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Research Article

Derivatives of Meromorphic Functions with Multiple Zeros and Small Functions

Pai Yang[1] and Peiyan Niu[2]

[1] *College of Applied Mathematics, Chengdu University of Information Technology, Chengdu 610225, China*
[2] *Department of Mathematics, Anhui Science and Technology University, Chuzhou 233100, China*

Correspondence should be addressed to Pai Yang; yangpai@cuit.edu.cn

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Let $f(z)$ be a meromorphic function in $\mathbb{C}$, and let $\alpha(z) = R(z)h(z) \not\equiv 0$, where $h(z)$ is a nonconstant elliptic function and $R(z)$ is a rational function. Suppose that all zeros of $f(z)$ are multiple except finitely many and $T(r,\alpha) = o\{T(r,f)\}$ as $r \to \infty$. Then $f'(z) = \alpha(z)$ has infinitely many solutions.

1. Introduction

The value distribution theory of meromorphic functions occupies one of the central places in complex analysis which now have been applied to complex dynamics, complex differential and functional equations, Diophantine equations, and others.

In his excellent paper [1], Hayman studied the value distribution of certain meromorphic functions and their derivatives under various conditions. Among other important results, he proved that if $f(z)$ is a transcendental meromorphic function in the plane, then either $f(z)$ assumes every finite value infinitely often or every derivative of $f(z)$ assumes every finite nonzero value infinitely often. This result is known as Hayman's alternative. Thereafter, the value distribution of derivatives of transcendental functions continued to be studied.

In 1998, Wang and Fang proved the following results.

Theorem A (see [2, Theorem 3]). *Let f be a transcendental meromorphic function in $\mathbb{C}$, all of whose zeros have multiplicity at least 3. Then f' assumes each nonzero complex value infinitely often.*

In 2006, Pang et al. proved the following result, which is a significant improvement of Theorem A.

Theorem B (see [3, Theorem 1]). *Let f be a transcendental meromorphic function in $\mathbb{C}$, all but finitely many of whose zeros are multiple, and let $R(\not\equiv 0)$ be a rational function. Then $f'-R$ has infinitely many zeros.*

Relative to f, R is a small function in Theorem B. Specifically, $T(r,R) = o\{T(r,f)\}$ as $r \to \infty$ in Theorem B. A natural problem arises: what can we say if the rational function R in Theorem B is replaced by a more general small function $\alpha(z)$? In this direction, we obtain the following result.

Theorem 1. *Let $f(z)$ be a meromorphic function in $\mathbb{C}$, and let $\alpha(z) = R(z)h(z) \not\equiv 0$, where $h(z)$ is a nonconstant elliptic function and $R(z)$ is a rational function. Suppose that all zeros of $f(z)$ are multiple except finitely many and $T(r,\alpha) = o\{T(r,f)\}$ as $r \to \infty$. Then $f'(z) = \alpha(z)$ has infinitely many solutions (including the possibility of infinitely many common poles of $f(z)$ and $\alpha(z)$).*

2. Notation and Preliminary Lemmas

We use the following notation. Let $\mathbb{C}$ be complex plane and let D be a domain in $\mathbb{C}$. For $z_0 \in \mathbb{C}$ and $r > 0$, $\Delta(z_0,r) = \{z \mid |z-z_0| < r\}$, $\Delta'(z_0,r) = \{z \mid 0 < |z-z_0| < r\}$, $\Delta = \Delta(0,1)$, and $\Gamma(0,r)=\{z:|z|=r\}$. We write $f_n \overset{\chi}{\Rightarrow} f$ in D to indicate that the sequence $\{f_n\}$ converges to f in the spherical metric uniformly on compact subsets of D and $f_n \Rightarrow f$ in D if the convergence is in the Euclidean metric. Let $n(D,f)$ denote the number of poles of $f(z)$ in D (counting multiplicities), and let $n(r,f)=n(\Delta(0,r),f)$.

For f meromorphic in D, we denote

$$f^{\#}(z)=\frac{|f'(z)|}{1+|f(z)|^2},$$
$$S(D,f)=\frac{1}{\pi}\iint_D\left[f^{\#}(z)\right]^2dx\,dy, \tag{1}$$
$$S(r,f)=S(\Delta(0,r),f).$$

The Ahlfors-Shimizu characteristic is defined by

$$T_0(r,f)=\int_0^r\frac{S(t,f)}{t}dt. \tag{2}$$

Remark 2. Let $T(r,f)$ denote the usual Nevanlinna characteristic function. Since $T(r,f)-T_0(r,f)$ is bounded as a function of r, we can replace $T_0(r,f)$ with $T(r,f)$ in the paper.

Recall that an elliptic function [4] is a meromorphic function h defined in $\mathbb{C}$ for which there exist two nonzero complex numbers ω_1 and ω_2 with ω_1/ω_2 not real such that $h(z+\omega_1)=h(z+\omega_2)=h(z)$ for all z in $\mathbb{C}$.

Lemma 3 (see [5, Lemma 2]). *Let $\mathcal{F}$ be a family of functions meromorphic in D, all of whose zeros have multiplicity at least k, and suppose that there exists $A\ge 1$ such that $|f^{(k)}(z)|\le A$ whenever $f(z)=0$. Then if $\mathcal{F}$ is not normal at z_0, there exist, for each $0\le\alpha\le k$,*

(a) *points z_n, $z_n\to z_0$;*

(b) *functions $f_n\in\mathcal{F}$;*

(c) *positive numbers $\rho_n\to 0$*

such that $\rho_n^{-\alpha}f_n(z_n+\rho_n\zeta)=g_n(\zeta)\overset{\chi}{\Rightarrow}g(\zeta)$ in $\mathbb{C}$, where g is a nonconstant meromorphic function in $\mathbb{C}$, all of whose zeros have multiplicity at least k.

Lemma 4 (see [6, Lemma 3.2]). *Let k be a positive integer, and let R be a rational function satisfying $R'(z)\ne z^k$ in $\mathbb{C}$. If all zeros of R are multiple, then*

$$R(z)=\frac{\prod_{i=1}^{n+k+1}(z-\alpha_i)}{(k+1)(z-\beta)^n}, \tag{3}$$

where n is a nonnegative integer, $\beta\in\mathbb{C}$, and $\alpha_i\ne 0,\beta$ $(1\le i\le n+k+1)$.

Lemma 5 (see [7, Lemma 6]). *Let l be a positive integer, and let $R(z)$ be a rational function and all of whose zeros are multiple. If $R'(z)\ne z^{-l}$ in $\mathbb{C}$, then $R(z)$ is a constant function.*

Lemma 6 (see [6, Lemma 3.6]). *Let $\{f_n\}$ be a sequence of functions meromorphic in $\Delta(z_0,r)$. Suppose that $f_n\overset{\chi}{\Rightarrow}f$ in $\Delta'(z_0,r)$, where f is a nonconstant meromorphic function or $f\equiv\infty$ in $\Delta'(z_0,r)$. If there exists $M_0>0$ such that for each n, $n(\Delta(z_0,r),1/f_n)<M_0$, then there exists $M_1>0$ such that $S(\Delta(z_0,r/2),f_n)<M_1$.*

Lemma 7 (see [8, Corollary 2]). *If $h(z)$ is a nonconstant elliptic function with primitive periods ω_1, ω_2, where ω_1/ω_2 is not real, then $T(r,h)=Ar^2(1+o(1))$ as $r\to\infty$, where $A>0$ is a constant.*

Lemma 8 (see [6, Lemma 3.4]). *Let $\{f_n\}$ and $\{\psi_n\}$ be two sequences of meromorphic functions in D, and let $f(z)$ and $\psi(z)$ be two meromorphic functions in D. Suppose that*

(a) *$f_n(z)\overset{\chi}{\Rightarrow}f(z)$ and $\psi_n(z)\overset{\chi}{\Rightarrow}\psi(z)$ in D;*

(b) *$f_n'(z)\ne\psi_n(z)$ in D.*

Then, either $f'(z)\equiv\psi(z)$ or $f'(z)\ne\psi(z)$ in D.

Lemma 9 (see [9, Lemma 3.1]). *Let $\{f_n\}$ be a sequence of meromorphic functions in D, and let $\{\psi_n\}$ be a sequence of holomorphic functions in D such that $\psi_n\Rightarrow\psi$, where $\psi(z)\ne 0,\infty$ in D. If for each n, $f_n(z)\ne 0$ and $f_n'(z)\ne\psi_n(z)$ for all $z\in D$, then $\{f_n\}$ is normal in D.*

Using the same proving method of Theorem 1.1 in [10], we can prove the following result without any difficulties. In fact, there is no essential distinction between Theorem 1.1 in [10] and the following result.

