Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 1, 1993, 9–14

ON SOME CONTRIBUTIONS OF KAROL BORSUK TO HOMOTOPY THEORY

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Dedicated to the memory of Karol Borsuk, a great topologist and a dear friend.

Introduction

So many and varied are the contributions of Karol Borsuk to topology in general and to homotopy theory in particular that it would be idle to attempt a global view in the space of a brief memoir.[1] I have therefore chosen to be highly selective, and discuss only two of his major contributions, each selected for a special attribute it possesses. First, I cite his invention of the *cohomotopy groups*. Of course Borsuk introduced them to analyse a topological problem—he was never a formalist. But it is important to remember that Borsuk was not an algebraic topologist and thus was not in the mainstream of development of algebraic topology; moreover, his circumstances at the time of this work (just before World War II) made it even more difficult for him to remain in even indirect contact with other topologists in Western Europe and the United States. Nevertheless, he invented a notion which, as I show, fitted into the mainstream of homotopy theory and made an essential contribution to its progress. Once the Borsuk cohomotopy groups are recognized as special but important cases of stable track groups, the full force of the apparatus of modern algebraic topology (exact sequences, functoriality, spectral sequences, ...) can be brought to bear on them automatically. That recognition owes much to the excellent account provided by Spanier [9]

[1]In any case, I would expect that other aspects of his immense contributions would feature in other articles in this volume.

My second example of Borsuk's contribution is chosen for its fundamental impact on my own development as a topologist; thus I should apologize in advance for the inevitably personal tone of my account. I was fortunate to meet Borsuk at the Amsterdam International Congress in 1954, and he invited me to visit Warsaw the following year. We had many discussions centering on his current interests, and I was tremendously stimulated by his ideas. Indeed, several of the principal directions of my own research owe their origins to Borsuk's ideas, not least the duality in homotopy theory which Beno Eckmann and I developed when I settled for a sabbatical leave in Zürich after my exciting visit to Warsaw (see [5,7]). It is a pleasure and a privilege to have this opportunity to acknowledge my debt to Karol Borsuk.

1. The cohomotopy groups

Borsuk announced his discovery of the cohomotopy groups in a paper in the Comptes Rendus in 1936 [1] , but most of us learnt about them by reading Spanier's definitive account [9] , published in 1949. In Spanier's version one considers a polyhedron X with $\dim X \leq 2n-2$ and considers the set $\pi^n X$ of (based) homotopy classes of maps of X into S^n. If f, g are any two such maps then a map $(f,g) : X \to S^n \times S^n$ is defined in the obvious way. Now $S^n \times S^n$ may be given the structure of a CW-complex in which a $2n$-cell e^{2n} is attached to $S^n \vee S^n$, the wedge (= one-point union) of two copies of S^n. Since $\dim X \leq 2n-2$, the map (f,g) may be deformed into a map $h' : X \to S^n \vee S^n$, and the homotopy class of h' is uniquely determined by that of (f,g), that is, by those of f and g. If one follows h' by the folding map $\nabla : S^n \vee S^n \to S^n$, which identifies the two copies of S^n, one obtains a map $h = \nabla h' : X \to S^n$ and one defines

$$[h] = [f] + [g], \tag{1.1}$$

yielding a binary operation in the set $\pi^n(X)$. Special arguments are then invoked to show that the addition rule (1.1) turns $\pi^n(X)$ into an additive abelian group, which is plainly contravariant in X on the category of pointed polyhedra of dimension $\leq 2n-2$. There is, moreover, a Hurewicz theorem showing that, if $\dim X = n$, then

$$\pi^n(X) \cong H^n(X).$$

We now recognize that, in a sense, Borsuk was really inventing stable homotopy theory. For let us embed S^n in ΩS^{n+1}, the space of loops on S^{n+1}, in the canonical manner. The induced map of homotopy sets $[X, S^n] \to [X, \Omega S^{n+1}]$ is bijective since $\dim X \leq 2n-2$; and the set $[X, \Omega S^{n+1}]$ acquires a natural group structure from the (homotopy-) group structure on the loop-space ΩS^{n+1}. Moreover, $[X, \Omega S^{n+1}] \to [X, \Omega^2 S^{n+2}]$, induced by the embedding $S^{n+1} \hookrightarrow \Omega S^{n+2}$, is a group-isomorphism, and $[X, \Omega^2 S^{n+2}]$ is abelian, so $[X, S^n]$ acquires the structure of an abelian group.

