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Research Article

Multiplicity of Solutions for Gradient Systems Using Landesman-Lazer Conditions

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We establish existence and multiplicity of solutions for an elliptic system which presents resonance at infinity of Landesman-Lazer type. In order to describe the resonance, we use an eigenvalue problem with indefinite weights. In all results, we use Variational Methods, Morse Theory and Critical Groups.

1. Introduction

In this paper, we discuss results on existence and multiplicity of solutions for the system

$$\begin{gathered} -\Delta u = a(x)u + b(x)v + f(x,u,v) - h_1(x) \quad \text{in } \Omega, \\ -\Delta v = b(x)u + d(x)v + g(x,u,v) - h_2(x) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$ with $a, b, d \in C^0(\overline{\Omega}, \mathbb{R})$ and $h_1, h_2 \in L^2(\Omega)$, $f, g \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$. We assume that the system (1.1) is of gradient type, that is, there is some function $F \in C^2(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ such that $\nabla F = (f, g)$. Throughout this paper, ∇F denotes the gradient in the variables u and v for each $x \in \Omega$ fixed.

From a variational standard point of view, to find weak solutions of (1.1) in $H = H_0^1(\Omega) \times H_0^1(\Omega)$ is equivalent to find critical points of the C^2 functional $J : H \to \mathbb{R}$

given by

$$J(z) = \frac{1}{2}\|z\|^2 - \frac{1}{2}\int_\Omega \langle A(x)(u,v),(u,v)\rangle dx - \int_\Omega F(x,u,v)dx + \int_\Omega h_1 u + h_2 v\, dx, \tag{1.2}$$

where $\| \ \|$ denotes the Dirichlet norm

$$\|z\|^2 = \int_\Omega |\nabla u|^2 + |\nabla v|^2 dx, \quad z = (u,v) \in H. \tag{1.3}$$

We observe that the problem (1.1) represents a steady state case of reaction-diffusion systems of interest in Biology, Chemistry, Physics, and Ecology; see [1, 2].

In order to define the resonance conditions, we need to consider eigenvalue problems for functions $A \in C(\overline{\Omega}, M_{2\times 2})$, where $M_{2\times 2}$ denotes the set of all matrices of order 2. Let us denote by $\mathcal{S}_2(\Omega)$ the set of all continuous, cooperative, and symmetric functions $A \in C(\overline{\Omega}, M_{2\times 2})$ of order 2 written as

$$A(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & d(x) \end{pmatrix}, \tag{1.4}$$

satisfying the following hypotheses.

(M_1) A is cooperative, that is, $b(x) \geq 0$ for all $x \in \overline{\Omega}$. Moreover, we assume that

$$\Omega_b := \{x \in \Omega : b(x) = 0\} \tag{1.5}$$

has zero Lebesgue measure.

(M_2) There is $x_1 \in \Omega$ such that $a(x_1) > 0$ or $d(x_1) > 0$.

In this way, given $A \in \mathcal{S}_2(\Omega)$, we consider the eigenvalue problem with weights as follows:

$$\begin{gathered} -\Delta \begin{pmatrix} u \\ v \end{pmatrix} = \lambda A(x) \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.6}$$

Using the conditions (M_1) and (M_2) above and applying the Spectral Theory for compact operators, we get a sequence of eigenvalues

$$0 < \lambda_1(A) < \lambda_2(A) \leq \lambda_3(A) \leq \cdots \tag{1.7}$$

such that $\lambda_k(A) \to +\infty$ as $k \to \infty$. Here, each eigenvalue $\lambda_k(A)$, $k \geq 1$; see [3–5].

We point out that the Problem (1.1) presents a resonance phenomenon depending on the behavior of the functions f and g at infinity. We assume all along this paper the following basic hypothesis.

(B^{∞}) There is $h \in C(\Omega, \mathbb{R})$ such that

$$\begin{aligned} |f(x,z)| &\leq h(x), \quad \forall (x,z) \in \Omega \times \mathbb{R}^2, \\ |g(x,z)| &\leq h(x), \quad \forall (x,z) \in \Omega \times \mathbb{R}^2. \end{aligned} \tag{1.8}$$

Under these hypotheses, system (1.1) is asymptotically quadratic at infinity due to the presence of a linear part given by the function $A \in \mathcal{S}_2(\Omega)$. In addition, when $\lambda_k(A) = 1$ for some $k \geq 1$ the problem (1.1) becomes resonant. In this case, in order to obtain existence and multiplicity of solutions for (1.1), we will assume conditions of the Landesman-Lazer type introduced in the scalar case in [6]. These famous conditions are well known in the scalar case. However for gradient systems, to the best our knowledge, these conditions have not been explored in our case.

In order to introduce our Landesman-Lazer conditions for system (1.1), we need the following auxiliary assumptions.

(f^{∞}) There are functions $f^{++}, f^{+-}, f^{-+}, f^{--} \in C(\Omega)$ such that

$$\begin{aligned} f^{++}(x) &= \lim_{\substack{u\to\infty \\ v\to\infty}} f(x,u,v), \qquad f^{+-}(x) = \lim_{\substack{u\to\infty \\ v\to-\infty}} f(x,u,v), \\ f^{-+}(x) &= \lim_{\substack{u\to-\infty \\ v\to\infty}} f(x,u,v), \qquad f^{--}(x) = \lim_{\substack{u\to-\infty \\ v\to-\infty}} f(x,u,v). \end{aligned} \tag{1.9}$$

Moreover,

(g^{∞}) there are functions $g^{++}, g^{+-}, g^{-+}, g^{--} \in C(\Omega)$ such that

$$\begin{aligned} g^{++}(x) &= \lim_{\substack{u\to\infty \\ v\to\infty}} g(x,u,v), \qquad g^{+-}(x) = \lim_{\substack{u\to\infty \\ v\to-\infty}} g(x,u,v), \\ g^{-+}(x) &= \lim_{\substack{u\to-\infty \\ v\to\infty}} g(x,u,v), \qquad g^{--}(x) = \lim_{\substack{u\to-\infty \\ v\to-\infty}} g(x,u,v), \end{aligned} \tag{1.10}$$

where the limits in (1.9) and (1.10) are taken uniformly and for all $x \in \Omega$.

So we can write the Landesman-Lazer conditions for our problem (1.1), when $k = 1$. It will be assumed either

$$(LL)_1^{+} \quad \int_{\Omega} f^{--}\phi_1 + g^{--}\psi_1 dx < \int_{\Omega} h_1\phi_1 + h_2\psi_1 dx < \int_{\Omega} f^{++}\phi_1 + g^{++}\psi_1 dx, \tag{1.11}$$

or

$$(LL)_1^{-} \quad \int_{\Omega} f^{--}\phi_1 + g^{--}\psi_1 dx > \int_{\Omega} h_1\phi_1 + h_2\psi_1 dx > \int_{\Omega} f^{++}\phi_1 + g^{++}\psi_1 dx, \tag{1.12}$$

where $\Phi_1 = (\phi_1, \psi_1)$ is the positive eigenfunction associated to the first positive eigenvalue for problem (1.6).

Similarly, we write the Landesman-Lazer conditions for $k > 1$. In that case, we denote $V(\lambda_k)$ the eigenspace associated to the eigenvalue $\lambda_k(A)$. Then, let $(u,v) \in V(\lambda_k)$ and define

$$\begin{aligned} L_k(u,v) := & \int_{u>0,v>0} f^{++}u + g^{++}v\,dx + \int_{u<0,v>0} f^{-+}u + g^{-+}v\,dx \\ & + \int_{u>0,v<0} f^{+-}u + g^{+-}v\,dx + \int_{u<0,v<0} f^{--}u + g^{--}v\,dx. \end{aligned} \tag{1.13}$$

So it will be assumed either

$$(LL)_k^+ \quad L_k(u,v) > \int_\Omega h_1 u + h_2 v\,dx, \quad \forall (u,v) \in V(\lambda_k) \setminus \{0\}, \tag{1.14}$$

or

$$(LL)_k^- \quad L_k(u,v) < \int_\Omega h_1 u + h_2 v\,dx, \quad \forall (u,v) \in V(\lambda_k) \setminus \{0\}. \tag{1.15}$$

Using these conditions, we will prove our main results. First, we consider the existence of solutions for the Problem (1.1). To do that, we prove that the functional J has an appropriate saddle point geometry given in [7] whenever $(LL)_k^+$ with $k \geq 1$ holds. So, we can prove the following result.

