PARAMETER ESTIMATION FOR STOCHASTIC PROCESSES

BY

T. S. PITCHER

Lincoln Laboratory (1), *Massachusets Institute of Technology*

1. Introduction

A stochastic process $[x(t), t \in I]$, or x for short, has associated with it a probability measure P_x defined on suitable subsets of the space of sample functions on I. The problems of determining when measures P_x and P_y associated with processes x and y are mutually absolutely continuous and of computing the Radon-Nikodym derivative dP_x/dP_y have been much investigated in recent years. In particular, a necessary and sufficient criterion has been given in case x and y are Gaussian for determining the mutual absolute continuity of P_x and P_y [3]. If we take I to be an interval and x and y to have zero means and correlation functions $R_x(s,t)$ and $R_y(s,t)$ whose associated integral operators on $L_2(dt, I)$ are compact, then the criterion is that $R_x^{-\frac{1}{2}} R_y R_x^{-\frac{1}{2}} - I$ have an extension to a Hilbert–Schmidt operator and under these circumstances dP_x/dP_y can be expressed in terms of the eigenfunctions and eigenvalues of this operator. In parameter estimation, however, where whole families (P_α) of measures must be considered, results of this type (which tend to involve separate calculations for each pair α_1 and α_2) often involve prohibitive amounts of calculation and also obscure the role played by the parameter itself.

In [8] we attacked this problem under the assumption that the processes x_α were gotten from each other by the application of a one-parameter group T_α of transformations acting on the sample functions of the process. Specifically, we assumed given an algebra F of bounded random variables on which T_α operated as a group of automorphisms (intuitively $(T_\alpha f)(x) = f(T_\alpha x)$) such that the derivative $DT_\alpha f(x) = \partial T_\alpha f(x)/\partial\alpha$ existed and was uniformly bounded in α and x. It was shown there that the existence of a random variable φ satisfying $\int \varphi f dP_x = \int Df dP_x$ for all f in F implied the existence of a strongly continuous one-parameter group $[V(\alpha) \mid \alpha \geqslant 0]$ of contractions on $L_1(P_x)$

(1) Operated with support from the U. S. Army, Navy, and Air Force.

given for f in F by $(V(\alpha)f)(x) = Q_\alpha(x)(T_{-\alpha}f)(x)$ and that, under further assumptions, the P_{x_α} were mutually absolutely continuous and $Q_\alpha = dP_{x_\alpha}/dP_x$.

The above setup is not restricted to Gaussian processes and is sufficiently general to handle, for example, the mean value problem, $(T_\alpha x)(t) = x(t) + \alpha m(t)$. The requirement that $DT_\alpha f(x)$ be bounded, however, rules out many other cases of interest (1) and section 2 of this paper is devoted to replacing it with the requirement that $DT_\alpha f$ be continuous in $L_1(P_x)$ and $O(e^{K|\alpha|})$ in $L_1(P_x)$ norm. This is not, strictly speaking, less restrictive than the previous set of requirements but seems to be much more practical in applications. All the examples used in [8] and [9] will be easily seen to apply to the new situation.

Section 3 carries over some results of [8] and all the results of [9] to this new context and ends with two new theorems expressing the effect of an inequality of the form

$$\int_{[x \mid |\varphi(x)| \geqslant N]} |\varphi| \, dP \leqslant Ce^{-\varepsilon N}$$

on the distribution of $\log(dP_\alpha/dP)$ and on the amount of information in P_α about P.

The results of sections 2 and 3 are applied in section 4 to the Gaussian case and section 5 consists of Gaussian examples. Section 5 as a whole is intended to show the wide range of parameter estimation problems which are associated with groups of transformations on the sample functions, but it is hoped that some of the examples (especially numbers 2 and 5) may be of interest in applications and that at least example 4 will be of interest in its own right.

2. The Semigroups $V_+(\alpha)$ and $V_-(\alpha)$

Let P be a probability measure defined on a σ-algebra S of subsets of a set X, F an algebra of bounded S-measurable functions dense in $L_1(P)$ and containing the constant functions, and T_α a one-parameter group of automorphisms of F which preserve bounds. We shall make the following assumptions throughout this section:

(A 1) For every f in F,

$$\lim_{\varepsilon \to 0} \frac{T_\varepsilon f - f}{\varepsilon} = Df$$

exists in $L_1(P)$, $DT_\alpha f$ is continuous and $\|DT_\alpha f\|_1 = O(e^{K|\alpha|})$ for some K independent of f,

(1) Example 1 of [8] does not satisfy this requirement and should not have been included there. It appears here as example 1 of section 5.

and

(A 2) There is a φ in $L_1(P)$ satisfying $\int \varphi f dP = \int Df dP$ for every f in F.

Throughout this section lim will mean limit in $L_1(P)$ norm unless otherwise specified and $\|f\|_q$ will mean the $L_q(P)$ norm of f. We note that $\bar{F}$, the uniform closure of F, contains $f \wedge g = \min(f, g)$ and $f \vee g = \max(f, g)$ whenever it contains f and g and that, since

$$-\sup_{x \in X} |f_n(x) - f_m(x)| \leqslant T_\alpha f_n - T_\alpha f_m \leqslant \sup_{x \in X} |f_n(x) - f_m(x)|,$$

$(T_\alpha f_n)$ is a uniformly convergent sequence whenever (f_n) is, from which it follows that T_α can be extended to $\bar{F}$ by setting $T_\alpha(\lim f_n) = \lim T_\alpha(f_n)$.

LEMMA 2.1. *D has an extension (which we also call D) to a domain Δ of bounded functions satisfying*

(i) *$\int \varphi f dP = \int Df dP$ for all f in Δ,*

(ii) *If f is in F and g is in Δ, then fg is in Δ and $D(fg) = fDg + gDf$,*

(iii) *If (f_n) is a sequence from Δ converging boundedly almost everywhere to some f, and if Df_n is $L_1(P)$ convergent to g, then f is in Δ and $Df = g$.*

Proof. If f and g are in F, then

$$D(fg) = \lim_{\varepsilon \to 0} \frac{(T_\varepsilon f)(T_\varepsilon g) - fg}{\varepsilon} = \lim_{\varepsilon \to 0} \left[\frac{T_\varepsilon f - f}{\varepsilon}(T_\varepsilon g - g) + f \frac{T_\varepsilon g - g}{\varepsilon} + g \frac{T_\varepsilon f - f}{\varepsilon} \right],$$

and

$$\left\| \frac{T_\varepsilon f - f}{\varepsilon}(T_\varepsilon g - g) \right\|_1 \leqslant \left\| \left(\frac{T_\varepsilon f - f}{\varepsilon} - Df \right)(T_\varepsilon g - g) \right\|_1 + \|(Df)(T_\varepsilon g - g)\|_1$$
$$\leqslant 2\|g\|_\infty \left\| \frac{T_\varepsilon f - f}{\varepsilon} - Df \right\|_1 + \int |Df| \, |T_\varepsilon g - g| \, dP.$$

The first term in the inequality goes to 0 as ε goes to 0 while for some subsequence ε_j, chosen so that $T_{\varepsilon_j} g$ converges to g almost everywhere, the second term goes to 0 as j goes to ∞ by the dominated convergence theorem. Thus $D(fg) = fDg + gDf$. Now consider the set of domains Δ, which contain only bounded functions, and onto which D can be extended so as to satisfy (i) and (ii) partially ordered by inclusion. If $\Delta_1 \subset \Delta_2$ and D_1 and D_2 are the corresponding extensions of D, then, for any f in F and g in Δ_1, $\int f D_1 g dP = \int \varphi f g dP - \int g Df dP = \int f D_2 g dP$ and since we can find a sequence from F to converge boundedly and almost everywhere to any bounded measurable function, this implies that $D_1 g = D_2 g$, i.e., that D_2 is an extension of D_1. Thus the union of a linearly ordered set of such domains is again a domain onto which D can be properly extended so, by Zorn's lemma, there is a maximal such domain Δ. If (f_n) is a sequence from Δ converging boundedly almost everywhere to 0 and Df_n is $L_1(P)$ convergent to g, then for any h in F,

$$\int hg\,dP = \lim \int hDf_n\,dP = \lim\left(\int \varphi f_n h\,dP - \int f_n Dh\,dP\right) = 0$$

by dominated convergence so $g=0$. Thus D can be extended to the set Δ' of all g which are bounded, almost everywhere limits of sequences (g_n) from Δ such that Dg_n is $L_1(P)$ convergent. For such g_n and g it is clear that (fg_n), which is in Δ by (ii), converges boundedly almost everywhere to fg and $D(fg_n) = fDg_n + g_n Df$ converges in $L_1(P)$ to $f(\lim Dg_n) + gDf$ so that (ii) holds for the extension of D to Δ'. Since, as is easily seen, (i) also holds for this extension, we must have $\Delta = \Delta'$ so that Δ satisfies all the requirements of the lemma.

Since $T_{-\beta} f$ is $L_1(P)$ continuous, $\int_0^\alpha T_{-\beta} f\,d\beta$ exists as an $L_1(P)$ integral for every $\alpha \geqslant 0$ and has $L_1(P)$ derivative equal to $T_{-\alpha} f$. $\int_0^\alpha DT_{-\beta} f\,d\beta$ also exists as an $L_1(P)$ integral and has $L_1(P)$ derivative equal to $DT_{-\alpha} f$, from which it follows that $\int_0^\alpha DT_{-\beta} f\,d\beta = f - T_{-\alpha} f$. For f and g in F and $\alpha \geqslant 0$ we define

$$V_f(\alpha)(g) = \exp\left(\int_0^\alpha T_{-\beta} f\,d\beta\right) T_{-\alpha} g.$$

LEMMA 2.2. *$\int_0^\alpha T_{-\beta} f\,d\beta$ is in Δ and $D\int_0^\alpha T_{-\beta} f\,d\beta = \int_0^\alpha DT_{-\beta} f\,d\beta = f - T_{-\alpha} f$. $V_f(\alpha)(g)$ is in Δ and $D(V_f(\alpha)(g)) = (f - T_{-\alpha} f)\, V_f(\alpha)(g) + (V_f(\alpha)(1))\, DT_{-\alpha} g$.*

Proof. For any f in F we can find numbers ν_i^n, β_i^n, and N_n for which $\sum_{i=1}^{N_n} \nu_i^n T_{-\beta_i^n} f$ converges boundedly almost everywhere to $\int_0^\alpha T_{-\beta} f\,d\beta$ and $\sum_{i=1}^{N_n} \nu_i^n DT_{-\beta_i^n} f$ converges in $L_1(P)$ to $\int_0^\alpha DT_{-\beta} f\,d\beta$ as n goes to ∞. Thus $\int_0^\alpha T_{-\beta} f\,d\beta$ is in Δ and $D\int_0^\alpha T_{-\beta} f\,d\beta = \int_0^\alpha DT_{-\beta} f\,d\beta$ which proves the first assertion. A straightforward induction argument shows that $(\int_0^\alpha T_{-\beta} f\,d\beta)^n$ is in Δ and that $D(\int_0^\alpha T_{-\beta} f\,d\beta)^n = n(\int_0^\alpha T_{-\beta} f\,d\beta)^{n-1}(f - T_{-\alpha} f)$. Finally,

$$\sum_{n=0}^{N} \frac{1}{n!}\left(\int_0^\alpha T_{-\beta} f\,d\beta\right)^n T_{-\alpha} g$$

converges boundedly almost everywhere to $V_f(\alpha)(g)$ and

$$D\left(\sum_{n=0}^{N}\frac{1}{n!}\left(\int_0^\alpha T_{-\beta}f\,d\beta\right)^n T_{-\alpha}g\right)$$

converges in $L_1(P)$ to $(f-T_{-\alpha}f)\,V_f(\alpha)\,(g)+V_f(\alpha)\,(1)\,DT_{-\alpha}g$ from which the last assertion follows.

LEMMA 2.3. *$V_f(\alpha)\,(g)$ has $L_1(P)$ derivative $T_{-\alpha}fV_f(\alpha)\,(g)-V_f(\alpha)\,(1)\,DT_{-\alpha}g$, and*

$$\frac{\partial}{\partial\alpha}\int V_f(\alpha)\,(g)\;dP=\int(f-\varphi)\,V_f(\alpha)\,(g)\,dP.$$

Proof.

$$\frac{V_f(\alpha+\varepsilon)\,(g)-V_f(\alpha)\,(g)}{\varepsilon}$$

$$=\exp\left(\int_0^\alpha T_{-\beta}f\,d\beta\right)\left\{\frac{\exp\left(\int_\alpha^{\alpha+\varepsilon}T_{-\beta}f\,d\beta\right)-1}{\varepsilon}\,(T_{-\alpha-\varepsilon}g-T_{-\alpha}g)\right.$$

$$\left.+\frac{\exp\left(\int_\alpha^{\alpha+\varepsilon}T_{-\beta}f\,d\beta\right)-1}{\varepsilon}\,T_{-\alpha}g+\frac{T_{-\alpha-\varepsilon}g-T_{-\alpha}g}{\varepsilon}\right\}.$$

The first term in the brackets is dominated by

$$2\,\|g\|_\infty\left|\frac{\exp\left(\int_\alpha^{\alpha+\varepsilon}T_{-\beta}f\,d\beta\right)-1}{\varepsilon}-T_{-\alpha}f\right|+\|f\|_\infty\,|T_{-\alpha-\varepsilon}g-T_{-\alpha}g|$$

which goes to 0, the second term differs from $T_{-\alpha}fT_{-\alpha}g$ by less than

$$\|g\|_\infty\frac{1}{\varepsilon}\int_\alpha^{\alpha+\varepsilon}|T_{-\alpha-\gamma}f-T_{-\alpha}f|\,d\gamma+\|g\|_\infty\frac{1}{\varepsilon}\,(e^{\varepsilon\|f\|_\infty}-\varepsilon\,\|f\|_\infty-1)$$

which goes to 0, and the third term goes to $-DT_{-\alpha}g$ so the first assertion is proved. We have, by Lemma 2.2,

$$\int(f-\varphi)\,V_f(\alpha)\,(g)\,dP=\int[fV_f(\alpha)\,(g)-D(V_f(\alpha)\,(g))]\,dP$$

$$=\int[(T_{-\alpha}f)\,V_f(\alpha)\,(g)-V_f(\alpha)\,(1)\,DT_{-\alpha}g]\,dP$$

and by the above argument this is

$$\frac{\partial}{\partial\alpha}\int V_f(\alpha)\,g\,dP.$$

LEMMA 2.4. *If* (f_n) *is a sequence from* F *converging in* $L_1(P)$ *to* $\varphi \wedge N$ *and* (f_n) *is bounded above, then* $V_{f_n}(\alpha)(g)$ *converges in* $L_1(P)$ *to a limit* $V_N(\alpha)(g)$. *The limit is independent of the sequence used. The* $V_N(\alpha)$ *have unique extensions to positivity preserving contractions on* $L_1(P)$ *which satisfy* $V_N(\alpha)(fg) = (V_N(\alpha)(g))T_{-\alpha}f$ *for all* f *in* F *and* g *in* $L_1(P)$, *and* $\|V_N(\alpha)(g)\|_\infty \leqslant e^{\alpha N}\|g\|_\infty$ *for all bounded* g. $V_N(0) = I$ *and the* $V_N(\alpha)$ *are strongly continuous in* α.

Proof. The proof is exactly the same as the proof of the corresponding parts of Lemma 2.2 of [8] except for the relation involving L_∞ norms. This relation is easily established for g in F and then can be extended to all bounded g by an approximation argument.

