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HOMOCLINIC SOLUTIONS FOR A CLASS OF SYSTEMS OF SECOND ORDER DIFFERENTIAL EQUATIONS

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Dedicated to Louis Nirenberg

1. Introduction

Variational methods have recently been applied to the search of homoclinic solutions for first and second order Hamiltonian systems (see e.g. [1, 2, 3, 6] and the references therein). Homoclinic solutions obtained there are mountain pass points for suitable functionals, namely either the Lagrangian functional or its dual with respect to the Legendre transform. In order to apply the mountain pass theorem to the Lagrangian functional, one requires that the spectrum of the operator describing the system linearized at zero is contained in $(0, \infty)$, while to switch to the dual functional it is necessary to assume that the potential (or the Hamiltonian) is the sum of a quadratic and a convex part. Furthermore, the potential of the second order systems treated in the above cited references is required to have a local maximum at zero.

In this paper we prove the existence of a nontrivial homoclinic solution of the system of second order differential equations

$$\ddot{q}(t) + A(t)q(t) = -W_q(q(t), t), \qquad q \in \mathbb{R}^N,\ t \in \mathbb{R}. \tag{1}$$

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We make the following assumptions:

(i) $W \in C^1(\mathbb{R}^{N+1}, \mathbb{R})$, $A(t)$ is a symmetric $N \times N$ matrix, continuous with respect to t, and both $A(t)$ and $W(q,t)$ are 2π-periodic in t.

(ii) There is $\alpha > 2$ such that $W_q(q,t) \cdot q \geq \alpha W(q,t)$ for all t and q.

(iii) There are $a_1, a_2 > 0$ such that $a_1|q|^\alpha \leq W(q,t)$ and $|W_q(q,t)| \leq a_2|q|^{\alpha-1}$ for all t and q.

Note that by the second inequality in (iii) there exists a constant a_3 such that $W(q,t) \leq a_3|q|^\alpha$.

The novelty of our result is that we neither require the potential $V(q,t) = \frac{1}{2}(A(t)q,q) + W(q,t)$ to have a local maximum at $q = 0$, nor the superquadratic part $W(q,t)$ to be convex with respect to q. However, we do require that the operator $q \mapsto \ddot{q} + A(t)q$ is invertible. More precisely, let $H = H^1(\mathbb{R}, \mathbb{R}^N)$ be the usual Sobolev space and let L be the self-adjoint operator defined by

$$(Lq,p) = \int_{\mathbb{R}} (\dot{q}(t)\dot{p}(t) - A(t)q(t)p(t))\,dt, \qquad q,p \in H.$$

Then we assume:

(iv) $0 \notin \sigma(L)$.

We prove the following theorem:

Theorem 1. *Assume* (i)–(iv). *Then equation* (1) *admits a nontrivial solution q which is homoclinic to* 0 *and such that* $q \in C^2(\mathbb{R}, \mathbb{R}^N) \cap H^1(\mathbb{R}, \mathbb{R}^N)$.

Assumption (iv) requires some comments: in [7] it is proved that if $N = 1$, then L has only continuous spectrum, and if $A(t)$ is not constant, there exists at least one spectral gap, i.e. an interval $[\nu_1, \nu_2]$ such that $[\nu_1, \nu_2] \cap \sigma(L) = \{\nu_1, \nu_2\}$; moreover, if $A(t)$ is not real analytic, then there exist infinitely many such gaps. If $N > 1$, then it can be seen that assumption (iv) holds for a large class of matrices $A(t)$; in such cases we can decompose H into two L-invariant subspaces on which L is positive respectively negative definite, that is, $H = H^+ \oplus H^-$ and there exists a constant $\widehat{\lambda} > 0$ such that $(Lq,q) \geq \widehat{\lambda}\|q\|^2$ for $q \in H^+$ and $(Lq,q) \leq -\widehat{\lambda}\|q\|^2$ for $q \in H^-$. Although we do not exclude the possibility of having $\sigma(L) \subset (0,\infty)$, in this case stronger results than ours have been obtained in [2, 6].