Theorem 1.1. *Suppose $h_1, h_2 \in L^2(\Omega)$, and $(B^\infty), (f^\infty), (g^\infty)$. In addition, suppose that $(LL)_k^+$ and $\lambda_k(A) = 1$ with $k \geq 1$ hold, then Problem (1.1) has at least one solution.*

Similarly, using the condition $(LL)_k^-$ instead of $(LL)_k^+$, we prove the following result:

Theorem 1.2. *Suppose $h_1, h_2 \in L^2(\Omega)$, and $(B^\infty), (f^\infty), (g^\infty)$. In addition, suppose that $(LL)_k^-$ and $\lambda_k(A) = 1$ with $k \geq 2$ hold, then Problem (1.1) has at least one solution.*

In the case that $k = 1$, using the condition $(LL)_1^-$ and the Ekeland's variational principle, we can prove the following result.

Theorem 1.3. *Suppose $h_1, h_2 \in L^2(\Omega)$, and $(B^\infty), (f^\infty), (g^\infty)$. In addition, suppose that $(LL)_1^-$ and $\lambda_1(A) = 1$ hold, then Problem (1.1) has at least one solution.*

Now, we assume that $\nabla F(x,0,0) \equiv 0$, $F(x,0,0) \equiv 0$ and $h_1 \equiv h_2 \equiv 0$ hold, then Problem (1.1) admits the trivial solution $(u,v) \equiv 0$. In this case, the main point is to ensure the existence of nontrivial solutions. The existence of these solutions depends mainly on the behavior of F at the origin and at infinity.

In this case, we make some assumptions at the origin. First, we define the function $\widetilde{F}(x,z) = F(x,z) + \langle A(x)z, z\rangle$, $(x,z) \in \Omega \times \mathbb{R}^2$. Then, we consider the following.

$(\widetilde{F}_0)$ There is $A_0 \in \mathcal{S}_2(\Omega)$ such that

$$\lim_{z \to 0} \frac{2\widetilde{F}(x,z) - \langle A_0(x)z, z\rangle}{|z|^2} = 0. \tag{1.16}$$

In fact, the function A_0 is the Hessian matrix at the origin in the variables u and v for each $x \in \Omega$ fixed. Under these assumptions, we consider the eigenvalue problem

$$-\Delta \begin{pmatrix} u \\ v \end{pmatrix} = \lambda A_0(x) \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{in } \Omega,$$
$$u = v = 0 \quad \text{on } \partial\Omega. \tag{1.17}$$

Thus, using the Spectral Theory for compact operators, we have a sequence of eigenvalues denoted by

$$0 < \lambda_1(A_0) < \lambda_2(A_0) \leq \lambda_3(A_0) \leq \cdots \tag{1.18}$$

such that $\lambda_m(A_0) \to +\infty$ as $m \to \infty$.

In the next result, we complement the statement of Theorem 1.1 by proving that the solution which was found in Theorem 1.1 is nonzero. Indeed, we prove the following multiplicity result.

Theorem 1.4. *Suppose that* $(B^\infty), (f^\infty), (g^\infty)$ *and* $\lambda_k(A) = 1, (LL)_k^+$ *with* $k \geq 1$ *hold. Assume also that* $(\widetilde{F}_0)$ *and* $\lambda_m(A_0) < 1 < \lambda_{m+1}(A_0)$ *hold for an integer number* $m \geq 1$ *such that* $m \neq k$, *then the solution given in Theorem 1.1 is nontrivial.*

For the next result, we will add further hypotheses on F'' and find other nontrivial solutions. Firstly, we consider the following definition.

Definition 1.5. Let $A, B \in \mathcal{S}_2(\Omega)$. We say the inequality $A \leq B$ holds when we have $\langle A(x)z, z\rangle \leq \langle B(x)z, z\rangle$, for all $(x, z) \in \Omega \times \mathbb{R}^2$. Moreover, we define $A \preceq B$, if $A \leq B$ and $B - A$ are positive definite on $\widetilde{\Omega} \subseteq \Omega$, where $|\widetilde{\Omega}| > 0$. Here, $|\ |$ denotes the Lebesgue measure.

Remark 1.6. Let $F \in C^2$ and $A, B \in \mathcal{S}_2(\Omega)$. Then the inequalities $A \leq F'' \leq B$ mean $\langle A(x)z, z\rangle \leq \langle F''(x)z, z\rangle \leq \langle B(x)z, z\rangle$ for all $(x, z) \in \Omega \times \mathbb{R}^2$. Here, F'' denotes the Hessian matrix of F in the variables u and v for each $x \in \Omega$ fixed.

In the next multiplicity result, we explore the Mountain Pass Theorem. More specifically, we find two mountain pass points which are different from the solution obtained by Theorem 1.1. In addition, we find all critical groups at infinity introduced in [8] using the Landesman-Lazer conditions. This last part is new complement and permits us to show the following result.

Theorem 1.7. *Suppose that* $(B^\infty), (f^\infty), (g^\infty), (LL)_k^+, (\widetilde{F}_0)$, *and* $\lambda_k(A) = 1$ *with* $k \geq 2$ *hold. In addition, suppose also that* $\lambda_1(A_0) > 1$ *and* $F'' \preceq (\delta - 1)A$ *for some* $\delta \in (0, \lambda_{k+1})$ *hold, then Problem* (1.1) *has at least four nontrivial solutions.*

We note that the Problem (1.1) has been studied by many authors in recent years since the appearance of the pioneering paper of Chang [3]. We refer the reader to [3, 5, 9–13] and references therein. In these works, the authors proved several results on existence and multiplicity for the problem (1.1). In [3], Chang considered the problem (1.1) with nonresonance conditions using Variational Methods and the Morse theory. In [9], Bartsch et al. obtained sign changing solutions under resonant conditions. More precisely, they considered the conditions of the Ahmad et al. type [14], in short $(ALP)_k^{\pm}$, written as follows:

$$(ALP)_k^{+} \quad \int_\Omega F(x,z)dx \longrightarrow \infty, \quad \text{as } \|z\| \longrightarrow \infty,\ z \in V(\lambda_k), \tag{1.19}$$

or

$$(ALP)_k^{-} \quad \int_\Omega F(x,z)dx \longrightarrow -\infty, \quad \text{as } \|z\| \longrightarrow \infty,\ z \in V(\lambda_k). \tag{1.20}$$

Recall that $V(\lambda_k)$ denotes the eigenspace associated to the eigenvalue $\lambda_k(A)$.

Remark 1.8. It is well known that the condition $(LL)_1^{\pm}$ implies $(ALP)_1^{\pm}$, respectively. However, the same property is not clear for higher eigenvalues, that is, it is not known that $(LL)_k^{\pm}$ implies the condition $(ALP)_k^{\pm}$ for $k \geq 2$, respectively.

In [10], Chang considered the problem (1.1) using Subsuper solutions and Degree Theory. In [5], Furtado and de Paiva used the nonquadraticity condition at infinity and the Morse theory.

In this paper, we explore the conditions of Landesman-Lazer type. These famous conditions imply interesting properties on geometry of J given by (1.2), see Propositions 3.2, 3.3, and 3.4. In addition, we calculate all the critical groups for a critical point $z_1 \in H$ given by a saddle theorem provided in [7]. Thus, we obtain further results on existence and multiplicity of solutions for problem (1.1) which complements the previous papers abovementioned.

In the proof of our main theorems, we study Problem (1.1) using Variational Methods, the Morse Theory, and some results related to the critical groups at an isolated critical point; see [8, 15].

The paper is organized as follows. In Section 2, we recall the abstract framework of problem (1.1) and highlight the properties for the eigenvalue problem (1.6). In Section 3, we prove some auxiliary results involving the Palais-Smale condition and some properties on the geometry for the functional J. In Section 4, we prove Theorems 1.1, 1.2, and 1.3. In Section 5, we prove Theorems 1.4 and 1.7. Section 6 is devoted to the proofs of further multiplicity results which are analogous to Theorems 1.4 and 1.7. However, in these theorems, we use the $(LL)_k^{-}$ condition instead of $(LL)_k^{+}$, where $k \geq 1$.

2. Abstract Framework and Eigenvalue Problem for the System (1.1)

Initially, we recall that $H = H_0^1(\Omega) \times H_0^1(\Omega)$ denotes the Hilbert space with the Dirichlet norm

$$\|z\|^2 = \int_\Omega |\nabla u|^2 + |\nabla v|^2 dx, \quad z = (u,v) \in H. \tag{2.1}$$

Moreover, we denote by $\langle , \rangle$ the scalar product in H which has given us the norm above.

Again, we recall the properties of the eigenvalue problem as follows:

$$-\Delta\begin{pmatrix}u\\v\end{pmatrix} = \lambda A(x)\begin{pmatrix}u\\v\end{pmatrix} \quad \text{in } \Omega, \qquad u = v = 0 \quad \text{on } \partial\Omega. \tag{2.2}$$

Let $A \in \mathcal{S}_2(\Omega)$, then there is a unique compact self-adjoint linear operator, which we denote by $T_A : H \longrightarrow H$ satisfying

$$\langle T_A z, w\rangle = \int_\Omega \langle A(x)z, w\rangle dx, \quad \forall z, w \in H. \tag{2.3}$$

This operator has the following propriety: λ is an nonzero eigenvalue of (2.2), if and only if $T_A z = (1/\lambda)z$ for some nonzero $z \in H$, that is, $1/\lambda$ is an eigenvalue for T_A.

