

Research Article

Conservation Laws for a Variable Coefficient Variant Boussinesq System

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We construct the conservation laws for a variable coefficient variant Boussinesq system, which is a third-order system of two partial differential equations. This system does not have a Lagrangian and so we transform it to a system of fourth-order, which admits a Lagrangian. Noether's approach is then utilized to obtain the conservation laws. Lastly, the conservation laws are presented in terms of the original variables. Infinite numbers of both local and nonlocal conserved quantities are derived for the underlying system.

1. Introduction

The first type of variant Boussinesq equations [1, 2] is given by

$$u_t + v_x + uu_x = 0, \quad (1a)$$

$$v_t + (uv)_x + u_{xxx} = 0, \quad (1b)$$

and was introduced as a model for water waves [3]. Wang in his paper [4] obtained the solitary wave solutions of ((1a) and (1b)) by using homogeneous balance method. The periodic wave solutions of ((1a) and (1b)) were derived in [5] by using ansatz method and the multisolitary wave solutions were obtained in [6] using the homogeneous balance method. Xu et al. [7] obtained traveling wave solutions of ((1a) and (1b)). Conservation laws for ((1a) and (1b)) were derived in [8].

Conservation laws play a vital role in the solution process of differential equations (DEs) because they describe physical properties that remain constant throughout the various processes that occur in the physical world. Thus it is very important to compute conservation laws for differential equations. One can see from the various published papers (see, e.g., [9–11]) that conservation laws have been used in studying the existence, uniqueness, and stability of solutions of nonlinear partial differential equations. They have also been applied in the development and use of numerical methods (see, e.g.,

[12, 13]). Most importantly, conserved vectors associated with Lie point symmetries have been used to derive exact solutions of some partial differential equations [14–16].

In this paper, we study the variable coefficient variant Boussinesq system:

$$u_t + \alpha(t) v_x + \beta(t) uu_x = 0, \quad (2a)$$

$$v_t + \beta(t) (uv)_x + \psi(t) u_{xxx} = 0, \quad (2b)$$

which generalizes the system ((1a) and (1b)). In ((2a) and (2b)), $\alpha(t)$, $\beta(t)$, and $\psi(t)$ are arbitrary functions of t , with $\psi(t)$ describing the different diffusion strength, $u = u(x, t)$ representing the field of a horizontal velocity, and $v = v(x, t)$ representing the amplitude describing the deviation from the equilibrium position of the liquid.

The objective of the present study is to construct conservation laws for the system ((2a) and (2b)).

The paper is organized as follows. In Section 2 we briefly give the preliminaries concerning the Noether symmetry approach. Section 3 obtains the conservation laws for the system ((2a) and (2b)). Finally, in Section 4 concluding remarks are presented.

2. Preliminaries

Here we present some salient features of Noether operators concerning the system of two partial differential equations. These results will be utilized in Section 3. The reader is referred to [8, 17–19] for further details.

Consider the vector field

$$X = \xi^1(t, x, U, V) \frac{\partial}{\partial t} + \xi^2(t, x, U, V) \frac{\partial}{\partial x} + \eta^1(t, x, U, V) \frac{\partial}{\partial U} + \eta^2(t, x, U, V) \frac{\partial}{\partial V}, \tag{3}$$

which has the second-order prolongation

$$X^{[2]} = \xi^1(t, x, U, V) \frac{\partial}{\partial t} + \xi^2(t, x, U, V) \frac{\partial}{\partial x} + \eta^1(t, x, U, V) \frac{\partial}{\partial u} + \eta^2(t, x, U, V) \frac{\partial}{\partial v} + \zeta_t^1 \frac{\partial}{\partial U_t} + \zeta_t^2 \frac{\partial}{\partial V_t} + \zeta_x^1 \frac{\partial}{\partial U_x} + \zeta_x^2 \frac{\partial}{\partial V_x} + \zeta_{xx}^1 \frac{\partial}{\partial U_{xx}} + \dots, \tag{4}$$

