s\right)\right)\right|d_{q_{k-1}}s\\
&\quad+\frac{1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|\left(t_{i+1}-t_i\right)\left|I_k\left(x\left(t_k\right)\right)\right|\\
&\quad+\frac{|\beta|+|\Omega|}{|\Omega|}\sum_{k=1}^{m}\left|I_k\left(x\left(t_k\right)\right)\right|\\
&\leq \frac{|\beta|+|\Omega|}{|\Omega|}\\
&\quad\times\sum_{k=1}^{m+1}\int_{t_{k-1}}^{t_k}\left(\xi_1\|x\|\right.\\
&\qquad\left.+\xi_2\int_{t_{k-1}}^{s}\left(\xi_3\|x\|+Q_2\right)d_{q_{k-1}}v+Q_1\right)d_{q_{k-1}}s\\
&\quad+\frac{1}{|\Omega|}\sum_{i=0}^{m}|\gamma_i|\\
&\quad\times\int_{t_i}^{t_{i+1}}\int_{t_i}^{u}\left(\xi_1\|x\|\right.\\
&\qquad\left.+\xi_2\int_{t_{k-1}}^{s}\left(\xi_3\|x\|+Q_2\right)d_{q_{k-1}}v+Q_1\right)d_{q_i}s\,d_{q_i}u\\
&\quad+\frac{1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|\left(t_{i+1}-t_i\right)\\
&\quad\times\int_{t_{k-1}}^{t_k}\left(\xi_1\|x\|+\xi_2\int_{t_{k-1}}^{s}\left(\xi_3\|x\|+Q_2\right)d_{q_{k-1}}v+Q_1\right)d_{q_{k-1}}s\\
&\quad+\frac{1}{|\Omega|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\gamma_i|\left(t_{i+1}-t_i\right)\left(\xi_4\|x\|+Q_3\right)\\
&\quad+\frac{|\beta|+|\Omega|}{|\Omega|}\sum_{k=1}^{m}\left(\xi_4\|x\|+Q_3\right)\\
&=\Lambda_5\|x\|+\Lambda_6,
\end{aligned}
\tag{51}
$$

which implies that

$$\|x\|\leq\frac{\Lambda_6}{1-\Lambda_5}. \tag{52}$$

If $\rho=\Lambda_6/(1-\Lambda_5)+1$, inequality (48) holds. This completes the proof. □

4. Examples

In this section, we will give some examples to illustrate our main results.

Example 1. Consider the following boundary value problem for nonlinear first-order impulsive functional q_k-integrodifference equation:

$$
\begin{aligned}
&D_{(1/2)\sin(((k+1)/6)\pi)}x\left(t\right)\\
&\quad=\frac{t^2\sin\pi t}{(2t+4)^2}\frac{|x|}{|x|+1}-\frac{3tx\left(t/2\right)}{2(t+3)^2}\\
&\qquad+\frac{t^2}{2(e^t+1)^2}\int_{t_k}^{t}\frac{2t-s}{4e^{t-s}}x\left(s\right)d_{(1/2)\sin(((k+1)/6)\pi)}s,\\
&\qquad\qquad t\in J,\quad t\neq t_k,\\
&\Delta x\left(t_k\right)=\frac{\left|x\left(t_k\right)\right|}{2\left(k+3\right)+\left|x\left(t_k\right)\right|},\quad t_k=\frac{k}{5},\ k=1,2,\ldots,4,\\
&x\left(0\right)=\frac{1}{3}x\left(1\right)+\sum_{i=0}^{4}\left(\frac{1}{i+2}\right)\int_{t_i}^{t_{i+1}}x\left(s\right)d_{(1/2)\sin(((i+1)/6)\pi)}s.
\end{aligned}
\tag{53}
$$

Set $J=[0,1]$, $q_k=(1/2)\sin((k+1)\pi/6)$ for $k=0,1,\ldots,4$, $\gamma_i=1/(i+2)$ for $i=0,1,\ldots,4$, $m=4$, $T=1$, $\theta(t)=t/2$,

$$
\begin{aligned}
&f\left(t,x,x\left(\theta\right),\left(S_{q_k}x\right)\right)\\
&\quad=\frac{t^2\sin\pi t}{(2t+4)^2}\frac{|x|}{|x|+1}-\frac{3tx\left(t/2\right)}{2(t+3)^2}\\
&\qquad+\frac{t^2}{2(e^t+1)^2}\int_{t_k}^{t}\frac{2t-s}{4e^{t-s}}x\left(s\right)d_{(1/2)\sin(((k+1)/6)\pi)}s,
\end{aligned}
\tag{54}
$$

and $I_k(x)=|x(t_k)|/(2(k+3)+|x(t_k)|)$.