So, for each matrix $A \in \mathcal{S}_2(\Omega)$, there exist a sequence of eigenvalues for problem (2.2) and a Hilbertian basis for H formed by eigenfunctions of (2.2). Let $\lambda_k(A)$ the eigenvalues of problem (2.2) and let $\Phi_k(A)$ be the associated eigenfunctions, we note that

$$0 < \lambda_1(A) < \lambda_2(A) \leq \cdots \leq \lambda_k(A) \longrightarrow \infty \quad \text{as } k \longrightarrow \infty. \tag{2.4}$$

We also note that

$$\frac{1}{\lambda_1(A)} = \sup\{\langle T_A z, z\rangle,\ \|z\| = 1,\ z \in H\}, \qquad \frac{1}{\lambda_k(A)} = \sup\left\{\langle T_A z, z\rangle,\ \|z\| = 1,\ z \in V_{k-1}^{\perp},\ k \geq 2\right\} \tag{2.5}$$

hold, where $V_{k-1} := \operatorname{span}\{\Phi_1(A), \ldots, \Phi_{k-1}(A)\}$. Thus, we have $H = V_k \oplus V_k^{\perp}$ for $k \geq 1$, and the following variational inequalities hold:

$$\|z\|^2 \geq \lambda_1(A)\langle T_A z, z\rangle, \quad \forall z \in H, \tag{2.6}$$

$$\|z\|^2 \leq \lambda_k(A)\langle T_A z, z\rangle, \quad \forall z \in V_k, \tag{2.7}$$

$$\|z\|^2 \geq \lambda_{k+1}(A)\langle T_A z, z\rangle, \quad \forall z \in V_k^{\perp}. \tag{2.8}$$

These inequalities will be used in the proof, our main theorems. We recall that the eigenvalue $\lambda_1(A)$ is positive and simple. Moreover, we have that the associated eigenfunction $\Phi_1(A)$ is positive in Ω. In other words, we have a Hess-Kato Theorem for eigenvalue problem (2.2) proved by Chang, see [3]. For more properties to the eigenvalue problem (2.2), see [4, 5, 10].

3. Preliminary Results

The critical groups in Morse Theory can be used to distinguish critical points and, hence, are very useful in critical point theory. Let J be a functional C^1 defined on a Hilbert Space H, then the critical groups of J at an isolated critical point u with $J(u) = c$ are given by

$$C_q(J,u) = H_q(J_c, J_c \setminus \{u\}, \mathcal{G}), \quad \forall q \in \mathbb{N}, \tag{3.1}$$

where H_q is the singular relative homology with coefficients in an Abelian Group $\mathcal{G}$ and $J_c = J^{-1}(-\infty, c]$, see [15].

We recall that $J : H \to \mathbb{R}$ is said to satisfy Palais-Smale condition at the level $c \in \mathbb{R}$ ($(\text{PS})_c$ in short), if any sequence $(z_n)_{n\in\mathbb{N}} \subseteq H$ such that

$$J(z_n) \longrightarrow c, \qquad J'(z_n) \longrightarrow 0 \tag{3.2}$$

as $n \to \infty$ possess a convergent subsequence in H. Moreover, we say that J satisfies (PS) condition when $(\text{PS})_c$ is satisfied for all $c \in \mathbb{R}$.

The critical groups at infinity are formally defined by

$$C_q(J,\infty) = H_q(H, J_c), \quad \forall q \in \mathbb{N}. \tag{3.3}$$

We observe that, by Excision Property, the critical groups at infinity are independent of $c \in \mathbb{R}$; see [8].

Now, we observe that condition (B^∞) implies the following growth condition: Let $\epsilon > 0$, then there exists $M_\epsilon > 0$ such that

$$|F(x,z)| \le \epsilon |z|^2, \quad \forall x \in \Omega \text{ whenever } |z| \ge M_\epsilon, \tag{3.4}$$

and, there exists $C_\epsilon > 0$ such that

$$|F(x,z)| \le C_\epsilon + \epsilon |z|^2, \quad \forall x \in \Omega, \; z \in \mathbb{R}^2, \tag{3.5}$$

where $|\cdot|$ denotes the Euclidian norm in $\mathbb{R}^2$. Moreover, since $h_1, h_2 \in L^2(\Omega)$, we use Cauchy-Schwartz's inequality obtaining the following estimate. For each $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$\left| \int_\Omega h_1 u + h_2 v \, dx \right| \le C_\epsilon + \epsilon \|z\|^2, \quad \forall z = (u,v) \in H. \tag{3.6}$$

In this way, we prove the following compactness result.

Proposition 3.1. *Suppose $(B^\infty), (f^\infty)$, and(g^∞). In addition, suppose $(LL)_k^+$ or $(LL)_k^-$ with $k \ge 1$, then the functional J satisfies the (PS) condition.*

Proof. Initially, we take $k = 1$. In this case, we have that $\lambda_1(A)$ is simple and it admits an eigenfunction $\Phi_1(A)$ with definite sign in Ω. For this reason, the proof in this case is standard. We will omit the details of the proof in this case.

Now, we consider the case $k > 1$. The proof of this case is by contradiction. We assume that there is a sequence $(z_n)_{n\in\mathbb{N}} \in H$ such that

(i) $J(z_n) \longrightarrow c$, where $c \in \mathbb{R}$,

(ii) $J'(z_n) \longrightarrow 0$,

(iii) $\|z_n\| \longrightarrow \infty$ as $n \longrightarrow \infty$.

Let us consider $\overline{z_n} = z_n/\|z_n\|$, then, we get $\|\overline{z_n}\| = 1$, and there exists $\overline{z} \in H$ such that

(i) $\overline{z_n} \rightharpoonup \overline{z}$ in H,

(ii) $\overline{z_n} \longrightarrow \overline{z}$ in $L^p(\Omega) \times L^p(\Omega)$ with $p \in [1, 2^*)$,

(iii) $\overline{z_n}(x) \longrightarrow \overline{z}(x)$ a.e. in Ω as $n \longrightarrow \infty$.

At the same time, given $\Phi = (\phi, \psi) \in H$, we get the following identity

$$\begin{aligned}\frac{J'(z_n)\Phi}{\|z_n\|} = {} & \int_\Omega \nabla\overline{u_n}\nabla\phi + \nabla\overline{v_n}\nabla\psi\, dx - \int_\Omega \langle A(x)(\overline{u_n}, \overline{v_n}), (\phi, \psi)\rangle dx \\ & - \int_\Omega \frac{\nabla F(x, u_n, v_n)(\phi, \psi)}{\|z_n\|} dx + \int_\Omega \frac{h_1\phi + h_2\psi}{\|z_n\|}, \quad \text{where } z_n = (u_n, v_n).\end{aligned} \tag{3.7}$$

In this way, using the last identity, we conclude that

$$\int_\Omega \nabla\overline{u}\nabla\phi + \nabla\overline{v}\nabla\psi\, dx - \int_\Omega \langle A(x)(\overline{u}, \overline{v}), (\phi, \psi)\rangle dx = 0, \quad \forall\Phi = (\phi, \psi) \in H, \tag{3.8}$$

where $\overline{z} = (\overline{u}, \overline{v})$. Choosing $\Phi = (\overline{u_n}, \overline{v_n})$ in (3.7) we obtain the following identity $\int_\Omega \langle A(x)(\overline{u}, \overline{v}), (\overline{u}, \overline{v})\rangle dx = 1$. Moreover, taking $\Phi = (\overline{u}, \overline{v})$ and using (3.8), we get that $\|\overline{z}\|^2 = \int_\Omega \langle A(x)(\overline{u}, \overline{v}), (\overline{u}, \overline{v})\rangle dx = 1$. Consequently, $\overline{z_n} \longrightarrow \overline{z}$ in H and $\overline{z}$ is an eigenfunction associated to the eigenvalue $\lambda_k(A) = 1$.

On the other hand, we define $A_n = ((1/2)J'(z_n)z_n - J(z_n))/\|z_n\|$. Then, $A_n \longrightarrow 0$ as $n \longrightarrow \infty$. More specifically, the (PS) sequence $(z_n)_{n\in\mathbb{N}}$ yields

$$A_n = \frac{1}{2}\int_\Omega 2\frac{F(x, u_n, v_n)}{\|z_n\|} - \nabla F(x, u_n, v_n)(\overline{u_n}, \overline{v_n})dx - \frac{1}{2}\int_\Omega h_1\overline{u_n} + h_2\overline{v_n}dx \longrightarrow 0, \tag{3.9}$$

as $n \longrightarrow \infty$.