Lemma 10. *Let $\{f_n\}$ be a family of meromorphic functions in D, all of whose zeros and poles are multiple, and let $\{h_n\}$ be a sequence of meromorphic functions in D such that $h_n\overset{\chi}{\Rightarrow}h$ in D, where $h\not\equiv\infty$ is meromorphic and zero-free in D. Suppose that h and h_n have the same poles with the same multiplicity and $f_n'(z)\ne h_n(z)$ for all $z\in D$. Then $\mathcal{F}$ is normal in D.*

Lemma 11 (see [6, Lemma 3.8]). *Let $\{f_n\}$ be a sequence of meromorphic functions in D, all of whose zeros are multiple, and let $\{\psi_n\}$ be a sequence of meromorphic functions in D such that $\psi_n\Rightarrow\psi$ in D, where ψ is a nonvanishing holomorphic function in D. Let E be a (countable) discrete set in D which has no accumulation points in D. Suppose that*

(a) *$f_n(z)\overset{\chi}{\Rightarrow}f(z)$ in $D\setminus E$;*

(b) *for some $a_1\in E$, no subsequence of $\{f_n\}$ is normal at a_1;*

(c) *for all $n\in\mathbb{N}$, $f_n'(z)\ne\psi_n(z)$ in D.*

Then,

(d) *there exists $r\ge 0$ such that for sufficiently large n, f_n has a single zero $z_{n,1}$ of order 2 and a single pole $z_{n,2}$ of order 1 in $\Delta(a_1,r)$, where $z_{n,i}\to a_1$ as $n\to\infty$, $i=1,2$;*

(e) *$f(z)=\int_{a_1}^z\psi(\zeta)d\zeta$.*

Lemma 12 (see [6, Lemma 3.9]). *Let $\{f_n\}$ be a family of meromorphic functions in D, all of whose zeros are multiple, and let $\{\psi_n\}$ be a sequence of meromorphic functions in D such that $\psi_n \overset{\chi}{\Rightarrow} \psi$ in D, where $\psi(z) \not\equiv 0, \infty$ in D. If for each $n \in \mathbb{N}$, $f_n'(z) \neq \psi_n(z)$ for all $z \in D$, then $\{f_n\}$ is quasinormal in D.*

3. Auxiliary Lemmas

Lemma 13. *Let $\{f_n\}$ be a family of meromorphic functions in Δ, all of whose zeros are multiple. Let $\{b_n\}$ be a sequence of meromorphic functions in Δ such that $b_n(z) \overset{\chi}{\Rightarrow} b(z)$ in Δ, where $b \not\equiv 0$ is a meromorphic function and $b(0) = 0$. Suppose that*

(a) *b and b_n have the same zeros and poles with the same multiplicity;*

(b) *for all $n \in \mathbb{N}$ and all $z \in D$, $f_n'(z) \neq b_n(z)$;*

(c) *there exist points z_n in Δ such that $f_n(z_n) = 0$ and $z_n \to 0$ as $n \to \infty$;*

(d) *$f_n(z) \overset{\chi}{\Rightarrow} f(z)$ in Δ', where $f(z)$ is a meromorphic function in Δ'.*

Then $f'(z) \equiv b(z)$ in Δ'.

Proof. Set $F_n(z) = (f_n'(z)/b_n(z))$. By (a), $f_n'(0) \neq b_n(0) = 0$ and hence $F_n(0) = \infty$. Since all zeros of $\{f_n(z)\}$ are multiple and $f_n'(0) \neq 0$, we have $f_n(0) \neq 0$. Hence, $z_n \neq 0$ and $F_n(z_n) = 0$ for sufficiently large n. Since $F_n(0) = \infty$ and $F_n(z_n) = 0$ for sufficiently large n, $\{F_n(\zeta)\}$ is not equicontinuous at 0 and hence $\{(f_n'(z)/b_n(z)) - 1\}$ is not normal at 0.

By (b), we have $0 \neq (f_n'(z)/b_n(z)) - 1 \Rightarrow (f'(z)/b(z)) - 1$ in E, where $E = \{z \mid f(z) \neq \infty, b(z) \neq 0 \text{ and } z \in \Delta'\}$. By Hurwitz's theorem, either $(f'(z)/b(z)) - 1 \equiv 0$ or $(f'(z)/b(z)) - 1 \neq 0$ in E. Suppose first that $(f'(z)/b(z)) - 1 \equiv 0$ in E. Obviously, $(f'(z)/b(z)) - 1$ is a meromorphic function in Δ', So $(f'(z)/b(z)) - 1 \equiv 0$ in Δ'. Suppose that $(f'(z)/b(z)) - 1 \neq 0$ in E. If $f(z) = \infty$ or $b(z) = 0$, then $(f'(z)/b(z)) - 1 = \infty$. Hence $(f'(z)/b(z)) - 1 \neq 0$ in Δ'. Suppose that $(f'(z)/b(z)) - 1 \neq 0$ in Δ'. By the assumptions, there exists $\delta > 0$ such that $f(z)$ has no poles on $\Gamma(0, \delta)$ and $b(z)$ has no zeros on $\Gamma(0, \delta)$. Thus, we have

$$\infty \neq \frac{1}{(f_n'(z)/b_n(z)) - 1} \Longrightarrow \frac{1}{(f'(z)/b(z)) - 1}, \quad z \in \Gamma(0, \delta). \tag{4}$$

By the maximum principle, (4) holds in $\Delta(0, \delta)$ and then $\{(f_n'(z)/b_n(z)) - 1\}$ is normal at 0. A contradiction. Thus, $f'(z) \equiv b(z)$ in Δ'. □

Lemma 14. *Let $f(z)$ be a meromorphic function in $\mathbb{C}$ satisfying $\overline{\lim}_{r\to\infty}(T(r, f)/r^2) = \infty$. Then there exist $a_n \to \infty$ and $\delta_n \to 0$ such that*

$$f^{\#}(a_n) \longrightarrow \infty, \quad S(\Delta(a_n, \delta_n), f) \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \tag{5}$$

Proof. We claim that there exist $t_n \to \infty$ and $\varepsilon_n \to 0$ such that

$$S(\Delta(t_n, \varepsilon_n), f) = \frac{1}{\pi} \iint_{|z-t_n|<\varepsilon_n} \left[f^{\#}(z)\right]^2 dx\, dy \longrightarrow \infty. \tag{6}$$

Otherwise, there would exist $\varepsilon > 0$ and $M > 0$ such that $S(\Delta(z_0, \varepsilon), f) < M$ for all $z_0 \in \mathbb{C}$. From this follows

$$S(r, f) = \frac{1}{\pi} \iint_{|z|<r} \left[f^{\#}(z)\right]^2 dx\, dy = O(r^2), \tag{7}$$

and hence

$$T_0(r, f) = \int_0^r \frac{S(t)}{t} dt = O(r^2). \tag{8}$$

Now, there exists $M > 0$ such that $\overline{\lim}_{r\to\infty}(T_0(r, f)/r^2) \leq M$, and hence $\overline{\lim}_{r\to\infty}(T(r, f)/r^2) \leq M$ which contradicts the hypothesis $\overline{\lim}_{r\to\infty}(T(r, f)/r^2) = \infty$.

By (6), there exists a sequence $\{a_n\}$ such that $|a_n - t_n| \to 0$ and $f^{\#}(a_n) \to \infty$ as $n \to \infty$. Set $\delta_n = \varepsilon_n + |a_n - t_n|$. Obviously, $\delta_n \to 0$ and $\Delta(t_n, \varepsilon_n) \subset \Delta(a_n, \delta_n)$, and hence $S(\Delta(a_n, \delta_n), f) \to \infty$ as $n \to \infty$. □

Lemma 15. *Let d be an integer, and let f be a transcendental meromorphic function, all of whose zeros are multiple. Set $g(z) := f(z)/z^d$ with $g := f$ if $d = 0$. If $\overline{\lim}_{r\to\infty}(T(r, f)/r^2) = \infty$, then there exist sequences $a_n \to \infty$ and $\delta_n \to 0$ such that as $n \to \infty$,*

$$\frac{f(a_n)}{a_n^d} \longrightarrow 0, \qquad \frac{f'(a_n)}{a_n^d} \longrightarrow \infty, \qquad S(\Delta(a_n, \delta_n), g) \longrightarrow \infty. \tag{9}$$

Proof. By standard results in Nevanlinna theory, $T(r, f) = T(r, z^{-d} g) \leq T(r, g) + T(r, z^{-d})$ and $T(r, z^{-d}) = O(\log r)$ as $r \to \infty$. Thus, $\overline{\lim}_{r\to\infty}(T(r, g)/r^2) = \infty$. By Lemma 14, there exist $b_n \to \infty$ and $\varepsilon_n \to 0$ such that

$$g^{\#}(b_n) \longrightarrow \infty, \qquad S(\Delta(b_n, \varepsilon_n), g) \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \tag{10}$$

Set $g_n(z) = g(z + b_n)$. Then $g_n^{\#}(0) = g^{\#}(b_n) \to \infty$ and hence $\{g_n\}$ is not normal at 0. Since all zeros of g are multiple in $\mathbb{C} \setminus \{0\}$, all zeros of $g_n(z)$ are multiple in Δ for sufficiently large n. Using Lemma 3 for $\alpha = 1/2$, there exist points $z_n \to 0$ and positive numbers $\rho_n \to 0$ and a subsequence of $\{g_n\}$ (still denoted by $\{g_n\}$) such that

$$G_n(\zeta) = \frac{g_n(z_n + \rho_n \zeta)}{\rho_n^{1/2}} \overset{\chi}{\Longrightarrow} G(\zeta) \tag{11}$$

in $\mathbb{C}$, where G is a nonconstant meromorphic function in $\mathbb{C}$, all of whose zeros are multiple.

