Fredholm pseudo-differential operators on weighted Sobolev spaces

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1. Introduction

Let $m \in (-\infty, \infty)$. Define S^m by

$$S^m = \{\sigma \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n): |D_x^\beta D_\xi^\alpha \sigma(x, \xi)| \leqq C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}\}.$$

If $\sigma \in S^m$, then we define the pseudo-differential operator T_σ with symbol σ on $\mathcal{S}$ (the Schwartz space) by

$$(T_\sigma f)(x) = \int_{\mathbf{R}^n} e^{-2\pi i x \cdot \xi} \sigma(x, \xi) \hat{f}(\xi)\, d\xi, \quad f \in \mathcal{S}.$$

It can be shown that T_σ can be extended to a linear operator from the space $\mathcal{S}'$ of tempered distributions into $\mathcal{S}'$.

Suppose that $\sigma \in S^0$. Then it is well known that T_σ is a bounded linear operator from $L^p(\mathbf{R}^n)$ into $L^p(\mathbf{R}^n)$ for $1<p<\infty$. An immediate consequence of this result is that every T_σ with $\sigma \in S^m$ is a bounded linear operator from $L^p_{s+m}(\mathbf{R}^n)$ into $L^p_s(\mathbf{R}^n)$ for $1<p<\infty$ and $-\infty<s<\infty$. Here $L^p_s(\mathbf{R}^n)$ stands for the Sobolev space of order s. See Calderón [2] or Stein [19, Chapter 5]. Prompted by the L^p-boundedness result, it is obviously of interest to characterize the nonnegative functions w on $\mathbf{R}^n$ for which every T_σ with $\sigma \in S^0$ is a bounded linear operator on $L^p(\mathbf{R}^n, w\,dx)$ for $1<p<\infty$.

Let $1<p<\infty$. A nonnegative function w is said to be in $A_p(\mathbf{R}^n)$ if $w \in L^1_{\text{loc}}(\mathbf{R}^n)$ and

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-\frac{1}{p-1}}\,dx\right)^{p-1} < \infty$$

where the supremum is taken over all cubes Q in $\mathbf{R}^n$. See Coifman and Fefferman [5] and Muckenhoupt [17] for basic properties of functions in $A_p(\mathbf{R}^n)$. Miller has recently shown in [16] that a necessary and sufficient condition for every T_σ

with $\sigma \in S^0$ be a bounded linear operator on $L^p(\mathbf{R}^n, w dx)$ for $1<p<\infty$ is that $w\in A_p(\mathbf{R}^n)$.

In [16] Miller has defined weighted Sobolev spaces $L_s^p(\mathbf{R}^n, w\,dx)$ and developed some basic properties of these spaces. As in the unweighted case, an immediate consequence of the weighted L^p-boundedness result for pseudo-differential operators is that every T_σ with $\sigma\in S^m$ is a bounded linear operator from $L^p_{s+m}(\mathbf{R}^n, w\,dx)$ into $L_s^p(\mathbf{R}^n, w\,dx)$ for $1<p<\infty$ and $-\infty<s<\infty$ if $w\in A_p(\mathbf{R}^n)$.

In this paper we obtain some more useful results for weighted Sobolev spaces and give, as an application, sufficient conditions on $\sigma\in S^m$ such that T_σ is Fredholm on weighted Sobolev spaces.

In Section 2 we study the weighted L^p-boundedness result for T_σ with $\sigma\in S^0$. Miller's proof of the sufficiency of the condition in [16] depends on the well known L^p-boundedness result, the Fefferman—Stein sharp function operator in [7] and various versions of the Hardy—Littlewood maximal function operator. Weighted norm inequalities for quite general singular integral operators including T_σ with $\sigma\in S^0$ have been derived in Coifman and Fefferman [5]. Suggested by the techniques in Stein [20], we give another proof of the sufficiency part of Miller's result. See Coifman and Meyer [6] for the use of similar techniques in studying pseudo-differential operators. Not only is our proof independent of the well known classical L^p-boundedness result, it also produces a more precise inequality which is useful for studying weighted Sobolev spaces in Section 4 and Fredholm opertors in Section 5. See Grushin [8, Theorem 3.1]. Our proof depends on two results on weighted norm inequalities. These are formulated in Theorems 1.1 and 1.2.

Theorem 1.1. *Let $w\in A_p(\mathbf{R}^n)$ for $1<p<\infty$. Then there is a constant $C>0$, depending only on p, w and n, such that*

$$\|Mf\|_p \leqq C\|f\|_p,\quad f\in\mathcal{S}.$$

Here Mf is the usual Hardy—Littlewood maximal function of f. The proof of Theorem 1.1 can be found in Muckenhoupt [17] or Coifman and Fefferman [5].

