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SOME REMARKS ON DEGREE THEORY FOR SO(2)-EQUIVARIANT TRANSVERSAL MAPS

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Abstract. The aim of this article is to introduce a new class SO(2)-equivariant transversal maps $\mathcal{TR}(\mathrm{cl}(\Omega), \partial\Omega)$ and to define degree theory for such maps. We define degree for SO(2)-equivariant transversal maps and prove some properties of this invariant. Moreover, we characterize SO(2)-equivariant transversal isomorphisms and derive formula for degree of such isomorphisms.

1. Introduction

The Brouwer degree plays important role in nonlinear analysis. This degree and its infinite-dimensional generalization, the Leray–Schauder degree, have been successfully applied to the studies of solutions of nonlinear problems. Significant contributions to the degree theory have been done by Borsuk. Borsuk established that the degree of an odd map of a sphere into itself is odd. He was the first who observed that symmetries can lead to the restriction of possible values of the mapping degree. Some computations of these degrees for equivariant maps have been done in [5], [9], [18], [19], [28], [29]. Degree theories for equivariant maps have been defined by many authors. The first degree theory for admissible SO(2)-equivariant gradient maps (this class is a subclass of the class $\mathcal{GRAD}(\mathrm{cl}(\Omega), \partial\Omega)$

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defined in our article), which is a rational number, is due to E. N. Dancer ([2]). Degree theories for equivariant maps with SO(2)-symmetries have been defined in [4], [7], [10], [11], [16]. The class of SO(2)-equivariant orthogonal maps (class $\mathcal{ORT}(\mathrm{cl}(\Omega), \partial\Omega)$ in our article) have been introduced in [20]. The first degree theory for SO(2)-equivariant orthogonal maps, which is an element of tom Dieck ring $U(\mathrm{SO}(2)) = \mathbb{Z} \oplus (\bigoplus_{i=1}^{\infty} \mathbb{Z})$, is due to the second author (see [20]). Applications of degree theory for SO(2)-equivariant orthogonal maps to differential equations one can find in [3], [17], [20]–[27]. Degree theories for abelian actions have been defined in [13]–[15]. A definition of degree theory for equivariant orthogonal maps (symmetries of any compact abelian Lie group are admitted) is due to J. Ize and A. Vignoli (see [15]). Finally, degree theory for G-equivariant gradient maps, where G is any compact Lie group, is due to K. Gęba (see [6]).

Let $\Omega \subset V$ be an open, bounded subset of a real, finite-dimensional, orthogonal representation V of the group SO(2). Since V is an orthogonal representation of the group SO(2), $\mathcal{GRAD}(\mathrm{cl}(\Omega), \partial\Omega) \subset \mathcal{ORT}(\mathrm{cl}(\Omega), \partial\Omega)$. The aim of this article is to define a class of SO(2)-equivariant transversal maps $\mathcal{TR}(\mathrm{cl}(\Omega), \partial\Omega)$ such that $\mathcal{ORT}(\mathrm{cl}(\Omega), \partial\Omega) \subset \mathcal{TR}(\mathrm{cl}(\Omega), \partial\Omega)$ and to define degree theory for these maps.

The paper is organized as follows. In Section 2 we set up notations, terminology and introduce classes of maps considered in this article. Moreover, we show the relation of symmetric SO(2)-equivariant isomorphisms to orthogonal SO(2)-equivariant isomorphisms, see Lemma 2.5. Additionally, we characterize SO(2)-equivariant transversal isomorphisms, see Lemma 2.8. In Section 3 we define degree for SO(2)-equivariant transversal maps, see Definition 3.5, and prove its properties, see Theorem 3.6. In Section 4 we derive an interesting formula for degree of transversal SO(2)-equivariant isomorphism, see Theorem 4.3. In Section 5 we discuss relation of degree for SO(2)-equivariant transversal maps to the Brouwer degree. Moreover, we indicate possible applications of this degree to nonlinear analysis.