$G'(\zeta)$ is not a constant function (otherwise, either $G(\zeta)$ is a constant function, or the zero of $G(\zeta)$ is not multiple).

Thus, we can assume ζ_0 is not a zero or pole of $G'(\zeta)$. Set $a_n = z_n + \rho_n\zeta_0 + b_n$. Now, we have

$$g^{(i)}(a_n) = g_n^{(i)}(z_n + \rho_n\zeta_0) = \rho_n^{1/2-i} G_n^{(i)}(\zeta_0), \tag{12}$$

where $i = 0, 1$. Since $\rho_n \to 0$ and ζ_0 is not a zero or pole of $G^{(k)}(\zeta)$, we have $a_n \to \infty$, $g(a_n) \to 0$, and $g'(a_n) \to \infty$ as $n \to \infty$.

Now, we have $f(a_n)/a_n^d = g(a_n) \to 0$ and

$$\begin{aligned}\frac{f'(a_n)}{a_n^d} &= \left.\frac{\left(z^d g(z)\right)'}{a_n^d}\right|_{z=a_n} = \left.\frac{dz^{d-1}g(z) + z^d g'(z)}{a_n^d}\right|_{z=a_n} \\ &= \frac{dg(a_n)}{a_n} + g'(a_n) \longrightarrow \infty.\end{aligned} \tag{13}$$

Set $\delta_n = \varepsilon_n + |a_n - b_n| = \varepsilon_n + |z_n + \rho_n\zeta_0|$. Obviously, $\delta_n \to 0$ and $\Delta(b_n, \varepsilon_n) \subset \Delta(a_n, \delta_n)$, and hence $S(\Delta(a_n, \delta_n), g) \to \infty$ as $n \to \infty$. □

Lemma 16. *Let $\{f_n\}$ be a sequence of meromorphic functions in D, and let $\{h_n\}$ be a sequence of meromorphic functions in D such that $h_n \overset{\chi}{\Rightarrow} h$ in D, where $h \not\equiv 0, \infty$. If $f_n(z) \neq 0$ and $f_n'(z) \neq h_n(z)$ for all z in D, then $\mathcal{F}$ is normal in D.*

Proof. By Lemma 9, it suffices to prove that $\{f_n\}$ is normal at points where h has poles or zeros. Without loss of generality, we assume that $D = \Delta$, $h(z) = z^l b(z)$, where $b \neq 0, \infty$ in Δ, and $l(\neq 0)$ is an integer. Then $\{f_n\}$ is normal in Δ'.

Suppose $\{f_n\}$ is not normal at 0. Since $f_n \neq 0$ in Δ, we have that there exists $r > 0$ such that $\Delta_{2r} \subset \Delta$ and $f_n \Rightarrow 0$ in Δ'_{2r}. By Argument Principle, for sufficiently large n, we have

$$\begin{aligned} &n\left(r, \frac{1}{f_n' - h_n}\right) - n\left(r, f_n' - h_n\right) \\ &\quad = \frac{1}{2\pi i}\int_{|z|=r} \frac{f_n'' - h_n'}{f_n' - h_n} dz = \frac{1}{2\pi i}\int_{|z|=r} \frac{h'}{h} dz = l. \end{aligned} \tag{14}$$

Since $f_n'(z) \neq h_n(z)$, $l = -n(r, f_n' - h_n) < 0$. Obviously, f_n has poles (otherwise $f_n \overset{\chi}{\Rightarrow} \infty$ in Δ') which are different from the poles of h_n, so $n(r, f_n' - h_n) > -l$. A contradiction. □

Lemma 17. *Let $\{f_n\}$ be a family of meromorphic functions in D, all of whose zeros are multiple. Let $\{h_n\}$ be a sequence of meromorphic functions in D such that $h_n(z) \overset{\chi}{\Rightarrow} h(z)$ in D, where $h \not\equiv 0, \infty$. Let $E \subset D$ be a set which has no accumulation points in D. Suppose that*

(*a*) *h and h_n have the same zeros and poles with the same multiplicity;*

(*b*) *for all $n \in \mathbb{N}$ and all $z \in D$, $f_n'(z) \neq h_n(z)$;*

(*c*) *for each $a \in E$, no subsequence of $\{f_n\}$ is normal at a;*

(*d*) *$f_n(z) \overset{\chi}{\Rightarrow} f(z)$ in $D \setminus E$.*

Then

(*e*) *for each $a \in E$, $h(a) \neq \infty$;*

(*f*) *for each $a \in E$, there exist $r_a > 0$ and $N_a > 0$ such that for sufficiently large n, $n(\Delta(a, r_a), 1/f_n) < N_a$, where r_a and N_a only depend on a;*

(*g*) *for each $a \in E$, $f(z) = \int_a^z h(\zeta)d\zeta$ in $D \setminus E$.*

4. Proof of Lemma 17

Proof. It suffices to prove that each subsequence of $\{f_n\}$ has a subsequence which satisfies that (f), and prove that (e) and (g) hold. So suppose we have a subsequence of $\{f_n\}$, which (to avoid complication in notation) we again call $\{f_n\}$.

Without loss of generality, for each $a \in E$, we may assume that $a = 0$, $\Delta \subset D$, $\Delta' \cap E = \emptyset$, and

$$h(z) = z^k + a_{k+1}z^{k+1} + \cdots = z^k \hat{h}(z), \tag{15}$$

where $\hat{h}(0) = 1$ and $\hat{h}(z) \neq 0, \infty$ in Δ.

We consider the following three cases.

Case 1 ($h(0) = \infty$). We will derive a contradiction in the case, and hence (e) holds. For convenience, we set $m = -k$. Thus, $h(z) = z^{-m} + a_{-m+1}z^{-m+1} + \cdots = (\hat{h}(z)/z^m)$, where m is a positive integer. Clearly, we have $h(z) \neq 0, \infty$ in Δ', $h_n(z) \neq 0, \infty$ in Δ', and $h(0) = h_n(0) = \infty$.

Subcase 1.1 (For sufficiently large n, $f_n(0) \neq 0$). We claim that for each $\delta > 0$, there exists at least one zero of f_n in $\Delta'(0, \delta)$ for sufficiently large n. Otherwise, there exists a subsequence of $\{f_n\}$ (still denoted by $\{f_n\}$) such that $f_n(z) \neq 0$ in $\Delta'(0, \delta)$. Since $f_n(0) \neq 0$, $f_n(z) \neq 0$ in $\Delta(0, \delta)$ for sufficiently large n. By Lemma 16, $\{f_n\}$ is normal at 0. A contradiction.

Taking a subsequence and renumbering if necessary, we may assume that $a_n(\neq 0)$ is the zero of $\{f_n\}$ of the smallest modulus. Obviously, $a_n \to 0$ as $n \to \infty$. Set $F_n(\zeta) = a_n^{m-1} f_n(a_n\zeta)$. We have

(A1) $F_n(\zeta) \neq 0$ in Δ;

(A2) all zeros of $F_n(\zeta)$ are multiple and $F_n(1) = 0$;

(A3) $F_n'(\zeta) \neq a_n^m h_n(a_n\zeta)$ and $a_n^m h_n(a_n\zeta) \overset{\chi}{\Rightarrow} 1/\zeta^m$ in $\mathbb{C}$.