Now, we study the limits of the three terms in (3.9). First, we get

$$\int_\Omega h_1\overline{u_n} + h_2\overline{v_n}dx \longrightarrow \int_\Omega h_1\overline{u} + h_2\overline{v}\, dx \quad \text{as } n \longrightarrow \infty. \tag{3.10}$$

In addition, using the functions in (1.9) and (1.10), we obtain

$$\begin{aligned}
\nabla F(x,u_n,v_n) &\longrightarrow \left(f_1^{++}(x), f_2^{++}(x)\right), \quad \text{if } x \in \{y \in \Omega : \overline{u}(y) > 0, \overline{v}(y) > 0\},\\
\nabla F(x,u_n,v_n) &\longrightarrow \left(f_1^{+-}(x), f_2^{+-}(x)\right), \quad \text{if } x \in \{y \in \Omega : \overline{u}(y) > 0, \overline{v}(y) < 0\},\\
\nabla F(x,u_n,v_n) &\longrightarrow \left(f_1^{-+}(x), f_2^{-+}(x)\right), \quad \text{if } x \in \{y \in \Omega : \overline{u}(y) < 0, \overline{v}(y) > 0\},\\
\nabla F(x,u_n,v_n) &\longrightarrow \left(f_1^{--}(x), f_2^{+-}(x)\right), \quad \text{if } x \in \{y \in \Omega : \overline{u}(y) < 0, \overline{v}(y) < 0\},
\end{aligned} \tag{3.11}$$

as $n \to \infty$.

We point out that $\Omega_0 = \{x \in \Omega : \overline{z}(x) = (\overline{u}(x), \overline{v}(x)) = 0\}$ has zero Lebesgue measure. Indeed, the eigenfunctions associated to the eigenvalue problem (1.6) enjoy the Strong Unique Continuation Property, in short (SUCP). For this property, we refer the reader to [16–21]. More specifically, for each solution $\overline{z}$ of (1.6) which is zero on $E \subset \Omega$ with positive Lebesgue measure, we obtain a zero of infinite order for some $x_\star \in \Omega$ similar to the scalar case. This property implies that $\overline{z}$ is zero in some neighborhood of $x_\star$; see Theorem 2.1 in [21]. In this way, the function $\overline{z} \equiv 0$ in Ω which is not an eigenfunction for (1.6). In other words, the eigenfunctions associated to (1.6) are not zero for any subset of Ω with positive Lebesgue measure.

Let $\widetilde{\Omega} \subset \Omega$ be such that $\overline{u} = 0$ and $\overline{v} \neq 0$ in $\widetilde{\Omega}$, then $\widetilde{\Omega}$ has zero Lebesgue measure. The proof of this claim is by contradiction. Suppose that $\widetilde{\Omega}$ has positive measure and recall that $\overline{z} = (\overline{u}, \overline{v})$ is an eigenfunction associated to $\lambda_k(A) = 1,\ k > 1$, thus, the problem (1.6) implies that

$$b(x)\overline{v} = 0 \quad \text{in } \widetilde{\Omega}. \tag{3.12}$$

Therefore, using the fact that $\overline{v} \neq 0$ in $\widetilde{\Omega}$, we obtain

$$b(x) = 0 \quad \text{in } \widetilde{\Omega}. \tag{3.13}$$

In that case, $\widetilde{\Omega} \subset \Omega_b$ and using the hypothesis (M_1), we have a contradiction. Summarizing, for all subsets $\widetilde{\Omega} \subset \Omega$ such that $\overline{u} = 0$ and $\overline{v} \neq 0$ in $\widetilde{\Omega}$ has zero Lebesgue measure. Analogously, the subsets of Ω, where $\overline{u} \neq 0$ and $\overline{v} = 0$, satisfy the same property.

Hence, we have

$$\begin{aligned}
\int_\Omega \nabla F(x,u_n,v_n)\overline{z_n}dx \longrightarrow & \int_{\overline{u}>0\ \overline{v}>0} (f_1^{++}, f_2^{++})\overline{z}\,dx + \int_{\overline{u}>0\ \overline{v}<0} (f_1^{+-}, f_2^{+-})\overline{z}\,dx\\
&+ \int_{\overline{u}<0\ \overline{v}>0} (f_1^{-+}, f_2^{-+})\overline{z}\,dx + \int_{\overline{u}<0\ \overline{v}<0} (f_1^{--}, f_2^{--})\overline{z}\,dx
\end{aligned} \tag{3.14}$$

as $n \to \infty$.

Finally, using L' Hospital's rule and $(LL)_k^+$ or $(LL)_k^-$, we also have

$$\int_\Omega \frac{F(x,u_n,v_n)}{\|z_n\|}dx \longrightarrow \int_{\overline{u}>0\ \overline{v}>0}(f_1^{++},f_2^{++})\overline{z}\,dx + \int_{\overline{u}>0\ \overline{v}<0}(f_1^{+-},f_2^{+-})\overline{z}\,dx$$
$$+\int_{\overline{u}<0\ \overline{v}>0}(f_1^{-+},f_2^{-+})\overline{z}\,dx + \int_{\overline{u}<0\ \overline{v}<0}(f_1^{--},f_2^{--})\overline{z}\,dx \tag{3.15}$$

as $n \to \infty$. Therefore (3.9), (3.10), (3.14), and (3.15) imply that

$$\int_{\overline{u}>0\ \overline{v}>0}(f_1^{++},f_2^{++})\overline{z}\,dx + \int_{\overline{u}>0\ \overline{v}<0}(f_1^{+-},f_2^{+-})\overline{z}\,dx$$
$$+\int_{\overline{u}<0\ \overline{v}>0}(f_1^{\mp},f_2^{\mp})\overline{z}\,dx + \int_{\overline{u}<0\ \overline{v}<0}(f_1^{--},f_2^{--})\overline{z}\,dx = \int_\Omega (h_1,h_2)\overline{z}\,dx. \tag{3.16}$$

However, $\overline{z} = (\overline{u},\overline{v})$ is an eigenfunction associated to the eigenvalue $\lambda_k(A) = 1$ with $\|\overline{z}\| = 1$. So, we have a contradiction with the conditions $(LL)_k^+$ or $(LL)_k^-$. Therefore all the $(\mathrm{PS})_c$ sequence is bounded. Then, by standard arguments, we conclude that all $(\mathrm{PS})_c$ sequence has a convergent subsequence. This statement finishes the proof of this proposition. □

Next, we prove some properties involving the geometry of the functional J. More specifically, we prove that the functional J has at least one of the following geometries: a special saddle geometry, mountain pass geometry, or a linking at the origin. First, we prove the following result.

Proposition 3.2. *Suppose* $(B^\infty), (f^\infty), (g^\infty)$, *and* $(LL)_k^+$ *with* $k \geq 1$, *then the functional* J *has the following saddle geometry:*

(a) $J(z) \to \infty$, *if* $\|z\| \to \infty$ *with* $z = (u,v) \in V_k^\perp$,

(b) *there is* $\alpha \in \mathbb{R}$ *such that* $J(z) \leq \alpha$, *for all* $z \in V_k$.

Proof. Initially, we check the proof of item (a). Let $z = (u,v) \in V_k^\perp$, then we have the following estimates:

$$\begin{aligned} J(z) &= \frac{1}{2}\|z\|^2 - \frac{1}{2}\int_\Omega \langle A(x)z,z\rangle dx - \int_\Omega F(x,z)dx + \int_\Omega (h_1u+h_2v)dx \\ &\geq \frac{1}{2}\left(1-\frac{1}{\lambda_{k+1}}\right)\|z\|^2 - \int_\Omega F(x,z)dx + \int_\Omega h_1u+h_2v\,dx \\ &\geq \frac{1}{2}\left(1-\frac{1}{\lambda_{k+1}}\right)\|z\|^2 - \frac{1}{2}\epsilon\|z\|^2 - C_\epsilon \longrightarrow \infty \quad \text{se } \|z\| \longrightarrow \infty \text{ com } z \in V_k^\perp, \end{aligned} \tag{3.17}$$

where we used (2.8), (3.5), (3.6), and Sobolev's embedding. So, the proof of item (a) is now complete.

Now, we prove the item (b). The proof in this case is by contradiction. We suppose that there exists a sequence $(z_n)_{n\in\mathbb{N}} \in V_k$ such that

(i) $J(z_n) > n$, for all $n \in \mathbb{N}$,

(ii) $\|z_n\| \to \infty$ as $n \to \infty$.

So, this information must lead us to a contradiction.

