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# BORSUK'S SHAPE THEORY

J. SEGAL

*Dedicated to the memory of Karol Borsuk*

In 1968, K. Borsuk introduced a new branch of topology called *shape theory.* Like homotopy theory, shape theory is devoted to the study of global properties of topological spaces. However, in order for homotopy theory to yield interesting results the spaces need to behave well locally as in the case of ANRs. Shape theory, on the other hand, yields interesting results in the case of complicated local behavior (e.g., on compacta). Moreover, it agrees with homotopy theory on ANRs. Thus shape theory can be considered as an extension of classical homotopy theory. In fact, it can be viewed as a Čech homotopy theory since its relation to homotopy is analogous to the relation of Čech homology to singular homology.

In [1] Borsuk described the fundamental notions of shape theory based on his notion of a fundamental sequence. The latter is more general than the notion of mapping (continuous function) which is basic to homotopy theory. Borsuk was able to generalize mappings and yet retain a great deal of the geometry inherent in the original notion.

Roughly, a fundamental sequence is based on having the metric compacta $X$ and $Y$ embedded in the Hilbert cube $Q$ and considering maps of $Q$ into itself which behave as follows on neighborhoods of $X$ and $Y$. A fundamental sequence $(\varphi_n) : X \to Y$ is a sequence of maps $\varphi_n : Q \to Q$, for all positive integers $n$, with the following property:

Every neighborhood $V$ of $Y$ admits a neighborhood $U$ of $X$ and a positive integer $n_V$ such that

$$\varphi_n|U \simeq \varphi_m|U \text{ in } V, n, m \geq n_V.$$

An equivalence relation $\simeq$ between fundamental sequences $(\varphi_n)$, $(\varphi'_n) : X \to Y$ is defined by taking $(\varphi_n) \simeq (\varphi'_n)$ provided every neighborhood $V$ of $Y$ admits a neighborhood $U$ of $X$ and a positive integer $n_V$ such that

$$\varphi_n|U \simeq \varphi'_n|U \text{ in } V, n \geq n_V.$$

Fundamental sequences turned out to the basic notion which allowed Borsuk to construct his theory. In particular, it produced the necessary morphisms (i.e., equivalence classes of fundamental sequences) which allowed him to globally compare metric compacta which are in different homotopy classes because of a lack of mappings due to local difficulties in one of the spaces (e.g., when $X = S^1$ and $Y$ is the Warsaw circle).

Shape theory represented an expression of Borsuk's deep geometric insight. It had an immediate impact on the work of other topologists. These included R. H. Fox, S. Mardešić and J. Segal, and T. A. Chapman.

Fox [10] saw shape theory as a way to remove some troublesome local conditions which had blocked a full classification of covering spaces. Fox wrote: "Over the past few years I have been investigating the relationship between fundamental group and covering spaces, and when I learned recently from Borsuk about this new concept I realized that it was exactly what was needed to complete the theory I had been developing." Fox specialized the notion of a covering space to that of an overlay. Using this notion he was able to extend the fundamental theory of covering space theory to metric space which need not be locally connected and semi-locally 1-connected.

On the other hand, Mardešić and Segal [14] saw it as a way to extend the classification of continua described in terms of inverse systems of ANRs. This allowed them to give a more categorical treatment of shape theory and extend it to the compact Hausdorff case.

Meanwhile Chapman had been working on the topology of the Hilbert cube or infinite-dimensional topology and in [6] he obtained a remarkable characterization of the shape of a compactum embedded in $Q$ as a Z-set in terms of the topological type of its complement. Moreover, he associated with every Z-set $X$ in $Q$ its complement $Q\backslash X$ and with every equivalence class of fundamental sequence of Z-sets $(\varphi_n) : X \to Y$ a class of weakly properly homotopic proper maps $Q\backslash X \to Z\backslash Y$ in such a manner as to obtain an isomorphism of categories.

Subsequently, topologists from all over the world contributed to this area. The reader is referred to [16] for description of these various aspects and an extensive bibliography which give an indication of the scope of this work. In addition, Borsuk influenced the work of his own students and other topologists in Poland. Among these were J. Dydak, S. Godlewski, W. Holsztyński, A. Kadlof,

K. Kuperberg, M. Moszyńska, S. Nowak, J. Olędzki, M. Orłowski, S. Spież, M. Strok and A. Trybulec.

Shape theory has been a very active research area with several hundred papers having been published in this area. Much of the activity has been inspired by the numerous questions raised by Borsuk. These questions appear throughout his book [5] and his various research publications. Shape theory has also made contact with several other areas of topology, e.g., geometric topology, infinite-dimensional topology and topological groups.

In discussing some of Borsuk's over 40 publications in shape theory, it will be convenient to divide them into several classes corresponding to various shape invariants, e.g., FANR, movability, fundamental dimension, etc. These are all entirely new notions due to Borsuk.