By Lemmas 16 and 12, $\{F_n(\zeta)\}$ is normal in Δ and quasinormal in $\mathbb{C}$. Thus, there exists a subsequence of $\{F_n(\zeta)\}$ (still denoted by $\{F_n(\zeta)\}$) and $D_1 \subset \mathbb{C}$ such that

(B1) D_1 has no accumulation point in $\mathbb{C}$;

(B2) for each $\zeta_0 \in D_1$, no subsequence of $\{F_n(\zeta)\}$ is normal at ζ_0;

(B3) $F_n(\zeta) \overset{\chi}{\Rightarrow} F(\zeta)$ in $\mathbb{C} \setminus D_1$.

Obviously, $D_1 \cap \Delta = \emptyset$ and all zeros of $F(\zeta)$ are multiple.

Subcase 1.1.1 ($1 \notin D_1$). By (A2), $F(1) = F'(1) = 0$. Let $\zeta_0 \in D_1$. By Lemma 11, $F'(\zeta) = 1/\zeta^m$ which contradicts $F'(1) = 0$. Hence, D_1 is an empty set. Since $F(1) = 0$, $F(\zeta)$ is a meromorphic function in $\mathbb{C}$. By Lemma 8 and (A3), either $F'(\zeta) \equiv 1/\zeta^m$ or $F'(\zeta) \neq 1/\zeta^m$ in $\mathbb{C}$. If $F'(\zeta) \equiv 1/\zeta^m$, we have $F'(1) = 1$ which contradicts $F'(1) = 0$. If $F'(\zeta) \neq 1/\zeta^m$, then by Theorem B and Lemma 5, F is a constant function. Since $F(1) = 0$, $F(\zeta) \equiv 0$ in $\mathbb{C}$. Now,

$$F_n(\zeta) = a_n^{m-1} f_n(a_n\zeta) \Longrightarrow 0 \quad \text{in } \mathbb{C}. \tag{16}$$

We claim that for each $\delta > 0$, there exists at least one pole of f_n in $\Delta'(0,\delta)$ for sufficiently large n. Otherwise, there exist $\delta > 0$ and a subsequence of $\{f_n\}$ (still denoted by $\{f_n\}$) such that $\{f_n(z)\}$ has no poles in $\Delta'(0,\delta)$. Since $f_n'(z) \neq h_n(z)$ and $h_n(0) = \infty$, we have $f(0) \neq \infty$. Thus, $\{f_n(z)\}$ is a sequence of holomorphic functions in $\Delta(0,\delta)$. By Lemma 10, $\{f_n\}$ is normal at 0. A contradiction.

Taking a subsequence and renumbering if necessary, we may assume that $y_n(\neq 0)$ is the pole of $f_n(z)$ of the smallest modulus. Obviously, $y_n \to 0$ as $n \to \infty$. By Hurwitz's theorem and (16), we have $a_n/y_n \to 0$ as $n \to 0$. Set $G_n(\zeta) = y_n^{m-1} f_n(y_n\zeta)$, and we have

(C1) $G_n(\zeta)$ is holomorphic in Δ;

(C2) $G_n(1) = \infty$;

(C3) all zeros of $G_n(\zeta)$ are multiple;

(C4) $G_n'(\zeta) \neq y_n^m h_n(y_n\zeta)$ and $y_n^m h_n(y_n\zeta) \overset{\chi}{\Rightarrow} 1/\zeta^m$ in $\mathbb{C}$.

By Lemma 10 and Lemma 12, $\{G_n(\zeta)\}$ is normal in Δ and quasinormal in $\mathbb{C}$. Thus, there exists a subsequence of $\{G_n(\zeta)\}$ (still denoted by $\{G_n(\zeta)\}$) and $D_2 \subset \mathbb{C}$ such that

(D1) D_2 has no accumulation point in $\mathbb{C}$;

(D2) for each $\zeta_0 \in D_2$, no subsequence of $\{G_n(\zeta)\}$ is normal at ζ_0;

(D3) $G_n(\zeta) \overset{\chi}{\Rightarrow} G(\zeta)$ in $\mathbb{C} \setminus D_2$.

Obviously, $D_2 \cap \Delta = \emptyset$ and all zeros of $G(\zeta)$ are multiple in $\mathbb{C} \setminus D_2$.

Clearly, $G(0) = \lim_{n\to\infty} G_n(a_n/y_n) = y_n^{m-1} f_n(a_n) = 0$, so $G(z)$ is meromorphic in $\mathbb{C} \setminus D_2$. By Lemma 8 and (C4), either $G'(\zeta) \equiv 1/\zeta^m$ or $G'(\zeta) \neq 1/\zeta^m$ in $\mathbb{C} \setminus D_2$.

(1) (D_2 is an empty set.) By (C2), we have $G(1) = \infty$. If $G'(\zeta) \neq 1/\zeta^m$ in $\mathbb{C}$, then by Theorem B and Lemma 5, we have $G(\zeta) = c$ which contradicts that $G(1) = \infty$. If $G'(\zeta) - 1/\zeta^m \equiv 0$ in $\mathbb{C}$, we have $G(0) = \infty$ which contradicts that $G(0) = 0$.

(2) (D_2 is not an empty set.) Let $\zeta_0 \in D_2$. Since $D_2 \cap \Delta = \emptyset$, by Lemma 11, we have $G'(\zeta) = 1/\zeta^m$ in $\mathbb{C} \setminus D_2 - \{0\}$. Clearly, $G'(\zeta)$ and $1/\zeta^m$ are meromorphic functions in $\mathbb{C} \setminus D_2$, so we have $G'(\zeta) = 1/\zeta^m$ in $\mathbb{C} \setminus D_2$ which contradicts $G(0) = 0$.

Subcase 1.1.2 ($1 \in D_1$). By Lemma 11, we have $F(\zeta) = \int_1^\zeta (1/\xi^m)d\xi$. If $m = 1$, $F(\zeta)$ is a multivalued function. A contradiction. Thus, $m > 1$ and we have

$$F(\zeta) = \int_1^\zeta \frac{1}{\xi^m} d\xi = \frac{1}{m-1}\left(\frac{\zeta^{m-1}-1}{\zeta^{m-1}}\right), \quad \zeta \in \mathbb{C} \setminus D_1. \tag{17}$$

Let ζ_i be the ith root of the equation $\zeta^{m-1} - 1 = 0$, where $i = 1, 2, \ldots, m-1$.

We claim that $D_1 = \{\zeta_1, \zeta_2, \ldots, \zeta_{m-1}\}$. Suppose that $\zeta_0 \notin D_1$ and $\zeta_0^{m-1} - 1 = 0$. Obviously, we have $F(\zeta_0) = 0$ and $F'(\zeta_0) = \zeta_0^{-m}$ which contradicts that all of zeros of $F(\zeta)$ are multiple. Suppose that $\zeta_0 \in D_1$ and $\zeta_0^{m-1} - 1 \neq 0$. Since $D_1 \cap \Delta = \emptyset$, by Lemma 11,

$$F(\zeta) = \int_{\zeta_0}^\zeta \frac{1}{\zeta^m} d\xi = \frac{1}{m-1}\left(\frac{\zeta^{m-1}-\zeta_0^{m-1}}{\zeta^{m-1}}\right), \quad \zeta \in \mathbb{C} \setminus D_1. \tag{18}$$

Comparing the coefficients of (17) and (18), we obtain that $\zeta_0^{m-1} = 1$. A contradiction.

Now, we have

$$F_n(\zeta) \overset{\chi}{\Longrightarrow} \frac{\zeta^{m-1}-1}{(m-1)\zeta^{m-1}} \quad \text{in } \Delta. \tag{19}$$

By Hurwitz's theorem, there exist $\gamma_{n,i}$ such that $\gamma_{n,i} \to 0$ and $F_n(\gamma_{n,i}) = \infty$, where $i = 1, 2, \ldots, m-1$. Observing that $F_n(0) \neq \infty$, we have $\gamma_{n,i} \neq 0$.

Set $U_n(\xi) = s_n^{m-1} F_n(s_n\xi)$, where s_n is one of $\{\gamma_{n,1}, \gamma_{n,2}, \ldots, \gamma_{n,m-1}\}$ of the largest modulus. Then, there exists a subsequence of $\{U_n(\xi)\}$ (still denoted by $\{U_n(\xi)\}$) such that

(E1) for each $R > 0$, $U_n(\xi) \neq 0$ in $\Delta(0,R)$ for sufficiently large n;

(E2) $U_n(1) = \infty$;

(E3) $U_n'(\xi) \neq s_n^m h_n(s_n\xi)$ and $s_n^m h_n(s_n\xi) \overset{\chi}{\Rightarrow} 1/\xi^m$ in $\mathbb{C}$;

(E4) $U_n(\xi)$ has only $m-1$ poles $\eta_{n,i} = \gamma_{n,i}/s_n$ on $\overline{\Delta}$, where $i = 1, 2, \ldots, m-1$.