Firstly, we write $z_n = z_n^0 + w_n \in V_k = V(\lambda_k) \oplus V_{k-1}$, with $z_n^0 \in V(\lambda_k)$ = eigenspace associated to eigenvalue $\lambda_k(A) = 1$ and $w_n \in V_{k-1}$. Consequently, for each $\epsilon > 0$, we obtain

$$\begin{aligned} J(z_n) &= \frac{1}{2}\|w_n\|^2 - \frac{1}{2}\langle T_A w_n, w_n\rangle - \int_\Omega F\left(x, z_n^0 + w_n\right)dx + \int_\Omega (h_1, h_2)\left(z_n^0 + w_n\right)dx \\ &\le \frac{1}{2}\left(1 - \frac{1}{\lambda_{k-1}}\right)\|w_n\|^2 - \int_\Omega F\left(x, z_n^0 + w_n\right)dx + \int_\Omega (h_1, h_2)\left(z_n^0 + w_n\right)dx \\ &\le \frac{1}{2}\left(1 - \frac{1}{\lambda_{k-1}}\right)\|w_n\|^2 + \frac{1}{2}\epsilon\left\|z_n^0 + w_n\right\|^2 + C_\epsilon, \end{aligned} \tag{3.18}$$

where we use (2.7) and the growths conditions (3.5) and (3.6). Now, we show the following claim.

Claim 1. We have that $\|z_n^0\| \to \infty$ as $n \to \infty$.

The proof of this claim is by contradiction. In this case, assuming that $\|z_n^0\|$ is bounded. Thus, we obtain that $\|w_n\| \to \infty$ as $n \to \infty$. In this case, using the estimate (3.18) we conclude that $J(z_n) \to -\infty$. Therefore, we have a contradiction because we have $J(z_n) > n$ by construction. Consequently the proof of Claim 1 it follows.

Now, we define $s_n = z_n^0/\|z_n^0\| = (u_n^0/\|z_n\|, v_n^0/\|z_n\|) \in V(\lambda_k)$. In this way, there exists $s_0 \in V(\lambda_k)$ satisfying $\|s_0\| = 1$ such that

(i) $s_n \to s_0$ in $V(\lambda_k)$,

(ii) $s_n \to s_0$ in $L^p(\Omega)^2$,

(iii) $s_n(x) \to s_0(x)$ a.e. in Ω.

Thus, for each $\epsilon > 0$, using (3.5) and (3.6), we obtain

$$\begin{aligned} J(z_n) &\le \frac{1}{2}\left(1 - \frac{1}{\lambda_{k-1}}\right)\|w_n\|^2 - \int_\Omega F\left(x, z_n^0 + w_n\right)dx - \int_\Omega (h_1, h_2)\left(z_n^0 + w_n\right)dx \\ &\le \frac{1}{2}\left(1 - \frac{1}{\lambda_{k-1}} + \epsilon\right)\|w_n\|^2 + \epsilon\left\|z_n^0\right\|^2 + C_\epsilon. \end{aligned} \tag{3.19}$$

Defining $L = \lim_{n\to\infty}(\|w_n\|/\|z_n^0\|)$, we will consider the following cases:

(1) $L = \infty$,

(2) $L \in (0, \infty)$,

(3) $L = 0$.

We will obtain a contradiction in the cases (1), (2), or (3). Initially, we consider case (1). In this case, for all $M > 0$, there exists $n_0 \in \mathbb{N}$ such that $\|w_n\| \geq M\|z_n^0\|$, for $n \geq n_0$. The last inequality shows that

$$\begin{aligned} J(z_n) &\leq \frac{1}{2}\left(1 - \frac{1}{\lambda_{k-1}} + \epsilon\right)\|w_n\|^2 + \epsilon\left\|z_n^0\right\|^2 + C_\epsilon \\ &\leq \frac{M^2}{2}\left(1 - \frac{1}{\lambda_{k-1}} + \epsilon\right)\left\|z_n^0\right\|^2 + \epsilon\left\|z_n^0\right\|^2 + C_\epsilon \longrightarrow -\infty \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.20}$$

Again we have a contradiction because $J(z_n) > n$ by construction. Therefore, case (1) does not occur.

Now, we consider case (2). In this case, for each $\epsilon > 0$ small enough there exists $n_0 \in \mathbb{N}$ such that $0 < L - \epsilon < \|w_n\|/\|z_n^0\| < L + \epsilon$ whenever $n \geq n_0$. In this way, we have the following inequalities:

$$\begin{aligned} J(z_n) &\leq \frac{1}{2}\left(1 - \frac{1}{\lambda_{k-1}} + \epsilon\right)\|w_n\|^2 + \epsilon\left\|z_n^0\right\|^2 + C_\epsilon \\ &\leq \frac{1}{2}\left(1 - \frac{1}{\lambda_{k-1}} + \epsilon\right)(L-\epsilon)^2\left\|z_n^0\right\|^2 + \epsilon\left\|z_n^0\right\|^2 + C_\epsilon \longrightarrow -\infty \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.21}$$

Again, we have a contradiction and case (2) does not occur too.

Finally, we consider case (3). In this case, using the Landesman-Lazer conditions, we obtain the following identity:

$$\lim_{n\to\infty}\int_\Omega \frac{F(x, z_n^0 + w_n)}{\|z_n^0\|}dx = \lim_{n\to\infty}\int_\Omega \frac{F(x, z_n^0)}{\|z_n^0\|}dx. \tag{3.22}$$

Moreover, by L'Hospital's rule, we get following inequality:

$$\lim_{n\to\infty}\int_\Omega \frac{F(x, z_n^0)}{\|z_n^0\|}dx = \lim_{n\to\infty}\int_\Omega \nabla F\left(x, z_n^0\right)s_n dx > \int_\Omega (h_1, h_2)s_0 dx, \tag{3.23}$$

where $s_n = z_n^0/\|z_n^0\| = (u_n^0/\|z_n\|, v_n^0/\|z_n\|) \in V(\lambda_k)$. Thus, for each $\epsilon > 0$ small, there is $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies that

$$\int_\Omega \frac{F(x, z_n^0 + w_n)}{\|z_n^0\|}dx > \int_\Omega (h_1, h_2)s_0 dx + 2\epsilon, \tag{3.24}$$

where we use $(LL)_k^+$, (3.22), and (3.23). Now, using (3.24), we get the following estimates:

$$\begin{aligned}
J(z_n) &\le \frac{1}{2}\left(1-\frac{1}{\lambda_{k-1}}\right)\|w_n\|^2-\int_\Omega F\left(x,z_n^0+w_n\right)dx+\int_\Omega (h_1,h_2)\left(z_n^0+w_n\right)dx \\
&\le \frac{1}{2}\left(1-\frac{1}{\lambda_{k-1}}\right)\|w_n\|^2-\left\|z_n^0\right\|\left(\int_\Omega (h_1,h_2)s_0+2\epsilon\right) \\
&\quad +\left\|z_n^0\right\|\int_\Omega (h_1,h_2)s_n dx+\int_\Omega (h_1,h_2)w_n dx \\
&\le \frac{1}{2}\left(1-\frac{1}{\lambda_{k-1}}+\epsilon\right)\|w_n\|^2-\left\|z_n^0\right\|\left(\int_\Omega (h_1,h_2)s_0+2\epsilon\right) \\
&\quad +\left\|z_n^0\right\|\int_\Omega (h_1,h_2)s_n dx+C_\epsilon \\
&\le \frac{1}{2}\left(1-\frac{1}{\lambda_{k-1}}+\epsilon\right)\|w_n\|^2-\left\|z_n^0\right\|\epsilon\le -\epsilon\left\|z_n^0\right\|,
\end{aligned} \tag{3.25}$$

where $\epsilon > 0$ is small enough. Therefore, we have $J(z_n) \to -\infty$ as $n \to \infty$. Again, we get a contradiction because $J(z_n) > n$ by construction. Consequently, there is a constant $\alpha \in \mathbb{R}$ such that $J(z) \le \alpha$, for all $z = (u, v) \in V_k$. This statement finishes the proof of this proposition. □

Now, we have an analogous geometry for J using the $(LL)_k^-$ condition instead of $(LL)_k^-$, where $k \ge 2$. In this case, we can prove the following result.

Proposition 3.3. *Suppose $(B^\infty), (f^\infty), (g^\infty)$, and $(LL)_k^-$ with $k \ge 2$, then the functional J has the following saddle geometry:*

(a) *$J(z) \to -\infty$, if $\|z\| \to \infty$ with $z = (u, v) \in V_{k-1}$,*

(b) *there is $\beta \in \mathbb{R}$ such that $J(z) \ge \beta$, for all $z \in V_{k-1}^\perp$.*

Proof. The proof of this result is similar to the proof of Proposition 3.2. Thus, we will omit the proof of this proposition. □

Finally, using the $(LL)_1^-$ condition, we will prove the following result.