LEMMA 2.5. $V_N(\alpha)$ *is a strongly continuous semigroup whose generator* A_N *contains* Δ *in its domain and is defined there by:*

$$A_N f = (\varphi \wedge N) f - Df.$$

Proof. By using Riemann approximations to the integrals involved we can show that

$$V_N(\alpha)\left(g\int_\alpha^\beta T_{-\gamma} f\,d\gamma\right) = V_N(\alpha)(g)\int_\alpha^{\alpha+\beta} T_{-\gamma} f\,d\gamma$$

for any bounded g. Repeating this argument we get, for g in F,

$$V_N(\alpha)\left(\left(\int_0^\beta T_{-\gamma} f\,d\gamma\right)^n T_{-\beta}g\right) = V_N(\alpha)(1)\left(\int_\alpha^{\alpha+\beta} T_{-\gamma} f\,d\gamma\right)^n T_{-\alpha-\beta}g$$

and hence
$$V_N(\alpha)\,V_f(\beta)(g) = V_N(\alpha)(1)\exp\left(\int_\alpha^{\alpha+\beta} T_{-\gamma} f\,d\gamma\right) T_{-\alpha-\beta}g.$$

If (f_n) is a sequence from F converging to $\varphi \wedge N$ and if $f_n \leqslant 2N$ for all n, we have

$$\begin{aligned}
&\|V_N(\alpha)(V_N(\beta)(g)) - V_N(\alpha+\beta)(g)\| \\
&= \lim_{n\to\infty}\|V_N(\alpha)(V_{f_n}(\beta)(g)) - V_{f_n}(\alpha+\beta)(g)\| \\
&= \lim_{n\to\infty}\left\| V_N(\alpha)(1)\exp\left(\int_\alpha^{\alpha+\beta} T_{-\gamma} f_n\,d\gamma\right) T_{-\alpha-\beta}g - V_{f_n}(\alpha+\beta)(g)\right\| \\
&\leqslant \limsup_{n\to\infty}\|g\|_\infty e^{2\beta N}\left\| V_N(\alpha)(1) - \exp\left(\int_0^\alpha T_{-\gamma} f_n\,d\gamma\right)\right\| = 0.
\end{aligned}$$

Again by using a straightforward Riemann approximation argument we can show that if f and g are in F and $\lambda > \|f\|_\infty + K$ then $\int_0^\infty e^{-\lambda\alpha} V_f(\alpha)(g)\, d\alpha$ is in Δ and

$$D\left(\int_0^\infty e^{-\lambda\alpha} V_f(\alpha)(g)\, d\alpha\right) = \int_0^\infty e^{-\lambda\alpha}\{(f - T_{-\alpha} f)\, V_f(\alpha)(g) + V_f(\alpha)(1)\, DT_{-\alpha} g\}\, d\alpha.$$

It is easy to verify that $e^{-\lambda\alpha} V_f(\alpha)(g)$ has $L_1(P)$ derivative

$$\begin{aligned} -\lambda e^{-\lambda\alpha} V_f(\alpha)(g) + e^{-\lambda\alpha} \frac{\partial}{\partial\alpha} V_f(\alpha)(g) \\ = -\lambda e^{-\lambda\alpha} V_f(\alpha)(g) + e^{-\lambda\alpha}(T_{-\alpha} f V_f(\alpha)(g) - V_f(\alpha)(1)\, DT_{-\alpha} g) \end{aligned}$$

and, since this is $L_1(P)$ continuous and integrable, that

$$\int_0^\infty \frac{\partial}{\partial\alpha}(e^{-\lambda\alpha} V_f(\alpha)(g))\, d\alpha = \lim_{n\to\infty} \int_0^n \frac{\partial}{\partial\alpha}(e^{-\lambda\alpha} V_f(\alpha)(g))\, d\alpha = \lim_{n\to\infty}(e^{-n\lambda} V_f(n)(g) - g) = -g.$$

Thus

$$\begin{aligned} (\lambda - f + D) \int_0^\infty e^{-\lambda\alpha} V_f(\alpha)(g)\, d\alpha \\ = \int_0^\infty \{\lambda e^{-\lambda\alpha} V_f(\alpha)(g) - e^{-\lambda\alpha} T_{-\alpha} f V_f(\alpha)(g) + e^{-\lambda\alpha} V_f(\alpha)(1)\, DT_{-\alpha} g\}\, d\alpha \\ = -\int_0^\infty \frac{\partial}{\partial\alpha}(e^{-\lambda\alpha} V_f(\alpha)(g))\, d\alpha = g. \end{aligned}$$

Now choosing a sequence (f_n) from F converging to $\varphi \wedge N$ and bounded above by $2N$ and taking $\lambda > 2N + K$, we have $\int_0^\infty e^{-\lambda\alpha} V_{f_n}(\alpha)(g)\, d\alpha$ uniformly bounded and

$$\left\| \int_0^\infty e^{-\lambda\alpha} V_{f_n}(\alpha)(g)\, d\alpha - \int_0^\infty e^{-\lambda\alpha} V_N(\alpha)(g)\, d\alpha \right\|_1 \leqslant \int_0^\infty e^{-\lambda\alpha} \| V_{f_n}(\alpha)(g) - V_N(\alpha)(g)\|_1\, d\alpha$$

which goes to 0 since the integrand is dominated by $e^{2N\alpha}\|g\|_\infty$ and goes to 0 everywhere. Hence there is some subsequence (which we also call (f_n)) for which

$$\int_0^\infty e^{-\lambda\alpha} V_{f_n}(\alpha)(g)\, d\alpha$$

converges boundedly almost everywhere to

$$\int_0^\infty e^{-\lambda\alpha} V_N(\alpha)(g)\, d\alpha$$

and it is easily seen that

$$D\left(\int_0^\infty e^{-\lambda\alpha} V_{f_n}(\alpha)(g)\,d\alpha\right)$$

converges to
$$g + (\varphi_N - \lambda)\int_0^\infty e^{-\lambda\alpha} V_N(\alpha)(g)\,d\alpha$$

in $L_1(P)$. Thus
$$\int_0^\infty e^{-\lambda\alpha} V_N(\alpha)(g)\,d\alpha$$

is in Δ and
$$(\lambda - (\varphi_N - D))\int_0^\infty e^{-\lambda\alpha} V_N(\alpha)(g)\,d\alpha = g.$$

It follows now for every g in $L_1(P)$ by a simple continuity argument that

$$\int_0^\infty e^{-\lambda\alpha} V_N(\alpha)(g)\,d\alpha$$

is in the domain of the closure $\bar{B}_N$ of the operator B_N defined on Δ by $B_N(f) = (\varphi \wedge N)f - Df$ and that

$$(\lambda - \bar{B}_N)\int_0^\infty e^{-\lambda\alpha} V_N(\alpha)(g)\,d\alpha = g$$

for all $\lambda > 2N + K$. The lemma follows from this [2; Cor. 16, p. 627].

THEOREM 2.1. *For any $\alpha \geqslant 0$, $V_N(\alpha)$ converges strongly to a limit $V(\alpha)$. The $V(\alpha)$ form a strongly continuous semigroup satisfying*

(1) $\|V(\alpha)\| \leqslant 1$,

(2) $V(\alpha)(fg) = V(\alpha)(f)T_{-\alpha}g$ *if g is in $\bar{F}$*,

(3) $V(\alpha)$ *preserves positivity*,

and

(4) *the generator A of $[V(\alpha)\,|\,\alpha \geqslant 0]$ contains Δ in its domain and is defined there by the equation $Af = \varphi f - Df$.*

Proof. The proof is exactly the same as the proof of Theorem 2.1 of [8] except for the size of the domain of A. It will be sufficient to show that

$$V(\alpha)(f) = f + \int_0^\alpha V(\beta)(Af)\,d\beta$$

for f in Δ since then we will have

$$\lim_{\varepsilon\to 0} \frac{V(\varepsilon)(f) - f}{\varepsilon} = \lim_{\varepsilon\to 0}\int_0^\varepsilon V(\beta)(Af)\,d\beta = Af.$$

However,

$$\begin{aligned}
&\left\| V(\alpha)(f) - f - \int_0^\alpha V(\beta)(Af)\, d\beta \right\|_1 \\
&\quad = \lim_{N\to\infty} \left\| V_N(\alpha)(f) - f - \int_0^\alpha V(\beta)(A)\, d\beta \right\|_1 \\
&\quad = \lim_{N\to\infty} \left\| \int_0^\alpha (V(\beta)(Af) - V_N(\beta)(A_N f))\, d\beta \right\|_1 \\
&\quad \leqslant \limsup_{N\to\infty} \left\{ \int_0^\alpha \| V(\beta)(Af) - V_N(\beta)(Af)\|_1 d\beta + \int_0^\alpha \| V_N(\beta)(Af - A_N f)\|_1 d\beta \right\},
\end{aligned}$$

and the first integrand is dominated by $2\|Af\|_1$ and goes to 0 everywhere while the second integral is dominated by $\int_0^\alpha \|Af - A_N f\|_1 d\beta = \alpha \|Af - A_N f\|_1$ which goes to 0.

We can also construct the 'backward' semigroups $[V_N(-\alpha) \mid \alpha \geqslant 0]$ and $[V(-\alpha) \mid \alpha \geqslant 0]$ (called $V_-(\alpha)$ in [8]) by replacing T_α, D, and φ by $T_{-\alpha}$, $-D$, and $-\varphi$. With this extended definition of $V(\alpha)$; (1), (2), and (3) of Theorem 2.1 are now satisfied for all α and (4) is supplemented by:

(4′) *the genenerator of* $[V(-\alpha) \mid \alpha \geqslant 0]$ *contains the operator* $-A$ *defined on* Δ *by* $-Af = -\varphi f + Df$.

Examples given in [8] show that $V(-\alpha)$ need not be $[V(\alpha)]^{-1}$ and that, in fact, $V(\alpha)$ may not have an inverse.

THEOREM 2.2. *$V(-\alpha)(V(\alpha)(f))(x) = e_\alpha(x) f(x)$ for all α where $e_\alpha = V(-\alpha)(V(\alpha)(1))$. e_α is L_1 continuous, nondecreasing for $\alpha \leqslant 0$ and nonincreasing for $\alpha \geqslant 0$, $0 \leqslant e_\alpha \leqslant e_0 = 1$. For $\alpha \geqslant 0$,*

$$\int_0^\alpha V(-\beta)([(\varphi \wedge N) - \varphi]\, V_N(\beta)(1))\, d\beta \textit{ increases to } e_\alpha - 1 \textit{ and}$$

$$\int_0^\alpha V(\beta)([\varphi - (\varphi \vee -N)]\, V_N(-\beta)(1))\, d\beta \textit{ increases to } e_{-\alpha} - 1 \textit{ as } N \textit{ goes to } \infty.$$

If $e_\alpha = 1$ for some $\alpha \neq 0$, then $V(\alpha)$ is a group.

Proof. By (2) of Theorem 2.1, if f is in F, then

$$V(-\alpha)(V(\alpha)(f)) = V(-\alpha)(V(\alpha)(1)\, T_{-\alpha} f) = V(-\alpha)(V(\alpha)(1)) f$$

from which the first assertion follows. Clearly, $e_0 = 1$ and $e_\alpha \geqslant 0$ and since $\|e_\alpha f\| \leqslant \|f\|$, we also have $e_\alpha \leqslant 1$. Assume now that $\alpha \geqslant 0$. From Lemma 2.3, $V_f(\alpha)(1)$ is in Δ and this coupled with Lemma 2.2 shows that $V(-\alpha)(V_f(\alpha)(1))$ has an $L_1(P)$ derivative and

$$\frac{\partial}{\partial\alpha} V(-\alpha)(V_f(\alpha)(1)) = V(-\alpha)(A(V_f(\alpha)(1))) + V(-\alpha)(T_{-\alpha} f V_f(\alpha)(1))$$
$$= V(-\alpha)((f-\varphi) V_f(\alpha)(1)).$$

Since this derivative is $L_1(P)$ continuous,

$$V(-\alpha)(V_f(\alpha)(1)) = 1 + \int_0^\alpha V(-\beta)((f-\varphi) V_f(\beta)(1))\, d\beta.$$

Choosing a sequence (f_n) from F which converges to $\varphi \wedge N$ almost everywhere and in $L_1(P)$ norm and satisfies $f_n \leqslant 2N$, and letting n go to ∞ yields;

$$V(-\alpha)(V_N(\alpha)(1)) = 1 + \int_0^\alpha V(-\beta)([(\varphi \wedge N) - \varphi] V_N(\beta)(1))\, d\beta$$

from which the limit relation for $e_\alpha - 1$ follows. If $0 \leqslant \alpha \leqslant \gamma$, then

$$e_\gamma - e_\alpha = \lim_{N\to\infty} \int_\alpha^\gamma V(-\beta)([(\varphi \wedge N) - \varphi] V_N(\beta)(1))\, d\beta \leqslant 0.$$

The corresponding facts for $e_{-\alpha}$ are similarly proved and then the remainder of the theorem is proved in the same way as Theorem 2.2 of [8] is.

THEOREM 2.3. *If $V(\alpha)$ is a group, then all the $V(\alpha)$ are isometries and there are probability measures P_α on S satisfying $\int f dP_\alpha = \int T_\alpha f dP$ for all f in F. The P_α are mutually absolutely continuous and $V(\alpha)(1) = dP_\alpha/dP$.*

Proof. Since both $V(\alpha)$ and $[V(\alpha)]^{-1} = V(-\alpha)$ are contractions, $V(\alpha)$ is an isometry. If (f_n) is a sequence from $\bar{F}$ decreasing to 0 everywhere, then $\int T_\alpha f_n dP = \int V(\alpha)(T_\alpha f_n) dP = \int V(\alpha)(1) f_n dP$ and this decreases to 0 by the dominated convergence theorem. Hence the linear functionals $l_\alpha(f) = \int T_\alpha f dP$ defined on the lattice $\bar{F}$ can be extended to Daniell integrals $\bar{l}_\alpha$ [7, chap. III] and we define P_α to be the associated measures. For any f in $\bar{F}$, $\int f dP_\alpha = \int T_\alpha f dP = \int V(\alpha)(T_\alpha f) dP = \int V(\alpha)(1) f dP$ from which it easily follows that the P_α are mutually absolutely continuous, that they are defined on the same field S and that $V(\alpha)(1) = dP_\alpha/dP$.

We can define mappings $V^p(\alpha)$ of F into $L_p(P)$ by setting $V^p(\alpha)(f) = [V(\alpha)(1)]^{1/p} T_{-\alpha} f$ and each of these clearly has a unique extension to a positivity preserving contraction operator on $L_p(P)$.

LEMMA 2.6. *For all nonnegative f in $L_p(P)$, $V^p(\alpha)(f) = [V(\alpha)(f^p)]^{1/p}$.*

Proof. We will only prove this for $\alpha \geqslant 0$ since the other case is essentially the same. For any nonnagative f in $\bar{F}$ we can find a set of polynomials Q_n such that

$Q_n(f)$ converges uniformly to f^p and hence also $T_{-\alpha} Q_n(f) = Q_n(T_{-\alpha} f)$ converges uniformly to $(T_{-\alpha} f)^p$. Then

$$V(\alpha)(f^p) = \lim_{n\to\infty} V(\alpha)(Q_n(f)) = \lim_{n\to\infty} V(\alpha)(1)T_{-\alpha}(Q_n(f)) = V(\alpha)(1)(T_{-\alpha} f)^p = [V^p(\alpha)(f)]^p.$$

If (f_n) is a sequence from $\overline{F}$ converging in $L_p(P)$ to a nonnegative f, then

$$V^p(\alpha)(f) = \lim_{n\to\infty} V^p(\alpha)(f_n) = \lim_{n\to\infty} [V(\alpha)(f_n^p)]^{1/p},$$

but $$\lim_{n\to\infty} \int |[V(\alpha)(f_n^p)]^{1/p} - [V(\alpha)(f^p)]^{1/p}|^p \, dP \leqslant \lim_{n\to\infty} \int |V(\alpha)(f_n^p) - V(\alpha)(f^p)| \, dP = 0$$

so the lemma is proved.

THEOREM 2.4. *$V^p(\alpha)$, $\alpha \geqslant 0$ and $V^p(\alpha)$, $\alpha \leqslant 0$ are strongly continuous semigroups of operators on $L_p(P)$ for every $1 < p < \infty$.*

Proof. The strong continuity of $V^p(\alpha)$ follows from the fact that, for nonnegative f in $L_p(P)$;

$$\int |V^p(\alpha)(f) - V^p(\beta)(f)|^p dP = \int |[V(\alpha)(f^p)]^{1/p} - [V(\beta)(f^p)]^{1/p}|^p dP \leqslant \int |V(\alpha)(f^p) - V(\beta)(f^p)| dP$$

and the semigroup property from the fact that (again for nonnegative f in $L_p(P)$);

$$\begin{aligned} V^p(\alpha)(V^p(\beta)(f)) &= [V(\alpha)([V^p(\beta)f]^p)]^{1/p} = [V(\alpha)(V(\beta)(f^p))]^{1/p} \\ &= [V(\alpha+\beta)(f^p)]^{1/p} = V^p(\alpha+\beta)(f). \end{aligned}$$

3. φ's of exponential bound and the smoothing of P_α with respect to a Gaussian kernel

The first theorem of this section is simply a restatement of Theorem 3.4 of [8] for this case.

THEOREM 3.1. *If* (A1) *and* (A2) *hold and either*

$$\int_{[x|\varphi(x)\geqslant N]} \varphi \, dP \leqslant C e^{-\varepsilon N} \quad \text{if} \quad N \geqslant N_0$$

or $$-\int_{[x|\varphi(x)\leqslant -N]} \varphi \, dP \leqslant C e^{-\varepsilon N} \quad \text{if} \quad N \geqslant N_0$$

for some positive numbers C, ε, and N_0, then $V(\alpha)$ is a group of isometries.