In fact, (E1) holds by (19). By Lemma 16, (E1), and (E3), we obtain that $U_n(\xi)$ is normal in $\mathbb{C}$. We assume that $U_n(\xi) \overset{\chi}{\Rightarrow} U(\xi)$ in $\mathbb{C}$. Obviously, $U(1) = \infty$ by (E2).

(1) ($U(\xi)$ is a meromorphic function in $\mathbb{C}$.) By Lemma 8 and (E3), either $U'(\zeta) \equiv 1/\xi^m$ or $U'(\xi) \neq 1/\xi^m$ in $\mathbb{C}$. If $U'(\xi) \equiv 1/\xi^m$, we have $U(\xi) = (1/(1-k))((1/\xi^{m-1}) + c)$ which contradicts that $U(1) = \infty$. If $U'(\xi) \neq 1/\xi^m$, then by Theorem B and Lemma 5, $U(\xi)$ is a constant function which contradicts $U(1) = \infty$.

(2) ($U(\xi) \equiv \infty$ in $\mathbb{C}$.) Set

$$U_n^*(\xi) = U_n(\xi) \cdot \prod_{i=1}^{m-1} (\xi - \eta_{n,i}). \tag{20}$$

By the maximum principle applied to $1/U_n^*(\xi)$, we get that

$$U_n^*(\xi) \overset{\chi}{\Longrightarrow} \infty \quad \text{in } \mathbb{C}. \tag{21}$$

Set

$$\begin{aligned} F_n^*(\zeta) &= F_n(\zeta) \cdot \prod_{i=1}^{m-1} (\zeta - \gamma_{n,i}) \\ &= F_n(\zeta) \cdot \prod_{i=1}^{m-1} (\zeta - s_n\eta_{n,i}). \end{aligned} \tag{22}$$

By (19), for sufficiently large n, $F_n^*(\zeta)$ has no pole in $\Delta(0,1/2)$, and by the maximum principle, $F_n^*(\zeta) \overset{\chi}{\rightrightarrows} ((\zeta^{m-1}-1)/(m-1))$ in $\Delta(0,1/2)$. Thus, we have

$$F_n^*(0) \longrightarrow \frac{1}{1-m} \quad \text{as } n \longrightarrow \infty. \tag{23}$$

By (21) and (22),

$$\begin{aligned} F_n^*(s_n\xi) &= F_n(s_n\xi) \cdot \prod_{i=1}^{m-1}(s_n\xi - s_n\eta_{n,i}) \\ &= s_n^{m-1}F_n(s_n\xi) \cdot \prod_{i=1}^{m-1}(\xi - \eta_{n,i}) \\ &= U_n^*(\xi) \overset{\chi}{\Longrightarrow} \infty \end{aligned} \tag{24}$$

in $\mathbb{C}$. Equation (24) implies that $F_n^*(0) \to \infty$ as $n \to \infty$ which contradicts (23).

Subcase 1.2. There exists a subsequence of $\{f_n(z)\}$ (still denoted by $\{f_n(z)\}$) such that $f_n(0) = 0$ for each n.

Doing as in Subcase 1.1.1, we may assume that $y_n(\neq 0)$ is the pole of $f_n(z)$ of the smallest modulus and $y_n \to 0$ as $n \to \infty$. Set $G_n(\zeta) = y_n^{m-1} f_n(y_n\zeta)$. We have

(F1) $G_n(\zeta)$ is holomorphic function in Δ;

(F2) $G_n(1) = \infty$;

(F3) all zeros of $G_n(\zeta)$ are multiple;

(F4) $G_n'(\zeta) \neq y_n^m h_n(y_n\zeta)$ and $y_n^m h_n(y_n\zeta) \overset{\chi}{\rightrightarrows} 1/\zeta^m$ in $\mathbb{C}$.

By Lemma 10 and Lemma 12, $\{G_n(\zeta)\}$ is normal in Δ and quasinormal in $\mathbb{C}$. Thus, there exists a subsequence of $\{G_n(\zeta)\}$ (still denoted by $\{G_n(\zeta)\}$) and $D_3 \subset \mathbb{C}$ such that

(G1) D_3 has no accumulation point in $\mathbb{C}$;

(G2) for each $\zeta_0 \in D_3$, no subsequence of $\{G_n(\zeta)\}$ is normal at ζ_0;

(G3) $G_n(\zeta) \overset{\chi}{\rightrightarrows} G(\zeta)$ in $\mathbb{C} \setminus D_3$.

Obviously, $D_3 \cap \Delta = \emptyset$ and all zeros of $G(\zeta)$ are multiple in $\mathbb{C} \setminus D_3$.

Clearly, $G(0) = \lim_{n\to\infty} G_n(0) = y_n^{m-1} f_n(0) = 0$, so $G(z)$ is a meromorphic function in $\mathbb{C} \setminus D_3$. By Lemma 8 and (F4), either $G'(\zeta) \equiv 1/\zeta^m$ or $G'(\zeta) \neq 1/\zeta^m$ in $\mathbb{C} \setminus D_3$.

Subcase 1.2.1 (D_3 is an empty set). By (F2), $G(1) = \infty$. If $G'(\zeta) \neq 1/\zeta^m$, then by Theorem B and Lemma 5, $G(\zeta)$ is a constant function which contradicts that $G(1) = \infty$. If $G'(\zeta) - (1/\zeta^m) \equiv 0$, we have $G(0) = \infty$ which contradicts that $G(0) = 0$.

Subcase 1.2.2 (D_3 is not an empty set). Let $\zeta_0 \in E$. Since $D_3 \cap \Delta = \emptyset$, by Lemma 11, we have $G'(\zeta) = 1/\zeta^m$ which contradicts $G(0) = 0$.

Case 2 ($h(0) = 0$). In this case, we will show that (e) and (f) hold. Clearly, we have $h(z) \neq 0, \infty$ in Δ', $h_n(z) \neq 0, \infty$ in Δ', and $h(0) = h_n(0) = 0$.

We claim that for each $\delta > 0$, there exists at least one zero of f_n in $\Delta'(0,\delta)$ for sufficiently large n. Otherwise, there exist $\delta > 0$ and a subsequence of $\{f_n\}$ (still denoted by $\{f_n\}$) such that $f_n(z) \neq 0$ in $\Delta'(0,\delta)$. Since $f_n'(0) \neq h_n(0)$ and all the zeros of $\{f_n\}$ are multiple, we have $f_n(0) \neq 0$, and hence $f_n(z) \neq 0$ in $\Delta(0,\delta)$. By Lemma 16, $\{f_n\}$ is normal at 0 which contradicts the condition (c).

Taking a subsequence and renumbering if necessary, we may assume that $a_n(\neq 0)$ is the zero of f_n of the smallest modulus. Obviously, $a_n \to 0$ as $n \to \infty$. Set $F_n(z) = f_n(a_n\zeta)/a_n^{k+1}$. We have that

(a1) $F_n(\zeta) \neq 0$ in Δ;

(a2) all zeros of $F_n(\zeta)$ are multiple and $F_n(1) = 0$;

(a3) $F_n'(\zeta) \neq h_n(a_n\zeta)/a_n^k$ and $h_n(a_n\zeta)/a_n^k \overset{\chi}{\rightrightarrows} \zeta^k$ in $\mathbb{C}$.

By Lemma 16 and Lemma 12, $\{F_n(\zeta)\}$ is normal in Δ and quasinormal in $\mathbb{C}$. Thus, there exists a subsequence of $\{F_n(\zeta)\}$ (still denoted by $\{F_n(\zeta)\}$) and $D_4 \subset \mathbb{C}$ such that

(b1) D_4 has no accumulation point in $\mathbb{C}$;

(b2) for each $\zeta_0 \in D_4$, no subsequence of $\{F_n(\zeta)\}$ is normal at ζ_0;

(b3) $F_n(\zeta) \overset{\chi}{\rightrightarrows} F(\zeta)$ in $\mathbb{C} \setminus D_4$.

Obviously, $D_4 \cap \Delta = \emptyset$ and all zeros of $F(\zeta)$ are multiple in $\mathbb{C} \setminus D_4$.

Subcase 2.1 ($1 \notin D_4$). By (a2), $F(1) = F'(1) = 0$, and hence $F(\zeta)$ is a meromorphic function in $\mathbb{C} \setminus D_4$.

We claim that $D_4 = \emptyset$. Otherwise, let $\zeta_0 \in D_4$. Since $D_4 \cap \Delta = \emptyset$, by Lemma 11, $F'(\zeta) = \zeta^k$ which contradicts that $F'(1) = 0$.