2. Preliminary results

In this section we set up notation and terminology. Throughout this article we let $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{Z} = \{0, \pm1, \pm2, \dots\}$ Moreover, $\mathrm{GL}(n, \mathbb{R}), \mathrm{O}(n, \mathbb{R})$ and $\mathrm{SO}(n, \mathbb{R})$ stands for the group of nonsingular matrices, orthogonal matrices and special orthogonal matrices, respectively. The cyclic subgroup of SO(2) of order $m \in \mathbb{N}$ will be denoted by $\mathbb{Z}_m$. From now on V stands for a finite-dimensional, real, orthogonal representation of the group SO(2) with an SO(2)-invariant scalar product $\langle \cdot\,, \cdot \rangle$, i.e. $V = (\mathbb{R}^n, \rho)$, where $\rho \colon \mathrm{SO}(2) \to \mathrm{O}(n, \mathbb{R})$ is a homomorphism. The linear action of SO(2) on $\mathbb{R}^n$ is given by $\varrho \colon \mathrm{SO}(2) \times \mathbb{R}^n \to \mathbb{R}^n$, where $\varrho(g, x) = \rho(g)x$. To shorten notations we will write gx instead of $\varrho(g, x)$. For $K = \mathrm{SO}(2)$

or $\mathbb{Z}_m$ put $V^K = \{v \in V : gv = v \text{ for all } g \in K\}$ and $(V^K)^\perp = \{v \in V : \langle v, w\rangle = 0 \text{ for all } w \in V^K\}$. Moreover, put $D_\alpha(V) = \{v \in V : \|v\| < \alpha\}$. Let $\Omega \subset V$ denote an open, bounded and SO(2)-invariant subset. The closure of Ω will be denoted by $\mathrm{cl}(\Omega)$. A continuous map $f: V \to V$ is said to be SO(2)-equivariant map, if $f(gv) = gf(v)$ for any $g \in \mathrm{SO}(2)$ and $v \in V$. Let $L: V \to V$ be an SO(2)-equivariant linear map. We will denote by $L^*: V \to V$ the adjoint operator to L, i.e. $\langle Lv, w\rangle = \langle v, L^*w\rangle$ for all $v, w \in V$. For $m \in \mathbb{N}$ define a map $\rho^m: \mathrm{SO}(2) \to GL(2, \mathbb{R})$ as follows

$$\rho^m(e^{i\theta}) = \begin{bmatrix} \cos m\cdot\theta & -\sin m\cdot\theta \\ \sin m\cdot\theta & \cos m\cdot\theta \end{bmatrix} \quad 0 \le \theta \le 2\pi.$$

For $k, m \in \mathbb{N}$ we denote by $\mathbb{R}[k, m]$ the direct sum of k copies of $(\mathbb{R}^2, \rho^m)$, we also denote by $\mathbb{R}[k, 0]$ the trivial k-dimensional representation of SO(2). We will use the symbol ∇f to denote the gradient of SO(2)-equivariant C^1-map $f: \Omega \to \mathbb{R}[1, 0]$. Since representation V is orthogonal, ∇f is SO(2)-equivariant continuous map. It is known that

$$\mathrm{SO}(2) = \left\{ \xi(\theta) = \begin{bmatrix} \cos(\theta) & -sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} : \theta \in [-\pi, \pi) \right\}.$$

Let $E: V \to V$ be an SO(2)-equivariant, linear map defined as follows:

$$E(v) = \lim_{\theta \to 0} \frac{\xi(\theta)v - v}{\theta}.$$

It is clear that that $E(v)$ is a tangent vector to the orbit $\mathrm{SO}(2)v$ at $v \in V$.

Definition 2.1. Let $Q \subset V$. Two SO(2)-equivariant continuous maps $f, g: Q \to V$ are said to be *orthogonal on* Q, if $\langle f(v), g(v)\rangle = 0$ for any $v \in Q$. The notation $f \perp_Q g$ means that maps f and g are orthogonal on Q.

Definition 2.2. Let $Q \subset V$. Two SO(2)-equivariant continuous maps $f, g: Q \to V$ are said to be *transversal on* Q, if for any $v \in Q$

$$f(v) \neq 0 \text{ and } g(v) \neq 0 \text{ implies } \mathrm{rank}\{f(v), g(v)\} = 2.$$

The notation $f \pitchfork_Q g$ means that maps f and g are transversal on Q.