Since

$$\begin{gathered}\left|\phi(t,s,y)-\phi(t,s,z)\right|\le\frac{1}{2}\left|y-z\right|,\\ \left|f\left(t,y_1,y_2,y_3\right)-f\left(t,z_1,z_2,z_3\right)\right|\\ \le\frac{1}{9}\left|y_1-z_1\right|+\frac{1}{6}\left|y_2-z_2\right|+\frac{1}{8}\left|y_3-z_3\right|,\\ \left|I_k(y)-I_k(z)\right|\le\frac{1}{8}\left|y-z\right|,\end{gathered}\tag{55}$$

then (H_1)–(H_3) are satisfied with $\phi_0 = 1/2$, $L_1 = 1/9$, $L_2 = 1/6$, $L_3 = 1/8$, and $L_4 = 1/8$. We can show that $\Lambda_1 \approx 0.9517257476 < 1$. Hence, by Theorem 5, the boundary value problem (53) has a unique solution on $[0, 1]$.

Example 2. Consider the following boundary value problem for nonlinear first-order impulsive functional q_k-integrodifference equation:

$$\begin{aligned}&D_{(k+1)/\sqrt{e^{k+1}}}x(t)\\ &\quad=\frac{t^2\cos\pi t}{(4t+3)^2}\frac{x}{x^2+2}+(t+2)^2\sin^2\left(x\left(\frac{2t}{3}\right)\right)\\ &\qquad+\frac{5\cos\pi t}{3(e^t+4)^2}\int_{t_k}^{t}\frac{\sin^2(t-s)}{(e^{t-s}+1)^2}x(s)\,d_{(k+1)/\sqrt{e^{k+1}}}s,\\ &\qquad\qquad t\in J=[0,1],\quad t\ne t_k,\\ &\Delta x(t_k)=\frac{\left|x(t_k)\right|}{5(k+4)+\left|x(t_k)\right|}+3\sin\pi x(t_k),\\ &\qquad\qquad t_k=\frac{k}{9},\quad k=1,2,\ldots,8,\\ &2x(0)=\frac{1}{4}x(1)+\sum_{i=0}^{8}\left(\frac{i+1}{i+4}\right)\int_{t_i}^{t_{i+1}}x(s)\,d_{(i+1)/\sqrt{e^{i+1}}}s.\end{aligned}\tag{56}$$

Set $q_k = (k+1)/(\sqrt{e^{k+1}})$ for $k = 0, 1, \ldots, 8$, $\gamma_i = (i+1)/(i+4)$ for $i = 0, 1, \ldots, 8$, $m = 8$, $T = 1$,

$$\begin{aligned}&f\left(t,x,x(\theta),\left(S_{q_k}x\right)\right)\\ &\quad=\frac{t^2\cos\pi t}{(4t+3)^2}\frac{x}{x^2+2}+(t+2)^2\sin^2\left(x\left(\frac{2t}{3}\right)\right)\\ &\qquad+\frac{5\cos\pi t}{3(e^t+4)^2}\int_{t_k}^{t}\frac{\sin^2(t-s)}{(e^{t-s}+1)^2}x(s)\,d_{(k+1)/\sqrt{e^{k+1}}}s,\end{aligned}\tag{57}$$

$\theta(t) = 2t/3$, and $I_k(x) = (|x(t_k)|/(5(k+5)+|x(t_k)|)) + 3\sin\pi x(t_k)$.

Since

$$\left|f\left(t,x,x(\theta),\left(S_{q_k}x\right)\right)\right|\le\frac{1}{18}|x|+\frac{1}{15}\left|S_{q_k}x\right|+9,\tag{58}$$

$|\phi(t,s,x)| \le (1/4)|x|$, and $|I_k(x)| \le (1/25)|x| + 3$, then (H_6)–(H_8) are satisfied with $\xi_1 = 1/18$, $\xi_2 = 1/15$, $\xi_3 = 1/4$, $\xi_4 = 1/25$, $Q_1 = 9$, $Q_2 = 0$, and $Q_3 = 3$. We can show that $\Lambda_5 \approx 0.9134109736 < 1$. Hence, by Theorem 8, the boundary value problem (56) has at least one solution on $[0, 1]$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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