The Lagrangian functional

$$J(q) = \frac{1}{2}(Lq,q) - \int_{\mathbb{R}} W(q,t)\,dt, \qquad q \in H, \tag{2}$$

has the property that $J(q) \geq a > 0$ for $q \in H^+$, $\|q\| = \varrho > 0$ and $J(q) \leq 0$ for $q \in H^-$ (the so-called linking geometry); however, since H^+ and H^- are infinite-dimensional spaces and the derivative of the second term on the right hand side of (2) is not a compact operator, we could not work directly in the space H. Instead we obtain a sequence of periodic solutions of (1) by a standard linking argument (see Th. 5.3 of [5]), and show via suitable estimates that this sequence converges to a homoclinic solution $q \neq 0$.

2. Periodic solutions

For all positive integers n, let $I_n = [-\pi n, \pi n]$, let $H_n = \{q \in H^1(I_n, \mathbb{R}^N) : q(-\pi n) = q(\pi n)\}$ endowed with the norm $\|\cdot\|_{H_n}$ and $L_n^p = L^p(I_n, \mathbb{R}^N)$ endowed with the norm $\|\cdot\|_{L_n^p}$ (we will write $\|\cdot\|$ and $\|\cdot\|_p$ for these norms when no ambiguity arises), and let $L_n : H_n \to H_n$ be the self-adjoint operator defined by

$$(L_n q, \varphi) = \int_{I_n} (\dot{q}(t)\dot{\varphi}(t) - A(t)q(t)\varphi(t))\,dt, \qquad q, \varphi \in H_n.$$

Critical points of the functional $J_n : H_n \to \mathbb{R}$ defined by

$$J_n(q) = \frac{1}{2}(L_n q, q) - \int_{I_n} W(q,t)\,dt \tag{3}$$

are weak $2\pi n$-periodic solutions of (1), and in fact they are classical solutions because of the continuity of $A(t)$ and $W_q(q,t)$ (here and in the following we identify the functions $q \in H_n$ with their $2\pi n$-periodic extensions).

To prove the existence of critical points q_n of J_n and to obtain necessary estimates, we need some technical lemmas:

Lemma 1. *There exists a constant $\widehat{c}$ independent of n such that $\|q\|_{L_n^p} \leq \widehat{c}\|q\|_{H_n}$ for all $q \in H_n$ and $p \in [2, \infty]$.*

Proof. By the definitions of the norms we have $\|q\|_{L_n^2} \leq \|q\|_{H_n}$ for all $q \in H_n$. In [6] (see eq. 2.18) it is proved that there exists $\widehat{c}$ such that $\|q\|_{L_n^\infty} \leq \widehat{c}\|q\|_{H_n}$ for all $q \in H_n$. Then if $p \in (2, \infty)$ we have

$$\|q\|_{L_n^p}^p = \int_{I_n} |q|^p \leq \int_{I_n} |q|^{p-2}|q|^2 \leq \|q\|_{L_n^\infty}^{p-2}\|q\|_{L_n^2}^2 \leq \widehat{c}^{p-2}\|q\|_{H_n}^p,$$

that is, $\|q\|_{L_n^p} \leq \widehat{c}^{(p-2)/p}\|q\|_{H_n} \leq \widehat{c}\|q\|_{H_n}$. □

Lemma 2. *For all $n \in \mathbb{N}$ there exists a bounded operator $i_n : H_n \to H$ such that $i_n q(t) = q(t)$ for $t \in [-\pi n, \pi n]$ and*

$$\|q\|_{H_n} \leq \|i_n q\|_H \leq \|q\|_{H_n} + c|q(\pi n)| \leq (1 + c\widehat{c})\|q\|_{H_n}, \tag{4}$$

where the constant c is independent of n.