Theorem 1.2. *Let $k>n$ and $w\in A_p(\mathbf{R}^n)$ for $1<p<\infty$. Suppose that $m\in C^k(\mathbf{R}^n-\{0\})$ satisfies*

$$|(D^\alpha m)(\xi)|\leqq B|\xi|^{-|\alpha|},\quad |\alpha|\leqq k.$$

Then the operator $f\to Tf$ defined on $\mathcal{S}$ by

$$(Tf)(x)=\int_{\mathbf{R}^n} e^{-2\pi i x\cdot\xi} m(\xi)\hat{f}(\xi)\,d\xi$$

can be extended to a bounded linear operator on $L^p(\mathbf{R}^n, w\,dx)$. Moreover, there is a constant $C>0$, depending only on n, p and w, such that

$$\|Tf\|_p\leqq CB\|f\|_p,\quad f\in\mathcal{S}.$$

Theorem 1.2 is in fact a special case of a weighted version of the Hörmander—Marcinkiewicz—Mihlin Multiplier Theorem obtained by Kurtz in [13]. See also Kurtz and Wheeden [14]. Our proof also depends on two fairly elementary properties of pseudo-differential operators with symbols in S^0. For the sake of completeness, they are proved in Section 3.

In Section 4 we obtain some results on weighted Sobolev spaces defined in Miller [16]. Specifically, we prove a version of the Sobolev's Theorem, an interpolation result by the complex method in Calderón [3, 4] and a compact embedding theorem.

In Section 5 we give sufficient conditions on $\sigma\in S^m$ such that T_σ is Fredholm on weighted Sobolev spaces. Our results include those given in Grushin [8]. Fredholm pseudo-differential operators have been studied in Beals [1], Kumano-go [11], Kumano-go and Taniguchi [12], Hörmander [9] and others. Information about the indices are also given in Kumano-go [11] and Hörmander [9].

2. A weighted norm inequality

Let $m\in(-\infty, \infty)$. Define S^m by

$$S^m = \{\sigma\in C^\infty(\mathbf{R}^n\times\mathbf{R}^n)\colon |D_x^\beta D_\xi^\alpha\sigma(x, \xi)| \leqq C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}\}.$$

If $\sigma\in S^0$, then for all multi-indices α and β, we let

$$K_{\alpha\beta}(\sigma) = \sup |D_x^\beta D_\xi^\alpha\sigma(x, \xi)|(1+|\xi|)^{-|\alpha|}.$$

Theorem 2.1. *Let $\sigma\in S^0$ and $w\in A_p(\mathbf{R}^n)$ for $1<p<\infty$. Then T_σ is a bounded linear operator from $L^p(\mathbf{R}^n, w\,dx)$ into $L^p(\mathbf{R}^n, w\,dx)$. Moreover, for any sufficiently large positive integer N, there is a constant $C_N>0$ such that*

$$\|T_\sigma f\|_p \leqq C_N\|f\|_p \sum_{|\alpha+\beta|\leqq N} K_{\alpha\beta}(\sigma)$$

for all f in $L^p(\mathbf{R}^n, w\,dx)$.

Remark. Using the density of $\mathcal{S}$ in $L^p(\mathbf{R}^n, w\,dx)$, it is sufficient to prove Theorem 2.1 for functions f in $\mathcal{S}$.

Proof of Theorem 2.1. Partition $\mathbf{R}^n$ into cubes $\mathbf{R}^n=U_m Q_m$, where Q_m is the cube with size one and centre at $m\in\mathbf{Z}^n$. Let $\eta\in C_0^\infty(\mathbf{R}^n)$ be such that $\eta(x)=1$ for $x\in Q_0$. For $m\in\mathbf{Z}^n$, set

$$\sigma_m(x, \xi) = \eta(x-m)\sigma(x, \xi).$$

Obviously, $T_{\sigma_m}=\eta(x-m)T_\sigma$ and

$$\int_{Q_m} |(T_\sigma f)(x)|^p w(x)\,dx \leqq \int_{\mathbf{R}^n} |(T_{\sigma_m} f)(x)|^p w(x)\,dx \tag{2.1}$$

for all f in $\mathscr{S}$. Since $\sigma_m(x, \xi)$ has compact support in x, Fubini's Theorem implies that

$$(2.2) \qquad (T_{\sigma_m} f)(x) = \int_{\mathbf{R}^n} \left\{ \int_{\mathbf{R}^n} \hat{\sigma}_m(\lambda, \xi) e^{-2\pi i x \cdot \xi} \hat{f}(\xi)\, d\xi \right\} e^{-2\pi i \lambda \cdot x}\, d\lambda$$

where

$$\hat{\sigma}_m(\lambda, \xi) = \int_{\mathbf{R}^n} e^{2\pi i \lambda \cdot x} \sigma_m(x, \xi)\, dx.$$

Claim 2.2. For all multi-indices α and positive integers N, there is a constant $C_N > 0$ such that

$$|\partial_\xi^\alpha \hat{\sigma}_m(\lambda, \xi)| \leqq C_N \left\{ \sum_{|\beta| \leqq N} K_{\alpha\beta}(\sigma) \right\} (1+|\xi|)^{-|\alpha|} (1+|\lambda|)^{-N}.$$

The proof of Claim 2.2, though easy, will be given in Section 3. This Claim and Theorem 1.2 imply that the operator $f \to T_\lambda f$ defined on $\mathscr{S}$ by

$$(2.3) \qquad (T_\lambda f)(x) = \int_{\mathbf{R}^n} \hat{\sigma}_m(\lambda, \xi) e^{-2\pi i x \cdot \xi} \hat{f}(\xi)\, d\xi$$

can be extended to a bounded linear operator on $L^p(\mathbf{R}^n, w\, dx)$. Moreover, for any sufficiently large positive integer N, there is a constant $C_N > 0$ such that

$$(2.4) \qquad \|T_\lambda f\|_p \leqq C_N (1+|\lambda|)^{-N} \left\{ \sum_{|\alpha+\beta| \leqq N} K_{\alpha\beta}(\sigma) \right\} \|f\|_p$$

for all f in $\mathscr{S}$. Using (2.2), (2.3), (2.4) and Minkowski's inequality in integral form,

$$\|T_{\sigma_m} f\|_p \leqq C_N \left\{ \sum_{|\alpha+\beta| \leqq N} K_{\alpha\beta}(\sigma) \right\} \int_{\mathbf{R}^n} (1+|\lambda|)^{-N}\, d\lambda \|f\|_p.$$