Proof. Under the first assumption we have, for $\alpha < \varepsilon$,

$$\|e_\alpha - 1\| = \lim_{N\to\infty} \left\| \int_0^\alpha V(-\beta)\,((\varphi_N - \varphi)\, V_N(\beta)\,(1))\, d\beta \right\|$$

$$\leqslant \limsup_{N\to\infty} \int_0^\alpha \|(\varphi_N - \varphi)\, V_N(\beta)\,(1)\|\, d\beta$$

$$\leqslant \limsup_{N\to\infty} C \int_0^\alpha e^{N\beta} e^{-N\varepsilon}\, d\beta = 0,$$

and by Theorem 2.2 this is sufficient. The other case is similar.

The next result is a generalization of Theorem 4.2 of [8]. We assume that X, S, P, F, and T_α are given as in section 2 and satisfy both (A 1) of that section and

(A 3) There exist probability measures P_α satisfying

$$\int T_\alpha f\, dP = \int f\, dP_\alpha$$

for all f in F.

(A 3) is equivalent to several other assumptions, for example, that $T_\alpha f_n$ decreases to 0 everywhere whenever f_n does but is generally the easiest one to verify in practice.

Let $K_\sigma(\alpha)$ for positive σ be given by $K_\sigma(\alpha) = (2\pi\sigma)^{-\frac{1}{2}} \exp(-\alpha^2/2\sigma)$ and l_σ be the linear functional on $\bar{F}$ given by $l_\sigma(f) = \int_{-\infty}^{\infty} K_\sigma(\alpha)\,(\int T_\alpha f\, dP)\, d\alpha$. $l_\sigma(f)$ exists because $\int T_\alpha f\, dP$ is continuous and bounded in α and l_σ is clearly order preserving in $\bar{F}$. If (f_n) is a sequence from $\bar{F}$ converging monotonely to 0, then $\int T_\alpha f_n\, dP = \int f_n\, dP_\alpha$ is bounded by $\|f_0\|_\infty$ and converges to 0 so $l_\sigma(f_n)$ converges to 0 and it follows that there is a probability measure P^σ satisfying

$$\int f\, dP^\sigma = \int_{-\infty}^{\infty} K_\sigma(\alpha) \left[\int T_\alpha f\, dP \right] d\alpha$$

for all f in $\bar{F}$. We will write

$$\|f\|_p^\sigma \text{ for } \left[\int |f|^p\, dP^\sigma \right]^{1/p} \quad \text{and} \quad \|f\|_p \text{ for } \left[\int |f|^p\, dP \right]^{1/p}.$$

THEOREM 3.2. *If X, S, P, F and T_α satisfy* (A 1) *and* (A 3), *then X, S. P^σ, F and T_α satisfy* (A 1) *for every positive σ. There is a φ^σ in $L_1(P^\sigma)$ satisfying $\int \varphi^\sigma f\, dP^\sigma = \int D^\sigma f\, dP^\sigma$ for every f in F and*

$$\int_{|\varphi^\sigma| \geqslant N} |\varphi^\sigma|\, dP^\sigma \leqslant \sqrt{\frac{2}{\pi\sigma}}\, N\, e^{-\frac{1}{2}\sigma(N-1)^2}.$$

For every real α and positive σ there is a probability measure P_α^σ satisfying $\int f\,dP_\alpha^\sigma = \int T_\alpha f\,dP^\sigma$ for f in F and these measures are mutually absolutely continuous. If there is a φ in $L_1(P)$ satisfying (A2), *then each P_α is absolutely continuous with respect to each P_β^σ and we have*

$$\int \left|\frac{dP_\alpha}{dP_\alpha^\sigma} - 1\right| dP_\alpha^\sigma \leqslant \sqrt{\frac{2\sigma}{\pi}}\,\|\varphi\|_1.$$

Proof. We first have to show that $T_\alpha f$ has an $L_1(P^\sigma)$ continuous derivative $D^\sigma T_\alpha f$ and that $\|D^\sigma T_\alpha f\|_1^\sigma = O(e^{K|\alpha|})$. If f is in F

$$\begin{aligned}
&\left\|\frac{1}{\alpha}(T_\alpha f - f) - \frac{1}{\beta}(T_\beta f - f)\right\|_1^\sigma \\
&= \int_{-\infty}^{\infty} K_\sigma(\gamma)\left[\int\left|\frac{1}{\alpha}(T_{\alpha+\gamma} f - T_\gamma f) - \frac{1}{\beta}(T_{\beta+\gamma} f - T_\gamma f)\right| dP\right] d\gamma \\
&= \int_{-\infty}^{\infty} K_\sigma(\gamma)\left[\int\left|\frac{1}{\alpha}\int_0^\alpha (DT_{\delta+\gamma} f - DT_\gamma f)\,d\delta - \frac{1}{\beta}\int_0^\beta (DT_{\delta+\gamma} f - DT_\gamma f)\,d\delta\right| dP\right] d\gamma \\
&\leqslant \int_{-\infty}^{\infty} K_\sigma(\gamma)\left[\frac{1}{\alpha}\int_0^\alpha \|DT_{\delta+\gamma} f - DT_\gamma f\|\,d\delta + \frac{1}{\beta}\int_0^\beta \|DT_{\delta+\gamma} f - DT_\gamma f\|\,d\delta\right] d\gamma.
\end{aligned}$$

The integrand above is dominated by $CK_\sigma(\gamma)\,e^{K|\gamma|}$ and goes to 0 as α and β do so the limit $D^\sigma f$ exists. Moreover,

$$\begin{aligned}
\|D^\sigma T_\alpha f - D^\sigma T_\beta f\|_1^\sigma &= \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} K_\sigma(\gamma)\left\{\int\left|\frac{1}{\varepsilon}\int_0^\varepsilon (DT_{\alpha+\gamma+\delta} f - DT_{\beta+\gamma+\delta} f)\,d\delta\right| dP\right\} d\gamma \\
&\leqslant \limsup_{\varepsilon\to 0}\int_{-\infty}^{\infty} K_\sigma(\gamma)\,\frac{1}{\varepsilon}\left[\int_0^\varepsilon \|DT_{\alpha+\gamma+\delta} f - DT_{\beta+\gamma+\delta} f\|_1\,d\delta\right] d\gamma \\
&= \int K_\sigma(\gamma)\,\|DT_{\alpha+\gamma} f - DT_{\beta+\gamma} f\|_1\,d\gamma
\end{aligned}$$

because of the $L_1(P)$ continuity and exponential bound of $DT_\alpha f$. Again the integrand is bounded by $CK_\sigma(\gamma)\,e^{K|\gamma|}$ and goes to 0 as α goes to β and the $L_1(P^\sigma)$ continuity of $D^\sigma T_\alpha$ follows from this by the dominated convergence theorem. A similar calculation shows that

$$\|D^\sigma T_\alpha f\| = \int_{-\infty}^{\infty} K_\sigma(\gamma)\left[\int |DT_{\alpha+\gamma} f|\,dP\right] d\gamma \leqslant A\int_{-\infty}^{\infty} K_\sigma(\gamma)\,e^{K|\alpha+\gamma|}\,d\gamma = O(e^{K|\gamma|}).$$

We can show as in Lemma 4.4 of [8] that a φ_σ exists in $L_1(P^\sigma)$ satisfying $\int \varphi^\sigma f\,dP^\sigma = \int D^\sigma f\,dP$ for every f in F and

$$NP^{\sigma}(|\varphi^{\sigma}|\geqslant N)\leqslant\int_{|\varphi^{\sigma}|\geqslant N}|\varphi^{\sigma}|\,dP^{\sigma}\leqslant\frac{B}{\sigma}P(|\varphi^{\sigma}|\geqslant N)+2K_{\sigma}(B)$$

for every positive N and B. Setting B equal to $\sigma(N-1)$ gives

$$P(|\varphi^{\sigma}|\geqslant N)\leqslant\sqrt{\frac{2}{\pi\sigma}}\,e^{-\frac{1}{2}\sigma(N-1)^2}$$

and using the same B again

$$\int_{|\varphi^{\sigma}|\geqslant N}|\varphi^{\sigma}|\,dP\leqslant(N-1)\sqrt{\frac{2}{\pi\sigma}}\,e^{-\frac{1}{2}\sigma(N-1)^2}+\frac{2}{\sqrt{2\pi\sigma}}\,e^{-\frac{1}{2}\sigma(N-1)^2}=\sqrt{\frac{2}{\pi\sigma}}\,N\,e^{-\frac{1}{2}\sigma(N-1)^2}.$$

The existence of the measures P^{σ}_{σ} now follows from Theorems 3.1 and 2.3 and the remainder of the theorem is proved exactly as in [8].

We are now in a position to generalize the theorems of [9]. The next four theorems are restatements of Theorems 1 through 4 of that paper.

THEOREM 3.3. *If X, S, P, T_{α}, and F satisfy* (A1) *and* (A3) *and the P_{α} are mutually absolutely continuous, then T_{α} can be extended to all finite S-measurable functions and the mappings $U(\alpha)$ of $L_1(P)$ defined by $U(\alpha)(f)=(dP_{\alpha}/dP)\,T_{-\alpha}f$ form a strongly continuous group of isometries. The extension of T_{α} is linear and positive and satisfies*

(1) *If f_n converges to 0 almost everywhere, so does $T_{\alpha}f_n$,*

(2) $T_{\alpha}(fg)=T_{\alpha}(f)\,T_{\alpha}(g)$,

(3) $T_{\alpha}(T_{\beta}f)=T_{\alpha+\beta}f$,

(4) $T_{\alpha}\left(\dfrac{dP^{\sigma}_{\beta}}{dP^{\tau}_{\gamma}}\right)=\dfrac{dP^{\sigma}_{\beta-\alpha}}{dP^{\tau}_{\gamma-\alpha}}$,

and

(5) *If either side of the equation $\int T_{\alpha}h\,dP^{\sigma}_{\beta}=\int h\,dP^{\sigma}_{\beta+\alpha}$ exists, so does the other side and they are equal.*

Proof. This theorem is proved in exactly the same way as is Theorem 1 of [9].

THEOREM 3.4. *If X, S, P, T_{α}, F, and φ satisfy* (A1), (A2), *and* (A3), *then the generator A of $[U(\alpha)\,|\,\alpha\geqslant 0]$ contains F in its domain and is defined there by: $Af=\varphi f-Df$. $U(\alpha)(\varphi)$ is almost always integrable on every finite interval and the equation $dP_{\alpha}/dP=1+\int_0^{\alpha}U(\beta)(\varphi)\,d\beta$ defines a continuous version of the stochastic process dP_{α}/dP.*

Proof. The only difficulty in applying the proof of Theorem 2 of [9] here arises from the fact that Df is not necessarily bounded. That proof can still be used, however, to show that $A(1)=\varphi$ since $D(1)=0$. For any f in F we can find a sequence α_i going to 0 such that $T_{-\alpha_i}f-f$ converges to 0 almost everywhere and we have then

$$\left\| \frac{U(\alpha_i)(f)-f}{\alpha_i} - (\varphi f - Df) \right\|_1 \leqslant \left\| \left[\frac{1}{\alpha_i}(U(\alpha_i)(1)-1)-\varphi \right] (T_{-\alpha_i}f-f) \right\|_1$$

$$+ \| \varphi (T_{-\alpha_i}f - f)\|_1 + \left\| \frac{T_{-\alpha_i}f-f}{\alpha_i} + Df \right\|_1 + \left\| \left[\frac{1}{\alpha_i}(U(\alpha_i)(1)-1)-\varphi \right] f \right\|_1.$$

The first and fourth terms are dominated by $2\|f\|_\infty \|\alpha_i^{-1}(U(\alpha_i)(1)-1)-\varphi\|_1$ which goes to 0, the second term goes to 0 by the dominated convergence theorem and the third term also goes to 0. Thus a subsequence of $(U(\alpha)f-f)/\alpha$ converges to $\varphi f - Df$ and this implies that $A(f)=\varphi f - Df$ [4; Theorem 10.5.4, p. 318]. The rest of the proof is exactly the same as the proof of Theorem 2 of [9].

If (A1) and (A2) hold for X, S, P, T_α, and F, then φ is uniquely determined in $L_1(P)$ but not in $L_1(P^\sigma)$. As in [9] we call φ a *normalized* solution of (A2) if φ vanishes almost everywhere with respect to P^σ on the set where dP/dP^σ vanishes. Since the P^σ_α are mutually absolutely continuous, the transformations T_α can be extended to all finite S-measurable functions, and, in particular, to φ.

THEOREM 3.5. *Let φ be a normalized solution of* (A2). *If, for some $\gamma>0$ (or $\delta<0$), $T_{-\beta}\varphi$ is integrable on $[0,\gamma]$ (or $[\delta,0]$) almost everywhere with respect to P^σ, then the P_α are mutually absolutely continuous, $T_{-\beta}\varphi$ is almost always integrable on every finite interval, and* $\log(dP_\alpha/dP)=\int_0^\alpha T_{-\beta}\varphi\, d\beta$.

Proof. The proof is the same as the proof of Theorem 3 of [9].

THEOREM 3.6. *Suppose that X, S, P, T_α, F, φ, and P_α satisfy* (A1), (A2), *and* (A3), *that φ is in $L_2(P)$ and that the P_α are mutually absolutely continuous. If e is any random variable with $\int_J [\int e^2 dP_\alpha]^{\frac{1}{2}} d\alpha < \infty$ for some interval J containing the origin, and if we define the bias $b(\alpha)$ of the estimate e by: $\alpha + b(\alpha) = \int e\, dP_\alpha$, then at almost every point of J, $b(\alpha)$ has a derivative and*

$$\int (e-\alpha)^2 dP_\alpha \geqslant \frac{1+\dfrac{db}{d\alpha}}{\displaystyle\int \varphi^2 dP}.$$

If, in addition, $T_\beta e$ is continuous in $L_2(P)$ on J, then $b(\alpha)$ has a continuous derivative and satisfies the inequality at every point.

Proof. The proof is the same as the proof of Theorem 4 of [9].

For the remaining theorems of this section we will assume that X, S, P, T_α, F, and φ satisfy (A1), (A2), and

(A4) There exist positive numbers C, ε, and N_0 such that

$$\int_{[x \mid |\varphi(x)| \geqslant N]} |\varphi| \, dP \leqslant C e^{-\varepsilon N}$$

for all $N \geqslant N_0$.

We will write $e_N(\alpha)$ for $V_N(-\alpha)(V_N(\alpha)(1))$. Clearly, $0 \leqslant e_N(\alpha) \leqslant e(\alpha) \leqslant 1$.

Lemma 3.1. *$0 \leqslant \int (1 - e_N(\alpha))\, dP \leqslant (C/N)\, e^{-(\varepsilon - |\alpha|)N}$ for all $N \geqslant N_0$.*

Proof. We will do the case $\alpha \geqslant 0$ and the other will follow from the symmetry of the problem. As in the proof of Theorem 2.2 we can show that

$$1 - V_N(-\alpha)\, V_N(\alpha)(1) = -\int_0^\alpha V_N(-\beta)\, [\varphi \wedge N + (-\varphi \wedge N)]\, V_N(\beta)(1)\, d\beta$$

so

$$\int (1 - e_N(\alpha))\, dP \leqslant \int_0^\alpha \int |\varphi \wedge N + (-\varphi \wedge N)|\, V_N(\beta)(1)\, dP\, d\beta$$

$$\leqslant \int_0^\alpha e^{\beta N} \int_{|\varphi| > N} |\varphi|\, dP\, d\beta \leqslant \frac{C}{N} e^{-(\varepsilon - \alpha)N}.$$

Lemma 3.2. *If the sequence (h_n) from F converges in $L_1(P)$ to φ, then*

$$e_N(\alpha) = \lim_{n \to \infty} \exp\left(\int_0^\alpha T_\beta(|h_n| \wedge N - N)\, d\beta\right) \quad \textit{if} \quad \alpha \geqslant 0$$

and

$$e_N(\alpha) = \lim_{n \to \infty} \exp\left(\int_0^\alpha T_\beta(|h_n| \wedge N - N)\, d\beta\right) \quad \textit{if} \quad \alpha \leqslant 0.$$

Proof. We can find sequences (f_n) and (g_n) in F to satisfy $\|f_n - h_n \wedge N\|_\infty \leqslant 1/n$ and $\|g_n - (h_n \vee -N)\|_\infty \leqslant 1/n$. (f_n) converges in $L_1(P)$ to $\varphi \wedge N$ and (g_n) to $-(\varphi \vee -N)$ so, if $\alpha \geqslant 0$,

$$e_N(\alpha) = \lim_{n \to \infty} V_N(-\alpha)\, V_{f_n}(\alpha)(1)) = \lim_{n \to \infty} V_N(-\alpha)(1) \exp\left(\int_0^\alpha T_\beta f_n\, d\beta\right)$$

$$= \lim_{n \to \infty} \left\{\exp\left(\int_0^\alpha T_\beta(f_n + g_n)\, d\beta\right) + \left(V_N(-\alpha)(1) - \exp\left(\int_0^\alpha T_\beta g_n\, d\beta\right)\right) \exp\left(T_\beta f_n\, d\beta\right)\right\}$$

but the second term in the brackets is bounded in norm by

$$e^{\alpha(N+1/n)}\left\| V_N(-\alpha)(1)-\exp\left(\int_0^\alpha T_\beta g_n\,d\beta\right)\right\|_1$$

which goes to 0 since $\int_0^\alpha T_\beta g_n\,d\beta=\int_0^\alpha T_{-(-\beta)} g_n\,d\beta$ and (g_n) converges to $-(\varphi\vee -N)=(-\varphi\wedge N)$, and the first term converges to $\lim_{n\to\infty}\exp\left(\int_0^\alpha T_\beta(|h_n|\wedge N-N)\,d\beta\right)$. The proof for $\alpha\leqslant 0$ is similar.