Now ΩS^{n+1} may be replaced by James' reduced product space S^n_∞ (see [8]), whose $2n$-skeleton S^n_2 has the cell-structure $S^n \cup e^{2n}$, the $2n$-cell being attached by the Whitehead product $[\iota, \iota]$. Thus there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "{(f,g)}"] & S^n \times S^n \arrow[r, "\tau"] & S^n_2 \\ & S^n \vee S^n \arrow[u] \arrow[r, "\nabla"] & S^n \arrow[u] \end{tikzcd} \tag{1.2}$$

where $\tau(x, y) = (x, y) \in S^n_2$. However maps $X \to S^n$ are composed in ΩS^{n+1} by actually composing them in S^n_2 by juxtaposition. This implies that the sum of f and g is obtained by deforming $\tau(f, g)$ down into S^n. Such a deformation, however, yields $\nabla h'$, so that the Borsuk-Spanier addition in $\pi^n(X)$ coincides with that induced by the bijection $[X, S^n] \to [X, \Omega S^{n+1}]$. This, of course, immediately establishes, from general principles, that $\pi^n(X)$, as originally defined, satisfies all the axioms of an abelian group.

However, it does much more. For the bijections $[X, S^n] \to [X, \Omega S^{n+1}] \to [X, \Omega^2 S^{n+2}]$ are equivalent to the bijections $[X, S^n] \xrightarrow{\Sigma} [\Sigma X, S^{n+1}] \xrightarrow{\Sigma} [\Sigma^2 X, S^{n+2}]$ induced by suspension. Now we see that there is no need to assume $\dim X \leq 2n - 2$; we may *define* $\pi^n X$, for *any* polyhedron X, by the rule

$$\pi^n X = \varinjlim_k [\Sigma^k X, S^{n+k}] \tag{1.3}$$

Then $\pi^n X$ is an abelian group, coinciding with the Borsuk-Spanier definition if $\dim X \leq 2n - 2$. If $\dim X = m$, then $[\Sigma^k X, S^{n+k}]$ stabilizes at $k = m - 2n + 2$, meaning that

$$\Sigma : [\Sigma^k X, S^{n+k}] \to [\Sigma^{k+1} X, S^{n+k+1}]$$

is an isomorphism if $k \geq m - 2n + 2$. Thus, if X is finite-dimensional, then $\pi^n X$, as defined by (1.3), is, in fact, a cohomotopy group in the original sense, namely, $\pi^{n+k}(\Sigma^k X)$, for k sufficiently large.

Of course, one may go much further. There is no reason to confine attention to the case in which the target space is a sphere. Thus we may define

$$\{X, Y\} = \varinjlim_k [\Sigma^k X, \Sigma^k Y]. \tag{1.4}$$

Just as in the special case $Y = S^n$, this gives an abelian group structure to the set $\{X, Y\}$ of *stable homotopy classes* of maps of X into Y. (Notice that the maps are also 'stable'; an element of $\{X, Y\}$ is represented by a map $\Sigma^k X \to \Sigma^k Y$

for some k.) Again, as in the special case, $\{X, Y\}$ stabilizes, provided X is finite-dimensional.

2. Divisors and multiples of maps; dependence of maps

Borsuk published a series of papers in 1955-56 [2,3,4] which had a profound impact on the author's own research. Let us again modify Borsuk's ideas slightly by talking of polyhedra, rather than of compacta or ANR's. Thus let f, g be maps from the polyhedron X to the polyhedron Y. Then f is said to be a *divisor* of g (and g a *multiple* of f) if, whenever f can be extended to a polyhedron X' containing X, g can also be extended to that same X'. Further, if f is replaced by a family Φ of maps from X to Y, then g is said to *depend* on Φ if, whenever f can be extended to X' for every f in Φ, then g can also be extended to X'.

Borsuk proved that g is a multiple of f if and only if $g \simeq hf$ for some $h : Y \to Y$. Before giving the proof, let me give a generalization of Borsuk's definition which proved very fruitful in my own development of his work. I will speak only of *dependence*, since the notions of *divisor* and *multiple* may be subsumed in the notions of dependence. The generalization consists of allowing f and g to have different targets. Thus we will say that $g : X \to Z$ depends on $f : X \to Y$ if, whenever f may be extended to $X' \supseteq X$, so may g. Notice that if Y^Φ is the Cartesian product of copies of Y indexed by Φ, then we may identify the family Φ of maps from X to Y with a single map $F : X \to Y^\Phi$; and g depends on Φ in the sense of Borsuk if and only if g depends on F in our sense. Thus, by our generalization, we avoid the need for any distinction between maps and families of maps. We now prove:

THEOREM 2.1. *The map $g : X \to Z$ depends on the map $f : X \to Y$ if and only if there exists $h : Y \to Z$ with $g \simeq hf$.*

PROOF. If h exists, and if f extends to $f' : X' \to Y$, then hf extends to $hf' : X' \to Z$. But, by the homotopy extension property, it follows that g extends to X'. Conversely, suppose that g depends on f. Let M be the mapping cylinder of f; thus M is obtained from $X \times I \cup Y$ by identifying $(x, 1)$ with fx. If f is cellular (and there is plainly no real loss of generality in supposing it is, since the extendability of f is a homotopy invariant property of f), then M is a polyhedron in which X is embedded by the map $x \mapsto (x, 0)$. Moreover, f extends to a map $M \to Y$ given by $(x, t) \mapsto fx, y \mapsto y$. Thus g extends to a map $G : M \to Z$. Since G extends $g, G(x, 0) = g(x)$. Thus $G|X \times I$ is a homotopy of g and $G(x, 1) = Gf(x) = hf(x)$, where $h : Y \to Z$ is the restriction of G to Y. We have $g \simeq hf$, proving the theorem.