where

$$\begin{aligned} \zeta_t^1 &= D_t(\eta^1) - U_t D_t(\xi^1) - U_x D_t(\xi^2), \\ \zeta_x^1 &= D_x(\eta^1) - U_t D_x(\xi^1) - U_x D_x(\xi^2), \\ \zeta_t^2 &= D_t(\eta^2) - V_t D_t(\xi^1) - V_x D_t(\xi^2), \\ \zeta_x^2 &= D_x(\eta^2) - V_t D_x(\xi^1) - V_x D_x(\xi^2), \\ \zeta_{xx}^1 &= D_x(\zeta_x^1) - U_t D_x(\xi^1) - U_x D_x(\xi^2), \end{aligned} \tag{5}$$

with

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + U_t \frac{\partial}{\partial U} + V_t \frac{\partial}{\partial V} + U_{tt} \frac{\partial}{\partial U_t} + V_{tt} \frac{\partial}{\partial V_t} \\ &\quad + U_{xt} \frac{\partial}{\partial U_x} + V_{xt} \frac{\partial}{\partial V_x} + \dots, \\ D_x &= \frac{\partial}{\partial x} + U_x \frac{\partial}{\partial U} + V_x \frac{\partial}{\partial V} + U_{xx} \frac{\partial}{\partial U_x} + V_{xx} \frac{\partial}{\partial V_x} \\ &\quad + U_{xt} \frac{\partial}{\partial U_t} + V_{xt} \frac{\partial}{\partial V_t} + \dots. \end{aligned} \tag{6}$$

The Euler-Lagrange operators are defined by

$$\begin{aligned} \frac{\delta}{\delta U} &= \frac{\partial}{\partial U} - D_t \frac{\partial}{\partial U_t} - D_x \frac{\partial}{\partial U_x} + D_t^2 \frac{\partial}{\partial U_{tt}} + D_x^2 \frac{\partial}{\partial U_{xx}} \\ &\quad + D_x D_t \frac{\partial}{\partial U_{xt}} - \dots, \\ \frac{\delta}{\delta V} &= \frac{\partial}{\partial V} - D_t \frac{\partial}{\partial V_t} - D_x \frac{\partial}{\partial V_x} + D_t^2 \frac{\partial}{\partial V_{tt}} + D_x^2 \frac{\partial}{\partial V_{xx}} \\ &\quad + D_x D_t \frac{\partial}{\partial V_{xt}} - \dots. \end{aligned} \tag{7}$$

Consider a system of two partial differential equations of two independent variables, t and x , namely,

$$E_1(t, x, U, V, U_x, V_x, U_t, V_t, U_{tt}, V_{tt}, U_{xx}, V_{xx}, \dots) = 0, \tag{8a}$$

$$E_2(t, x, U, V, U_x, V_x, U_t, V_t, U_{tt}, V_{tt}, U_{xx}, V_{xx}, \dots) = 0, \tag{8b}$$

which has a second-order Lagrangian L ; that is, ((8a) and (8b)) are equivalent to the Euler-Lagrange equations:

$$\frac{\delta L}{\delta U} = 0, \quad \frac{\delta L}{\delta V} = 0. \tag{9}$$

Definition 1. The vector field X , of the form (3), is called a Noether operator corresponding to a Lagrangian L of ((8a) and (8b)) if

$$X^{[2]}(L) + \{D_t(\xi^1) + D_x(\xi^2)\}L = D_t(B^1) + D_x(B^2) \tag{10}$$

for some gauge functions $B^1(t, x, U, V)$ and $B^2(t, x, U, V)$.

We recall the following theorem.

Theorem 2 (Noether [17]). *If X , as given in (3), is a Noether point symmetry generator corresponding to a Lagrangian L of (8a) and (8b), then the vector $T = (T^1, T^2)$ with components,*

$$\begin{aligned} T^1 &= \xi^1 L + W^1 \frac{\delta L}{\delta U_t} + W^2 \frac{\delta L}{\delta V_t} - D_t(W^1) \frac{\delta L}{\delta U_{tt}} \\ &\quad - D_t(W^2) \frac{\delta L}{\delta V_{tt}} - B^1, \\ T^2 &= \xi^2 L + W^1 \frac{\delta L}{\delta U_x} + W^2 \frac{\delta L}{\delta V_x} - D_x(W^1) \frac{\delta L}{\delta U_{xx}} \\ &\quad - D_x(W^2) \frac{\delta L}{\delta V_{xx}} - B^2, \end{aligned} \tag{11}$$

is a conserved vector for ((8a) and (8b)) associated with the operator X , where $W^1 = \eta^1 - U_t \xi^1 - U_x \xi^2$ and $W^2 = \eta^2 - V_t \xi^1 - V_x \xi^2$ are the Lie characteristics functions.