LEMMA 3.3.

$$\log\frac{dP_\alpha}{dP}-\left(\log\frac{dP_\alpha}{dP}\right)\wedge\alpha N\leqslant -\log e_N(-\alpha).$$

Proof. We take sequences (f_n) and (h_n) as in Lemma 3.2 and then refine them so that $\int_0^\alpha T_{-\beta}f_n\,d\beta$ converges almost everywhere to $\log V_N(\alpha)(1)$ and $\int_{-\alpha}^0 T_\beta(h_n-h_n\wedge N)\,d\beta$ converges almost everywhere to $-\log e_N(-\alpha)$ (α being taken positive). We will still write (f_n) and (h_n) for the new sequences. Since $\int_0^\alpha T_\beta f_n\,d\beta\leqslant\alpha N+\varepsilon_n$ where ε_n goes to 0 as n goes to ∞,

$$\left(\log\frac{dP_\alpha}{dP}\right)\wedge\alpha N\geqslant(\log V_N(\alpha)(1))\wedge\alpha N\geqslant\lim_{n\to\infty}\int_0^\alpha T_{-\beta}f_n\,d\beta=\lim_{n\to\infty}\int_{-\alpha}^0 T_\beta(h_n\wedge N)\,d\beta.$$

Hence, for any positive M,

$$\log V_{N+M}(\alpha)(1)-\left(\log\frac{dP_\alpha}{dP}\right)\wedge\alpha N\leqslant\lim_{n\to\infty}\int_{-\alpha}^0 T_\beta(h_n-h_n\wedge N)\,d\beta=-\log e_N(-\alpha).$$

The proof now follows from Theorem 3.1 on letting M go to ∞. The proof for $\alpha\leqslant 0$ is similar.

THEOREM 3.7. *If X, S, P, T_α, F, and φ satisfy* (A1), (A2), *and* (A4), *then*

$$P\left(\log\frac{dP_\alpha}{dP}<M\right)\leqslant C_1\,e^{-(\varepsilon/|\alpha|-1)(M-1)}$$

if $|\alpha|<\varepsilon$ and $M\geqslant|\alpha|N_0+1$, and

$$P\left(\log\frac{dP_\alpha}{dP}<-M\right)\leqslant D_1\,e^{-(\varepsilon/|\alpha|-2)(M-1)}$$

if $|\alpha|<\frac{1}{2}\varepsilon$ and $M\geqslant|\alpha|N_0+1$. For any p, $1<p<\infty$, $\int|dP_\alpha/dP|^p\,dP$ is bounded in any interval $|\alpha|\leqslant\alpha_0<\varepsilon/(p+1)$.

Proof. By Lemma 3.3,

$$\left[\log\frac{dP_\alpha}{dP}-\left(\log\frac{dP_\alpha}{dP}\right)\wedge\alpha N\right]\wedge 1\leqslant[-\log e_N(-\alpha)]\wedge 1 \quad \textit{if} \quad N\geqslant N_0.$$

Using this inequality, Lemma 3.1, and the fact that

$$[-\log e_N(-\alpha)]\wedge 1\leqslant(1-e_N(-\alpha))/(1-e^{-1})$$

gives

$$P\left(\log\frac{dP_\alpha}{dP}\geqslant\alpha N+1\right)\leqslant\frac{1}{1-e^{-1}}\int(1-e_N(-\alpha))\,dP$$

$$\leqslant\frac{C}{1-e^{-1}}\frac{1}{N}e^{-(\varepsilon-|\alpha|)N}=\frac{C}{1-e^{-1}}\frac{1}{N}e^{-(\varepsilon/|\alpha|-1)(\alpha N)}.$$

Setting $$C_1=\frac{C}{1-e^{-1}}\frac{1}{N_0}\quad\text{and}\quad M=\alpha N+1$$

yields the first formula of the theorem.

By Theorem 3.3

$$P\left(\log\frac{dP_\alpha}{dP}<-M\right)=P\left(\log\frac{dP}{dP_\alpha}>M\right)$$

$$=P\left(T_{-\alpha}\left(\log\frac{dP_{-\alpha}}{dP}\right)>M\right)=P_{-\alpha}\left(\log\frac{dP_{-\alpha}}{dP}>M\right).$$

If f is the characteristic function of the set where $\log(dP_{-\alpha}/dP)>M$, then

$$P\left(\log\frac{dP_\alpha}{dP}<-M\right)=\int f\frac{dP_{-\alpha}}{dP}\,dP\leqslant\sum_{k=0}^{\infty}e^{M+k+1}P\left(\log\frac{dP_{-\alpha}}{dP}>M+k\right)$$

$$\leqslant C_1\sum_{k=0}^{\infty}e^{M+k+1}e^{-(\varepsilon/|\alpha|-1)(M+k+1)}=D_1e^{-(\varepsilon/|\alpha|-2)(M-1)}$$

which is the second formula of the theorem.

Finally, if $|\alpha|\leqslant\alpha_0<\varepsilon/(p+1)$,

$$\int\left(\frac{dP_\alpha}{dP}\right)^p dP\leqslant e^{p(|\alpha|N_0+1)}+\sum_{k=0}^{\infty}e^{p(|\alpha|N_0+k+2)}P\left(\log\frac{dP_\alpha}{dP}>|\alpha|N_0+k+1\right)$$

$$\leqslant A_p+\sum_{k=0}^{\infty}e^{p(|\alpha|N_0+k+2)}C_1e^{-(\varepsilon/|\alpha|-1)(\alpha N_0+k)}$$

$$=A_p+B\,e^{p(|\alpha|N_0+2)}e^{-(\varepsilon-|\alpha|)N_0}\sum_{k=0}^{\infty}e^{-(\varepsilon/|\alpha|-(p+1))k}$$

$$\leqslant A_p+B_pe^{p|\alpha_0|N_0}e^{-(\varepsilon-|\alpha_0|)N_0}\sum_{k=0}^{\infty}e^{-(\varepsilon/|\alpha_0|-(p+1))k}$$

where A_p and B_p are independent of α and this completes the proof.

The information contained in a probability measure P about a probability measure Q, written $I(P, Q)$ is given by:

$$I(P, Q) = \int \log \frac{dQ}{dP} dP + \int \log \frac{dP}{dQ} dQ.$$

THEOREM 3.8. *If X, S, P, T_α, F, and φ satisfy* (A 1), (A 2), *and* (A 4), *then for any α, β, and γ, $I(P_\alpha, P_\beta) = I(P_{\alpha+\gamma}, P_{\beta+\gamma})$. $I(P_\alpha, P_\beta)$ is finite whenever $|\alpha - \beta| < \frac{1}{3}\varepsilon$ and $I(P_\alpha, P_\beta) = O((\alpha-\beta)^2)$ as α converges to β.*

Proof. From Theorem 3.3

$$T_\alpha(e^f) = \lim_{n\to\infty} T_\alpha \left(\sum_{k=0}^{n} \frac{f^k}{k!}\right) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(T_\alpha f)^k}{k!} = e^{T_\alpha f}$$

for any measurable f so $\exp(T_\alpha \log g) = T_\alpha g = \exp(\log T_\alpha g)$, i.e., $T_\alpha \log g = \log T_\alpha g$ for any measurable g. Hence,

$$\begin{aligned} I(P_{\alpha+\gamma}, P_{\beta+\gamma}) &= \int \log \frac{dP_{\alpha+\gamma}}{dP_{\beta+\gamma}} dP_{\beta+\gamma} + \int \log \frac{dP_{\beta+\gamma}}{dP_{\alpha+\gamma}} dP_{\alpha+\gamma} \\ &= \int T_\gamma \log \frac{dP_{\alpha+\gamma}}{dP_{\beta+\gamma}} dP_\beta + \int T_\gamma \log \frac{dP_{\beta+\gamma}}{dP_{\alpha+\gamma}} dP_\alpha \\ &= \int \log \frac{dP_\alpha}{dP_\beta} dP_\beta + \int \log \frac{dP_\beta}{dP_\alpha} dP_\alpha = I(P_\alpha, P_\beta). \end{aligned}$$

Now, by Theorem 3.7, if $|\alpha| \leqslant \alpha_0 < \frac{1}{3}\varepsilon$, then

$$\int |T_\alpha \varphi|^2 dP = \int |\varphi|^2 \frac{dP_\alpha}{dP} dP \leqslant \left\{\int |\varphi|^4 dP \int \left(\frac{dP_\alpha}{dP}\right)^2 dP\right\}^{\frac{1}{2}} = C < \infty.$$

Since the $L_1(P)$ norm is dominated by the $L_2(P)$ norm, $T_{-\beta}\varphi$ is integrable on $[0, \alpha]$ and by Theorem 3.5, $|\log(dP_\alpha/dP)| = |\int_0^\alpha T_{-\beta}\varphi \, d\beta| \leqslant \int_0^\alpha |T_{-\beta}\varphi| \, d\beta$. $(dP_\alpha/dP)\, T_{-\gamma}\varphi$ is also integrable on $[0, \alpha]$ so, almost everywhere,

$$\left|\frac{dP_\alpha}{dP} - 1\right| = \left|\int_0^\alpha \frac{dP_\gamma}{dP} T_{-\gamma}\varphi \, d\gamma\right| \leqslant \int_0^\alpha \frac{dP_\gamma}{dP} T_{-\gamma} |\varphi| \, d\gamma.$$

Hence,
$$\begin{aligned} 0 \leqslant I(P, P_\alpha) &\leqslant \int \left\{\int_0^\alpha T_{-\beta}|\varphi| \, d\beta\right\}\left\{\int_0^\alpha \frac{dP_\gamma}{dP} T_{-\gamma}|\varphi| \, d\gamma\right\} dP \\ &= \int_0^\alpha d\beta \int_0^\alpha d\gamma \int (T_{-\beta+\gamma}|\varphi|)\, |\varphi| \, dP \leqslant C\alpha^2. \end{aligned}$$

4. The Gaussian case

Let (X, S, P) be a set, a σ-algebra of subsets and a probability measure, and let L be a linear set of real-valued S-measurable random variables whose joint distributions with respect to P are Gaussian. We will write H for the L_2-closure of L. All limit operations in this section will be with respect to the $L_2(P)$ norm unless otherwise specified. Let T_α be a one-parameter group of linear transformations of L into itself. We make the following assumptions:

(i) 1 (the constant function) is in L and $T_\alpha 1 = 1$ for all α.

(ii) H is separable.

(iii) For every x in L, $Dx = \lim_{\varepsilon \to 0} (T_\varepsilon x - x)/\varepsilon$ exists and $DT_\alpha x$ is L_2 continuous in α.

(iv) There exists a ψ in $L_2(P)$ satisfying $\int \psi xy\, dP = \int (xDy + yDx)\, dP$ for all x and y in L.

LEMMA 4.1. *If $x_0 = 1, x_1, x_2, \ldots$ is a complete orthonormal set in L, then*

$$\varphi = \lim_{n\to\infty} \sum_{0 \leqslant i < j \leqslant n} \left(\int (x_i Dx_j + x_j Dx_i)\, dP \right) x_i x_j + \sum_{i=1}^{n} \left(\int x_i Dx_i dP \right) (x_i^2 - 1)$$

$$= \lim_{n\to\infty} \mathsf{E} \left(\sum_{i=0}^{n} \left(x_i Dx_i - \int x_i Dx_i dP \right) \middle| x_1, \ldots, x_n \right)$$

exists and satisfies $\int \varphi xy\, dP = \int (xDy + yDx)\, dP$ for every x and y in L. φ is independent of the particular sequence $x_1, \ldots$.

Proof. The random variables $x_i x_j - \delta_{ij}$ are orthogonal and the first expression for φ is just the Fourier expansion for ψ with respect to this orthogonal set which guarantees its L_2 convergence. The equality of the two expressions is proved by computing the Fourier coefficients of the second with respect to this orthogonal set noting that $\mathsf{E}(x_i x_j | x_1, \ldots, x_n) = 0$ if $i \leqslant n < j$. If $i < j$, then clearly $\int \varphi x_i x_j dP = \int (x_i Dx_j + x_j Dx_i) dP$ and

$$\int \varphi x_i^2\, dP = \int \varphi (x_i^2 - 1)\, dP = \sqrt{2} \int \varphi \frac{(x_i^2 - 1)}{\sqrt{2}}\, dP = 2 \int x_i Dx_i dP$$

so $\int \varphi zw dP = \int (zDw + wDz) dP$ holds for z and w which are finite linear combinations of the x_i. By the same argument any other complete orthonormal set $y_0 = 1, y_1, \ldots$

in L gives rise to a φ' which is the expansion of ψ over the orthogonal set $y_i y_j - \delta_{ij}$ and satisfies the desired equation for z and w which are finite linear combinations of the y_i. However, if $\sum_{k=1}^n a_{ik} x_k$ converges to y_i then, since all the random variables involved are Gaussian $(\sum_{k=1}^n a_{ik} x_k)(\sum_{l=1}^n a_{jl} x_l)$ converges to $y_i y_j$ so the spaces spanned by the $x_i x_j - \delta_{ij}$ and the $y_i y_j - \delta_{ij}$ are the same and $\varphi = \varphi'$. It now follows, on applying the Gram–Schmidt procedure to the sequence $1, z, w, x_1, x_2, \ldots$ and forming φ with respect to the resulting complete orthonormal sequence that $\int \varphi zw\,dP = \int (zDw + wDz)\,dP$ for all z and w in L.