By choosing N large enough, there is a constant $C_N > 0$ such that $\|T_{\sigma_m} f\|_p \leqq C_N \left\{ \sum_{|\alpha+\beta| \leqq N} K_{\alpha\beta}(\sigma) \right\} \|f\|_p$ and hence by (2.1),

$$(2.5) \qquad \int_{Q_m} |(T_\sigma f)(x)|^p w(x)\, dx \leqq C_N^p \left\{ \sum_{|\alpha+\beta| \leqq N} K_{\alpha\beta}(\sigma) \right\}^p \|f\|_p^p$$

for all f in $\mathscr{S}$. Now we need to represent T_σ as a singular integral operator. Precisely, we give

Claim 2.3. *Let* $K(x, z) = \int_{\mathbf{R}^n} e^{-2\pi i z \cdot \xi} \sigma(x, \xi)\, d\xi$ *in distribution sense. Then*

(i) $K(x, z)$ *is a function when* $|z| \neq 0$;

(ii) $|K(x, z)| \leqq C_N |z|^{-N} \sum_{|\alpha| \leqq N} K_{\alpha\beta}(\sigma)$ *for* N *large enough*;

(iii) *for* $x_0 \in \mathbf{R}^n$ *and* $f \in \mathscr{S}$ *vanishin g in a neighbourhood of* x_0,

$$(T_\sigma f)(x_0) = \int_{\mathbf{R}^n} K(x_0, x_0 - z) f(z)\, dz.$$

The proof of Claim 2.3 will also be given in Section 3.

Let Q_m^* be the double of Q_m. Let $\varphi \in C_0^\infty(\mathbf{R}^n)$ be such that $0 \leqq \varphi(x) \leqq 1$ for all x in $\mathbf{R}^n$ and $\varphi(x) = 1$ in a neighbourhood of Q_m^*. Write $f = f_1 + f_2$, where $f_1 = \varphi f$ and $f_2 = (1-\varphi) f$. Then $T_\sigma f = T_\sigma f_1 + T_\sigma f_2$. Let $I_m = \int_{Q_m} |(T_\sigma f)(x)|^p w(x) dx$ and $I_m' = \int_{Q_m} |(T_\sigma f_2)(x)|^p w(x) dx$. Then for any sufficiently large positive integer N, (2.5) and Claim 2.3 imply that there are positive constants C_N and C_{2N} such that

$$(2.6) \qquad I_m \leqq 2^p C_N^p \{\textstyle\sum_{|\alpha+\beta|\leqq N} K_{\alpha\beta}(\sigma)\}^p \|f_1\|_p^p + 2^p I_m'$$

and

$$(2.7) \qquad |T_\sigma f_2)(x)| \leqq C_{2N}\{\textstyle\sum_{|\alpha|\leqq 2N} K_{\alpha 0}(\sigma)\}\int_{\mathbf{R}^n - Q_m^*} \frac{(1+|m-z|)^{-N}|f_2(z)|}{(1+|x-z|)^N}\,dz.$$

Let $(1+|m-z|)^{-N} f_2(z) = f_{2,m,N}(z)$. Then by (2.7), there is a constant $C>0$ such that

$$|T_\sigma f_2)(x)| \leqq CC_{2N}\{\textstyle\sum_{|\alpha|\leqq 2N} K_{\alpha 0}(\sigma)\}(Mf_{2,m,N})(x)$$

where $Mf_{2,m,N}$ is the Hardy—Littlewood maximal function of $f_{2,m,N}$. See Stein [19, p. 62—64]. Hence by Theorem 1.1, there is a constant $C'>0$ such that

$$(2.8) \qquad I_m' \leqq C' C_{2N}\{\textstyle\sum_{|\alpha|\leqq 2N} K_{\alpha 0}(\sigma)\}^p \int_{\mathbf{R}^n} \frac{|f_2(z)|^p w(z)}{(1+|m-z|)^{Np}}\,dz.$$

So for any sufficiently large positive integer N, (2.6) and (2.8) imply that there is a constant $C_N>0$ such that

$$I_m \leqq C_N^p \{\textstyle\sum_{|\alpha+\beta|\leqq N} K_{\alpha\beta}(\sigma)\}^p \left\{\int_{Q_m^*} |f(x)|^p w(x)\,dx + \int_{\mathbf{R}^n} \frac{|f_2(x)|^p w(x)}{(1+|m-x|)^{Np}}\,dx\right\}.$$

Hence by summing over $\mathbf{Z}^n$ and choosing N large enough, we complete the proof.

3. Proofs of the Claims

We prove in this section the two claims in Section 2.