By Lemma 8 and (a3), either $F'(\zeta) \equiv \zeta^k$ or $F'(\zeta) \neq \zeta^k$ in $\mathbb{C}$. Since $F'(1) = 0$, we have $F'(\zeta) \neq \zeta^k$ in $\mathbb{C}$. By Theorem B, $F(\zeta)$ must be rational, and then by Lemma 4,

$$F(\zeta) = \frac{\prod_{i=1}^{m+k+1}(\zeta - \alpha_i)}{(k+1)(\zeta - \beta)^m}, \tag{25}$$

where m is a nonnegative integer, $\beta \in \mathbb{C}$, and $\alpha_i \neq 0, \beta$ $(1 \leq i \leq m+k+1)$. Now, we have

$$F_n(\zeta) \overset{\chi}{\Longrightarrow} \frac{\prod_{i=1}^{m+k+1}(\zeta - \alpha_i)}{(k+1)(\zeta - \beta)^m} \quad \text{in } \mathbb{C}. \tag{26}$$

By Hurwitz's theorem, there exist sequences $\zeta_{n,i} \to \alpha_i$ and $\eta_{n,j} \to \beta$ as $n \to \infty$ (counting multiplicities of zeros and poles, resp.) such that for sufficiently large n, $F_n(\zeta_{n,i}) = 0$ and $F_n(\eta_{n,j}) = \infty$, where $i = 1, 2, \ldots, m+k+1$ and $j = 1, 2, \ldots, m$. Write $z_{n,i} = a_n\zeta_{n,i}$. Thus, $f_n(z_{n,i}) = 0$ and $z_{n,i} \to 0$ as $n \to \infty$. Set $B_n = \{z_{n,1}, z_{n,2}, \ldots, z_{n,m+k+1}\}$.

Subcase 2.1.1. For each $\delta > 0$, f_n has at least $m+k+2$ zeros (counting multiplicities) in $\Delta(0,\delta)$ for sufficiently large n.

Taking a subsequence and renumbering if necessary, we may assume that $b_n(\neq 0)$ is the zero of f_n of the smallest modulus in $\Delta \setminus B_n$. Obviously, $b_n \to 0$ as $n \to \infty$. Observing that $F_n(b_n/a_n) = 0$ and $b_n/a_n \neq \zeta_{n,i}$, where $i = 1, 2, \ldots, m+k+1$, by Hurwitz's theorem and (26), we have $a_n/b_n \to 0$ as $n \to \infty$. Let $G_n(\zeta) = f_n(b_n\zeta)/b_n^{k+1}$. We have that for sufficiently large n,

(c1) $G_n(\zeta)$ has only $m+k+1$ zeros $a_n\zeta_{n,i}/b_n$ in Δ. Obviously, $|a_n\zeta_{n,i}/b_n| \to 0$ as $n \to \infty$;

(c2) all zeros of $G_n(\zeta)$ are multiple and $G_n(1) = 0$;

(c3) $G_n'(\zeta) \neq h_n(b_n\zeta)/b_n^k$ and $h_n(b_n\zeta)/b_n^k \overset{\chi}{\Rightarrow} \zeta^k$ in $\mathbb{C}$.

By Lemma 16 and Lemma 12, $\{G_n(\zeta)\}$ is normal in Δ' and quasinormal in $\mathbb{C}$. Thus, there exists a subsequence of $\{G_n(\zeta)\}$ (still denoted by $\{G_n(\zeta)\}$) and $D_5 \subset \mathbb{C}$ such that

(d1) D_5 has no accumulation point in $\mathbb{C}$;

(d2) for each $\zeta_0 \in D_5$, no subsequence of $\{G_n(\zeta)\}$ is normal at ζ_0;

(d3) $G_n(\zeta) \overset{\chi}{\Rightarrow} G(\zeta)$ in $\mathbb{C} \setminus D_5$.

Obviously, $D_5 \cap \Delta' = \emptyset$ and all zeros of $G(\zeta)$ are multiple in $\mathbb{C} \setminus D_5$.

Let

$$G_n^*(\zeta) = G_n(\zeta)\frac{\prod_{j=1}^{m}\left(\zeta - \left(a_n\eta_{n,j}/b_n\right)\right)}{\prod_{i=1}^{m+k+1}\left(\zeta - \left(a_n\zeta_{n,i}/b_n\right)\right)},$$
$$F_n^*(\zeta) = F_n(\zeta)\frac{\prod_{j=1}^{m}\left(\zeta - \eta_{n,j}\right)}{\prod_{i=1}^{m+k+1}\left(\zeta - \zeta_{n,i}\right)}. \tag{27}$$

By (26),

$$G_n^*\left(\frac{a_n\zeta}{b_n}\right) = F_n^*(\zeta) \Longrightarrow \frac{1}{k+1} \tag{28}$$

in $\mathbb{C}$. Hence

$$G_n^*(0) \longrightarrow \frac{1}{k+1}. \tag{29}$$

(1) ($G(\zeta) \equiv \infty$ in $\mathbb{C} \setminus D_5$.) Obviously, $G_n^*(\zeta)$ has no zeros in Δ for sufficiently large n. Applying the maximum principle to the sequence $1/G_n^*(\zeta)$ of analytic functions, we see that $G_n^*(\zeta) \overset{\chi}{\Rightarrow} \infty$ in Δ which contradict (29).

(2) $G(\zeta)$ is a meromorphic function in $\mathbb{C} \setminus D_5$.

We claim that $G(\zeta) = \zeta^{k+1}/(k+1)$ in $\mathbb{C} \setminus D_5$. By Lemma 13, $G(\zeta) = (\zeta^{k+1}+c)/(k+1)$, where c is a constant. Since $G_n^*(\zeta)$ has no zeros in Δ for sufficiently large n, applying the maximum principle to the sequence $1/G_n^*(\zeta)$ of analytic functions, we have $G_n^*(\zeta) \overset{\chi}{\Rightarrow} ((\zeta^{k+1} + c)/(k+1))(1/\zeta^{k+1})$ in Δ. Hence, $G_n^*(0) \to (((\zeta^{k+1} + c)/(k+1))(1/\zeta^{k+1}))|_{\zeta=0}$, and then we get that $c = 0$ by (29).

Suppose that $1 \notin D_5$. By ($c2$), $G(1) = 0$ which contradicts $G(\zeta) = (\zeta^{k+1}/(k+1))$. Suppose that $1 \in D_5$. By Lemma 11, $G(\zeta) = \int_1^{\zeta} \xi^k d\xi = (\zeta^{k+1} - 1)/(k+1)$ which contradicts $G(\zeta) = \zeta^{k+1}/(k+1)$.

Subcase 2.1.2. There exists $\delta > 0$ such that f_n has exactly $m + k + 1$ zeros (counting multiplicities) in $\Delta(0, \delta)$ for sufficiently large n.

Now, (f) holds with $r_a = \delta$ and $N_a = m + k + 2$. Next, we will show that (g) also holds.

Set

$$f_n^*(z) = f_n(z)\frac{\prod_{j=1}^{m}\left(z - a_n\eta_{n,j}\right)}{\prod_{i=1}^{m+k+1}\left(z - a_n\zeta_{n,i}\right)},$$
$$F_n^*(\zeta) = F_n(\zeta)\frac{\prod_{j=1}^{m}\left(\zeta - \eta_{n,j}\right)}{\prod_{i=1}^{m+k+1}\left(\zeta - \zeta_{n,i}\right)}. \tag{30}$$

By (26), $f_n^*(a_n\zeta) = F_n^*(\zeta) \Rightarrow 1/(k+1)$ in $\mathbb{C}$, and hence we have

$$f_n^*(0) \longrightarrow \frac{1}{k+1}. \tag{31}$$

(1) ($f(z) \equiv \infty$ in $D \setminus E$.) Since f_n has exactly $m+k+1$ zeros in $\Delta(0, \delta)$ for sufficiently large n, $f_n^*(z)$ has no zeros in $\Delta(0, \delta)$ for sufficiently large n. By the maximum principle applied to $1/f_n^*(z)$, we have $f_n^*(z) \overset{\chi}{\Rightarrow} \infty$ which contradicts (31).

(2) ($f(z)$ is a meromorphic function in $D \setminus E$.) By Lemma 13, $f'(z) = h(z)$ in $\Delta'(0, \delta)$, and hence $f(z) = \int_0^z h(\zeta)d\zeta + c$ in $D \setminus E$, where c is a constant. Since $f_n^*(z)$ has no zeros in $\Delta(0, \delta)$ for sufficiently large n, by the maximum principle applied to $1/f_n^*(z)$, $f_n^*(z) \overset{\chi}{\Rightarrow} (\int_0^z h(\zeta)d\zeta + c)/\zeta^{k+1}$ in $\Delta(0, \delta)$, and hence $f_n^*(0) \to ((\int_0^z h(\zeta)d\zeta + c)/\zeta^{k+1})|_{\zeta=0}$. By (31), $c = 0$. Now, $f(z) = \int_0^z h(\zeta)d\zeta$ in $D \setminus E$.