Metric compacta shape dominated by compact polyhedra are called FANRs and represent the shape theoretic analogues of ANRs. Borsuk introduced and developed this notion in [130], [134] and [136].

Movable compacta were defined by Borsuk in [2] and [3] as a far-reaching generalization of space having the shape of an ANR. The name comes from a geometric interpretation of the definition. While more general than FANRs, movable compacta are still special enough so that when it is present various theorems (e.g., the shape version of the Whitehead theorem) remain valid with the homotopy pro-groups replaced by the corresponding shape groups. A compactum $X$ in $Q$ is movable provided every neighborhood $U$ of $X$ admits a neighborhood $U'$ of $X$, $U' \subset U$, such that for every neighborhood $U''$ of $X$, $U'' \subset U$, there exists a homotopy $H : U' \times I \to U$ with $H(x,0) = x$ and $H(x,1) \in U''$, for any $x \in U'$. Borsuk in [4] also introduced $n$-movability which is an $n$-dimensional stratification of movability.

In [2] Borsuk introduced a simple numerical invariant called fundamental dimension. More recently, this property has been referred to as shape dimension. This notion has proved to be useful since it is often necessary to distinguish between the finite and infinite dimensional case in shape theory. S. Nowak [17] has provided a homological characterization of shape dimension.

In [1] Borsuk showed that the fundamental class of a fundamental sequence determined a homomorphism of the Vietoris homology groups of compacta. He went on to show that the Vietoris homology groups were a shape invariant. In this paper he also introduced the notion of shape group (which he called the fundamental group). For movable compacta these play the role in shape theory that the homotopy groups play in homotopy theory.

Some recent trends in geometric topology show that shape theoretic properties lend themselves to a geometric analysis of mappings or decompositions. For more detailed surveys of these developments the reader is referred to [15] as a general reference.

A map $f : X \to Y$ between metric spaces is called a cell-like map if, for all $y$ in $Y$, $Sh(f^{-1}(y)) = Sh(point)$, i.e., is of trivial shape. The class of cell-like mappings is of central importance in geometric topology. Between ANRs and, in particular, manifolds the importance of cell-like mappings is seen from their role in the work of L. C. Siebenmann [18], R. D. Edwards, and J. E. West [20]. However, J. L. Taylor's example [19] of a cell-like map which is not a shape equivalence showed the need to limit cell-like maps in this more general setting. G. Kozlowski [12] did this by introducing the notion of a hereditary shape equivalence as a generalization of cell-like mappings. Kozlowski used this shape-theoretic notion to show that if $f : X \to Y$ is a cell-like map and $X$ is an ANR, then $Y$ is an ANR if and only if $f$ is a hereditary shape equivalence. In [21], J E West showed that every compact ANR is the image of a compact $Q$-manifold under a cell-like map. Consequently every compact ANR is homotopy equivalent to a compact polyhedron. The latter answered a question raised by Borsuk in his 1954 address to the International Congress.

In [7] D. Coram describes approximate fibrations $p : E \to B$ between ANRs which were introduced by him and P. F. Duvall. They differ from Hurewicz fibrations between ANRs in that the usual homotopy lifting property is replaced by an approximate homotopy lifting property. This was motivated by the work of R. C. Lacher [13] and G. Kozlowski [11] on cell-like mappings. A cell-like mapping between compact ANRs is an approximate fibration. Coram and Duvall show that approximate fibrations have several shape theoretic properties analogous to the corresponding homotopic theoretic properties of fibrations. Another important fact they obtain is that the fibers of an approximate fibration are FANRs. If, in addition, $B$ is connected, then the fibers are all of the same shape.

In [8] R. J. Daverman surveys recent results on upper semicontinuous (usc) decompositions $G$ of $(n+k)$-manifolds $M$ into continua of the shape of closed connected $n$-manifolds. He considers what can be said about the decomposition space $B = M/G$. When $k < 3$, the decomposition space $B$ is necessarily a $k$-manifold, as long as certain orientability requirements are met. More specifically, in [9] Daverman and Walsh show that if $G$ is an usc decomposition of an orientable $(n+2)$-manifold $M$ into continua the shape of closed, orientable $n$-manifolds, then $B = M/G$ is a 2-manifold. On the other hand, if $M$ is non-orientable, then $B$ is a 2-manifold with boundary. As a corollary, they show there is no usc decomposition of $E^{n+2} (n > 0)$ into closed, connected, orientable $n$-manifolds. When $k = 3$, $B$ need not be a manifold but it must be finite-dimensional, in fact, $B$ is 3-dimensional. Here the question of whether $B$ is an

ANR becomes central. Approximate fibrations prove to be useful since they provide relationships among the structures on $M$, $B$ and $G$, the elements of which all have the same shape.

In 1931 Borsuk introduced ANRs, and in 1968 shape theory. It is interesting to note that these two contributions have in the past greatly influenced geometric topology and continue to do so. This only confirms Borsuk's deep geometric insight and originality.

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JACK SEGAL
University of Washington
Seattle, WA