The following classes of maps will be considered in this article:

- $\mathcal{GL}_{\mathrm{SO}(2)}(V) = \{L: V \to V : L \text{ is an SO(2)-equivariant linear isomorphism}\}$,
- $\mathcal{GRAD}_{GL}(V) = \{L \in \mathcal{GL}_{\mathrm{SO}(2)}(V) : L = L^*\}$,
- $\mathcal{ORT}_{GL}(V) = \{L \in \mathcal{GL}_{\mathrm{SO}(2)}(V) : L \perp_V E\}$,
- $\mathcal{TR}_{GL}(V) = \{L \in \mathcal{GL}_{\mathrm{SO}(2)}(V) : L \pitchfork_V E\}$,
- $\mathcal{C}_{\mathrm{SO}(2)}(\mathrm{cl}(\Omega), V) = \{f: \mathrm{cl}(\Omega) \to V : f \text{ is SO(2)-equivariant } C^0\text{-map}\}$,
- $\mathcal{C}_{\mathrm{SO}(2)}(\mathrm{cl}(\Omega), \partial\Omega) = \{f \in \mathcal{C}_{\mathrm{SO}(2)}(\mathrm{cl}(\Omega), V) : f^{-1}(0) \cap \partial\Omega = \emptyset\}$,

- $\mathcal{C}^1_{\mathrm{SO}(2)}(\mathrm{cl}(\Omega),\mathbb{R}[1,0]) = \{f\colon \mathrm{cl}(\Omega)\to\mathbb{R}[1,0] : f$ is SO(2)-equivariant C^1-map$\}$,
- $\mathcal{GRAD}(\mathrm{cl}(\Omega),\partial\Omega) = \{\nabla f \in \mathcal{C}_{\mathrm{SO}(2)}(\mathrm{cl}(\Omega),\partial\Omega) : f\in\mathcal{C}^1_{\mathrm{SO}(2)}(\mathrm{cl}(\Omega),\mathbb{R}[1,0])\}$,
- $\mathcal{ORT}(\mathrm{cl}(\Omega),\partial\Omega) = \{f \in \mathcal{C}_{\mathrm{SO}(2)}(\mathrm{cl}(\Omega),\partial\Omega) : f \perp_{\mathrm{cl}(\Omega)} E\}$,
- $\mathcal{TR}(\mathrm{cl}(\Omega),\partial\Omega) = \{f \in \mathcal{C}_{\mathrm{SO}(2)}(\mathrm{cl}(\Omega),\partial\Omega) : f \pitchfork_{\mathrm{cl}(\Omega)} E\}$.

In the sequel we call:

- $\mathcal{GRAD}(\mathrm{cl}(\Omega),\partial\Omega)$ – the class of admissible SO(2)-equivariant gradient maps,
- $\mathcal{ORT}(\mathrm{cl}(\Omega),\partial\Omega)$ – the class of admissible SO(2)-equivariant orthogonal maps,
- $\mathcal{TR}(\mathrm{cl}(\Omega),\partial\Omega)$ – the class of admissible SO(2)-equivariant transversal maps.

Let $\mathbb{C}^n = \mathbb{R}^n + \sqrt{-1}\mathbb{R}^n$ be a complexification of $\mathbb{R}^n$. Define complexification $L^c\colon\mathbb{C}^n\to\mathbb{C}^n$ of a linear map $L\colon\mathbb{R}^n\to\mathbb{R}^n$ as follows $L^c(x+\sqrt{-1}y) = L(x) + \sqrt{-1}L(y)$. Define $\chi(L) = \sigma(L^c)$, where $\sigma(L^c)\subset\mathbb{C}$ denotes the spectrum of L^c.

Fact 2.3. *Under the above assumptions the following holds true:*

(a) $\mathcal{GRAD}_{GL}(V)\subset\mathcal{ORT}_{GL}(V)\subset\mathcal{TR}_{GL}(V)\subset\mathcal{GL}_{\mathrm{SO}(2)}(V)$,
(b) $\mathcal{GRAD}(\mathrm{cl}(\Omega),\partial\Omega)\subset\mathcal{ORT}(\mathrm{cl}(\Omega),\partial\Omega)\subset\mathcal{TR}(\mathrm{cl}(\Omega),\partial\Omega)$
$\subset\mathcal{C}_{\mathrm{SO}(2)}(\mathrm{cl}(\Omega),V)$.

The easy proof of this fact is left to the reader.

Lemma 2.4. *Let $L\in\mathcal{GL}_{\mathrm{SO}(2)}(V)$. Then,*

(a) $L^*\in\mathcal{GL}_{\mathrm{SO}(2)}(V)$,
(b) $(L-L^*)^* = -(L-L^*)$,
(c) $(L-L^*)(L-L^*)^* = (L-L^*)^*(L-L^*)$,
(d) $\chi(L-L^*)\subset\sqrt{-1}\mathbb{R}$,
(e) $E^* = -E$,
(f) $EE^* = E^*E$,
(g) $\chi(E)\subset\sqrt{-1}\mathbb{R}$,
(h) $EL = LE$, $EL^* = L^*E$,
(i) $E(L-L^*) = (L-L^*)E$.