LEMMA 4.2. *If x is in L, $\|DT_\alpha x\| = O(e^{K\alpha})$ for $K = \sqrt{3}\,\|\varphi\|_2$.*

Proof. If $f(\alpha) = \int (T_\alpha x)^2 dP$, then

$$f'(\alpha) = 2\int (T_\alpha x) DT_\alpha x\,dP = \int \varphi (T_\alpha x)^2 dP \leqslant 3^{-\frac{1}{2}} K \left(\int (T_\alpha x)^4 dP\right)^{\frac{1}{2}}.$$

Writing m and σ for the mean and variance of $T_\alpha x$,

$$|f'(\alpha)| \leqslant 3^{-\frac{1}{2}} K (3\sigma^2 + 6\sigma m^2 + m^4)^{\frac{1}{2}} \leqslant K(\sigma + m^2) = Kf(\alpha).$$

Hence
$$\int (T_\alpha x)^2 dP \leqslant \left(\int x^2 dP\right) e^{K|\alpha|},$$

and
$$\|DT_\alpha x\|^2 = \lim_{\varepsilon \to 0} \left\| T_\alpha \left(\frac{T_\varepsilon x - x}{\varepsilon}\right)\right\|^2 \leqslant \limsup_{\varepsilon \to 0} \int \left(\frac{T_\varepsilon x - x}{\varepsilon}\right)^2 dP e^{K|\alpha|} = \|Dx\|^2 e^{K|\alpha|}.$$

LEMMA 4.3. *There exist independent normalized Gaussian random variables y_n in H and numbers λ_n and μ_n for which $\varphi = \sum_{n=1}^\infty \lambda_n (y_n^2 - 1) + \sum_{n=1}^\infty \mu_n y_n$*

Proof. We may write $\varphi = \varphi_0 + \varphi_1$ where $\varphi_0 = \sum_{1 \leqslant i < j} a_{ij} x_i x_j + \sum_{i=1}^\infty a_i (x_i^2 - 1)$ and $\varphi_1 = \sum_{i=1}^\infty b_i x_i$. For any Gaussian random variable x, $(\int x^4 dP)^{\frac{1}{4}} \leqslant 3^{\frac{1}{4}} (\int x^2 dP)^{\frac{1}{2}}$ so

$$\left|\int \varphi_0\, xy\,dP\right| \leqslant \|\varphi_0\|_2 \left(\int x^2 y^2 dP\right)^{\frac{1}{2}} \leqslant \|\varphi_0\|_2 \left(\int x^4 dP\right)^{\frac{1}{4}} \left(\int y^4 dP\right)^{\frac{1}{4}} \leqslant K\|x\|\,\|y\|$$

if x and y are in H. Thus the equation $\int (Tx) y\,dP = \int \varphi_0 xy\,dP$ defines a bounded self-adjoint operator on H. Also, for any complete orthonormal set (x_i),

$$\sum_{i,j=0}^\infty \left(\int (Tx_i) x_j\,dP\right)^2 = 2\sum_{i<j} \left(\int \varphi_0 x_i x_j\,dP\right)^2 + \sum_{i=1}^\infty \int \varphi_0 (x_i^2 - 1)\,dP \leqslant 2\,\|\varphi_0\|_2^2$$

so T is a Hilbert–Schmidt operator and hence is completely continuous. Let (y_n) be the eigenvectors and $(2\lambda_n)$ the corresponding eigenvalues of T. By the same argument used in Lemma 4.1, the random variables $y_i y_j - \delta_{ij}$ span the subspace of $L_2(P)$ containing φ_0 so we have

$$\varphi_0 = \sum_{i<j} \left(\int \varphi_0 y_i y_j dP \right) y_i y_j + \sum_{n=1}^{\infty} \left(\int \varphi_0 \frac{(y_n^2 - 1)}{\sqrt{2}} dP \right) \frac{y_n^2 - 1}{\sqrt{2}} = \sum_{n=1}^{\infty} \lambda_n (y_n^2 - 1).$$

Since φ_1 can be expanded over any complete orthonormal set, the lemma is proved.

LEMMA 4.4. *There exist positive numbers ε, C, and N_0 such that $\int_{|\varphi| \geq N} |\varphi| dP \leq Ce^{-\varepsilon N}$ whenever $N \geq N_0$.*

Proof. It will be sufficient to show that $\int e^{\delta|\varphi|} dP < \infty$ for some $\delta > 0$ since then

$$\int_{[x|\,|\varphi(x)| \geq N]} |\varphi| \, dP \leq Ne^{-\delta N} \int_{[x|\,|\varphi(x)| \geq N]} e^{\delta|\varphi|} dP \leq \left(\int e^{\delta|\varphi|} dP \right) e^{-\frac{1}{2}\delta N}$$

for large enough N. Also, since $e^{\delta|\varphi|} \leq e^{\delta\varphi} + e^{-\delta\varphi}$, it will be sufficient to show that $e^{\delta\varphi}$ is integrable for small $|\delta|$. Writing $\varphi = \varphi_0 + \varphi_1$ as in the preceeding lemma,

$$\int e^{\delta\varphi} dP = \int e^{\delta\varphi_0 + \delta\varphi_1} dP \leq \int e^{2\delta\varphi_0} dP + \int e^{2\delta\varphi_1} dP$$

and the second term on the right is finite because φ_1 is Gaussian so it only remains to show that $e^{\delta\varphi_0}$ is integrable. Taking $|\delta| < \inf \frac{1}{2} \lambda_n^{-1}$, we have

$$\int \exp \left(\delta \sum_{j=1}^{N} \lambda_j (y_j^2 - 1) \right) dP = \prod_{j=1}^{N} \int \exp \left(\delta (y_j^2 - 1) \right) dP = \prod_{j=1}^{N} (1 - 2\delta\lambda_j)^{\frac{1}{2}} \exp(-\delta\lambda_j).$$

The infinite product converges to a finite limit because

$$|1 - (1 - 2\delta\lambda_j)^{\frac{1}{2}} \exp(-\delta\lambda_j)| = O(\delta^2\lambda_j^2) \quad \text{and} \quad \sum_{j=1}^{\infty} \delta^2 \lambda_j^2 \leq 2\delta^2 \|\varphi\|_2^2$$

so, taking a subsequence N_k for which $\sum_{j=1}^{N_k} \lambda_j (y_j^2 - 1)$ converges almost everywhere to φ_0 and applying Fatou's lemma,

$$\int e^{\delta\varphi_0} dP \leq \lim_{k \to \infty} \prod_{j=1}^{N_k} (1 - 2\delta\lambda_j)^{\frac{1}{2}} \exp(-\delta\lambda_j) < \infty.$$

Let F be the set of random variables of the form $f(x_1, \ldots, x_n)$ where f is a bounded, real-valued function of n real variables with bounded first and second derivatives and the x_i belong to L.

LEMMA 4.5. *T_α is well-defined on F by: $T_\alpha f(x_1, \ldots, x_n) = f(T_\alpha x_1, \ldots, T_\alpha x_n)$. Writing f_i for the partial derivative of f with respect to the* ith *appearing variable, we have*

$$Df = \lim_{\alpha \to 0} \frac{T_\alpha f - f}{\alpha} = \sum_{i=1}^{n} f_i(x_1, \ldots, x_n)\, Dx_i.$$

$DT_\alpha f$ is continuous in $L_2(P)$ and hence in $L_1(P)$ and $\|DT_\alpha f\|_1 \leqslant \|DT_\alpha f\|_2 = O(e^{K|\alpha|})$.

Proof. We have to show that if $f(x_1, \ldots, x_n) = g(y_1, \ldots, y_m)$, then

$$f(T_\alpha x_1, \ldots, T_\alpha x_n) = g(T_\alpha y_1, \ldots, T_\alpha y_m).$$

After eliminating those variables on which f or g has only a constant dependence, the remaining sets of variables $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_m)$ must clearly span the same subspace of L. Hence each y_i can be written as a linear combination of the x_i and the first assertion of the lemma follows from this. By Taylor's theorem

$$\left|\frac{T_\alpha f - f}{\alpha} - \sum_{i=1}^{n} f_i(x_1, \ldots, x_n)\, Dx_i\right| \leqslant \sup_{1 \leqslant i, j \leqslant n} \|f_{ij}\|_\infty \sum_{k=1}^{n} \frac{|T_\alpha x_k - x_k|^2}{\alpha}$$

and this goes to zero in $L_2(P)$ since the x_k are Gaussian. A similar argument proves the L_2 continuity of $DT_\alpha f$. Finally,

$$\|DT_\alpha f\|_2 \leqslant \sup_i \|f_i\|_\infty \sum_{j=1}^{n} \|DT_\alpha x_j\| = O(e^{K|\alpha|}).$$

LEMMA 4.6. *If f is in F, then $\int \varphi f\, dP = \int Df\, dP$.*

Proof. We can assume, after making the appropriate linear change of variables, that $f = f(x_1, \ldots, x_n)$ where $x_0 = 1, x_1, \ldots, x_n$ are the first $n+1$ terms in a complete orthonormal set. If $Dx_j = \sum_{k=1}^{\infty} \alpha_{jk} x_k + \beta_j$, then

$$\mathsf{E}\left(x_j Dx_j - \int x_j Dx_j\, dP \,\Big|\, x_1, \ldots, x_n\right) = \sum_{k=1}^{n} \alpha_{jk} x_j x_k - \alpha_{jj} + \beta_j x_j \quad \text{if} \quad j \leqslant n.$$

We have

$$\int f\varphi\, dP = \int f\, \mathsf{E}\left(\sum_{j=1}^{n}\left(x_j Dx_j - \int x_j Dx_j\, dP\right) \Big|\, x_1, \ldots, x_n\right) dP$$
$$= (2\pi)^{-\frac{1}{2}n} \int f(a_1, \ldots, a_n) \sum_{j=1}^{n}\left(\sum_{k=1}^{n} \alpha_{jk} a_j a_k - \alpha_{jj} + \beta_j a_j\right) \exp\left(-\frac{1}{2}\sum_{l=1}^{n} a_l^2\right) da_1 \ldots da_n.$$

Using

$$\left(\sum_{k=1}^{n} \alpha_{jk} a_j a_k - \alpha_{jj} + \beta_j a_j\right) \exp\left(-\tfrac{1}{2} a_j^2\right) = -\frac{\partial}{\partial a_j}\left[\left(\sum_{k=1}^{n} \alpha_{jk} a_k + \beta_j\right) \exp\left(-\tfrac{1}{2} a_j^2\right)\right]$$

and integrating by parts leads to the desired equation.

THEOREM 4.1. *If* (i) *through* (iv) *are satisfied, then the measures* P_α *are mutually absolutely continuous and* T_α *can be extended to all S-measurable functions.* $T_{-\alpha}\varphi(x)$ *is integrable on every finite interval for almost every* x *and* $\log(dP_\alpha/dP) = \int_0^\alpha T_{-\beta}\varphi d\beta$. *For some* ε *and* N_0, *the results of Theorem 3.7 hold for* $\log(dP_\alpha/dP)$.

Proof. The lemmas of this section prove that the hypothesis of Theorems 3.1 and 3.7 hold for X, S, P, F, T_α, and φ. For ε as in Lemma 4.4 and $|\alpha| \leqslant \frac{1}{3}\varepsilon$, Theorem 3.7 implies that $\int (dP_\alpha/dP)^2 dP \leqslant C^2 < \infty$, so $\int T_\alpha|\varphi| dP = \int |\varphi| (dP_\alpha/dP) dP \leqslant C \|\varphi\|_2$. Hence $\int_{-\frac{1}{3}\varepsilon}^{\frac{1}{3}\varepsilon} \int T_\alpha |\varphi| d\alpha dP \leqslant \frac{2}{3}\varepsilon C \|\varphi\|_2 < \infty$, which proves that $T_{-\alpha}\varphi$ is almost always integrable on $[-\varepsilon/3, \varepsilon/3]$. The remainder of the theorem now follows from Theorem 3.5.

THEOREM 4.2. *If* (i) *through* (iv) *are satisfied and the sequence* (φ_n) *converges to* φ *in* $L_2(P)$, *then* $T_{-\beta}\varphi_n$ *is almost always integrable on* $[0, \alpha]$ *and* $\int_0^\alpha T_{-\beta}\varphi_n d\beta$ *converges in* $L_1(P)$ *to* $\log(dP_\alpha/dP)$ *for almost all* α *in some nondegenerate interval* $[-\alpha_0, \alpha_0]$. *For some subsequence* (n_j), $\int_0^\alpha T_{-\beta}\varphi_{n_j} d\beta$ *converges to* $\log(dP_\alpha/dP)$ *almost everywhere with respect to* $dPd\alpha$. *In particular if, for some complete orthonormal sequence* $x_0 = 1, x_1, \ldots$ *from* L, $\sum_{i=1}^n (x_i Dx_i - \int x_i Dx_i dP)$ *is* L_2-*convergent, then*

$$\sum_{i=1}^{n} \left(\tfrac{1}{2} x_i^2 - \tfrac{1}{2}(T_{-\alpha} x_i)^2 - \alpha \int x_i Dx_i \, dP\right)$$

converges in $L_1(P)$ *to* $\log(dP_\alpha/dP)$ *for* α *in* $[-\alpha_0, \alpha_0]$ *and for some subsequence* (n_j),

$$\sum_{i=1}^{n_j} \left(\tfrac{1}{2} x_i^2 - \tfrac{1}{2}(T_{-\alpha} x_i)^2 - \alpha \int x_i Dx_i \, dP\right)$$

converges to $\log(dP_\alpha/dP)$ *almost everywhere with respect to* $dPd\alpha$.

Proof. For $|\alpha| < \varepsilon/3$.

$$\int \left| \int_0^\alpha T_{-\beta}(\varphi - \varphi_n) d\beta \right| dP \leqslant \int_0^\alpha \left(\int |\varphi - \varphi_n| \frac{dP_{-\beta}}{dP} dP \right) d\beta \leqslant C \|\varphi - \varphi_n\|_2 \alpha$$

which proves the almost everywhere integrability of $T_{-\beta}\varphi_n$ on $[-\varepsilon/3, \varepsilon/3]$ and the $L_1(dPd\alpha)$ convergence of $\int_0^\alpha T_{-\beta}\varphi_n d\beta$ to $\int_0^\alpha T_{-\beta}\varphi d\beta$ for α in this interval. For some subsequence (n_j), $\int_0^\alpha T_{-\beta}\varphi_{n_j} d\beta$ converges almost everywhere $dPd\alpha$ on the interval and thus, because the P_α are all equivalent,

$$\int_0^{N\alpha} T_{-\beta}\varphi_{n_j} d\beta = \sum_{k=0}^{N-1} T_{-k\alpha} \int_0^\alpha T_{-\beta}\varphi_{n_j} d\beta$$

converges almost everywhere to $\int_0^{N\alpha} T_{-\beta}\varphi d\beta$ with respect to $dPd\alpha$. The remainder of the theorem will be proved if we show that $\int_0^\alpha T_{-\beta}(xDx)d\beta = \frac{1}{2}x^2 - \frac{1}{2}(T_{-\alpha}x)^2$ for x in L. For some sequence (ε_n) converging to 0, $(T_{-\beta+\varepsilon_n}x - T_{-\beta}x)/\varepsilon_n$ converges almost everywhere to $DT_{-\beta}x$ and $(T_{\varepsilon_n}x - x)/\varepsilon_n$ converges almost everywhere to Dx but then, since $P_{-\beta}$ is equivalent to P, $T_{-\beta}(T_{\varepsilon_n}x - x)/\varepsilon_n$ converges almost everywhere to $T_{-\beta}Dx$ so $T_{-\beta}Dx = DT_{-\beta}x$. Thus the integrand $T_{-\beta}(xDx) = (T_{-\beta}x)(DT_{-\beta}x)$ is L_2 continuous and $\int_0^\alpha T_{-\beta}(xDx)d\beta$ has $(T_{-\alpha}x)(DT_{-\alpha}x)$ as L_2 derivative. The L_2 derivative of $\frac{1}{2}x^2 - \frac{1}{2}(T_{-\alpha}x)^2$ is given by

$$\lim_{\varepsilon\to 0} -\tfrac{1}{2}[(T_{-\alpha-\varepsilon}x)^2 - (T_{-\alpha}x)^2] = \lim_{\varepsilon\to 0}\left[-T_{-\alpha}x\left(\frac{T_{-\alpha-\varepsilon}x - T_{-\alpha}x}{\varepsilon}\right) - \left(\frac{T_{-\alpha-\varepsilon}x - T_{-\alpha}x}{\varepsilon}\right)^2\right]$$
$$= (T_{-\alpha}x)(DT_{-\alpha}x)$$

(using again the fact that the random variables are Gaussian) and this proves the validity of the desired equation since both sides vanish for $\alpha = 0$. Example 4 of the next section shows that $\sum_{i=1}^\infty (x_i Dx_i - \int x_i Dx_i dP)$ need not converge to φ.

Before going to the next section we wish to discuss assumptions (i) through (iv) made at the beginning of this one. (i) which is simply a normalization and (iii) which expresses the continuity of T_α seem necessary in this context but (ii), the separability of H, could have been avoided. We have not thought it worth-while to make the minor changes in proofs and notation required for the nonseparable case since it is of infrequent occurrence in applications. Assumption (iv) is rather awkward as stated. In practice one generally chooses a complete orthonormal set $x_0 = 1, x_1, x_2, \dots$ from L; computes $\varphi = \sum_{i=0}^\infty x_i Dx_i$, which satisfies the desired equation when x and y are finite linear combinations of the x_i's; and then shows by a continuity argument that the equation is satisfied for all x and y in L. The following example shows that this continuity is not automatic.