Proof. We give an explicit definition of the operator i_n:

$$i_n(q)(t) = \begin{cases} (\pi n + 1 + t)q(\pi n) & \text{if } t \in [-\pi n - 1, -\pi n], \\ q(t) & \text{if } t \in [-\pi n, \pi n], \\ (\pi n + 1 - t)q(\pi n) & \text{if } t \in [\pi n, \pi n + 1], \\ 0 & \text{if } t \in (-\infty, -\pi n - 1) \cup (\pi n + 1, \infty); \end{cases}$$

then the first inequality in (4) is trivial, the second follows by direct computation and the third by Lemma 1. □

Lemma 3. *There exist* $\overline{n} \in \mathbb{N}$ *and* $\overline{\lambda} > 0$ *such that each space* H_n *with* $n \geq \overline{n}$ *splits into two* L_n*-invariant subspaces* H_n^+ *and* H_n^- *such that* $(L_n q, q) \geq \overline{\lambda}\|q\|^2$ *for* $q \in H_n^+$ *and* $(L_n q, q) \leq -\overline{\lambda}\|q\|^2$ *for* $q \in H_n^-$.

Proof. We prove the first statement, the proof of the second being similar. H_n admits the decomposition $H_n = H_n^+ \oplus \ker L_n \oplus H_n^-$ into subspaces where L_n is respectively positive definite, zero and negative definite because it is self-adjoint. Now by contradiction, assume that there exists a sequence $\{q_n\}$ of eigenfunctions such that $q_n \in H_{k_n}^+ \oplus \ker L_{k_n}$ for some $k_n \geq n$, $\|q_n\| = 1$, $L_{k_n} q_n = \lambda_n q_n$ and $\lambda_n \to 0$; such a sequence exists because each L_{k_n} is a compact perturbation of the identity. Then q_n satisfies

$$-\ddot{q}_n - Aq_n = \lambda_n(-\ddot{q}_n + q_n) \tag{5}$$

and $i_{k_n} q_n \rightharpoonup q$ in H up to a subsequence because by Lemma 2 the sequence $\{i_{k_n} q_n\}$ is bounded; by the Sobolev embedding $i_{k_n} q_n \to q$ in $C^0_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$, and by equation (5), $i_{k_n} q_n \to q$ in $C^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$; therefore q satisfies $-\ddot{q}(t) - A(t)q(t) = 0$ for all $t \in \mathbb{R}$ and as 0 is not an eigenvalue of L in H we infer $q \equiv 0$. Let $\widehat{q}_n(t) = q_n(t - \pi k_n)$; the sequence $\{\widehat{q}_n\}$ has the same properties as $\{q_n\}$, therefore the same conclusions apply, namely $i_{k_n} \widehat{q}_n \to 0$ in $C^2_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^N)$, hence $\widehat{q}_n(0) = q_n(\pm\pi k_n) \to 0$ and $\dot{\widehat{q}}_n(0) = \dot{q}_n(\pm\pi k_n) \to 0$. By the definition of i_n we get

$$\|L i_{k_n} q_n\| \leq \sup_{\|\varphi\|=1} \int_{I_{k_n}} [\dot{q}_n \dot{\varphi} - Aq_n \varphi] + \sup_{\|\varphi\|=1} \int_{\widehat{I}_{k_n}} \left[\frac{d}{dt}(i_{k_n} q_n) \frac{d}{dt}\varphi - A i_{k_n} q_n \varphi\right],$$

where $\widehat{I}_{k_n} = [-\pi k_n - 1, -\pi k_n] \cup [\pi k_n, \pi k_n + 1]$; integrating by parts twice we have

$$\begin{aligned}
\sup_{\|\varphi\|=1} \int_{I_{k_n}} [\dot{q}_n \dot{\varphi} - Aq_n \varphi] &\leq \sup_{\|\varphi\|=1} [\dot{q}_n \varphi|_{-\pi k_n}^{\pi k_n}] + \sup_{\|\varphi\|=1} \int_{I_{k_n}} (-\ddot{q}_n - Aq_n)\varphi \\
&\leq 2 \sup_{\|\varphi\|=1} [\dot{q}_n \varphi|_{-\pi k_n}^{\pi k_n}] + \lambda_n \sup_{\|\varphi\|=1} \int_{I_{k_n}} [\dot{q}_n \dot{\varphi} + q_n \varphi] \\
&\leq 4\widehat{c}\dot{q}_n(\pi k_n) + \lambda_n \to 0.
\end{aligned}$$