3. Conservation Laws of System ((2a) and (2b))

Consider the variable coefficient variant Boussinesq system ((2a) and (2b)); namely,

$$\begin{aligned} u_t + \alpha(t) v_x + \beta(t) u u_x &= 0, \\ v_t + \beta(t) u v_x + \beta(t) v u_x + \psi(t) u_{xxx} &= 0. \end{aligned} \tag{12}$$

Here we note that the system ((2a) and (2b)) does not admit a Lagrangian. Nevertheless, we can transform the system ((2a) and (2b)) into a variational form by setting $u = U_x$ and $v = V_x$. Thus, the system ((2a) and (2b)), with this transformation, becomes a fourth-order system, namely

$$U_{tx} + \alpha(t) V_{xx} + \beta(t) U_x U_{xx} = 0, \tag{13a}$$

$$V_{tx} + \beta(t) U_x V_{xx} + \beta(t) V_x U_{xx} + \psi(t) U_{xxxx} = 0 \tag{13b}$$

and has a second-order Lagrangian given by

$$L = \frac{1}{2} [\psi(t) U_{xx}^2 - \alpha(t) V_x^2 - \beta(t) U_x^2 V_x - V_t U_x - U_t V_x]. \tag{14}$$

Substituting the value of L from (14) to (1) and splitting with respect to the derivatives of u and v yield the linear overdetermined system of PDEs; namely

$$\begin{aligned} \xi_U^1 &= 0, & \xi_V^1 &= 0, & \xi_U^2 &= 0, & \xi_V^2 &= 0, \\ \xi_x^1 &= 0, & \eta_U^2 &= 0, & \eta_V^1 &= 0, & \eta_x^2 &= 0, \\ 2\beta(t) \eta_U^1 + \beta(t) \xi_t^1 + \beta(t) \eta_V^2 + \beta'(t) \xi^1 - 2\beta(t) \xi_x^2 &= 0, \\ \psi'(t) \xi^1 + 2\psi(t) \eta_U^1 - 3\psi(t) \xi_x^2 + \psi(t) \xi_t^1 &= 0, & \eta_{UU}^1 &= 0, \\ \alpha'(t) \xi^1 + 2\alpha(t) \eta_V^2 - \alpha(t) \xi_x^2 + \alpha(t) \xi_t^1 &= 0, \\ 2\eta_{xU}^1 - \xi_{xx}^2 &= 0, \\ \eta_{xx}^1 &= 0, & \eta_V^2 + \eta_U^1 &= 0, & \xi_t^2 - \beta(t) \eta_x^1 &= 0, \\ -\eta_t^2 &= 2B_U^2, & B_U^1 &= 0, & -\eta_t^1 &= 2B_V^2, \\ -\eta_x^1 &= 2B_V^1, & B_t^1 + B_x^2 &= 0. \end{aligned} \tag{15}$$

After some tedious and lengthy calculations, the above system yields

$$\begin{aligned} \xi^1 &= a(t), \\ \xi^2 &= c_1 x + c_2 \int \beta(t) dt + c_3, \\ \eta^1 &= h(t) U + c_2 x + k(t), \\ \eta^2 &= -h(t) V + m(t), \end{aligned} \tag{16}$$

$$B^1 = -\frac{c_2}{2} V + z(t, x),$$

$$B^2 = -\frac{1}{2} h'(t) UV - \frac{1}{2} m'(t) U - \frac{1}{2} k'(t) V + w(t, x),$$

$$z_t + w_x = 0,$$

$$\beta(t) h(t) + \beta(t) a'(t) + \beta'(t) a(t) - 2c_1 \beta(t) = 0, \tag{17}$$

$$2\psi(t) h(t) + \psi'(t) a(t) + \psi(t) a'(t) - 3c_1 \psi(t) = 0, \tag{18}$$

$$2\alpha(t) h(t) - \alpha(t) a'(t) - \alpha'(t) a(t) + c_1 \alpha(t) = 0. \tag{19}$$

The analysis of (17), (18), and (19) prompts the following two cases.