Proof of Claim 2.2. Let β be an arbitrary multi-index. Then by integration by parts and Leibnitz's rule, $(2\pi i\lambda)^\beta (\partial_\xi^\alpha \hat{\sigma}_m)(\lambda, \xi)$ is equal to

$$(-1)^\beta \textstyle\sum_{\gamma\leqq\beta} \binom{\beta}{\gamma} \int_{\mathbf{R}^n} e^{2\pi i\lambda\cdot x} (\partial_x^\gamma \eta)(x-m)(\partial_x^{\beta-\gamma}\partial_\xi^\alpha \sigma)(x,\xi)\,dx.$$

Using the properties of η and the fact that $\sigma\in S^0$, there is a constant $C_\beta>0$ such that

$$|(2\pi i\lambda)^\beta(\partial_\xi^\alpha \hat{\sigma}_m)(\lambda,\xi)| \leqq C_\beta(1+|\xi|)^{-|\alpha|} \textstyle\sum_{\gamma\leqq\beta} K_{\alpha\gamma}(\sigma).$$

The claim then follows easily from the preceding estimate.

Proof of Claim 2.3. Let α be an arbitrary multi-index. Then

$$(2\pi i z)^\alpha K(x,z) = \int_{\mathbf{R}^n} e^{-2\pi i\xi\cdot z} \partial_\xi^\alpha \sigma(x,\xi)\,d\xi$$

in distribution sense. Since $\sigma \in S^0$, there is a constant $C_\alpha > 0$ such that

$$\int_{\mathbf{R}^n} |(\partial_\xi^\alpha \sigma)(x, \xi)|\, d\xi \leqq K_{\alpha 0}(\sigma) \int_{\mathbf{R}^n} (1+|\xi|)^{-|\alpha|}\, d\xi \leqq C_\alpha K_{\alpha 0}(\sigma)$$

if $|\alpha|$ is large enough. Thus (i) and (ii) follow easily. To prove (iii), note that $K(x_0, x_0-z)f(z)$ is absolutely integrable in z and hence Fubini's Theorem implies the result.

4. Weighted Sobolev spaces

Let $z \in \mathbf{C}$. We denote by J_z the pseudo-differential operator with symbol $(1+4\pi^2|\xi|^2)^{-\frac{z}{2}}$. Since

$$w \in A_p(\mathbf{R}^n) \quad \text{for} \quad 1 < p < \infty \Rightarrow L^p(\mathbf{R}^n,\ w\,dx) \subset \mathscr{S}',$$

it follows that $J_z(L^p(\mathbf{R}^n, w\,dx)) \subset \mathscr{S}'$. For $-\infty < s < \infty$ and $1 < p < \infty$, Miller has defined in [16] the weighted Sobolev space of order s, denoted by $L_s^p(\mathbf{R}^n, w\,dx)$, by

$$L_s^p(\mathbf{R}^n, w\,dx) = J_s(L^p(\mathbf{R}^n, w\,dx)).$$

If $f \in L_s^p(\mathbf{R}^n, w\,dx)$, then the norm of $\|f\|_{s,p}$ of f is defined by $\|f\|_{s,p} = \|J_{-s}f\|_p$. $L_s^p(\mathbf{R}^n, w\,dx)$ is a Banach space with norm $\|f\|_{s,p}$. For elementary properties of weighted Sobolev spaces, see Miller [16]. Using Theorem 2.1 and the proof of Theorem 3.1 in Grushin [8], we get

Theorem 4.1. *Let $\sigma \in S^m$ for $-\infty < m < \infty$ and $w \in A_p(\mathbf{R}^n)$ for $1 < p < \infty$. Then for any $s \in (-\infty, \infty)$, T_σ is a bounded linear operator from $L_s^p(\mathbf{R}^n, w\,dx)$ into $L_{s-m}^p(\mathbf{R}^n, w\,dx)$ and there exist a constant $C > 0$ and a positive integer N such that*

$$\|T_\sigma f\|_{s-m,p} \leqq C \|f\|_{s,p} \sum_{|\alpha+\beta| \leqq N} \sup \frac{|\partial_x^\beta \partial_\xi^\alpha \sigma(x, \xi)|}{(1+|\xi|)^{m-|\alpha|}}$$

for all f in $L_s^p(\mathbf{R}^n, w\,dx)$.

Remark. Theorem 4.1 is a generalization of Theorem 3.1 in Grushin [8].

Miller in [16] has obtained a version of the Sobolev's Theorem for weighted Sobolev spaces. In order to give another weighted version of the Sobolev's Theorem, the following two lemmas are necessary.

Lemma 4.2. *Let $w \in A_p(\mathbf{R}^n)$ for $1 < p < \infty$. Then for sufficiently large $k > 0$, $\int_{\mathbf{R}^n} (1+|x|)^{-k} w(x)\, dx < \infty$.*

Proof. See for example Lemma 1 in Hunt, Muckenhoupt and Wheeden [10].

Lemma 4.3. *Let $w \in A_p(\mathbf{R}^n)$ for $1 < p < \infty$. Then $w \in A_q(\mathbf{R}^n)$ for some $q \in (1, p)$.*

Proof. See Lemma 2 in Coifman and Fefferman [5].

Let $w \in A_p(\mathbf{R}^n)$ for $1<p<\infty$. Then by Lemma 4.3, it is possible to define $q_{w,p}$ by

$$q_{w,p} = \inf\{q\colon 1 < q < p \text{ and } w \in A_q(\mathbf{R}^n)\}. \tag{4.1}$$

We can now give another version of the Sobolev's Theorem for weighted Sobolev spaces.