I learnt of Borsuk's notion of dependence on a visit I made to Warsaw, at Borsuk's invitation, in 1955. I obtained some results on dependence (see [6])—as well as introducing the generalization described—and was also able to disprove a conjecture of Borsuk (it was probably a question rather than a conjecture) that if two maps $f, g : X \to Y$ are dependent on each other, then $g \simeq hf$ for some homotopy equivalence $h : Y \simeq Y$. In fact, if we consider

$$
\begin{tikzcd}
 & S^3 \arrow[dd, shift right, "5"'] \\
S^6 \arrow[ur, "\omega"] \arrow[dr, "5\omega"'] & \\
 & S^3 \arrow[uu, shift right, "5"']
\end{tikzcd}
\tag{2.1}
$$

where ω generates $\pi_6(S^3) = \mathbb{Z}/12$, then we see that $\omega, 5\omega$ depend on each other, but, of course, there is no homotopy self-equivalence h of S^3 such that $h\omega = 5\omega$.

From Warsaw I went to Zürich to work with Beno Eckmann. We studied the homological algebra of Λ-modules, and it occurred to us that one could study the notion of dependence in the category of homomorphisms of Λ-modules, just as in the topological category. Indeed, the analogue of Theorem 2.1 remained valid, provided one introduced the appropriate notion of homotopy. Thus if A, B are Λ-modules, one says that $\phi : A \to B$ is *i-nullhomotopic* if ϕ extends to an injective container of A; and $\phi, \psi : A \to B$ are *i-homotopic*, $\phi \simeq_i \psi$, if $\phi - \psi$ is i-nullhomotopic. Then the homotopy extension property holds for i-homotopy and there is a *mapping cylinder* for a homomorphism $\phi : A \to B$; namely, one factors ϕ as $A \overset{\mu}{\rightarrowtail} CA \oplus B \overset{\varepsilon}{\twoheadrightarrow} B$, where CA is any injective container of A, $\mu(a) = (a, \phi(a))$ and $\varepsilon(c, b) = b$. One then essentially reproduces the proof of Theorem 2.1 in the category of Λ-modules.

However, in this category, there is a built-in duality. This leads to the notions of *p-homotopy* and *p-dependence*. Thus $\phi : A \to B$ is *p-nullhomotopic* if ϕ may be lifted to a projective module sitting over B, and $\phi, \psi : A \to B$ are *p-homotopic* if $\phi - \psi$ is p-nullhomotopic. Further $\psi : B \to C$ *p-depends* on $\phi : A \to C$ if ψ may be lifted to C' over C whenever ϕ may be lifted to C'. There is then a dual, in the category of Λ-modules, of the analogue of Theorem 2.1.

Now came the decisive step—there is an analogue of the dual! This analogue, however, relates to the same notion of homotopy as that used in Theorem 2.1—one may say that i-homotopy and p-homotopy coincide in topology. Precisely, we say that the map $g : Z \to X$ *co-depends* on the map $f : Y \to X$ if, whenever f lifts to the total space E of a fibration over X, g also lifts to E. Then the dual of Theorem 2.1 reads

THEOREM 2.1C. *The map $g : Z \to X$ co-depends on the map $f : Y \to X$ if and only if there exists $h : Z \to Y$ such that $g \simeq fh$.*

Theorem 2.1c holds—and marks the birth of *Eckmann-Hilton duality*. Thus my debt to Borsuk's inspiration is enormous!

However, there is one further item to mention in measuring this debt. Borsuk asked me, while I was in Warsaw, whether it would be possible to construct two compact polyhedra with isomorphic homology groups, isomorphic homotopy groups but of different homotopy types. In answering this question in the affirmative, I effectively exploited diagram (2.1), invented for a totally different purpose. Borsuk's question also led me to undertake a systematic study of non-cancellation phenomena and, later, of the Mislin genus.

But this is another story. Let me close with one reminiscence which epitomizes the remarkable modesty of the great mathematician to whom this brief memoir is dedicated. While I was in Warsaw, Borsuk told me of a result he had obtained, relating to dependence for maps of an n-complex into S^n, in which the n^{th} cohomology group appeared. I looked at his argument and pointed out that he could generalize his result to maps of an m-complex for $m \leq 2n-2$, provided he replaced the cohomology group by the cohomotopy group. 'How interesting', said Borsuk, 'but, you see, I never did really understand the cohomotopy groups'.

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