Subcase 2.2 ($1 \in D_4$). By Lemma 11,

$$F(\zeta) = \int_1^{\zeta} \xi^k d\xi = \frac{\zeta^{k+1} - 1}{k+1}, \quad \zeta \in \mathbb{C} \setminus D_4. \tag{32}$$

Let e_j be the jth root of the equation $\zeta^{k+1} - 1 = 0$, where $j = 1, 2, \ldots, k+1$.

We claim that $D_4 = \{e_1, e_2, \ldots, e_{k+1}\}$. Suppose that $\zeta_0 \notin D_4$, where $\zeta_0^{k+1} - 1 = 0$. Obviously, we have $F(\zeta_0) = 0$ and $F'(\zeta_0) = \zeta_0^k$ which contradict that all of zeros of $F(\zeta)$ are multiple. Suppose that $\zeta_0 \in D_4$, where $\zeta_0^{k+1} - 1 \neq 0$. By Lemma 11, $F(\zeta) = \int_{\zeta=\zeta_0}^{\zeta} \xi^{k+1} d\xi = (\zeta^{k+1} - \zeta_0^{k+1})/(k+1)$ in $\mathbb{C} \setminus D_4$. By (32), $\zeta_0^{k+1} = 1$. A contradiction.

By Lemma 11, there exists $\delta_j > 0$ such that for sufficiently large n, $F_n(\zeta)$ has a single zero $\zeta_{n,j} \to e_j$ of order 2 and a single pole $\eta_{n,j} \to e_j$ of order 1 in $\Delta(e_j, \delta_j)$. Set $z_{n,j} = a_n\zeta_{n,j}$. Thus, $f_n(z_{n,j}) = 0$ and $z_{n,j} \to 0$ as $n \to \infty$, where $j = 1, 2, \ldots, k+1$. Set $B_n = \{z_{n,1}, z_{n,2}, \ldots, z_{n,k+1}\}$.

Subcase 2.2.1. For each $\delta > 0$, f_n has at least $k + 2$ zeros (not counting multiplicities) in $\Delta(0, \delta)$ for sufficiently large n.

Taking a subsequence and renumbering if necessary, we may assume that $b_n \to 0$ is the zero of f_n of the smallest modulus in $\Delta \backslash B_n$. Obviously, $F_n(b_n/a_n) = 0$. Since $b_n \notin B_n$, we have $b_n/a_n \neq \zeta_{n,j}$, where $j = 1, 2, \ldots, k+1$. Since $F_n(\zeta)$ has a single zero $\zeta_{n,j} \to e_j$ of order 2 in $\Delta(e_j, \delta_j)$, by Hurwitz's theorem and (32), we have $a_n/b_n \to 0$ as $n \to \infty$. Let $G_n(\zeta) = f_n(b_n\zeta)/b_n^{k+1}$. We have that for sufficiently large n

(e1) $G_n(\zeta)$ has only $k+1$ zeros $a_n\zeta_{n,i}/b_n$ of order 2 and at least $k+1$ poles $a_n\eta_{n,i}/b_n$ of order 1 in Δ, and obviously, $|a_n\zeta_{n,i}/b_n| \to 0$ and $|a_n\eta_{n,i}/b_n| \to 0$ as $n \to \infty$;

(e2) all zeros of $G_n(\zeta)$ are multiple and $G_n(1) = 0$;

(e3) $G_n'(\zeta) \neq \zeta^k \widehat{h}(b_n\zeta)$ and $\zeta^k \widehat{h}(b_n\zeta) \overset{\chi}{\Rightarrow} \zeta^k$ in $\mathbb{C}$.

By Lemma 16 and Lemma 12, $\{G_n\}$ is normal in Δ' and quasinormal in $\mathbb{C}$. Thus, there exist a subsequence of $\{G_n(\zeta)\}$ (still denoted by $\{G_n(\zeta)\}$) and $D_6 \subset \mathbb{C}$ such that

(f1) D_6 has no accumulation point in $\mathbb{C}$;

(f2) for each $\zeta_0 \in D_6$, no subsequence of $\{G_n(\zeta)\}$ is normal at ζ_0;

(f3) $G_n(\zeta) \overset{\chi}{\Rightarrow} G(\zeta)$ in $\mathbb{C} \setminus D_6$.

Obviously, $D_6 \cap \Delta' = \emptyset$ and all zeros of $G(\zeta)$ are multiple in $\mathbb{C} \setminus D_6$.

Let

$$G_n^*(\zeta) = G_n(\zeta) \frac{\prod_{j=1}^{k+1}\left(\zeta - \left(a_n\eta_{n,j}/b_n\right)\right)}{\prod_{j=1}^{k+1}\left(\zeta - \left(a_n\zeta_{n,j}/b_n\right)\right)^2},$$
$$F_n^*(\zeta) = F_n(\zeta) \frac{\prod_{j=1}^{k+1}\left(\zeta - \eta_{n,j}\right)}{\prod_{j=1}^{k+1}\left(\zeta - \zeta_{n,j}\right)^2}. \quad (33)$$

By (32),

$$G_n^*\left(\frac{a_n\zeta}{b_n}\right) = F_n^*(\zeta) \Longrightarrow \frac{1}{k+1} \quad \text{in } \mathbb{C}. \quad (34)$$

Hence

$$G_n^*(0) \longrightarrow \frac{1}{k+1}. \quad (35)$$

(1) ($G(\zeta) \equiv \infty$ in $\mathbb{C} \setminus D_6$.) Obviously, $G_n^*(\zeta)$ has no zeros in Δ for sufficiently large n. Applying the maximum principle to the sequence $1/G_n^*(\zeta)$ of analytic functions, we have that $G_n^*(\zeta) \overset{\chi}{\Rightarrow} \infty$ in Δ which contradict (35).

(2) ($G(\zeta)$ is a meromorphic function in $\mathbb{C} \setminus D_6$.) We claim that $G(\zeta) = \zeta^{k+1}/(k+1)$ in $\mathbb{C} \backslash D_6$. By Lemma 13, $G(\zeta) = (\zeta^{k+1} + c)/(k+1)$, where c is a constant. Since $G_n^*(\zeta)$ has no zeros in Δ for sufficiently large n, applying the maximum principle to the sequence $1/G_n^*(\zeta)$ of analytic functions, we have $G_n^*(\zeta) \overset{\chi}{\Rightarrow} ((\zeta^{k+1} + c)/(k+1))(1/\zeta^{k+1})$ in Δ. Hence, $G_n^*(0) \to (((\zeta^{k+1} + c)/(k+1))(1/\zeta^{k+1}))|_{\zeta=0}$, and then $c = 0$ by (35).

Suppose that $1 \notin D_6$. By (e2), $G(1) = 0$ which contradicts that $G(\zeta) = \zeta^{k+1}/(k+1)$. Suppose that $1 \in D_6$. By Lemma 11, $G(\zeta) = \int_1^\zeta \zeta^k d\xi = (\zeta^{k+1} - 1)/(k+1)$ which contradicts that $G(\zeta) = \zeta^{k+1}/(k+1)$.

Subcase 2.2.2. There exists $\delta > 0$ such that f_n has exactly $k+1$ zeros (not counting multiplicities) for sufficiently large n.

Similar to the previous treatment in Subcase 2.1.2, we finally can show that (e) and (f) hold.

Case 3 ($h(0) \neq 0, \infty$). Obviously, (e) and (f) hold by Lemma 11. $\square$

5. Proof of Theorem 1

Proof. We assume that $f'(z) = \alpha(z)$ has at most finitely many solutions and derive a contradiction. Let $R(z) \sim c_0 z^d$ as $z \to \infty$, where $c_0 \in \mathbb{C} \setminus \{0\}$ and $d \in \mathbb{Z}$.

Clearly, $T(r, R) = O(\log r)$ and $T(r, 1/R) = O(\log r)$ as $r \to \infty$. By Lemma 7, $T(r, h) = Ar^2(1 + o(1))$ as $r \to \infty$, where $A > 0$ is a constant. By standard results in Nevanlinna theory, $T(r, h) = T(r, \alpha/R) \leq T(r, \alpha) + T(r, 1/R)$ and $T(r, \alpha) \leq T(r, R) + T(r, h)$ as $r \to \infty$. Thus, $T(r, \alpha) = Ar^2(1 + o(1))$ as $r \to \infty$. Since $T(r, \alpha) = o\{T(r, f)\}$ as $r \to \infty$, we obtain that $\overline{\lim}_{r\to\infty}(T(r, f)/r^2) = \infty$.