Let $x_1, x_2, \dots$ be an orthonormal set and define $y_n = A_n \sum_{k=1}^\infty k^{-n} x_k$ where A_n is chosen to make $\|y_n\| = 1$. The x_i's and y_j's are linearly independent since

$$z = \sum_{i=1}^N \alpha_i x_i + \sum_{j=1}^M \beta_j y_j = 0$$

implies that $\beta_1 = \lim_{n\to\infty} n\int zx_n dP = 0$, $\beta_2 = \lim_{n\to\infty} n^2 \int zx_n dP = 0$, etc. L is to be all finite linear combinations of $x_0 = 1$, the x_i's, and the y_i's, and T_α is defined by:

$$T_\alpha x_n = x_n + C_n(e^{a_n\alpha} - 1)y_n \quad \text{and} \quad T_\alpha y_n = e^{a_n\alpha}y_n.$$

This gives $Dx_n = a_n C_n y_n$ and $DT_\alpha y_n = a_n e^{a_n\alpha} y_n$. Choosing $C_n = 0$ gives

$$\varphi=\sum_{i=1}^{\infty} x_i Dx_i=0 \quad \text{but} \quad \int (x_k Dy_n + y_n Dx_k)\, dP = a_n \int x_k y_n \, dP = a_n A_n k^{-n} \neq 0 = \int \varphi x_k y_n \, dP.$$

This example can be patched up by cutting down the size of L but the following one can't. This time we take $a_n = n$ and $C_n = n^{-2}$ giving $\varphi = \sum_{i=1}^{\infty} x_i Dx_i = \sum_{i=1}^{\infty} i^{-1} y_i x_i$ which is L_2 convergent but cannot satisfy (iv) because $\|DT_\alpha y_n\|_2 = ne^{n\alpha} + O(e^{K|\alpha|})$ contradicting Lemma 4.2.

It would be very interesting to have a converse to Theorem 4.1, that is, a theorem asserting that if (i), (ii), and (iii) hold and if mutually absolutely continuous measures P_α satisfying $\int T_\alpha x dP = \int x dP_\alpha$ exist, then a ψ satisfying (iv) must exist. Under these assumptions Theorem 3.3 implies that $V(\alpha)f = (dP_\alpha/dP)T_{-\alpha}f$ is a strongly continuous group with generator A and the desired theorem is easily seen to be equivalent to the assertion that the constant function 1 is in the domain of A and $A(1)$ is in $L_2(P)$.

Finally, it should be pointed out that the relation between φ and T_α is not one to one. This shows up even in the finite dimensional case as the following example shows. Let y_1 and y_2 be independent normalized Gaussian random variables and let L be all finite linear combinations of $y_0 = 1$, y_1, and y_2. For each real ν let D_ν be the transformation given by:

$$D_\nu \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ \lambda_1 y_1 + \nu y_2 \\ -\nu y_1 + \lambda_2 y_2 \end{pmatrix}$$

and let $T_\alpha^{(\nu)}$ be the group given by $T_\alpha^{(\nu)} = e^{\alpha D_\nu}$. Then y_1 and y_2 are the variables whose existence is proved in Lemma 4.3 and $\varphi_\nu = y_1 D_\nu y_1 - \int y_1 D_\nu y_1 dP + y_2 Dy_2 - \int y_2 Dy_2 dP = \lambda_1 (y_1^2 - 1) + \lambda_2 (y_2^2 - 1)$ which is independent of ν.

5. Gaussian examples

Example 1. Translation of a random entire function.

Let (a_n) be a sequence of independent normalized (mean 0 and variance 1) Gaussian random variables and (ζ_n) a sequence of real numbers satisfying $\sum_{n=0}^{\infty} (\zeta_{n+1}/\zeta_n)^2 < \infty$. For each t, $-\infty < t < \infty$ and $k = 0, 1, \ldots$ the series $x^{(k)}(t) = \sum_{n=0}^{\infty} (\zeta_{n+k} a_{n+k}/n!) t^n$ converges with probability 1 because the independent random variables

$$y_n = \frac{\zeta_{n+k} a_{n+k}}{n!} t^n \quad \text{satisfy} \quad \sum_{n=0}^{\infty} \int y_n^2 \, dP \log^2 n < \infty$$

([1], Theorem 4.2, p. 157) and applying this result to an unbounded sequence t_n shows that the defining series for each $x^{(k)}$ has infinite radius of convergence with probability 1. It is also easy to see that $x^{(k)}(t)$ is the $L_2(P)$-limit of $\sum_{n=0}^{N}(\zeta_{n+k}a_{n+k}/n!)\,t^n$ for each k and t. We take L to be all finite linear combinations of the constant function and the random variables $x^{(k)}(t)$ for $-\infty<t<\infty$ and $k=0,1,\dots$ and define T_α by:

$$T_\alpha x^{(k)}(t)=x^{(k)}(t+\alpha).$$

The set $x^{(k)}(0)/\zeta_k=a_k$ is orthonormal and dense in L so H is separable.

Now
$$\int\left|\frac{T_\alpha x^{(k)}(t)-x^{(k)}(t)}{\alpha}-x^{(k+1)}(t)\right|^2 dP$$
$$=\int\left|\sum_{n=0}^{\infty}\frac{\zeta_{n+k+1}a_{n+k+1}}{(n+1)!}\left[\frac{(t+\alpha)^{n+1}-t^{n+1}}{\alpha}-(n+1)t^n\right]\right|^2 dP$$
$$=\sum_{n=0}^{\infty}\frac{(\zeta_{n+k+1})^2}{((n+1)!)^2}\left|\frac{(t+\alpha)^{n+1}-t^{n+1}}{\alpha}-(n+1)t^n\right|^2\to 0$$

so $Dx^{(k)}(t)=x^{(k+1)}(t)$. The continuity of $DT_\alpha x^{(k)}(t)=x^{(k+1)}(t+\alpha)$ is guaranteed by the fact that it has L_2 derivative $x^{(k+2)}(t+\alpha)$.

The assumption on the ζ_n's implies the $L_2(P)$ convergence of

$$\varphi=\sum_{k=0}^{\infty}\frac{x^{(k)}(0)}{\zeta_k}\,D\left(\frac{x^{(k)}(0)}{\zeta_k}\right)=\sum_{k=0}^{\infty}\frac{\zeta_{k+1}}{\zeta_k}\left(\frac{x^{(k)}(0)}{\zeta_k}\right)\left(\frac{x^{(k+1)}(0)}{\zeta_{k+1}}\right).$$

From its definition φ satisfies $\int\varphi yz\,dP=\int(yDz+zDy)\,dP$ for all y and z which are finite linear combinations of the $x^{(k)}(0)$. For arbitrary s and t

$$\int\varphi x^{(i)}(s)\,x^{(j)}(t)\,dP=\lim_{n\to\infty}\int\varphi\left(\sum_{m=0}^{N}\frac{x^{(m+i)}(0)}{m!}s^m\right)\left(\sum_{n=0}^{N}\frac{x^{(n+j)}(0)}{n!}t^n\right)dP$$
$$=\lim_{n\to\infty}\int\left[\left(\sum_{m=0}^{N}\frac{x^{(m+1)}(0)}{m!}s^m\right)\left(\sum_{n=0}^{N}\frac{x^{(n+j+1)}(0)}{n!}t^n\right)\right.$$
$$\left.+\left(\sum_{m=0}^{N}\frac{x^{(m+i+1)}(0)}{m!}s^m\right)\left(\sum_{n=0}^{N}\frac{x^{(n+j)}(0)}{n!}t^n\right)\right]dP$$
$$=\int[x^{(i)}(s)\,x^{(j+1)}(t)+x^{(i+1)}(s)\,x^{(j)}(t)]\,dP$$

so φ satisfies condition (iv) of section 4 and the theorems of that section are applicable here.

THEOREM 5.1. *Under the stated assumptions the measures P_α associated with the stochastic processes $x_\alpha(t) = x(t+\alpha) = \sum_{n=0}^{\infty} (\zeta_n a_n / n!)\,(t+\alpha)^n$ are mutually absolutely continuous. Some subsequence of $\frac{1}{2}\sum_{n=0}^{N} [(x^{(n)}(0)/\zeta_n)^2 - (x^{(n)}(\alpha)/\zeta_n)^2]$ converges almost everywhere $(dPd\alpha)$ to $\log\,(dP_\alpha/dP)$.*

Example 2. The Doppler shift.

Let $z(t)$ be a complex Gaussian process on an interval I with mean value f in $L_2(dt)$ and correlation function $R(s,t)$ in $L_2(ds\times dt)$. The integral operator R on $L_2(dt)$ associated with the kernel $R(s,t)$ is completely continuous, hence has a complete set (ξ_k) of eigenvectors with corresponding eigenvalues (λ_k). The λ_k are nonnegative and satisfy $\sum_{k=1}^{\infty}\lambda_k^2 < \infty$. We further assume that all the λ_k are strictly positive and that the real-valued, Gaussian random variables x_k and y_k given by:

$$x_k + iy_k = \sqrt{\frac{2}{\lambda_k}}\int_I (z(t) - f(t))\,\bar{\xi}_k(t)\,dt$$

are independent of each other and of all the other x_l and y_l. For a bounded function $a(t)$ on I and a real α the transformation $z(t) \to e^{\alpha a(t)} z(t)$ is called the Doppler shift of z by α [6].

We take L to be the set of all finite linear combinations of the constant function and the real-valued random variables u_g and v_g given by $u_g + iv_g = \int_I z(t) g(t)\,dt$ for g's in $L_2(dt)$, and define T_α by the equation $T_\alpha(u_g + iv_g) = u_{e^{\alpha a}g} + iv_{e^{\alpha a}g}$. T_α is well defined since

$$g(t) = \sum_{k=1}^{\infty}\sqrt{\frac{2}{\lambda_k}}\left(\int u_g x_k\,dP - i\int u_g y_k\,dP\right)\xi_k = \sum_{k=1}^{\infty}\sqrt{\frac{2}{\lambda_k}}\left(\int v_g y_k\,dP + i\int v_g x_k\,dP\right)\xi_k,$$

and the T_α obviously form a group. We have

$$\begin{aligned}&\left\|\int_I z(t)\left(\frac{e^{\varepsilon a(t)}-1}{\varepsilon} - a(t)\right) g(t)\,dt\right\|^2\\ &\quad= \int_I \left| R^{\frac{1}{2}}\left(\left(\frac{e^{\varepsilon a}-1}{\varepsilon} - a\right) g\right)(t)\right|^2 dt + \left[\int_I f(t)\left(\overline{\frac{e^{\varepsilon a(t)}-1}{\varepsilon} - a(t)}\right)\overline{g(t)}\,dt\right]^2\\ &\quad\leqslant C\int_I \left|\left(\frac{e^{\varepsilon a(t)}-1}{\varepsilon} - a(t)\right) g(t)\right|^2 dt\end{aligned}$$

which goes to 0 by the dominated convergence theorem as ε goes to 0 so $Du_g = u_{ag}$ and $Dv_g = v_{ag}$. It now follows from the fact that $DT_\alpha u_g = u_{ae^{\alpha a}g}$ and $DT_\alpha v_g = v_{ae^{\alpha a}g}$ have $L_2(P)$ derivatives that they are $L_2(P)$ continuous.

The set comprising 1, the x_k's, and the y_k's is a complete orthonormal subset of L. Elementary but tedious calculations yield the following equations in which $\mathfrak{R}(c)$ and $\mathfrak{J}(c)$ stand for the real and imaginary parts of c.

$$\int (x_k Dx_l + x_l Dx_k)\, dP = \int (y_k Dy_l + y_l Dy_k)\, dP$$

$$= \mathcal{R}\left(\sqrt{\frac{\lambda_k}{\lambda_l}} \int_I a(t)\, \xi_k(t)\, \bar{\xi}_l(t)\, dt + \sqrt{\frac{\lambda_l}{\lambda_k}} \int_I a(t)\, \bar{\xi}_k(t)\, \xi_l(t)\, dt\right) \quad \text{if } k \neq l,$$

$$\int (x_k Dy_l + y_l Dx_k)\, dP = \mathfrak{J}\left(\sqrt{\frac{\lambda_k}{\lambda_l}} \int_I a(t)\, \xi_k(t)\, \bar{\xi}_l(t)\, dt - \sqrt{\frac{\lambda_l}{\lambda_k}} \int_I a(t)\, \bar{\xi}_k(t)\, \xi_l(t)\, dt\right),$$

$$\int x_k Dx_k\, dP = \int y_k Dy_k\, dP = 2 \int_I \mathcal{R}(a(t))\, |\xi_k(t)|^2\, dt,$$

$$\int Dx_k\, dP = \sqrt{\frac{2}{\lambda_k}}\, \mathcal{R}\left(\int_I f(t)\, a(t)\, \bar{\xi}_k(t)\, dt\right),$$

$$\int Dy_k\, dP = \sqrt{\frac{2}{\lambda_k}}\, \mathfrak{J}\left(\int_I f(t)\, a(t)\, \bar{\xi}_k(t)\, dt\right).$$

THEOREM 5.2. *If*

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \left| \sqrt{\frac{\lambda_k}{\lambda_l}} \int_I a(t)\, \xi_k(t)\, \bar{\xi}_l(t)\, dt + \sqrt{\frac{\lambda_l}{\lambda_k}} \int_I a(t)\, \bar{\xi}_k(t)\, \xi_l(t)\, dt \right|^2 < \infty$$

and

$$\sum_{k=1}^{\infty} \frac{1}{\lambda_k} \left| \int a(t)\, f(t)\, \bar{\xi}_k(t)\, dt \right|^2 < \infty,$$

then the conclusions of Theorem 4.1 hold for this case. In particular, the measures P_α associated with the processes $e^{\alpha a(t)} z(t)$ are mutually absolutely continuous.

Proof. The hypotheses of the theorem and the computations immediately preceeding the theorem imply the existence of

$$\varphi = \sum_{j=1}^{\infty} \left[\left(\int Dx_j\, dP\right) x_j + \left(\int Dy_j\, dP\right) y_j\right]$$

$$+ \sum_{1 \leq j < k} \left[\left(\int (x_k Dx_j + x_j Dx_k)\, dP\right) x_j x_k + \int \left((y_k Dy_j + y_j Dy_k)\, dP\right) y_j y_k\right]$$

$$+ 2 \sum_{j=1}^{\infty} \left[\left(\int x_j Dx_j\, dP\right) (x_j^2 - 1) + \left(\int y_j Dy_j\, dP\right) (y_j^2 - 1)\right]$$

$$+ \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \left(\int (x_j Dy_k + y_k Dx_j)\, dP\right) x_j y_k$$

in $L_2(P)$. We can show, exactly as in Lemma 4.6, that

$$\int \varphi w_1 w_2 \, dP = \int (w_1 Dw_2 + w_2 Dw_1) \, dP$$

if the w_j are finite linear combinations of the constant function and the x_k's and y_k's. If (g_n) is any sequence in $L_2(dt)$ converging to g, then $\int_I (z-f)(t) g_n(t) dt$ is $L_2(P)$ convergent to $\int_I (z-f)(t) g(t) dt$. Hence for g and h in $L_2(dt)$ if we set $\alpha_k = \int_I g(t) \xi_k(t) \, dt$ and $\beta_k = \int_I h(t) \xi_k(t) \, dt$ we have

$$\begin{aligned}
\int \varphi u_g u_h \, dP &= \lim_{N \to \infty} \int \varphi \left[\mathcal{R} \left(\sum_{k=1}^{N} \sqrt{\frac{\lambda_k}{2}} \alpha_k (x_k + iy_k) + \int_I f(s) g(s) \, ds \right) \right] \\
&\quad \times \left[\mathcal{R} \left(\sum_{k=1}^{n} \sqrt{\frac{\lambda_k}{2}} \beta_k (x_k + iy_k) + \int f(t) h(t) \, dt \right] dP \\
&= \int \mathcal{R} \left(\int_I z(s) g(s) \, ds \right) \mathcal{R} \left(\int z(t) a(t) h(t) \, dt \right) dP \\
&\quad + \int \mathcal{R} \left(\int_I z(s) a(s) g(s) \, ds \right) \mathcal{R} \left(\int_I z(t) h(t) \, dt \right) dP \\
&= \int (u_g Du_h + u_h Du_g) \, dP.
\end{aligned}$$

The proofs that $\int \varphi u_g v_h \, dP = \int (u_g Dv_h + v_h Du_g) \, dP$ and that $\int \varphi v_g v_h dP = \int (v_g Dv_h + v_h Dv_g) dP$ are similar. This shows that the hypotheses of Theorem 4.1 are satisfied and thus proves the theorem.

THEOREM 5.3. *If*

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\lambda_j}{\lambda_k} \left| \int_I |a(t)| \, \overline{\xi_j(t)} \, \xi_k(t) \, dt \right|^2 < \infty$$

and

$$\sum_{j=1}^{\infty} \frac{1}{\lambda_j} \left| \int_I a(t) f(t) \bar{\xi}_k(t) \, dt \right|^2 < \infty$$

then some subsequence of

$$\sum_{k=1}^{N} \left\{ \frac{1}{\lambda_k} \left| \int_I (z-f)(t) \bar{\xi}_k(t) \, dt \right|^2 - \frac{1}{\lambda_k} \left| \int_I (e^{\alpha a(t)} z(t) - f(t)) \bar{\xi}_k(t) \, dt \right|^2 - \alpha \int 2 \mathcal{R}(a)(t) \, |\xi_k(t)|^2 \, dt \right\}$$

converges almost everywhere $(dP d\alpha)$ *to* $\log dP_\alpha / dP$.