Proposition 3.4. *Suppose $(B^\infty), (f^\infty), (g^\infty)$, and $(LL)_1^-$, then the functional J is coercive, that is, we have that $J(z) \to \infty$ as $\|z\| \to \infty$ for all $z \in H$.*

Proof. First, we must show that $J(z) \to \infty$, if $\|z\| \to \infty$. Suppose, by contradiction, that this information is false. Thus, there is a sequence $(z_n)_{n\in\mathbb{N}} \in H$ such that

(i) $J(z_n) \le C$, for all $n \in \mathbb{N}$,

(ii) $\|z_n\| \to \infty$ as $n \to \infty$.

However, the sequence (z_n) has the following form $z_n = t_n\Phi_1 + w_n$ where $(t_n)_{n\in\mathbb{N}} \in \mathbb{R}$ and $(w_n)_{n\in\mathbb{N}} \in V_1^{\perp}$. Hence, we obtain

$$\begin{aligned} J(z_n) &= \|w_n\|^2 - \int_\Omega \langle A(x)w_n, w_n\rangle dx - \int_\Omega F(x,z_n)dx + \int_\Omega (h_1,h_2)z_n dx \\ &\geq \frac{1}{2}\left(1-\frac{1}{\lambda_2}\right)\|w_n\|^2 - \int_\Omega F(x,t_n\Phi_1 + w_n)dx + \int_\Omega (h_1,h_2)(t_n\Phi_1 + w_n)dx \\ &\geq \frac{1}{2}\left(1-\frac{1}{\lambda_2}-\epsilon\right)\|w_n\|^2 - \frac{\epsilon}{2}\|t_n\Phi_1\|^2 - C_\epsilon, \end{aligned} \tag{3.26}$$

where we use Sobolev's embedding, (2.8), (3.5), and (3.6). In this way, we have the following claim.

Claim 2. $|t_n| \to \infty$ *as* $n \to \infty$.

The proof of this claim is similar to the proof of the Claim 1. We will omit the proof of this claim.

Now, we define $L = \lim_{n\to\infty}(\|w_n\|/|t_n|)$. In this case, using the same ideas developed in Proposition 3.2, it is easy to see that $L = 0$. Thus, we obtain the following information:

$$\lim_{n\to\infty}\int_\Omega \frac{F(x,t_n\Phi_1 + w_n)}{t_n}dx = \lim_{n\to\infty}\int_\Omega \nabla F(x,t_n\Phi_1)\Phi_1 dx < \int_\Omega (h_1,h_2)\Phi_1 dx, \tag{3.27}$$

where we are assuming that $t_n \to \infty$ and we use the condition $(LL)_1^-$. The case where $t_n \to -\infty$ is similar. Therefore, given $\epsilon > 0$ small, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies that

$$\int_\Omega \frac{F(x,t_n\Phi_1 + w_n)}{t_n}dx < \int_\Omega (h_1,h_2)\Phi_1 dx - \epsilon. \tag{3.28}$$

Now, using the estimates (3.26) and (3.28), we obtain

$$\begin{aligned} J(z_n) &\geq \frac{1}{2}\left(1-\frac{1}{\lambda_2}\right)\|w_n\|^2 - t_n\int_\Omega \frac{F(x,z_n)}{t_n}dx + \int_\Omega (h_1,h_2)z_n dx \\ &\geq \frac{1}{2}\left(1-\frac{1}{\lambda_2}-\epsilon\right)\|w_n\|^2 - t_n\left(\int_\Omega (h_1,h_2)\Phi_1 dx - \epsilon\right) \\ &\quad + t_n\int_\Omega (h_1,h_2)\Phi_1 dx - C_\epsilon = \frac{1}{2}\left(1-\frac{1}{\lambda_2}-\epsilon\right)\|w_n\|^2 + t_n\epsilon - C_\epsilon \geq t_n\epsilon - C_\epsilon, \end{aligned} \tag{3.29}$$

where $\epsilon > 0$ is small enough. In the above estimates, we use the growth condition (3.6). Consequently, $J(z_n) \to \infty$ and $n \to \infty$, and we have a contradiction. Therefore, the functional J is coercive. This affirmation finishes the proof of this proposition. □

Now, we prove some two auxiliary results related to the mountain pass geometry for the functional J. First, we prove the following result.

Proposition 3.5. *Suppose* $(B^\infty), (f^\infty),$ *and* (g^∞). *In addition, suppose that* $(\widetilde{F}_0)$ *and* $\lambda_1(A_0) > 1$, *then the origin is a local minimum for the functional* J.

Proof. First, using $(\widetilde{F}_0)$, we chose $p \in (2, 2^*)$ and a constant $C_\epsilon > 0$ for all $\epsilon > 0$ such that

$$\widetilde{F}(x,z) \le \frac{\epsilon}{2}|z|^2 + \frac{1}{2}\langle A(x)z, z\rangle + C_\epsilon |z|^p, \quad \forall (x,z) \in \Omega \times \mathbb{R}^2. \tag{3.30}$$

We recall that $\widetilde{F}(x,z) = (1/2)\langle A(x)z, z\rangle + F(x,z)$, $(x,z) \in \Omega \times \mathbb{R}^2$. So, we obtain

$$\begin{aligned} J(z) &= \frac{1}{2}\|z\|^2 - \frac{1}{2}\langle A(x)z, z\rangle - \int_\Omega F(x,z)dx = \frac{1}{2}\|z\|^2 - \int_\Omega \widetilde{F}(x,z)dx \\ &\ge \frac{1}{2}\left(1 - \frac{1}{\lambda_1(A_0)} + \epsilon C\right)\|z\|^2 - C_\epsilon \int_\Omega |z|^p dx \\ &\ge \frac{1}{2}\left(1 - \frac{1}{\lambda_1(A_0)} + \epsilon C\right)\|z\|^2 - C_\epsilon \|z\|^p \ge \frac{1}{4}\left(1 - \frac{1}{\lambda_1(A_0)} + \epsilon C\right)\|z\|^2 > 0, \end{aligned} \tag{3.31}$$

where $z \in B_\rho(0) \setminus \{0\}$ and $0 < \rho \le \rho_0$ with ρ_0 small enough. Here, $B_\rho(0)$ denotes the open ball in H centered at the origin with radius ρ. Therefore, the proof of this propositions is now complete. □

The next result, whose proof is standard and will be omitted, completes the mountain pass geometry for functional J.

Proposition 3.6. *Suppose* $(B^\infty), (f^\infty),$ *and* (g^∞). *In addition, suppose* $(LL)_k^+$ *or* $(LL)_k^-$ *with* $k \ge 2$. *Then* $J(t\Phi_1) \to -\infty$ *as* $|t| \to \infty$.

Now, we compute the critical groups at infinity. Initially, we need to prove an auxiliary result given by the following proposition.

Proposition 3.7. *Suppose* $(B^\infty), (f^\infty),$ *and* (g^∞). *In addition, suppose* $(LL)_k^+$ *or* $(LL)_k^-$ *holds with* $k \ge 1$. *Let* $R > 0$ *and* $\epsilon \in (0,1)$ *and define* $C(R,\epsilon) = \{z \in H : z = z^- + z^0 + z^+ \in H = V_{k-1} \oplus V(\lambda_k) \oplus V_k^\perp, \|z\| \ge R,$ *and* $\|z^+ + z^-\| \le \epsilon\|z\|\}$. *Then we have the following alternatives.*

(a) $(LL)_k^+$ *implies that there are constants* $R > 0, \epsilon \in (0,1),$ *and* $\delta > 0$ *such that*

$$\left\langle J'(z), z^0 \right\rangle \le -\delta, \quad \forall z \in C(R,\epsilon). \tag{3.32}$$

(b) $(LL)_k^-$ *implies that there are constants* $R > 0, \epsilon \in (0,1),$ *and* $\delta > 0$ *such that*

$$\left\langle J'(z), z^0 \right\rangle \ge \delta, \quad \forall z \in C(R,\epsilon). \tag{3.33}$$

Proof. First, we prove case (a), where $k \ge 2$. The proof of this proposition when $k = 1$ is similar. Let us assume, by contradiction, that for any $\epsilon = \delta = 1/n$, there exists a sequence

$(z_n)_{n\in\mathbb{N}}$ written as $z_n = z_n^- + z_n^0 + z_n^+ \in H = V_{k-1} \oplus V(\lambda_k) \oplus V_k^\perp$ such that $\|z_n\| \geq n$, $\|z_n^+ + z_n^-\| \leq (1/n)\|z_n\|$ and

$$\left\langle J(z_n), z_n^0 \right\rangle > -\frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{3.34}$$