Using the definition of i_n given in the proof of Lemma 2 we see that there exists a constant c such that

$$\sup_{\|\varphi\|=1} \int_{\widehat{I}_{k_n}} \left[\frac{d}{dt}(i_{k_n} q_n) \frac{d}{dt}\varphi - A i_{k_n} q_n \varphi\right] \leq c q_n(\pi k_n) \to 0;$$

hence $L i_{k_n} q_n \to 0$ and $\|i_{k_n} q_n\| \geq 1$, contradicting the invertibility of L on H. □

Lemma 4. *For all n the functional J_n satisfies the Palais–Smale condition.*

Proof. Let $\{q_k\}$ be a Palais–Smale sequence for J_n; then there exist a constant c and a sequence of positive numbers $\{\varepsilon_k\}$ decreasing to 0 such that

$$c + \varepsilon_k \|q_k\| \geq 2J_n(q_k) - J_n'(q_k)[q_k],$$

hence by (ii) and (iii),

$$\begin{aligned} c + \varepsilon_k \|q_k\| &\geq \int_{I_n} W_q(q_k, t) \cdot q_k - 2\int_{I_n} W(q_k, t) \\ &\geq (\alpha - 2)\int_{I_n} W(q_k, t) \geq a_1(\alpha - 2)\|q_k\|_\alpha^\alpha; \end{aligned} \tag{6}$$

for all k let q_k^+ and q_k^- be the projections of q_k on H_n^+ and H_n^-; then

$$\begin{aligned} \overline{\lambda}\|q_k^\pm\|^2 &\leq |(L_n q_k^\pm, q_k^\pm)| \leq \varepsilon_k \|q_k^\pm\| + \left| \int_{I_n} W_q(q_k, t) q_k^\pm \right| \\ &\leq \varepsilon_k \|q_k^\pm\| + a_2 \int_{I_n} |q_k|^{\alpha-1} |q_k^\pm| \leq \varepsilon_k \|q_k^\pm\| + a_2 \|q_k\|_\alpha^{\alpha-1} \|q_k^\pm\|_\alpha. \end{aligned}$$

By Lemma 1, $\|q_k^\pm\|_\alpha \leq \widehat{c}\,\|q_k^\pm\|$ and therefore $\|q_k^\pm\| \leq \overline{\lambda}^{-1}(\varepsilon_k + a_2\widehat{c}\,\|q_k\|_\alpha^{\alpha-1})$; by the last inequality and (6) we finally infer

$$\|q_k\| \leq \|q_k^+\| + \|q_k^-\| \leq 2\overline{\lambda}^{-1}(\varepsilon_k + a_2\widehat{c}\,\|q_k\|_\alpha^{\alpha-1}) \leq c(\|q_k\|^{(\alpha-1)/\alpha} + 1),$$

where c is some constant, and hence the sequence is bounded. The conclusion is standard by a compactness argument (see e.g. [5]). $\square$

In order to apply the linking theorem to the spaces H_n we need the following estimates:

Lemma 5. *There exist $\varrho, a > 0$ such that $J_n(q) \geq a$ for all $q \in H_n^+$ with $\|q\|_{H_n} = \varrho$ and for almost all $n \in \mathbb{N}$.*