Proof. If we set $A_k = x_k Dx_k + y_k Dy_k - \int (x_k Dx_k + y_k Dy_k) \, dP$, then we get, after a lengthy calculation

$$\int A_k A_j dP = \delta_{jk} \frac{2}{\lambda_j} \left| \int_I a(t) f(t) \bar{\xi}_j(t)\, dt \right|^2 + \delta_{jk} \frac{2}{\lambda_j} \sum_{l=1}^{\infty} \lambda_l \left| \int_I a(t) \bar{\xi}_j(t) \xi_l(t)\, dt \right|^2$$

$$+ \delta_{jk} 8 \left| \int_I a(t) |\xi_j(t)|^2 dt \right|^2 + 8 \int \mathcal{R}(a(t)) |\xi(t)|^2 dt \int \mathcal{R}(a(s)) |\xi_k(s)|^2 ds$$

$$+ \int \bar{a}(s) \xi_j(s) \bar{\xi}_k(s) ds \int \bar{a}(t) \bar{\xi}_j(t) \xi_k(t)\, dt + \int a(s) \xi_j(s) \bar{\xi}_k(s) ds \int a(t) \bar{\xi}_j(t) \xi_k(t)\, dt.$$

The hypotheses of the theorem imply the convergence of $\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |\int A_j A_k dP|$ so that $\|\sum_{j=l}^{m} A_j\|^2$ which is dominated by $\sum_{j=l}^{\infty} \sum_{k=l}^{\infty} |\int A_j A_k dP|$ goes to 0 as l goes to ∞. Thus $\sum_{k=1}^{N} A_k$ converges to φ and Theorem 4.2 applies.

Example 3. Rotation of a random periodic function.

We consider the process $x(t)$ for $-\pi \leqslant t < \pi$ given by

$$x(t) = \sum_{n=1}^{\infty} \sigma_n x_n \sin nt + \tau_n y_n \cos nt$$

where (x_n) and (y_n) are sets of independent normalized Gaussian random variables which are independent of each other and (σ_n) and (τ_n) are sets of real numbers such that $\sum_{n=1}^{\infty} (\sigma_n^2 + \tau_n^2) < \infty$. We take L to be the set of all finite linear combinations of 1 and the x_n's and y_n's and set

$$T_\alpha x_n = (\cos n\alpha) x_n - \frac{\tau_n}{\sigma_n} (\sin n\alpha) y_n \quad \text{and} \quad T_\alpha y_n = \frac{\sigma_n}{\tau_n} (\sin n\alpha) x_n + (\cos n\alpha) y_n.$$

Trivially, $Dx_n = -n(\tau_n/\sigma_n) y_n$ and $Dy_n = n(\sigma_n/\tau_n) x_n$ so that both $DT_\alpha x_n$ and $DT_\alpha y_n$ again have $L_2(P)$ derivatives and hence are $L_2(P)$ continuous.

THEOREM 5.4. *If*

$$\sum n^2 \left(\frac{\tau_n}{\sigma_n} - \frac{\sigma_n}{\tau_n} \right)^2 < \infty,$$

then the measures P_α associated with the processes $x(t+\alpha)$ are mutually absolutely continuous and some subsequence of

$$\sum_{n=1}^{N} \tfrac{1}{2} (\sin^2 n\alpha) \left(1 - \frac{\sigma_n^2}{\tau_n^2}\right) x_n^2 + \tfrac{1}{2} (\sin^2 n\alpha) \left(1 - \frac{\tau_n^2}{\sigma_n^2}\right) y_n^2 - (\sin n\alpha \cos n\alpha) \left(\frac{\tau_n}{\sigma_n} - \frac{\sigma_n}{\tau_n} \right) x_n y_n$$

converges to $\log dP_\alpha/dP$ *almost everywhere* $(dPd\alpha)$.

Proof. The condition insures the $L_2(P)$ convergence of

$$\varphi = \sum_{n=1}^{N} (x_n D x_n + y_n D y_n) = \sum_{n=1}^{N} n \left(\frac{\tau_n}{\sigma_n} - \frac{\sigma_n}{\tau_n}\right) x_n y_n$$

and we can show as in the proof of Lemma 4.6 that $\int \varphi z w dP = \int (z D w + w D z) dP$ for z and w which are finite linear combinations of 1 and the x_n's and y_n's, i.e., for z and w in L. The theorem now follows from Theorems 4.1 and 4.2 plus the fact that

$$\tfrac{1}{2} x_n^2 + \tfrac{1}{2} y_n^2 - \tfrac{1}{2} (T_{-\alpha} x_n)^2 - \tfrac{1}{2} (T_{-\alpha} y_n)^2 = \tfrac{1}{2} (\sin^2 n\alpha) \left(1 - \frac{\sigma_n^2}{\tau_n^2}\right) x_n^2$$

$$+ \tfrac{1}{2} (\sin^2 n\alpha) \left(1 - \frac{\tau_n^2}{\sigma_n^2}\right) y_n^2 - (\sin n\alpha \cos n\alpha) \left(\frac{\tau_n}{\sigma_n} - \frac{\sigma_n}{\tau_n}\right) x_n y_n.$$

Example 4. Linear fractional transformations of random analytic functions.

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ where $c_n = \sigma_n x_n + i \tau_n y_n$, (σ_n) and (τ_n) are bounded sequences of positive real numbers, and (x_n) and (y_n) are sets of independent normalized Gaussian random variables which are independent of each other. For any z with $|z| < 1$ the series defining $f(z)$ is almost everywhere convergent since

$$\sum_{n=0}^{\infty} |z|^{2n} \int |c_n|^2 dP \log^2 n \leqslant C \sum_{n=0}^{\infty} |z|^{2n} \log^2 n < \infty$$

[1, Theorem 4.2, p. 157] so f has radius of convergence at least 1 almost always.

If we set
$$L_\alpha(z) = \frac{(\cosh \alpha) z + \sinh \alpha}{(\sinh \alpha) z + \cosh \alpha},$$

then, for any fixed θ the linear fractional transformations $T_{\theta,\alpha}$: $T_{\theta,\alpha}(z) = e^{i\theta} L_\alpha(e^{-i\theta} z)$ form a one-parameter group each member of which takes the sets $[z \mid |z| < 1]$ and $[z \mid |z| = 1]$ into themselves. Furthermore, any linear fractional transformation preserving these sets is of this form. We shall find necessary conditions on the coefficients σ_n and τ_n for the mutual absolute continuity of the measures associated with the processes $f_{\theta,\alpha}$:

$$f_{\theta,\alpha}(z) = f(T_{\theta,\alpha} z).$$

We take for L all finite linear combinations of the constant function and random variables of the form $u_k(z) = \mathfrak{R}(f^{(k)}(z))$ and $v_k(z) = \mathfrak{J}(f^{(k)}(z))$ for $k = 0, 1, \ldots$ and $|z| < 1$ (we have written $\mathfrak{R}$ and $\mathfrak{J}$ for the real and imaginary parts of a number and $f^{(k)}$ for the kth derivative of f). T_α is defined, for fixed θ, by:

$$T_\alpha u_k(z)=\mathcal{R}\left(\left(\frac{d}{dz}\right)^k f(T_{\theta,\alpha}z)\right) \quad \text{and} \quad T_\alpha v_k(z)=\mathcal{J}\left(\left(\frac{d}{dz}\right)^k f(T_{\theta,\alpha}z)\right).$$

$T_\alpha u_k(z)$ and $T_\alpha v_k(z)$ are linear combinations of

$$u_0(T_{\theta,\alpha}z), \ldots, u_k(T_{\theta,\alpha}z), \; v_0(T_{\theta,\alpha}z), \ldots, v_k(T_{\theta,\alpha}z)$$

with coefficients which are functions of z analytic for $|z|\leqslant 1$.

We wish to show that

$$\begin{aligned}D(T_\alpha u_k(z)+iT_\alpha v_k(z)) &\\ &=(e^{i\theta}-e^{-i\theta}z^2)(T_\alpha u_{k+1}(z)+iT_\alpha v_{k+1}(z))\\ &\quad -2ke^{-i\theta}z(T_\alpha u_k(z)+iT_\alpha v_k(z))-k(k-1)e^{-i\theta}(T_\alpha u_{k-1}(z)+iT_\alpha v_{k-1}(z))\end{aligned}$$

the last term being replaced by 0 if $k=0$. It will follow from this, as in the previous examples, that $DT_\alpha u_k(z)$ and $DT_\alpha v_k(z)$ are themselves in L, hence $L_2(P)$ differentiable in α, hence $L_2(P)$ continuous in α. We note first that

$$\begin{aligned}&\left\|\frac{f(T_{\theta,\alpha+\varepsilon}w)-f(T_{\theta,\alpha}w)}{\varepsilon}-f^{(1)}(T_{\theta,\alpha}w)\frac{(e^{i\theta}-e^{-i\theta}w^2)}{((\sinh\alpha)we^{-i\theta}+\cosh\alpha)^2}\right\|^2\\ &\qquad\leqslant C\sum_{n=1}^{\infty}\left|\frac{(T_{\theta,\alpha+\varepsilon}w)^n-(T_{\theta,\alpha}w)^n}{\varepsilon}-n\frac{(T_{\theta,\alpha}w)^{n-1}(e^{i\theta}-e^{-i\theta}w^2)}{((\sinh\alpha)e^{-i\theta}w+\cosh\alpha)^2}\right|^2\end{aligned}$$

which goes to 0 uniformly for $|w|\leqslant 1$. Hence, taking Γ to be a circle of sufficiently small radius r about z,

$$\begin{aligned}&\left\|\frac{T_{\alpha+\varepsilon}(u_k(z)+iv_k(z))-T_\alpha(u_k(z)+iv_k(z))}{\varepsilon}-\left(\frac{d}{dz}\right)^k\left(\frac{f^{(1)}(T_{\theta,\alpha}z)(e^{i\theta}-e^{-i\theta}z^2)}{((\sinh\alpha)ze^{-i\theta}+\cosh\alpha)^2}\right)\right\|\\ &=\left\|\frac{k!}{2\pi i}\int_\Gamma\left(\frac{f(T_{\theta,\alpha+\varepsilon}w)-f(T_{\theta,\alpha}w)}{\varepsilon}-\frac{f^{(1)}(T_{\theta,\alpha}w)(e^{i\theta}-e^{-i\theta}w^2)}{((\sinh\alpha)we^{-i\theta}+\cosh\alpha)^2}\right)\frac{1}{(w-z)^{k+1}}\,dw\right\|\\ &\leqslant\frac{k!}{2\pi}\frac{2\pi r}{r^{k+1}}\sup_{w\in\Gamma}\left\|\frac{f(T_{\theta,\alpha+\varepsilon}w)-f(T_{\theta,\alpha}w)}{\varepsilon}-f^{(1)}(T_{\theta,\alpha}w)\frac{(e^{i\theta}-e^{-i\theta}w^2)}{((\sinh\alpha)we^{-i\theta}+\cosh\alpha)^2}\right\|\end{aligned}$$

and this goes to 0 as ε goes to 0 so

$$\begin{aligned}DT_\alpha(u_k(z)+iv_k(z))&=\left(\frac{d}{dz}\right)^k\left[\frac{f^{(1)}(T_{\theta,\alpha}z)(e^{i\theta}-e^{-i\theta}z^2)}{((\sinh\alpha)ze^{-i\theta}+\cosh\alpha)^2}\right]\\ &=\left(\frac{d}{dz}\right)^k\left[\left(\frac{d}{dz}f(T_{\theta,\alpha}z)\right)(e^{i\theta}-e^{-i\theta}z^2)\right]\end{aligned}$$

from which the desired formula follows.

The sets (x_k) and (y_k) form a complete orthonormal set and are contained in L since

$$x_k = \frac{1}{\sigma_k\, k!}\, u_k(0) \text{ and } y_k = \frac{1}{\tau_k\, k!}\, v_k(0).$$

We have

$$\int Dx_k\, dP = \int Dy_k\, dP = 0 \quad \text{and}$$

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left\{ \left(\int (x_j Dx_k + x_k Dx_j)\, dP \right)^2 + \left(\int (x_j Dy_k + y_k Dx_j)\, dP \right)^2 \right.$$
$$\left. + \left(\int (y_j Dx_k + x_k Dy_j)\, dP \right)^2 + \left(\int (y_j Dy_k + y_k Dy_j)\, dP \right)^2 \right\}$$
$$= \sum_{j=0}^{\infty} \left\{ \cos^2\theta \left[(j+1)\frac{\sigma_{j+1}}{\sigma_j} - j\frac{\sigma_j}{\sigma_{j+1}} \right]^2 + \sin^2\theta \left[(j+1)\frac{\tau_{j+1}}{\sigma_j} - j\frac{\sigma_j}{\tau_{j+1}} \right]^2 \right.$$
$$\left. + \sin^2\theta \left[(j+1)\frac{\sigma_{j+1}}{\tau_j} - j\frac{\tau_j}{\sigma_{j+1}} \right]^2 + \cos^2\theta \left[(j+1)\frac{\tau_{j+1}}{\tau_j} - j\frac{\tau_j}{\tau_{j+1}} \right]^2 \right\}.$$

THEOREM 5.5. *If the four series,*

$$\sum_{j=0}^{\infty} \left[(j+1)\frac{\sigma_{j+1}}{\sigma_j} - j\frac{\sigma_j}{\sigma_{j+1}} \right]^2, \quad \sum_{j=1}^{\infty} \left[(j+1)\frac{\tau_{j+1}}{\sigma_j} - j\frac{\sigma_j}{\tau_{j+1}} \right]^2,$$
$$\sum_{j=1}^{\infty} \left[(j+1)\frac{\sigma_{j+1}}{\tau_j} - j\frac{\tau_j}{\sigma_{j+1}} \right]^2, \text{ and } \sum_{j=1}^{\infty} \left[(j+1)\frac{\tau_{j+1}}{\tau_j} - j\frac{\tau_j}{\tau_{j+1}} \right]^2.$$

all converge, then the measures associated with the processes $f_{\theta,\alpha}(z)$ are mutually absolutely continuous, i.e., $f(z)$ is equivalent to any process gotten from it by applying a linear fractional transformation taking $|z| < 1$ into itself and $|z| = 1$ into itself.

Proof. For each θ the function

$$\varphi = \sum_{j=0} \left\{ \cos\theta \left[(j+1)\frac{\sigma_{j+1}}{\sigma_j} - j\frac{\sigma_j}{\sigma_{j+1}} \right] x_j x_{j+1} + \sin\theta \left[-(j+1)\frac{\tau_{j+1}}{\sigma_j} + i\frac{\sigma_j}{\tau_{j+1}} \right] x_j y_{j+1} \right.$$
$$\left. + \sin\theta \left[(j+1)\frac{\sigma_{j+1}}{\tau_j} - j\frac{\tau_j}{\sigma_{j+1}} \right] y_j x_{j+1} + \cos\theta \left[(j+1)\frac{\tau_{j+1}}{\tau_j} - j\frac{\tau_j}{\tau_{j+1}} \right] y_j y_{j+1} \right\}$$

is in $L_2(P)$ by hypothesis and we can show as in the proof of Lemma 4.6 that $\int \varphi w_1 w_2\, dP = \int (w_1 Dw_2 + w_2 Dw_1)\, dP$ for all w_i which are finite linear combinations of the x_k's and y_j's. It follows from this by straightforward calculations that

$$\int \varphi u_0(z)\, u_0(w)\, dP = \int D(u_0(z)\, u_0(w))\, dP,$$

$$\int \varphi u_0(z)\, v_0(w)\, dP = \int D(u_0(z)\, v_0(w))\, dP \quad \text{and}$$

$$\int \varphi v_0(z)\, v_0(w)\, dP = \int D(v_0(z)\, v_0(w))\, dP.$$

Because of the $L_2(P)$ continuity of $f(z)$ and the fact that all the $f(z)$ are (complex) Gaussian, we have, for properly chosen contours Γ_1 and Γ_2

$$\begin{aligned}
\int \varphi u_k(z)\, u_l(w)\, dP &= \frac{k!\, l!}{(2\pi i)^2} \int \varphi \int_{\Gamma_1} \mathcal{R}\left(\frac{f(z')\,dz'}{(z'-z)^{k+1}}\right) \int_{\Gamma_2} \mathcal{R}\left(\frac{f(w')\,dw'}{(w'-w)^{l+1}}\right) dP \\
&= \frac{k!\, l!}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \int \varphi\, \mathcal{R}\left(\frac{f(z')\,dz'}{(z'-z)^{k+1}}\right) \mathcal{R}\left(\frac{f(w')\,dw'}{(w'-w)^{l+1}}\right) dP \\
&= \frac{k!\, l!}{(2\pi i)^2} \int_{\Gamma_1} \int_{\Gamma_2} \left\{ \mathcal{R}\left(\frac{f(z')\,dz'}{(z'-z)^{k+1}}\right) \mathcal{R}\left(\frac{(e^{i\theta} - e^{-i\theta}(w')^2) f^{(1)}(w')\,dw'}{(w'-w)^{l+1}((\sinh\alpha)\, w e^{-i\theta} + \cosh\alpha)^2}\right) \right. \\
&\quad \left. + \mathcal{R}\left(\frac{(e^{i\theta} - e^{-i\theta}(z')^2) f^{(1)}(z')\,dz'}{(z'-z)^{k+1}((\sinh\alpha)\, z e^{-i\theta} + \cosh\alpha)^2}\right) \mathcal{R}\left(\frac{f(w')\,dw'}{(w'-w)^{l+1}}\right) \right\} dP \\
&= \int \left\{ u_k(z)\, Du_l(w) + (Du_k(z))\, u_l(w) \right\} dP.
\end{aligned}$$

Similar arguments show that $\int \varphi u_k(z)\, v_l(w)\, dP = \int (u_k(z)\, Dv_l(w) + v_l(w)\, Du_k(z))\, dP$ and $\int \varphi v_k(z)\, v_l(w)\, dP = \int (v_k(z)\, Dv_l(w) + v_l(w)\, Dv_k(z))\, dP$ and this completes the proof of the theorem.