Therefore, we have

$$-\frac{1}{n} < \left\langle J'(z_n), z_n^0 \right\rangle = \left\langle (I - T_A) z_n, z_n^0 \right\rangle - \int_\Omega \nabla F(x, z_n) z_n^0 dx \leq -\int_\Omega \nabla F(x, z_n) z_n^0 dx. \tag{3.35}$$

So we see that

$$\limsup_{n\to\infty} \int_\Omega \frac{\nabla F(x, z_n) z_n^0}{\|z_n\|} dx \leq 0. \tag{3.36}$$

On the other hand, by Hölder's inequality and Sobolev's embedding, we show that

$$\begin{aligned} &\left| \int_\Omega \frac{\nabla F(x, z_n)(z_n^+ + z_n^-)}{\|z_n\|} dx \right| \\ &\quad \leq \int_\Omega \left| \frac{\nabla F(x, z_n)(z_n^+ + z_n^-)}{\|z_n\|} \right| dx \leq C \int_\Omega |h| \frac{|z_n^+ + z_n^-|}{\|z_n\|} dx \\ &\quad \leq C\|h\|_{L^2(\Omega)} \frac{\|z_n^+ + z_n^-\|}{\|z_n\|} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.37}$$

Then, we get

$$\limsup_{n\to\infty} \int_\Omega \frac{\nabla F(x, z_n) z_n}{\|z_n\|} dx = \limsup_{n\to\infty} \int_\Omega \frac{\nabla F(x, z_n) z_n^0}{\|z_n\|} dx \leq 0. \tag{3.38}$$

Next, we define $\overline{z_n} = z_n / \|z_n\|$. Thus, there is $\overline{z} \in H$ such that

(i) $\overline{z_n} \rightharpoonup \overline{z}$ in H,

(ii) $\overline{z_n} \to \overline{z}$ in $L^p(\Omega) \times L^p(\Omega)$ with $p \in [1, 2^*)$,

(iii) $\overline{z_n}(x) \to \overline{z}(x)$ a.e. in Ω as $n \to \infty$.

We recall that $(z_n^- + z_n^+)/\|z_n\| \to 0$ as $n \to \infty$. Thus, we obtain that $\overline{z} \in V(\lambda_k)$ and $\|\overline{z}\| = 1$. In other words, we have that $\overline{z}$ is an eigenfunction associated to the eigenvalue $\lambda_k(A) = 1$. In conclusion, using the condition $(LL)_k^+$, we obtain

$$\liminf_{n\to\infty} \int_\Omega \frac{\nabla F(x, z_n) z_n}{\|z_n\|} dx = L_k(\overline{u}, \overline{v}) > 0, \tag{3.39}$$

where $\overline{z} = (\overline{u}, \overline{v})$ and $L_k(\overline{u}, \overline{v})$ is provided in (1.13). Therefore, we obtain a contradiction with (3.38). Finally, there are $R > 0$ large enough and $\epsilon \in (0,1)$ such that

$\langle J'(z), z_0 \rangle \leq -\delta$, for all $z \in C(R, \epsilon)$ for some $\delta > 0$. The proof of case (b) is similar to the previous one, and we will be omit the details in this case. □

Now, using the previous proposition, we compute the critical groups at infinity. More specifically, we can prove the following result.

Proposition 3.8. *Suppose* $(B^\infty), (f^\infty)$, *and* (g^∞). *In addition, suppose* $(LL)_k^+$ *or* $(LL)_k^-$ *holds with* $k \geq 1$, *then we have the following alternatives.*

(a) $(LL)_k^+$ *implies that* $C_q(J, \infty) = \delta_{qk}\mathbb{Z}$, *for all* $q \in \mathbb{N}$.

(b) $(LL)_k^-$ *implies that* $C_q(J, \infty) = \delta_{q(k-1)}\mathbb{Z}$, *for all* $q \in \mathbb{N}$.

Proof. The proof of this result follows using Proposition 3.7. More specifically, we have the well-known angle conditions at infinity introduced by Bartsch and Li in [8]. □

Next, we prove a result involving the behavior of J at the origin. This result is important because it implies that the functional J has a local linking at the origin. More precisely, we can prove the following result.

Proposition 3.9. *Suppose* $(B^\infty), (f^\infty), (g^\infty)$ *and* $(\widetilde{F}_0)$. *In addition, suppose that* $\lambda_m(A_0) < 1 < \lambda_{m+1}(A_0)$ *holds with* $m \geq 1$, *then the functional* J *has a linking at the origin. Moreover, we have* $C_q(J, 0) = \delta_{qm}\mathbb{Z}$, *for all* $q \in \mathbb{N}$.

Proof. First, we take $V_1 = \bigoplus_{j \geq m+1} V(\lambda_j), V_2 = \bigoplus_{j=1}^m V(\lambda_j)$. Let $R > 0$, we will show that J satisfies the following properties:

(i) $J(z) > 0$, for all $0 < \|z\| \leq R$, $z \in V_1$,

(ii) $J(z) \leq 0$, for all $\|z\| \leq R$, $z \in V_2$.

Initially, we prove the item (i). Thus, by $(\widetilde{F}_0)$, taking $p \in (2, 2^*)$, we obtain

$$\widetilde{F}(x, z) \leq \frac{1}{2}\langle A_0(x)z, z\rangle + \frac{\epsilon}{2}|z|^2 + C|z|^p, \quad \forall (x, z) \in \Omega \times \mathbb{R}^2. \tag{3.40}$$

Let $z \in V_1$, $0 < \|z\| \leq \delta_1$, with $\delta_1 > 0$ small enough, then (3.40) yields

$$\begin{aligned} J(z) &\geq \frac{1}{2}\|z\|^2 - \frac{1}{2}\langle T_{A_0}z, z\rangle - \frac{\epsilon}{2}|z|^2 - C\|z\|^p \\ &= \frac{1}{2}\left(1 - \frac{1}{\lambda_{m+1}(A_0)} - \epsilon C\right)\|z\|^2 - C\|z\|^p > 0. \end{aligned} \tag{3.41}$$

The estimates above finish the proof of item (i).

Now, we prove the item (ii). We recall that the norms $\| \|_\infty, \| \|$ are equivalents on V_2. Here, $\| \|_\infty$ denotes the usual norm in $L^\infty(\Omega)^2$. Thus, given $\epsilon > 0$, there are $r > 0$ and $\delta_2 > 0$ such that $\|z\|_\infty \leq r$ implies that

$$\|z\| \leq \delta_2, \quad z \in V_2. \tag{3.42}$$

Consequently, by $(\widetilde{F}_0)$, we have

$$\widetilde{F}(x,z) \geq \frac{1}{2}\langle A(x)z,z\rangle - \frac{\epsilon}{2}|z|^2, \quad \forall \|z\| \leq \delta_2,\ z \in V_2. \tag{3.43}$$

So, using (3.43) and (2.7), for $\delta_2 > 0$ small enough, we obtain

$$\begin{aligned} J(z) &= \frac{1}{2}\|z\|^2 - \int_\Omega \widetilde{F}(x,z)dx \leq \frac{1}{2}\|z\|^2 - \frac{1}{2}\langle T_{A_0}z,z\rangle + \frac{\epsilon}{2}\int_\Omega |z|^2 dx \\ &= \frac{1}{2}\left(1 - \frac{1}{\lambda_m(A_0)}\right)\|z\|^2 + \frac{\epsilon C}{2}\|z\|^2 \leq 0, \quad \forall \|z\| \leq \delta_2,\ z \in V_2. \end{aligned} \tag{3.44}$$

The estimates just above conclude the proof of item (ii). Therefore, choosing $R = \min\{\delta_1,\delta_2\} > 0$, J has a local linking at the origin; see [22]. In this case, we obtain that $C_m(J,0) \neq 0$ with $m \geq 1$. In addition, using the inequalities $\lambda_m(A_0) < 1 < \lambda_{m+1}(A_0)$, we have that the Morse index at the origin is m and the nullity at the origin is zero. In particular, we obtain $C_q(J,0) = \delta_{qm}\mathbb{Z}$, for all $q \in \mathbb{N}$; see [23]. So the proof of this proposition is now complete. □

4. Proof of Theorems 1.1, 1.2, and 1.3

First, we prove Theorem 1.1. Initially, we have the (PS) condition given by Proposition 3.1. In addition, taking $H = V_k \oplus V_k^{\perp}$, where $V_k = \operatorname{span}\{\Phi_1,\ldots,\Phi_k\}$ with $k \geq 1$, Proposition 3.2 shows that J has saddle point geometry given by Theorem 1.8 in [7]. Therefore, we have a critical point for J, and problem (1.1) admits at least one solution. This statement concludes the proof of Theorem 1.1.