Theorem 4.4. *Let $w \in A_p(\mathbf{R}^n)$ for $1<p<\infty$. Suppose that $s \in (-\infty, \infty)$ is such that $sp > nq_{w,p}$. Then for any compact subset K of $\mathbf{R}^n$, there is a constant $C_K > 0$ such that*

$$\sup_{x \in K} |v(x)| \leqq C_K \|v\|_{s,p}, \quad v \in \mathcal{S}.$$

Proof. Let $v \in \mathcal{S}$. Setting $f = \mathcal{F}^{-1}\{(1+4\pi^2|\xi|^2)^{\frac{s}{2}} \hat{v}\}$, then $v = J_s f = G_s * f$, where G_s is the Bessel potential of order s. See Schechter [18, Chapter 6] or Stein [19, Chapter 5] for properties of Bessel potentials. By Hölder's inequality,

$$|v(x)| \leqq \left\{\int_{\mathbf{R}^n} |G_s(x-y)|^{p'} w(y)^{-\frac{1}{p-1}} dy\right\}^{\frac{1}{p'}} \|f\|_p. \tag{4.2}$$

Now the p'th power of the first term in the right hand side of (4.2) is equal to

$$\left(\int_{|y| \leqq 1} + \int_{|y| \geqq 1}\right) |G_s(x-y)|^{p'} w(y)^{-\frac{1}{p-1}} dy = I_1(x) + I_2(x). \tag{4.3}$$

Since $w \in A_q(\mathbf{R}^n)$ where $q = q_{w,p}$ by Lemma 4.3 and (4.1), Hölder's inequality with $r = \dfrac{p-1}{q-1}$ implies that

$$I_1(x) \leqq \left\{\int_{|y| \leqq 1} w(y)^{-\frac{1}{q-1}} dy\right\}^{\frac{1}{r}} \left\{\int_{\mathbf{R}^n} |G_s(y)|^{\frac{p}{p-q}} dy\right\}^{\frac{1}{r'}}. \tag{4.4}$$

Using the estimates of G_s at the origin and at infinity, $\int_{\mathbf{R}^n} |G_s(y)|^{-\frac{p}{p-q}} dy < \infty$. It is easy to see that $w^{-\frac{p}{q-1}}$ is in $L^1_{\mathrm{loc}}(\mathbf{R}^n)$. Thus $I_1(x) < \infty$ uniformly in $x \in \mathbf{R}^n$ by (4.4). To estimate $I_2(x)$, first note that $w^{-\frac{p'}{p-1}} \in A_{p'}, (\mathbf{R}^n)$. Hence the estimates of G_s at infinity imply that, for every $k>0$, there is a constant $C_{K,k} > 0$ such that

$$I_2(x) \leqq C_{K,k} \int_{|y| \geqq 1} (1+|y|)^{-kp'} w(y)^{-\frac{1}{p-1}} dy, \; x \in K. \tag{4.5}$$

If we choose k large enough, then Lemma 4.2 implies that

$$\int_{|y| \geqq 1} (1+|y|)^{-kp'} w(y)^{-\frac{1}{p-1}} dy < \infty.$$

By (4.2), (4.3), (4.4) and (4.5), the proof is complete.

For $s_0 < s_1$, $0 \leqq \theta \leqq 1$ and $s = (1-\theta)s_0 + \theta s_1$, it is well known that the interpolation space $[L^p_{s_0}(\mathbf{R}^n), L^p_{s_1}(\mathbf{R}^n)]_\theta$ defined by the complex method in Calderón [3,4] is $L^p_s(\mathbf{R}^n)$ with equivalent norms. For a very good and rapid introduction of the complex method, see Schechter [18, Section 5 of Chapter 1]. In order to give a weighted version of the above mentioned interpolation result, we need Lemmas 4.5 and 4.6.

Lemma 4.5. *Let* $w\in A_p(\mathbf{R}^n)$ *for* $1<p<\infty$, $f\in L^p_s(\mathbf{R}^n, w\,dx)$ *for* $-\infty<s<\infty$ *and* $\psi\in\mathscr{S}$. *Then* $|f(\bar{\psi})|\leqq\|f\|_{s,p}\|\psi\|$, *where* $\|\psi\|_{-s,p',w'}$ *is the norm of* ψ *in* $L^{p'}_{-s}(\mathbf{R}^n, w^{-\frac{1}{p-1}}\,dx)$.

Proof. Since $\mathscr{S}$ is dense in $L^p_s(\mathbf{R}^n, w\,dx)$, it is sufficient to prove that $\left|\int_{\mathbf{R}^n}\varphi(x)\overline{\psi(x)}\,dx\right|\leqq\|\varphi\|_{s,p}\|\psi\|_{-s,p',w'}$ for all $\varphi\in\mathscr{S}$. But this follows easily from Plancherel's Theorem and Hölder's inequality.

By Theorem 1.2 or Theorem 4.1, we easily obtain

Lemma 4.6. *Let* $w\in A_p(\mathbf{R}^n)$ *for* $1<p<\infty$. *Suppose that* a, b *are real numbers such that* $a<b$. *Then for any* $s\in(-\infty,\infty)$, *there exist a constant* $C>0$ *and positive integer* N *such that the norm of the operator*

$$J_{-\mu-i\nu}\colon L^p_s(\mathbf{R}^n, w\,dx)\to L^p_{s-\mu}(\mathbf{R}^n, w\,dx)$$

is $\leqq C(1+|\nu|)^N$ *for* $a\leqq\mu\leqq b$ *and* $-\infty<\nu<\infty$.