Proof. (a) Suppose contrary to our claim that there is $w\in V$ such that $w\neq 0$ and $L^*w = 0$. Thus we obtain $0\neq\langle w,w\rangle = \langle LL^{-1}w,w\rangle = \langle L^{-1}w, L^*w\rangle = \langle L^{-1}w,0\rangle = 0$, a contradiction. What is left is to show that L^* is an SO(2)-equivariant map. Fix $g\in\mathrm{SO}(2)$ and $v,w\in V$. Since V is an orthogonal representation of the group SO(2) we obtain the following

$$\begin{aligned}\langle w, g^{-1}L^*v\rangle &= \langle gw, L^*v\rangle = \langle L(gw), v\rangle = \langle gLw, v\rangle \\ &= \langle Lw, g^*v\rangle = \langle Lw, g^{-1}v\rangle = \langle w, L^*(g^{-1}v)\rangle.\end{aligned}$$

From the above it follows that for any $g \in \mathrm{SO}(2)$ and $v \in V$ we have $g^{-1}L^*v = L^*(g^{-1}v)$, which completes the proof of (a).

(b) Obvious. (c) Direct consequence of (b).

(d) Since operator $L - L^*$ is skew-adjoint, i.e. $(L - L^*)^* = -(L - L^*)$, $\chi(L - L^*) \subset \sqrt{-1}\mathbb{R}$.

(e) Fix $v, w \in V$. Since V is an orthogonal representation of the group $\mathrm{SO}(2)$ we obtain the following

$$\begin{aligned}\langle v, E^*w\rangle = \langle Ev, w\rangle &= \left\langle \lim_{\theta\to 0} \frac{\xi(\theta)v - v}{\theta}, w\right\rangle = \lim_{\theta\to 0}\left\langle \frac{\xi(\theta)v - v}{\theta}, w\right\rangle \\ &= \lim_{\theta\to 0}\left\langle v, \frac{\xi(\theta)^*w - w}{\theta}\right\rangle = \lim_{\theta\to 0}\left\langle v, \frac{\xi(\theta)^{-1}w - w}{\theta}\right\rangle \\ &= \lim_{\theta\to 0}\left\langle v, \frac{\xi(-\theta)w - w}{\theta}\right\rangle = \langle v, -Ew\rangle,\end{aligned}$$

which completes the proof of (e).

(f) Direct consequence of (e).

(g) Since operator E is skew-adjoint, i.e. $E^* = -E$, $\chi(E) \subset \sqrt{-1}\mathbb{R}$.

(h) Fix $v \in V$. Then

$$ELv = \lim_{\theta\to 0} \frac{\xi(\theta)Lv - Lv}{\theta} = \lim_{\theta\to 0} \frac{L\xi(\theta)v - Lv}{\theta} = LEv,$$

which completes the proof (h).

The same reasoning applies to L^*. (i) Direct consequence of (h). $\square$

Let $L\colon V \to V$ be an $\mathrm{SO}(2)$-equivariant linear map and let $v = (v_{\mathrm{SO}(2)}, v_\perp) \in V = V^{\mathrm{SO}(2)} \oplus (V^{\mathrm{SO}(2)})^\perp$. Then,

$$L(v) = L(v_{\mathrm{SO}(2)}, v_\perp) = (L_{|V^{\mathrm{SO}(2)}}(v_{\mathrm{SO}(2)}), L_{|(V^{\mathrm{SO}(2)})^\perp}(v_\perp)).$$

For simplicity of notation we denote $L_{\mathrm{SO}(2)} = L_{|V^{\mathrm{SO}(2)}}$ and $L_\perp = L_{|(V^{\mathrm{SO}(2)})^\perp}$.

Lemma 2.5. *The following conditions are equivalent:*

(a) $\dim V^{\mathrm{SO}(2)} \le 1$,

(b) $\mathcal{GRAD}_{GL}(V) = \mathcal{ORT}_{GL}(V)$.