Next, we prove Theorem 1.2. Similarly, we have (PS) condition given by Proposition 3.1. Thus, we write $H = V_{k-1} \oplus V_{k-1}^{\perp}$, where $V_{k-1} = \operatorname{span}\{\Phi_1,\ldots,\Phi_{k-1}\}$ and $k \geq 2$. Again, we have the saddle point geometry required in Theorem 1.8 in [7], see Proposition 3.3. Therefore, we have a critical point, and problem (1.1) has at least one solution. So, the proof of Theorem 1.2 is complete.

Now, we prove Theorem 1.3. In this case the functional J is coercive, see Proposition 3.4. Therefore, using Ekeland's Variational Principle, we have a critical point $z_\star \in H$ such that $J(z_\star) = \inf\{J(z) : z \in H\}$, and Problem (1.1) has at least one solution. This statement finishes the proof of Theorem 1.3.

5. Proof of Theorems 1.4 and 1.7

First, we prove Theorem 1.4. Initially, we have one critical point $z_1 \in H$ given by Theorem 1.1 such that $C_k(J,z_1) \neq 0$, see [15]. Moreover, by Proposition 3.9, we obtain a local linking at origin. So $C_m(J,0) = \delta_{qm}\mathbb{Z}$, for all $q \in \mathbb{N}$ using that the origin is a nondegenerate critical point for J. Consequently, we have that $z_1 \neq 0$ because $m \neq k$. Thus problem (1.1) admits at least one nontrivial solution and the proof of Theorem 1.4 is now complete.

Now, we prove Theorem 1.7. Firstly, we have one critical point z_1 given by Theorem 1.1. Moreover, we have $C_k(J,z_1) \neq 0$ and $k \in [m(z_1), m(z_1)+n(z_1)]$, where $m(z_1)$ and $n(z_1)$ denote the Morse index and the nullity at the critical point z_1, respectively. But, using the inequality $F'' \preceq (\delta - 1)A$, we conclude that $m(z_1) + n(z_1) = k$. Consequently, by Shifting Theorem [15], we have that $C_q(J, z_1) = \delta_{qk}\mathbb{Z}$, for all $q \in \mathbb{N}$. Moreover, using Proposition 3.8, the critical groups at infinity are $C_q(J, \infty) = \delta_{qk}\mathbb{Z}$, for all $q \in \mathbb{N}$.

On the other hand, using Propositions 3.5 and 3.6, we have the mountain pass geometry for the functional J. Let P be the cone of positive functions in $C_0^1(\Omega)^2$, then, we consider the functional $J^{\pm}$ obtained from J by restriction on P and $-P$, respectively. In this case, we have two mountain pass points $z_2^{\pm} \in \pm P$ such that $C_q(J, z_2^{\pm}) = \delta_{q1}\mathbb{Z}$, for all $q \in \mathbb{N}$; see [3, 9]. Now, we assume, by contradiction, that J admits only 0, z_1 and $z_2^{\pm}$ as critical points. Then, Morse's identity implies that

$$(-1)^0 + 2(-1)^1 + (-1)^k = (-1)^k. \tag{5.1}$$

Therefore, we have a contradiction and there is another critical point for J which is different from 0, z_1, and $z_2^{\pm}$. Hence, the problem (1.1) admits at least four nontrivial solutions. This statement completes the proof of Theorem 1.7.

6. Further Multiplicity Results

In this section, we state and prove further multiplicity results using the condition $(LL)_k^-$ instead of $(LL)_k^+$, where $k \geq 1$. These results complement the theorems enunciated in the introduction and they have a similar proof. However, in this case, the functional J has a geometry different from to the geometry described in Section 5. Thus, we enunciate and prove the following multiplicity results.

Theorem 6.1. *Suppose that* $(B^\infty), (f^\infty), (g^\infty), (LL)_k^-$, *and* $\lambda_k(A) = 1$ *with* $k \geq 3$ *hold. In addition, suppose that* $(\widetilde{F}_0)$ *and* $\lambda_m(A_0) < 1 < \lambda_{m+1}(A_0)$ *hold with* $m \neq k - 1$, *then Problem* (1.1) *has at least one nontrivial solution. Moreover, when* $k = 1$ *and* $m \geq 1$ *and* $(\widetilde{F}_0)$ *and* $\lambda_m(A_0) < 1 < \lambda_{m+1}(A_0)$ *hold, then Problem* (1.1) *has at least two nontrivial solutions.*

Theorem 6.2. *Suppose that* $(B^\infty), (f^\infty), (g^\infty), (\widetilde{F}_0), (LL)_k^-$ *and* $\lambda_k(A) = 1$ *hold with* $k \geq 3$. *In addition, suppose that* $\lambda_1(A_0) > 1$ *and* $F'' \succeq (\delta - 1)A$ *for some* $\delta \in (\lambda_{k-1}, \infty)$ *holds, then Problem* (1.1) *has at least four nontrivial solutions.*

Finally, we use only the condition $(LL)_1^-$. In this case, the functional J is coercive. So, we will prove the following multiplicity result.

Theorem 6.3. *Suppose that* $(B^\infty), (f^\infty), (g^\infty), (LL)_1^-$ *and* $\lambda_1(A) = 1$ *hold. In addition, suppose that* $(\widetilde{F}_0)$ *and* $\lambda_m(A_0) < 1 < \lambda_{m+1}(A_0)$ *hold, where* $m \geq 1$ *is even, then Problem* (1.1) *has at least three nontrivial solutions.*

Now, we check the theorems stated in this section. Initially, we prove Theorem 6.1. In this case, we have a critical point $z_1 \in H$ given by Theorem 1.2 such that $C_{k-1}(J, z_1) \neq 0$ whenever $k \geq 3$. In addition, we have a local linking at origin given by Proposition 3.9. Then, we obtain $C_q(J, 0) = \delta_{qm}\mathbb{Z}$, for all $q \in \mathbb{N}$, where zero is a nondegenerate critical point and $m \neq k - 1$, see Proposition 3.9. Consequently, $z_1 \neq 0$, and Problem (1.1) has at least one nontrivial solution.

On the other hand, if $k = 1$, we use Theorem 1.3. Therefore, we obtain one critical point $z_\star \in H$ such that $C_q(J, z_\star) = \delta_{q0}\mathbb{Z}$, for all $q \in \mathbb{N}$. Moreover, using Proposition 3.9 we get $C_m(J,0) \neq 0$, with $m \geq 1$. Thus, applying the Three Critical Points Theorem [22], problem (1.1) has at least two nontrivial solutions and the proof of Theorem 6.1 is now complete.

Now, we prove Theorem 6.2. First, we have one critical point $z_1 \in H$ given by Theorem 1.2. Thus, we get $C_{k-1}(J,z_1) \neq 0$ and $k-1 \in [m(z_1), m(z_1)+n(z_1)]$ where $m(z_1)$ and $n(z_1)$ denote the Morse index and the nullity at the critical point z_1, respectively. But the inequality $F'' \geq (\delta - 1)A$ shows that $m(z_1) = k-1$. Consequently, by Shifting Theorem [15], we have $C_q(J,z_1) = \delta_{q(k-1)}\mathbb{Z}$, for all $q \in \mathbb{N}$. Moreover, using Proposition 3.8, we have $C_q(J,\infty) = \delta_{q(k-1)}\mathbb{Z}$, for all $q \in \mathbb{N}$.

On the other hand, we have the mountain pass geometry given by Propositions 3.5 and 3.6. Therefore, we have two mountain pass points $z_2^{\pm} \in \pm P$ such that $C_q(J,z_2^{\pm}) = \delta_{q1}\mathbb{Z}$, for all $q \in \mathbb{N}$. Moreover, we have $C_q(J,0) = \delta_{q0}\mathbb{Z}$, for all $q \in \mathbb{N}$. Then, applying the Morse's identity, we conclude that problem (1.1) has at least four nontrivial solutions. So we finish the proof of Theorem 6.2.

Finally, we prove Theorem 6.3. In this case, the functional J is bounded below and we have two critical points $z_0^{\pm}$ given by minimization. Indeed, it is sufficient to minimize the functional J on P and $-P$. Thus, we have that $C_q(J,z_0^{\pm}) = \delta_{q0}\mathbb{Z}$, for all $q \in \mathbb{N}$. Here, we assume that m is even. Moreover, by Proposition 3.9, we conclude that $C_q(J,0) = \delta_{qm}\mathbb{Z}$, for all $q \in \mathbb{N}$ where zero is a nondegenerate critical point. In addition, the critical groups at infinity are $C_q(J,\infty) = \delta_{q0}\mathbb{Z}$, for all $q \in \mathbb{N}$. Again, using Morse's identity, we conclude that the functional J has another critical point and problem (1.1) has at least three nontrivial solutions. This statement finishes the proof of this theorem.

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