Theorem 4.7. *Let* $w\in A_p(\mathbf{R}^n)$ *for* $1<p<\infty$. *Then for* $s_0<s_1$, $0\leqq\theta\leqq1$ *and* $s=(1-\theta)s_0+\theta s_1$,

$$[L^p_{s_0}(\mathbf{R}^n, w\,dx), L^p_{s_1}(\mathbf{R}^n, w\,dx)]_\theta=L^p_s(\mathbf{R}^n, w\,dx)$$

with equivalent norms.

Remark. In the proof of Theorem 4.7 given below, we shall use the terminology and notations in Schechter [18, Section 5 of Chapter 1].

Proof of Theorem 4.7. Let $H_\theta=[L^p_{s_0}(\mathbf{R}^n, w\,dx), L^p_{s_1}(\mathbf{R}^n, w\,dx)]_\theta$. Suppose that $u\in H_\theta$ and $\varepsilon>0$ are given. Then there is an f in $H\big(L^p_{s_0}(\mathbf{R}^n, w\,dx), L^p_{s_1}(\mathbf{R}^n, w\,dx)\big)$ such that $f(\theta)=u$ and $\|f\|_H\leqq\|u\|_\theta+\varepsilon$. Let $\lambda_j(\xi)=(1+4\pi^2|\xi|^2)^{\frac{s_j}{2}}$ for $j=0, 1$; $\lambda=\dfrac{\lambda_1}{\lambda_0}$ and $g(z)=\mathscr{F}^{-1}\{\lambda_0\lambda^z\hat{f}(z)\}$. Now $\lambda_0\lambda^z=J_{-(1-z)s_0-zs_1}$. Since $f(z)\in L^p_{s_0}(\mathbf{R}^n, w\,dx)$ for $0\leqq\operatorname{Re} z\leqq1$, it follows from Theorem 4.1 and Lemma 4.6 that $g(z)\in L^p_{(s_0-s_1)\operatorname{Re} z}(\mathbf{R}^n, w\,dx)$ and there exist a constant $C>0$ and a positive integer N such that

$$\|g(\mu+i\nu)\|_{(s_0-s_1)\mu,p}\leqq C(1+|\nu|)^N\|f(\mu+i\nu)\|_{s_0,p} \tag{4.6}$$

for $0\leqq\mu\leqq1$ and $-\infty<\nu<\infty$. Since $f\in H\big(L^p_{s_0}(\mathbf{R}^n, w\,dx), L^p_{s_1}(\mathbf{R}^n, w\,dx)\big)$, it follows that $f(1+iy)\in L^p_{s_1}(\mathbf{R}^n, w\,dx)$ and hence by Theorem 4.1 and Lemma 4.6 again, there exist a constant $C_1>0$ and a positive integer N_1 such that

(4.7) $$\|g(1+iv)\|_p \leqq C_1(1+|v|)^{N_1}\|f(1+iv)\|_{s_1,p}$$

for $-\infty<v<\infty$. Let $v\in\mathscr{S}$. Then $h(z)=\int_{\mathbf{R}^n} g(z)(x)v(x)\,dx$ is continuous in $0\leqq \operatorname{Re} z\leqq 1$ and analytic in $0<\operatorname{Re} z<1$. Moreover by (4.6), (4.7), Lemmas 4.5 and 4.6, there exist a constant $C'>0$ and a positive integer N' such that

$$|h(\mu+iv)| \leqq C'(1+|v|)^{N'}\|f\|_H\|v\|_{p',w'}$$

for $0\leqq\mu\leqq 1$ and $-\infty<v<\infty$. Thus $F(z)=\dfrac{h(z)}{\|f\|_H\|v\|_{p',w'}}$ satisfies the hypotheses of Lemma 4.2 in Stein and Weiss [21, Chapter 5]. Hence there is a constant $C_\theta>0$ such that $|F(\theta)|\leqq C_\theta$, i.e.,

$$\left|\int_{\mathbf{R}^n} g(\theta)(x)v(x)\right| dx \leqq C_\theta\|f\|_H\|v\|_{p',w'}.$$

By Lemma 4.5, $g(\theta)\in L^p(\mathbf{R}^n, w\,dx)$ and $\|g(\theta)\|_p\leqq C_\theta(\|u\|_\theta+\varepsilon)$. Since $\varepsilon>0$ is arbitrary, it follows that $\|g(\theta)\|_p\leqq C_\theta\|u\|_\theta$. But $g(\theta)=\mathscr{F}^{-1}\{(1+4\pi^2|\xi|^2)^{\frac{s}{2}}\hat{u}\}$. Thus $u\in L^p_s(\mathbf{R}^n, w\,dx)$ and $\|u\|_{s,p}\leqq C_\theta\|u\|_\theta$.