Proof. (a)$\Rightarrow$(b). From Fact 2.3(a) we obtain that

$$\mathcal{GRAD}_{GL}(V) \subset \mathcal{ORT}_{GL}(V).$$

What is left is to show that $\mathcal{ORT}_{GL}(V) \subset \mathcal{GRAD}_{GL}(V)$, i.e. $L = L^*$ for any $L \in \mathcal{ORT}_{GL}(V)$. Since V is an orthogonal $\mathrm{SO}(2)$-representation we have

(1) $V = V^{\mathrm{SO}(2)} \oplus (V^{\mathrm{SO}(2)})^\perp$,

(2) $L = \mathrm{diag}(L_{\mathrm{SO}(2)}, L_\perp)\colon V = V^{\mathrm{SO}(2)} \oplus (V^{\mathrm{SO}(2)})^\perp \to V^{\mathrm{SO}(2)} \oplus (V^{\mathrm{SO}(2)})^\perp$,

(3) $L^* = \mathrm{diag}(L^*_{\mathrm{SO}(2)}, L^*_\perp)\colon V = V^{\mathrm{SO}(2)} \oplus (V^{\mathrm{SO}(2)})^\perp \to V^{\mathrm{SO}(2)} \oplus (V^{\mathrm{SO}(2)})^\perp$,

(4) $E = \mathrm{diag}(\Theta, E_\perp) : V = V^{\mathrm{SO}(2)} \oplus (V^{\mathrm{SO}(2)})^\perp \to V^{\mathrm{SO}(2)} \oplus (V^{\mathrm{SO}(2)})^\perp$.

From Lemma 2.4 it follows that operators E and $L-L^*$ are normal, skew-adjoint and mutually commuting. Moreover, since $\dim V^{\mathrm{SO}(2)} \leq 1$, $L_{\mathrm{SO}(2)} = L^*_{\mathrm{SO}(2)}$. Hence there is $Q \in O(\dim V, \mathbb{R})$ such that

$$E = \begin{cases} Q\mathrm{diag}\left\{\left[\begin{smallmatrix} 0 & -\nu_1 \\ \nu_1 & 0 \end{smallmatrix}\right], \dots, \left[\begin{smallmatrix} 0 & -\nu_{(\dim V)/2} \\ \nu_{(\dim V)/2} & 0 \end{smallmatrix}\right]\right\} Q^{-1} & \text{if } \dim V^{\mathrm{SO}(2)} = 0, \\ Q\mathrm{diag}\left\{0, \left[\begin{smallmatrix} 0 & -\nu_1 \\ \nu_1 & 0 \end{smallmatrix}\right], \dots, \left[\begin{smallmatrix} 0 & -\nu_{(\dim V-1)/2} \\ \nu_{(\dim V-1)/2} & 0 \end{smallmatrix}\right]\right\} Q^{-1}, & \\ & \text{if } \dim V^{\mathrm{SO}(2)} = 1, \end{cases}$$

where $\mu_i \neq 0$ for any $i = 1, \dots, (\dim V)/2$ and that

$$L - L^* = \begin{cases} Q\mathrm{diag}\left\{\left[\begin{smallmatrix} 0 & -\mu_1 \\ \mu_1 & 0 \end{smallmatrix}\right], \dots, \left[\begin{smallmatrix} 0 & -\mu_{(\dim V)/2} \\ \mu_{(\dim V)/2} & 0 \end{smallmatrix}\right]\right\} Q^{-1} & \\ & \text{if } \dim V^{\mathrm{SO}(2)} = 0, \\ Q\mathrm{diag}\left\{0, \left[\begin{smallmatrix} 0 & -\mu_1 \\ \mu_1 & 0 \end{smallmatrix}\right], \dots, \left[\begin{smallmatrix} 0 & -\mu_{(\dim V-1)/2} \\ \mu_{(\dim V-1)/2} & 0 \end{smallmatrix}\right]\right\} Q^{-1} & \\ & \text{if } \dim V^{\mathrm{SO}(2)} = 1. \end{cases}$$

Suppose contrary to our claim that $L \neq L^*$. Therefore there is $\mu_{j_0} \neq 0$. Hence there is $v \in (V^{\mathrm{SO}(2)})^\perp$ such that $\langle (L-L^*)v, Ev\rangle \neq 0$. Taking into account that $L \in \mathcal{ORT}_{GL}(V)$ and Lemma 2.4 (e), (h) we obtain the following

$$\begin{aligned} 0 \neq \langle (L-L^*)v, Ev\rangle &= \langle -L^*v, Ev\rangle = \langle L^*v, -Ev\rangle \\ &= \langle v, -LEv\rangle = \langle v, -ELv\rangle = \langle v, E^*Lv\rangle = \langle Ev, Lv\rangle = 0, \end{aligned}$$

a contradiction.