Proof. By Lemma 3, for all $q \in H_n^+$ with $\|q\|_{H_n} = \varrho$ and $n \geq \overline{n}$ we have $(L_n q, q) \geq \overline{\lambda}\varrho^2$ and by Lemma 1, $\|q\|_\alpha \leq \widehat{c}\varrho$; therefore

$$J_n(q) = \frac{1}{2}(L_n q, q) - \int_{I_n} W(q, t) \geq \frac{\overline{\lambda}}{2}\|q\|^2 - a_3\|q\|_\alpha^\alpha \geq \frac{\overline{\lambda}}{2}\varrho^2 - a_3\widehat{c}^\alpha \varrho^\alpha,$$

and the assertion follows on choosing a suitable value for ϱ. $\square$

The spaces H_n^- have finite dimension and since $W \geq 0$ we have $J_n(q) \leq 0$ for all $q \in H_n^-$, therefore the standard linking theorem (see e.g. [5]) provides a $2\pi n$-periodic solution q_n of equation (1).

More precisely, let $Q_n = (B_R \cap H_n^-) \oplus [0, se]$, where $B_R = \{q \in H_n : \|q\| \le R\}$ and the constants $R, s > 0$ and $e \in H_n \setminus H_n^-$ are such that $J_n(q) \le 0$ for all $q \in \partial Q_n$. Let $\Gamma_n = \{\gamma \in C(Q_n, H_n) : \gamma|_{\partial Q} = \mathrm{id}\}$; then

$$b_n = \inf_{\gamma \in \Gamma_n} \max_{q \in Q_n} J(\gamma(q)) \tag{7}$$

is a critical level for J_n. During the course of the proof of Lemma 6 below we will show that R, s and e as above indeed exist and may be chosen independently of n. Note that by Lemma 5 and the definition of b_n we have $b_n \ge a > 0$ for almost all n.

3. Proof of Theorem 1

To consider the limit of the sequence $\{q_n\}$ of periodic solutions as $n \to \infty$ we need an upper estimate on the critical levels:

Lemma 6. *There exist $\overline{n} \in \mathbb{N}$, $e \in H_{\overline{n}} \setminus H_{\overline{n}}^-$ and $R, s, A > 0$ such that if b_n is defined as in* (7)*, then $J_n \le 0$ on ∂Q_n and $b_n \le A$ for all $n \ge \overline{n}$.*

Proof. The proof requires four steps.

(a) Choose $e \in H$ such that $\|e\| = 1$, $\mathrm{supp}[e] \subset I_{\overline{n}}$ and $(Le, e) > 0$; by the density of the functions with compact support in H there exists $\overline{n}$ satisfying these requirements; note that as $e \in H_0^1(I_{\overline{n}}, \mathbb{R}^N)$, we can assume $e \in H_n$ for all $n \ge \overline{n}$ as well as $e \in H$, furthermore $\|e\|_{H_n} = 1$. Choosing a larger $\overline{n}$ if necessary we may also assume that the conclusion of Lemma 3 is valid for $n \ge \overline{n}$.

(b) *There exists $R > 0$ such that $J_k(q + se) \le 0$ for all $k \ge \overline{n}$, all $q \in H_k^-$ with $\|q\| = R$, and all $s \ge 0$.* By contradiction, if such an R does not exist, then there exist sequences $\{R_n\} \subset \mathbb{R}$, $\{s_n\} \subset \mathbb{R}$ and $\{q_n\}$, $q_n \in H_{k_n}^-$ for some k_n, such that $\|q_n\| = R_n$, $R_n \to \infty$ and

$$J_{k_n}(q_n + s_n e) > 0. \tag{8}$$

We have

$$J_{k_n}(q_n + s_n e) \le \frac{1}{2}(L_{k_n}(q_n + s_n e), q_n + s_n e) - a_1 \int_{I_{k_n}} |q_n + s_n e|^\alpha. \tag{9}$$

Let $p_n = q_n / R_n$ and $\sigma_n = s_n / R_n$. Then by (8) and (9) we infer

$$\begin{aligned} &\frac{1}{2}(L_{k_n} p_n, p_n) + \sigma_n (L_{k_n} p_n, e) \\ &\qquad + \frac{1}{2}\sigma_n^2 (Le, e) - R_n^{\alpha-2} a_1 \int_{I_{k_n}} |p_n + \sigma_n e|^\alpha > 0. \end{aligned} \tag{10}$$