Set $g(z) = f(z)/z^d$. By Lemma 15, there exist sequences $t_n \to \infty$ and $\varepsilon_n \to 0$ such that

$$S(\Delta(t_n, \varepsilon_n), g) \longrightarrow \infty \quad \text{as } n \longrightarrow \infty,$$
$$\frac{f(t_n)}{t_n^d} \longrightarrow 0, \quad \frac{f'(t_n)}{t_n^d} \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \quad (36)$$

Let ω_1, ω_2 be the two fundamental periods of $h(z)$ and let $P(0 \in P)$ be a fundamental parallelogram of $h(z)$. There exist integers i_n and j_n such that $z_n \in P$, where $z_n = t_n - i_n\omega_1 - j_n\omega_2$. There exists a subsequence of $\{z_n\}$ (still denoted by $\{z_n\}$) such that $z_n \to z_0$ as $n \to \infty$. Set

$$g_n(z) = g(z + i_n\omega_1 + j_n\omega_2),$$
$$f_n(z) = \frac{f(z + i_n\omega_1 + j_n\omega_2)}{t_n^d}. \quad (37)$$

Clearly, we have $S(\Delta(z_n, \varepsilon_n), g_n) = S(\Delta(t_n, \varepsilon_n), g)$, $f_n(z_n) = f(t_n)/t_n^d$, and $f_n'(z_n) = f'(t_n)/t_n^d$. By (36), we have

$$S(\Delta(z_n, \varepsilon_n), g_n) \longrightarrow \infty \quad \text{as } n \longrightarrow \infty, \quad (38)$$

$$f_n(z_n) \longrightarrow 0, \quad f_n'(z_n) \longrightarrow \infty \quad \text{as } n \longrightarrow \infty. \quad (39)$$

There exists $R > 0$ such that $\overline{P} \subset \Delta(0, R)$ and $\Delta(z_n, \varepsilon_n) \subset \Delta(0, R)$ for each n. Set $D = \Delta(0, R)$. Obviously, we have $z_0 \in D$. By assumption, for sufficiently large n,

$$f_n'(z) = \frac{f'(z + i_n\omega_1 + j_n\omega_2)}{t_n^d}$$
$$\neq \frac{R(z + i_n\omega_1 + j_n\omega_2)\, h(z)}{t_n^d}, \quad z \in D. \quad (40)$$

For each $z \in D$,

$$\begin{aligned} &\left|t_n - (z + i_n\omega_1 + j_n\omega_2)\right| \\ &\quad = \left|(z_n + i_n\omega_1 + j_n\omega_2) - (z + i_n\omega_1 + j_n\omega_2)\right| \\ &\quad = \left|z_n - z\right| < 2R. \end{aligned} \tag{41}$$

So we have

$$\frac{R(z + i_n\omega_1 + j_n\omega_2)}{t_n^d} \longrightarrow c_0 \quad \text{as } n \longrightarrow \infty. \tag{42}$$

Set

$$T_n(z) = \frac{R(z + i_n\omega_1 + j_n\omega_2)\, h(z)}{t_n^d}. \tag{43}$$

Obviously, $T_n(z) \overset{\chi}{\rightrightarrows} c_0 h(z)$ in D, and for sufficiently large n, $c_0 h$ and T_n have the same zeros and poles with the same multiplicity in D.

Now, $\{f_n\}$ is a family of meromorphic functions in D such that for sufficiently large n,

(a1) all zeros of $\{f_n\}$ are multiple in D;

(a2) $T_n(z) \overset{\chi}{\rightrightarrows} c_0 h(z)$ in D, where $c_0 h(z) \not\equiv 0, \infty$ in D;

(a3) $f_n'(z) \neq T_n(z)$ in D.

It follows from Lemma 12 that $\{f_n\}$ is quasinormal in D. Hence there exists $\tau > 0$ such that $\Delta(z_0, \tau) \subset D$ and $\{f_n\}$ is normal in $\Delta'(z_0, \tau)$. Then there exists a subsequence of $\{f_n\}$ (still denoted by $\{f_n\}$) such that

(b1) $c_0 h(z)$ and $T_n(z)$ have the same zeros and poles with the same multiplicity in $\Delta(z_0, \tau)$;

(b2) for all $n \in \mathbb{N}$, $f_n'(z) \neq T_n(z)$ in $\Delta(z_0, \tau)$;

(b3) no subsequence of $\{f_n\}$ is normal at z_0;

(b4) all zeros of $\{f_n\}$ are multiple in $\Delta(z_0, \tau)$, and $f_n(z) \overset{\chi}{\rightrightarrows} f(z)$ in $\Delta'(z_0, \tau)$.

By (39), (b3) holds. By Lemma 17, we have

(c1) $h(z_0) \neq \infty$;

(c2) there exist $\tau^* \in (0, \tau)$ and $M^* > 0$ such that for sufficiently large n, $n(\tau^*, 1/f_n) < M^*$;

(c3) $f(z) = \int_{z_0}^{z} c_0 h(\zeta)\, d\zeta$ in $\Delta'(z_0, \tau)$.

By Lemma 6 and (c2) and (c3), there exists $M_1 > 0$ such that, for sufficiently large n,

$$S\left(\Delta\left(z_0, \frac{\tau^*}{2}\right), f_n\right) < M_1. \tag{44}$$

Next, we will derive a contradiction with (38).

By (37), $g_n(z) = f_n(z)(1 + ((z - z_n)/t_n))^{-d}$. Then

$$\begin{aligned} g_n^{\#}(z) = {} & \left(\left|\left(1 + \frac{z - z_n}{t_n}\right)^d f_n'(z) - \left(1 + \frac{z - z_n}{t_n}\right)^d \right.\right. \\ & \left.\left. \times \left(\frac{d}{t_n + z - z_n}\right) f_n(z)\right|\right) \\ & \times \left(\left|\left(1 + \frac{z - z_n}{t_n}\right)^d\right|^2 + \left|f_n(z)\right|^2\right)^{-1}, \end{aligned} \tag{45}$$

so

$$\begin{aligned} \left[g_n^{\#}(z)\right]^2 \le {} & \frac{2\left|(1 + (z - z_n)/t_n)^d f_n'(z)\right|^2}{\left(\left|(1 + (z - z_n)/t_n)^d\right|^2 + \left|f_n(z)\right|^2\right)^2} \\ & + \frac{2\left|(1 + (z - z_n)/t_n)^d (d/(t_n + z - z_n)) f_n(z)\right|^2}{\left(\left|(1 + (z - z_n)/t_n)^d\right|^2 + \left|f_n(z)\right|^2\right)^2}. \end{aligned} \tag{46}$$

Using the simple inequality

$$\frac{C}{C^2 + x^2} \le 2\max\left(C, \frac{1}{C}\right)\frac{1}{1 + x^2} \tag{47}$$

for $C > 0$, we have

$$\begin{aligned} & \frac{2\left|(1 + (z - z_n)/t_n)^d f_n'(z)\right|^2}{\left(\left|(1 + (z - z_n)/t_n)^d\right|^2 + \left|f_n(z)\right|^2\right)^2} \\ & \quad \le 2\max\left(\left|\left(1 + \frac{z - z_n}{t_n}\right)\right|^{2d}, \frac{1}{\left|(1 + (z - z_n)/t_n)\right|^{2d}}\right) \\ & \quad \times \left[f_n^{\#}(z)\right]^2. \end{aligned} \tag{48}$$

The second term on the right of (46) is

$$\begin{aligned} & \frac{1}{2}\left|\frac{d}{t_n + z - z_n}\right|^2 \left(\frac{2\left|(1 + (z - z_n)/t_n)^d f_n(z)\right|}{\left|(1 + (z - z_n)/t_n)^d\right|^2 + \left|f_n(z)\right|^2}\right)^2 \\ & \quad \le \frac{1}{2}\left|\frac{d}{t_n + z - z_n}\right|^2. \end{aligned} \tag{49}$$

Putting (46), (48), and (49) together, we have for $z \in \Delta(z_0, (\tau^*/2))$ and sufficiently large n,

$$\left[g_n^{\#}(z)\right]^2 \le 4\left[f_n^{\#}(z)\right]^2 + 1. \tag{50}$$

It follows from (44) and (50) that

$$S\left(\Delta\left(z_0, \frac{\tau^*}{2}\right), g_n\right) \le 4M_1 + \left(\frac{\tau^*}{2}\right)^2 := M_2 \tag{51}$$

which contradicts (38). □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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