Next suppose that $u\in L^p_s(\mathbf{R}^n, w\,dx)$. Putting $f(z)=e^{z^2-\theta^2}\mathscr{F}^{-1}\{\lambda^{\theta-z}\hat{u}\}$, then $f(\theta)=u$. Since $\lambda^{\theta-z}=J_{-(s_1-s_0)(\theta-z)}$ for $0\leqq\operatorname{Re} z\leqq 1$, it follows from Theorem 4.1 and Lemma 4.6 that $f(\mu+iv)\in L^p_{s_0}(\mathbf{R}^n, w\,dx)$ and there exist a constant $C>0$ and a positive integer N such that

$$\|f(\mu+iv)\|_{s_0,p} \leqq Ce^{-v^2}(1+|v|)^N\|u\|_{s,p}$$

for $0\leqq\mu\leqq 1$ and $-\infty<v<\infty$. Furthermore, by Theorem 4.1 and Lemma 4.6 again, $f(1+iy)\in L^p_{s_1}(\mathbf{R}^n, w\,dx)$ and there exist a constant $C_1>0$ and a positive integer N_1 such that

$$\|f(1+iv)\|_{s_1,p} \leqq C_1e^{-v^2}(1+|v|)^{N_1}\|u\|_{s,p}$$

for $-\infty<v<\infty$. Thus $f\in H\big(L^p_{s_0}(\mathbf{R}^n, w\,dx), L^p_{s_1}(\mathbf{R}^n, w\,dx)\big)$ and there is a constant $C_2>0$ such that $\|f\|_H\leqq C_2\|u\|_{s,p}$, i.e., $u\in H_\theta$ and $\|u\|_\theta\leqq C_2\|u\|_{s,p}$. This completes the proof.

The following proposition is a special case of a result in Lions and Peetre [15].

Proposition 4.8. *Let $1<p<\infty$ and $-\infty<s_0<s_1<\infty$ be given. Suppose that $w\in A_p(\mathbf{R}^n)$. Let S be a bounded linear operator from $L^p_{s_0}(\mathbf{R}^n, w\,dx)$ into $L^p_{s_0}(\mathbf{R}^n, w\,dx)$ such that $S: L^p_{s_1}(\mathbf{R}^n, w\,dx)\to L^p_{s_0}(\mathbf{R}^n, w\,dx)$ is compact. Then for $0\leqq\theta\leqq 1$,*

$$S\colon [L^p_{s_0}(\mathbf{R}^n, w\,dx),\quad L^p_{s_1}(\mathbf{R}^n, w\,dx)]_\theta \to L^p_{s_0}(\mathbf{R}^n, w\,dx)$$

is a compact operator. Similarly, if T is a bounded linear operator from $L^p_{s_1}(\mathbf{R}^n, w\,dx)$ into $L^p_{s_1}(\mathbf{R}^n, w\,dx)$ such that $T: L^p_{s_1}(\mathbf{R}^n, w\,dx)\to L^p_{s_0}(\mathbf{R}^n, w\,dx)$ is compact, then *for* $0 \leqq \theta \leqq 1$,

$$T\colon L^p_{s_1}(\mathbf{R}^n, w\,dx) \to [L^p_{s_0}(\mathbf{R}^n, w\,dx),\ L^p_{s_1}(\mathbf{R}^n, w\,dx)]_\theta$$

is a compact operator.

We can now give a weighted version of the compact embedding theorem for Sobolev spaces.

Theorem 4.9. *Let* $w \in A_p(\mathbf{R}^n)$ *for* $1 < p < \infty$. *Suppose that* $\{v_k\}$ *is a bounded sequence of elements in* $L^p_s(\mathbf{R}^n, w\,dx)$. *If* $\varphi \in C_0^\infty(\mathbf{R}^n)$ *and* $t < s$, *then there is a subsequence* $\{u_j\}$ *of* $\{v_k\}$ *such that* $\{\varphi u_j\}$ *converges in* $L^p_t(\mathbf{R}^n, w\,dx)$.

Proof. There are three cases to be considered, namely, (i) $0 \leqq t < s$, (ii) $s > 0$ and $t < 0$ and (iii) $t < s < 0$. It is clear that if Theorem 4.9 is true for case (i), then it is also true for Case (ii) since for $t < 0$, the inclusion from $L^p(\mathbf{R}^n, w\,dx)$ into $L^p_t(\mathbf{R}^n, w\,dx)$ is bounded. So we first suppose that $0 \leqq t < s$. Let $r > s$ be such that $(r-1)p > nq_{w,p}$ where $q_{w,p}$ is given in (4.1). Let μ be a multi-index. Then by Theorem 4.1, there is a constant $C_\mu > 0$ such that

$$\|D^\mu v\|_{r-|\mu|,p} \leqq C_\mu \|v\|_{r,p}, \quad v \in \mathcal{S}. \tag{4.8}$$