Case 1. $\alpha(t)$, $\beta(t)$, and $\psi(t)$ are arbitrary but not of the form contained in Case 2.

In this case we obtain four Noether point symmetries. These are given below together with their corresponding gauge functions:

$$X_1 = \frac{\partial}{\partial x}, \quad B^1 = z, \quad B^2 = w, \quad z_t + w_x = 0, \tag{20}$$

$$X_2 = k(t) \frac{\partial}{\partial U}, \quad B^1 = z, \quad B^2 = -\frac{1}{2} k'(t) V + w, \tag{21}$$

$$z_t + w_x = 0,$$

$$X_3 = m(t) \frac{\partial}{\partial V}, \quad B^1 = -\frac{1}{2} m'(t) U + z, \quad B^2 = w, \tag{22}$$

$$z_t + w_x = 0,$$

$$X_4 = \int \beta(t) dt \frac{\partial}{\partial x} + x \frac{\partial}{\partial U}, \quad B^1 = -\frac{1}{2} V + z, \tag{23}$$

$$B^2 = w, \quad z_t + w_x = 0.$$

Invoking Theorem 2, the four nontrivial conserved vectors associated with these four Noether point symmetries are, respectively,

$$T_1^1 = uv - z, \tag{24}$$

$$T_1^2 = \frac{\alpha(t)}{2} v^2 - \frac{\Psi(t)}{2} u_x^2 + \psi(t) uu_{xx} + \beta(t) u^2 v - w; \tag{25}$$

$$T_2^1 = -k(t) v + D_x \left(\frac{k(t)}{2} \int v dx \right) - z, \tag{26}$$

$$T_2^2 = -\beta(t) k(t) uv - \psi(t) k(t) u_{xx} + k'(t) \int v dx \tag{27}$$

$$- D_t \left(\frac{k(t)}{2} \int v dx \right) - w;$$

$$T_3^1 = -m(t) u + D_x \left(\frac{m(t)}{2} \int u dx \right) - z, \tag{28}$$

$$T_3^2 = -\alpha(t) m(t) v - \frac{1}{2} \beta(t) m(t) u^2 + m'(t) \int u dx \tag{29}$$

$$- D_t \left(\frac{m(t)}{2} \int u dx \right) - w;$$

$$T_4^1 = uv \int \beta(t) dt - xv + D_x \left(\frac{x}{2} \int v dx \right) - z, \tag{30}$$

$$T_4^2 = \frac{\alpha(t)}{2} v^2 \int \beta(t) dt - \frac{\Psi(t)}{2} u_x^2 \int \beta(t) dt$$

$$- \beta(t) xuv - \psi(t) xu_{xx} + \beta(t) u^2 v \int \beta(t) dt$$

$$+ \psi(t) uu_{xx} \int \beta(t) dt + \psi(t) u_x - D_t \left(\frac{x}{2} \int v dx \right) - w. \tag{31}$$

From the above we observe that the conserved vector (24)-(25) is a local conserved vector. In (30)-(31) one can see that the nonlocal part within the parenthesis gives the trivial part of the conserved vector and therefore can be set to zero. Thus, the conserved vector (30)-(31) is a local conserved vector. It is also interesting to notice that the conserved vectors (26)-(27) and (28)-(29) for $k(t) = 1$, $z(x, t) = 0$, $w(x, t) = 0$, and $m(t) = 1$ yield the local conserved vectors:

$$\begin{aligned} T_5^1 &= v, & T_5^2 &= \beta(t) uv + \psi(t) u_{xx}; \\ T_6^1 &= u, & T_6^2 &= \alpha(t) v + \frac{1}{2} \beta(t) u^2. \end{aligned} \quad (32)$$

Remark 3. We note that for arbitrary values of $k(t)$ and $m(t)$ infinitely many nonlocal conservation laws exist for the variable coefficient variant Boussinesq system.