The sequence $\{\sigma_n\}$ is bounded above and it is bounded away from zero; indeed, inequality (10) cannot hold when σ_n is too small because $(L_{k_n} p_n, p_n) \le -\overline{\lambda}$, while if $\sigma_n \to \infty$ we get a contradiction dividing the whole inequality by σ_n^α. Hence $\sigma_n \to \sigma > 0$ up to a subsequence, and the first three terms of (10) are bounded. Note that by the definition of e and L_n we have for all $n \geq \overline{n}$ and for all q in H_n,

$$(L_n q, e) = \int_{I_n} [\dot{q}\dot{e} - Aqe] = \int_{I_{\overline{n}}} [\dot{q}\dot{e} - Aqe] \leq c\|q\|,$$

hence dividing again (10) by $R_n^{\alpha-2}$ we get

$$\int_{I_{k_n}} |p_n + \sigma_n e|^\alpha < \frac{c}{R_n^{\alpha-2}} \tag{11}$$

for some constant c. Let $\widehat{p}_n$ be the restriction of p_n to the interval $I_{\overline{n}}$; we have $\widehat{p}_n \in H^1(I_{\overline{n}}, \mathbb{R}^N)$ and $\|\widehat{p}_n\|_{H^1(I_{\overline{n}}, \mathbb{R}^N)} \leq 1$, therefore $\widehat{p}_n$ converges weakly, up to a subsequence, to some function $p \in H^1(I_{\overline{n}}, \mathbb{R}^N)$. By (11), $\widehat{p}_n \to -\sigma e$ in $L^\alpha(I_{\overline{n}}, \mathbb{R}^N)$, therefore $\widehat{p}_n \rightharpoonup -\sigma e$ in $H^1(I_{\overline{n}}, \mathbb{R}^N)$ and by the definition of L_n we get

$$(L_{k_n} p_n, e) = (L_{\overline{n}} \widehat{p}_n, e) = (\widehat{p}_n, Le) \to -\sigma(Le, e),$$

and finally

$$\limsup_{n\to\infty} \frac{1}{2}(L_{k_n} p_n, p_n) + \sigma_n (L_{k_n} p_n, e) + \frac{1}{2}\sigma_n^2 (Le, e) - R_n^{\alpha-2} a_1 \int_{I_{k_n}} |p_n + \sigma_n e|^\alpha \\ \leq -\frac{\overline{\lambda}}{2} - \frac{1}{2}\sigma^2 (Le, e) < 0,$$

contradicting (10).

(c) *There exists $s > 0$ such that $J_n(q + se) \leq 0$ for all $q \in H_n^-$ with $\|q\| \leq R$ and all $n \geq \overline{n}$.* By the inequality

$$a_1 \int_{I_n} |q + se|^\alpha \geq \delta\left(s^\alpha \int_{I_n} |e|^\alpha - \int_{I_n} |q|^\alpha\right)$$

(which holds for some $\delta > 0$ and all $s \geq 2\|q\|_\alpha / \|e\|_\alpha$) we get

$$J_n(q + se) \leq \frac{1}{2}(L_n q, q) + s(L_n q, e) + \frac{1}{2}s^2(Le, e) - \delta s^\alpha \int_{I_n} |e|^\alpha + \delta \int_{I_n} |q|^\alpha$$

and the result follows for large s because $\|q\|_\alpha^\alpha \leq \widehat{c}^\alpha \|q\|^\alpha \leq \widehat{c}^\alpha R^\alpha$.