Let $K = \operatorname{supp} \varphi$. Then (4.8), Theorem 4.4 and bounded inclusion between weighted Sobolev spaces imply that there is a constant $C_K > 0$ such that

$$\sup_{x \in K} |v(x)| + \textstyle\sum_{j=1}^n \sup_{x \in K} |(D_j v)(x)| \leqq C_K \|v\|_{r,p} \tag{4.9}$$

for all $v \in \mathcal{S}$. Hence if $\{v_k\}$ is a bounded sequence of elements in $L^p_r(\mathbf{R}^n, w\,dx)$, then (4.9) implies that $\{v_k\}$ is a bounded equicontinuous sequence of functions on K. Using Ascoli—Arzela Theorem, there is a subsequence of $\{v_k\}$ which converges uniformly on K. Hence φ is a compact operator from $L^p_r(\mathbf{R}^n, w\,dx)$ into $L^p(\mathbf{R}^n, w\,dx)$. Since φ is also a pseudo-differential operator with symbol in S^0, φ is bounded from $L^p_r(\mathbf{R}^n, w\,dx)$ into $L^p_r(\mathbf{R}^n, w\,dx)$. By Theorem 4.7 and Proposition 4.8, φ is then a compact operator from $L^p_r(\mathbf{R}^n, w\,dx)$ into

$$[L^p(\mathbf{R}^n, w\,dx),\ L^p_r(\mathbf{R}^n, w\,dx)]_{\frac{t}{r}} = L^p_t(\mathbf{R}^n, w\,dx).$$

But φ is also bounded from $L^p_t(\mathbf{R}^n, w\,dx)$ into $L^p_t(\mathbf{R}^n, w\,dx)$. Hence by Theorem 4.7 and Proposition 4.8 again, φ is a compact operator from

$$[L^p_t(\mathbf{R}^n, w\,dx),\ L^p_r(\mathbf{R}^n, w\,dx)]_{\frac{s-t}{r-t}} = L^p_s(\mathbf{R}^n, w\,dx)$$

into $L^p_t(\mathbf{R}^n, w\,dx)$. This completes the proof of Theorem 4.9 for Cases (i) and (ii). Using Lemma 4.5, the result is also true for Case (iii) by duality.

5. Fredholm operators

We can now use the results obtained so far to give conditions on $\sigma \in S^m$ such that T_σ is Fredholm on weighted Sobolev spaces. We first introduce the following class S_0^m of slowly varying symbols. See Grushin [8], Kumano-go [11] and Kumano-go and Taniguchi [12].

Definition. A *symbol* σ *in* S^m *is said to be in* S_0^m *if for all multi-indices* α *and* β, *there is a constant* $C_{\alpha\beta}(x)>0$ *such that*

$$|D_x^\beta D_\xi^\alpha \sigma(x, \xi)| \leqq C_{\alpha\beta}(x)(1+|\xi|)^{m-|\alpha|}$$

for all $x, \xi \in \mathbf{R}^n$ *and* $\lim_{|x|\to\infty} C_{\alpha\beta}(x)=0$, *for* $|\beta| \neq 0$.

The following theorem on composition of two pseudo-differential operators can be found in Grushin [8].

Theorem 5.1. *Let* $\sigma_1 \in S^{m_1}$ *and* $\sigma_2 \in S_0^{m_2}$. *Then there is a symbol* $\sigma \in S_0^{m_1+m_2}$ *such that* $T_{\sigma_1} T_{\sigma_2} = T_\sigma$ *and for any positive integer* N,

$$\sigma(x, \xi) - \sum_{|\alpha|<N} \frac{(-2\pi i)^{-|\alpha|}}{\alpha!} (\partial_\xi^\alpha \sigma_1)(x, \xi)(\partial_x^\alpha \sigma_2)(x, \xi) \in S_0^{m_1+m_2-N}.$$

Using Theorems 4.1, 4.9 and the proof of Theorem 3.2 in Grushin [8], we obtain

Theorem 5.2. *Let* $\sigma \in S_0^m$ *for* $-\infty < m < \infty$ *and* $w \in A_p(\mathbf{R}^n)$ *for* $1<p<\infty$. *Then for any* $s \in (-\infty, \infty)$, T_σ *is a compact operator from* $L_{s+m}^p(\mathbf{R}^n, w\,dx)$ *into* $L_{s-1}^p(\mathbf{R}^n, w\,dx)$.

From Theorems 5.1 and 5.2, we easily obtain the following generalization of Theorem 3.3 in Grushin [8].

Theorem 5.3. *Let* $-\infty < m_1, m_2 < \infty, \sigma_1 \in S^{m_1}$ *and* $\sigma_2 \in S_0^{m_2}$. *Suppose that* $w \in A_p(\mathbf{R}^n)$ *for* $1<p<\infty$. *Then for any* $s \in (-\infty, \infty)$, $T_{\sigma_1} T_{\sigma_2} - T_{\sigma_1 \sigma_2}$ *is a compact operator from* $L_{s+m_1+m_2}^p(\mathbf{R}^n, w\,dx)$ *into* $L_s^p(\mathbf{R}^n, w\,dx)$.

The following theorem gives sufficient conditions under which a pseudo-differential operator is Fredholm on weighted Sobolev spaces. It is a generalization of Theorem 3.4 in Grushin [8]. See also Theorem 7.2 in Beals [1].

Theorem 5.4. *Let* $\sigma \in S_0^m$ *for* $-\infty < m < \infty$ *and* $w \in A_p(\mathbf{R}^n)$ *for* $1<p<\infty$. *Suppose that* $\liminf_{(x,\xi)\to\infty} |\sigma(x, \xi)|(1+|\xi|)^{-m}>0$. *Then for any* $s \in (-\infty, \infty)$, T_σ *s a Fredholm operator from* $L_{s+m}^p(\mathbf{R}^n, w\,dx)$ *into* $L_s^p(\mathbf{R}^n, w\,dx)$.

The proof of Theorem 5.4 depends on Theorem 5.3 and is the same as the proof of Theorem 3.4 in Grushin [8].

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