Case 2. $\alpha(t) = \alpha_1$, $\beta(t) = \beta_1$, and $\psi(t) = \psi_1$, where α_1 , β_1 , and ψ_1 are constants.

This case gives us five Noether point symmetries, namely, X_1 , X_2 , and X_3 , given by the generators (20)–(22) and X_4 , X_5 given by

$$\begin{aligned} X_4 &= \frac{\partial}{\partial t}, & B^1 &= z, & B^2 &= w, & z_t + w_x &= 0, \\ X_5 &= \beta_1 t \frac{\partial}{\partial x} + x \frac{\partial}{\partial U}, & B^1 &= -\frac{1}{2} V + z, \\ B^2 &= w, & z_t + w_x &= 0. \end{aligned} \quad (33)$$

The application of Theorem 2, due to Noether, gives the five nontrivial conserved vectors:

$$T_1^1 = uv - z, \quad (34)$$

$$T_1^2 = \frac{1}{2} \alpha_1 v^2 - \frac{1}{2} \psi_1 u_x^2 + \psi_1 uu_{xx} + \beta_1 u^2 v - w; \quad (35)$$

$$T_2^1 = -k(t) v + D_x \left(\frac{k(t)}{2} \int v dx \right) - z, \quad (36)$$

$$\begin{aligned} T_2^2 &= -\beta_1 k(t) uv - \psi_1 k(t) u_{xx} + k'(t) \int v dx \\ &\quad - D_t \left(\frac{k(t)}{2} \int v dx \right) - w; \end{aligned} \quad (37)$$

$$T_3^1 = -m(t) u + D_x \left(\frac{m(t)}{2} \int u dx \right) - z, \quad (38)$$

$$\begin{aligned} T_3^2 &= -\alpha_1 m(t) v - \frac{1}{2} \beta_1 m(t) u^2 + m'(t) \int u dx \\ &\quad - D_t \left(\frac{m(t)}{2} \int u dx \right) - w, \end{aligned} \quad (39)$$

$$T_4^1 = \frac{1}{2} \psi_1 u_x^2 - \frac{1}{2} \alpha_1 v^2 - \frac{1}{2} \beta_1 u^2 v - z, \quad (40)$$

$$T_4^2 = -\psi_1 u_t u_x + \left(\alpha_1 v + \frac{1}{2} \beta_1 u^2 \right) \int v_t dx \quad (41)$$

$$+ (\beta_1 uv + \psi_1 u_{xx}) \int u_t dx + \int u_t dx \int v_t dx - w;$$

$$T_5^1 = \beta_1 uvt - xv + D_x \left(\frac{x}{2} \int v dx \right) - z, \quad (42)$$

$$\begin{aligned} T_5^2 &= \frac{1}{2} \alpha_1 \beta_1 v^2 t - \frac{1}{2} \psi_1 \beta_1 u_x^2 t - \beta_1 xuv - \psi_1 x u_{xx} + \beta_1 u^2 vt \\ &\quad + \psi_1 uu_{xx} t + \psi_1 u_x - D_t \left(\frac{x}{2} \int v dx \right) - w, \end{aligned} \quad (43)$$

respectively, corresponding to the above five Noether point symmetries. We note that in this case we obtain an extra Noether operator and hence an extra conserved vector, which is given by (40)-(41).

Remark 4. When $\alpha_1 = \beta_1 = \psi_1 = 1$, we recover the results obtained in [8].

4. Concluding Remarks

In this paper we studied the variable coefficient variant Boussinesq system ((2a) and (2b)). This system does not have a Lagrangian. Therefore we converted it to a fourth-order system ((13a) and (13b)) which has a Lagrangian. Thereafter, we utilized the Noether's theorem to construct the conservation laws of system ((13a) and (13b)). Finally, by reverting back to our original variables u and v we constructed the conservation laws for the third-order variable coefficient variant Boussinesq system. The conservation laws obtained consisted of infinite number of local and nonlocal conserved vectors.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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