(d) As the identity map is in Γ_n, we have

$$b_n = \inf_{\gamma\in\Gamma_n} \max_{\tau\in Q_n} J_n(\gamma(\tau)) \leq \max_{q\in Q_n} J_n(q) \leq s(L_n q, e) + \frac{1}{2}s^2(Le, e) \leq c(Rs + s^2),$$

where R and s are the n-independent constants obtained in the previous steps. □

Lemma 7. *There exist two positive constants l and l' such that $l \leq \|q_n\|_{H_n} \leq l'$ for almost all n.*

Proof. The lower bound follows by $J_n(q_n) \geq a > 0$.

By Lemma 6 we have $J_n(q_n) \le A$ for almost all n, therefore $2A \ge 2J_n(q_n) - J_n'(q_n)[q_n]$ and

$$\begin{aligned}(12)\qquad 2A &\ge \int_{I_n} W_q(q_n,t)\cdot q_n - 2W(q_n,t) \ge (\alpha-2)\int_{I_n} W(q_n,t)\\ &\ge (\alpha-2)a_1\|q_n\|_\alpha^\alpha.\end{aligned}$$

Let q_n^+ and q_n^- be the projections of q_n on H_n^+ and H_n^-; then

$$\begin{aligned}\overline{\lambda}\|q_n^\pm\|^2 \le |(L_n q_n^\pm, q_n^\pm)| &= \left|\int_{I_n} W_q(q_n,t)q_n^\pm\right|\\ &\le a_2\int_{I_n}|q_n|^{\alpha-1}|q_n^\pm| \le a_2\|q_n\|_\alpha^{\alpha-1}\|q_n^\pm\|_\alpha;\end{aligned}$$

but by Lemma 1, $\|q_n^\pm\|_\alpha \le \widehat{c}\|q_n^\pm\|$, so by (12), $\|q_n\|_\alpha^{\alpha-1}$ is bounded and the claim follows. $\square$

Having obtained a sequence of $2\pi n$-periodic solutions of equation (1), we can prove the existence of a homoclinic solution: by using Lemmas 2, 7 and the fact that q_n satisfies (1) we find that $i_n q_n \rightharpoonup q$ in H up to a subsequence and $i_n q_n \to q$ in $C^2_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^N)$ (cf. the proof of Lemma 3). Hence q is a classical homoclinic solution of (1). We have to prove that $q \ne 0$. First note that $\|q_n\|_\infty \ge c > 0$ by the following lemma, which is a special case of Lemma I.1 of [4]:

Lemma 8. *Let $1 < p \le \infty$ and $1 \le q < \infty$. Let $\{f_n\}$ be a bounded sequence in $L^q(\mathbb{R})$ such that $\{f_n'\}$ is bounded in $L^p(\mathbb{R})$. If there exists R such that*

$$\lim_{n\to\infty}\left[\sup_{y\in\mathbb{R}}\int_{y-R}^{y+R}|f_n(x)|^q\,dx\right] = 0$$

then $f_n \to 0$ in $L^r(\mathbb{R})$ for all $r \in (q,\infty)$.

If $\|q_n\|_\infty \to 0$, it follows from this lemma that $\|q_n\|_\alpha \to 0$. Hence $\int_{I_n} W(q_n,t) \to 0$ and $\int_{I_n} W_q(q_n,t)q_n \to 0$, and from $J_n'(q_n)[q_n] = 0$ we would infer $(Lq_n,q_n) \to 0$ and $J(q_n) \to 0$, contradicting the lower bound of b_n.

The functionals J_n are invariant by translations of t by integer multiples of 2π, therefore as $\|q_n\|_\infty \ge c > 0$ for almost all n, we can assume that there exists $t_n \in [0,2\pi]$ such that $|q_n(t_n)| \ge c$ and as q_n converges uniformly in $[0,2\pi]$, we obtain $q \ne 0$.

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In [8, 9] elliptic partial differential equations of a type similar to (1) are considered. Also there 0 is in a gap of the spectrum of the linear part, but the nonlinearity is the derivative of a convex function. In [10] a first order Hamiltonian system without any convexity assumption is studied. The general idea of the proof is similar to ours: existence of a sequence of solutions is obtained by a linking argument and it is shown that this sequence tends to a homoclinic.

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