Theorem 4.2 is not applicable to this example since

$$\begin{aligned}
&\left\| \sum_{j=0}^{n-1} \cos\theta \left[(j+1)\frac{\sigma_{j+1}}{\sigma_j} - j\frac{\sigma_j}{\sigma_{j+1}}\right] x_j x_{j+1} + \sin\theta \left[-(j+1)\frac{\tau_{j+1}}{\sigma_j} + j\frac{\sigma_j}{\tau_j}\right] x_j y_{j+1} \right. \\
&\quad + \sin\theta \left[(j+1)\frac{\sigma_{j+1}}{\tau_j} - j\frac{\tau_j}{\sigma_{j+1}}\right] y_j x_{j+1} + \cos\theta \left[(j+1)\frac{\tau_{j+1}}{\tau_j} - j\frac{\tau_j}{\tau_{j+1}}\right] y_j y_{j+1} \\
&\quad \left. - \sum_{j=0}^{n} (x_j\, Dx_j + y_j\, Dy_j) \right\|^2 \\
&= \left\| -\cos\theta\, (n+1)\frac{\sigma_{n+1}}{\sigma_n} x_n x_{n+1} + \sin\theta\, (n+1)\frac{\tau_{n+1}}{\sigma_n} x_n y_{n+1} \right. \\
&\quad \left. - \sin\theta\, (n+1)\frac{\sigma_{n+1}}{\tau_n} y_n x_{n+1} + \cos\theta\, (n+1)\frac{\tau_{n+1}}{\tau_n} y_n y_{n+1} \right\|^2 \\
&= \cos^2\theta\, (n+1)^2 \left[\left(\frac{\sigma_{n+1}}{\sigma_n}\right)^2 + \left(\frac{\tau_{n+1}}{\tau_n}\right)^2\right] + \sin^2\theta\, (n+1)^2 \left[\left(\frac{\tau_{n+1}}{\sigma_n}\right)^2 + \left(\frac{\sigma_{n+1}}{\tau_n}\right)^2\right]
\end{aligned}$$

and in order to have this go to 0 and the series in the theorem converge, we would need both

$$\lim_{n\to\infty} (n+1)\frac{\sigma_{n+1}}{\sigma_n}=0 \quad \text{and} \quad \lim_{n\to\infty}\left|(n+1)\frac{\sigma_{n+1}}{\sigma_n}-n\frac{\sigma_n}{\sigma_{n+1}}\right|=0$$

which would imply

$$\lim_{n\to\infty} n(n+1)=\lim_{n\to\infty}\left((n+1)\frac{\sigma_{n+1}}{\sigma_n}\right)\left(n\frac{\sigma_n}{\sigma_{n+1}}\right)=0.$$

If we set $\sigma_k=\tau_k=\varrho_k k^{-\frac{1}{2}}$, we get the process $f(z)=\sum_{k=0}^{\infty}\varrho_k k^{-\frac{1}{2}}(a_k+ib_k)z^k$ whose boundary behavior has been extensively studied (see [5] for example). The measure associated with this $f(z)$ is rotationally invariant and the conditions of Theorem 5.5 boil down to the convergence of $\sum_{k=1}^{\infty}k(k+1)(\varrho_{k+1}/\varrho_k-\varrho_k/\varrho_{k+1})^2$ in this case. If $\varrho_k=k^{\varepsilon}$, the terms in this series converge to $4\varepsilon^2$ so the series diverges unless $\varepsilon=0$, i.e., the only process of the form $f(z)=\sum_{k=0}^{\infty}k^{-\gamma}(x_k+iy_k)z^k$ to which Theorem 4.1 applies is the one with $\gamma=\frac{1}{2}$.

Example 5. A test for the independence of processes.

Let $z(t)$ be a vector process $\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}$ defined on an interval I, $x(t)$ and $y(t)$ being independent Gaussian processes with means $m_x(t)$ and $m_y(t)$ which are square integrable on I and correlation functions $R_x(s,t)$ and $R_y(s,t)$ which are square integrable on $I\times I$. Let $(\xi_k(t))$ be the eigenfunctions of the integral operator R_x with kernel $R_x(s,t)$ and (λ_k) be the associated eigenvalues. Let $(\eta_k(t))$ and (μ_k) be the eigenfunctions and eigenvalues of R_y. We assume that all λ_k and μ_k are strictly positive in order to avoid some inessential complexities. We want to compare $z(t)$ with the 'mixed' process $z_\alpha(t)=\begin{pmatrix}x(t)\cos\alpha+y(t)\sin\alpha\\ -x(t)\sin\alpha+y(t)\cos\alpha\end{pmatrix}$.

The set L is to comprise all finite linear combinations of 1 and functions of the form $x_f=\int_I x(t)f(t)\,dt$ and $y_f=\int_I y(t)f(t)\,dt$ for square integrable f. T_α is defined on L by: $T_\alpha x_f=x_f\cos\alpha+y_f\sin\alpha$ and $T_\alpha y_f=-x_f\sin\alpha+y_f\cos\alpha$. It is evident that $DT_\alpha x_f=-x_f\sin\alpha+y_f\cos\alpha$ and $DT_\alpha y_f=-x_f\cos\alpha-y_f\sin\alpha$ and that these are $L_2(P)$ continuou in α. The random variables

$$x_k=\frac{1}{\sqrt{\lambda_k}}\int_I (x-m_x)(t)\,\xi_k(t)\,dt$$

and

$$y_k=\frac{1}{\sqrt{\mu_k}}\int_I (y-m_y)(t)\,\eta_k(t)\,dt$$

for $k=1,2,\dots$ form a complete orthonormal subset of L and the following formulas, in which we have written (f,g) for $\int_I f(t)g(t)\,dt$, are easily verified:

(1) $$Dx_k = \frac{1}{\sqrt{\lambda_k}}(y,\xi_k)$$

(2) $$Dy_k = \frac{-1}{\sqrt{\mu_k}}(x,\eta_k)$$

(3) $$\int Dx_k\,dP = \frac{1}{\sqrt{\lambda_k}}(m_y,\xi_k)$$

(4) $$\int Dy_k\,dP = \frac{-1}{\sqrt{\mu_k}}(m_x,\eta_k)$$

(5) $$\int (x_k Dx_l + x_l Dx_k)\,dP = \int (y_k Dy_l + y_l Dy_k)\,dP = 0$$

(6) $$\int (x_k Dy_l + y_l Dx_k)\,dP = \left(\sqrt{\frac{\mu_l}{\lambda_k}} - \sqrt{\frac{\lambda_k}{\mu_l}}\right)(\eta_l,\xi_k).$$

THEOREM 5.6. *If the series,*

$$\sum_{j=1}^{\infty}\frac{1}{\lambda_j}(m_y,\xi_j)^2,\quad \sum_{j=1}^{\infty}\frac{1}{\mu_j}(m_x,\eta_j)^2,\quad \textit{and}\quad \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left(\sqrt{\frac{\mu_j}{\lambda_k}} - \sqrt{\frac{\lambda_k}{\mu_j}}\right)^2(\eta_j,\xi_k)^2$$

all converge, then the measures associated with the vector processes z_α are mutually absolutely continuous and Theorem 4.1 holds for this example.

Proof. As in the previous examples the convergence of the three series implies the existence of a φ in $L_2(P)$ satisfying $\int \varphi w_1 w_2\,dP = \int (w_1 Dw_2 + w_2 Dw_1)\,dP$ for w_i which are finite linear combinations of 1 and the x_k's and y_k's. For any f and g in $L_2(dt)$,

$$\begin{aligned}\int \varphi x_f x_g\,dP &= \lim_{N\to\infty}\int \varphi\left(\sum_{n=1}^{N}\sqrt{\lambda_n}(f,\xi_n)x_n + (m_x,f)\right)\left(\sum_{m=1}^{N}\sqrt{\lambda_n}(g,\xi_n)x_n + (m_x,g)\right)dP\\ &= \lim_{N\to\infty}\left\{\int\left(\sum_{n=1}^{N}\sqrt{\lambda_n}(f,\xi_n)x_n + (m_x,f)\right)\left(\sum_{m=1}^{N}(g,\xi_n)y_n\right)dP\right.\\ &\qquad\left. + \int\left(\sum_{n=1}^{N}(f,\xi_n)y_n\right)\left(\sum_{m=1}^{N}\sqrt{\lambda_n}(g,\xi_n)x_n + (m_x,g)\right)dP\right\}\\ &= \int (x_f Dx_g + x_g Dx_f)\,dP.\end{aligned}$$

We can show by similar calculations that $\int \varphi x_f y_g \, dP = \int (x_f Dy_g + y_g Dx_f)$ and that $\int \varphi y_f y_g \, dP = \int (y_f Dy_g + y_g Dy_f) \, dP$. Hence Theorem 4.1 applies and the theorem is proved.

It is interesting to note that the convergence of the first series is equivalent to the mutual absolute continuity of the measures associated with the processes $x(t) - m_x(t) + \alpha m_y(t)$ and the convergence of the second series is equivalent to the mutual absolute continuity of the measures associated with the processes $y(t) - m_y(t) + \alpha m_x(t)$ [10].

Example 6. Adding independent Gaussian processes.

Consider the vector process $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ where x and y are independent Gaussian processes on an interval I with mean 0 and correlation functions R_x and R_y which are square integrable on $I \times I$. We wish to compare this process with $\begin{pmatrix} x(t) + \alpha y(t) \\ y(t) \end{pmatrix}$. We define L to be all random variables of the form

$$c + x_f + y_g = c + \int_I x(t) f(t) \, dt + \int_I y(s) g(s) \, ds$$

for square integrable functions f and g and real numbers c. T_α is defined by: $T_\alpha(c + x_f + y_g) = c + x_f + \alpha y_f + y_g$, giving $Dx_f = y_f$ and $Dy_g = 0$. If the integral operator R_x has eigenfunctions (ξ_k) and eigenvalues (λ_k) and the operator R_y has eigenfunctions (η_k) and eigenvalues (μ_k), then the random variables

$$1, \quad x_k = \frac{1}{\sqrt{\lambda_k}} x_{\xi_k}, \quad k = 1, \ldots \quad \text{and} \quad y_k = \frac{1}{\sqrt{\mu_k}} y_{\eta_k}, \quad k = 1, \ldots$$

form a complete orthonormal set.

THEOREM 5.7. *If*

$$\sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \frac{\mu_j}{\lambda_k} \left[\int_I \eta_j(t) \xi_k(t) \, dt \right]^2 < \infty$$

then the measures P_α associated with the vector stochastic processes $\begin{pmatrix} x(t) + \alpha y(t) \\ y(t) \end{pmatrix}$ are mutually absolutely continuous and some subsequence of

$$\sum_{k=1}^{n} \left(\frac{\alpha x_k}{\sqrt{\lambda_k}} y_{\xi_k} - \frac{1}{2} \frac{\alpha^2}{\lambda_k} y_{\xi_k}^2 \right)$$

converges almost everywhere $(dP d\alpha)$ to $\log dP_\alpha / dP$.

Proof. The convergence of the double series guarantees the convergence of $\varphi = \lim \sum_{k=1}^{n} x_k D x_k$ since

$$\int \left| \sum_{k=n+1}^{m} x_k D x_k \right|^2 dP = \sum_{k=n+1}^{m} \int \left(\frac{1}{\sqrt{\lambda_k}} x_k y_{\xi_k} \right)^2 dP$$

$$= \sum_{k=n+1}^{m} \frac{1}{\lambda_k} \int_I \int_I R_y(s,t)\, \xi_k(s) \xi_k(t)\, ds\, dt$$

$$= \sum_{j=1}^{\infty} \sum_{k=n+1}^{m} \frac{\mu_j}{\lambda_k} (\eta_j, \xi_k)^2,$$

where we have written (η_j, ξ_k) for $\int_I \eta_j(t) \xi_k(t)\, dt$. We can show as in Lemma 4.6 that $\int \varphi x_f x_g dP = \int (y_f x_g + x_f y_g)\, dP$ whenever f and g are finite linear combinations of the ξ_k's. x_{f_n} converges to x_f and y_{f_n} to y_f in $L_2(P)$ whenever f_n converges to f in $L_2(dt)$ and it follows easily that assumption (iv) is satisfied in this case.

The theorem now follows from Theorems 4.1 and 4.2 since

$$\frac{1}{2} \sum_{k=1}^{n} (x_k^2 - (T_{-\alpha} x_k)^2 + y_k^2 - (T_{-\alpha} y_k)^2) = \sum_{k=1}^{n} \left(\frac{\alpha x_k}{\sqrt{\lambda_k}} y_{\xi_k} - \frac{1}{2} \frac{\alpha^2}{\lambda_k} y_{\xi_k}^2 \right).$$

Now
$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \frac{\mu_j}{\lambda_k} (\xi_k, \eta_j)^2 = \sum_{k=1}^{\infty} \frac{1}{\lambda_k} (R_y \xi_k, \xi_k) = \int \sum_{k=1}^{\infty} \frac{1}{\lambda_k} (y_{\xi_k})^2 dP$$

so the convergence of the double series implies that the y sample functions are in the range of $R_x^{\frac{1}{2}}$ with probability 1, i.e., that the measures Q_y associated with the processes $x(t) + y(t)$ are absolutely continuous with respect to P almost always. The expression for the likelihood ratio is exactly $dP_\alpha / dP\ (x, y) = dQ_y / dP\ (x)$ as one would expect. Conversely, as was shown in [11], the condition that the y sample functions be in the range of $R_x^{\frac{1}{2}}$ with probability 1 is necessary for the mutual absolute continuity of the P_α.

References

[1]. Doob, J. L., *Stochastic Processes.* New York, John Wiley and Sons, 1953.

[2]. Dunford, N. & Schwartz, J. T., *Linear Operators.* New York, Interscience Publishers, Inc., 1958.

[3]. Feldman, J., Equivalence and perpendicularity of Gaussian processes. *Pacific J. Math.*, 8 (1958), 699–708.

[4]. Hille, E. & Phillips, R. S., *Functional Analysis and Semigroups.* Providence, R. I., American Mathematical Society, 1957.

[5]. Hunt, G. A., Random Fourier transforms. *Trans. Amer. Math. Soc.* 71 (1951), 38–69.

[6]. Kelly, E., Reed, I. S., & Root, W. L., The detection of radar echoes in noise. *J. Soc. Indust. Appl. Math.*, 8 (1960), 309–341.

[7]. Loomis, L., *An Introduction to Abstract Harmonic Analysis*. New York, Van Nostrand, 1953.

[8]. Pitcher, T. S., Likelihood ratios for stochastic processes related by groups of transformations. *Illinois J. Math.*, 7 (1963), 396–414.

[9]. —— Likelihood ratios for stochastic processes related by groups of transformations, II. *Illinois J. Math.* To appear.

[10]. —— Likelihood ratios of Gaussian processes. *Ark Mat.*, 4 (1959), 35–44.

[11]. —— On the sample functions of processes which can be added to a Gaussian process. *Ann. Math. Statist.*, 34 (1963), 329–333.

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