(b)$\Rightarrow$(a) Suppose, contrary to our claim, that $\mathcal{GRAD}_{GL}(V) = \mathcal{ORT}_{GL}(V)$ and that $\dim V^{\mathrm{SO}(2)} \geq 2$. To obtain a contradiction we will construct a map $L \in \mathcal{ORT}_{GL}(V)$ such that $L \neq L^*$. Since $\dim V^{\mathrm{SO}(2)} \geq 2$, there is a linear map $L_{\mathrm{SO}(2)}: V^{\mathrm{SO}(2)} \to V^{\mathrm{SO}(2)}$ such that $L_{\mathrm{SO}(2)} \neq L^*_{\mathrm{SO}(2)}$. Moreover, put $L_\perp = \mathrm{id}: \big(V^{\mathrm{SO}(2)}\big)^\perp \to (V^{\mathrm{SO}(2)})^\perp$. Finally define $L := (L_{\mathrm{SO}(2)}, L_\perp): V = V^{\mathrm{SO}(2)} \oplus (V^{\mathrm{SO}(2)})^\perp \to V$. It is evident that $L \in \mathcal{ORT}_{GL}(V)$ and that $L \neq L^*$, a contradiction. $\square$

We say that two representations V and W are equivalent if there exists an equivariant, linear isomorphism $T: V \to W$. The following classic result gives a complete classification (up to equivalence) of finite-dimensional representations of the group SO(2) (see [1]).

Theorem 2.6. *If V is a representation of* SO(2) *then there exist finite sequences $\{k_i\}$, $\{m_i\}$ satisfying*

$$(*) \qquad m_i \in \{0\} \cup \mathbb{N}, \quad k_i \in \mathbb{N}, \quad 1 \leq i \leq r, \quad m_1 < \dots < m_r$$

such that V is equivalent to $\bigoplus_{i=1}^r \mathbb{R}[k_i, m_i]$. Moreover, the equivalence class of V ($V \approx \bigoplus_{i=1}^r \mathbb{R}[k_i, m_i]$) is uniquely determined by $\{m_i\}$, $\{k_i\}$ satisfying $(*)$.

Let $J_{\mathbb{R}}(L)$ denote the real Jordan form of $L \in \mathcal{GL}_{\mathrm{SO}(2)}(\mathbb{R}[k,m])$ and $J_{\mathbb{C}}(T)$ denote the complex Jordan form of $T \in GL(\mathbb{C},k)$.

Lemma 2.7. *Let $k, m \in \mathbb{N}$. Then,*

(a) *there is isomorphism $h: \mathcal{GL}_{\mathrm{SO}(2)}(\mathbb{R}[k,m]) \to GL(\mathbb{C},k)$,*
(b) *group $\mathcal{GL}_{\mathrm{SO}(2)}(\mathbb{R}[k,m])$ is arc-connected,*
(c) *if $L \in \mathcal{GL}_{\mathrm{SO}(2)}(\mathbb{R}[k,m])$, then $h(J_{\mathbb{R}}(L)) = J_{\mathbb{C}}(h(L))$,*
(d) *if $L \in \mathcal{GL}_{\mathrm{SO}(2)}(\mathbb{R}[k,m])$, then there is $P \in \mathcal{GL}_{\mathrm{SO}(2)}(\mathbb{R}[k,m])$ such that*
  (d1) *$P^{-1}LP = J_{\mathbb{R}}(L)$,*
  (d2) *$P^{-1}EP = E$.*

Proof. (a) It is clear that if $L \in \mathcal{GL}_{\mathrm{SO}(2)}(\mathbb{R}[k,m])$, then

$$L = \begin{bmatrix} \alpha_{ij} & -\beta_{ij} \\ \beta_{ij} & \alpha_{ij} \end{bmatrix}_{i,j=1}^{k}.$$

Therefore isomorphism $h: \mathcal{GL}_{\mathrm{SO}(2)}(\mathbb{R}[k,m]) \to GL(\mathbb{C},k)$ is given by

$$h\left(\begin{bmatrix} \alpha_{ij} & -\beta_{ij} \\ \beta_{ij} & \alpha_{ij} \end{bmatrix}_{i,j=1}^{k} \right) = [(\alpha_{ij} + \beta_{ij}\sqrt{-1})]_{i,j=1}^{k}.$$

(b) Since $GL(\mathbb{C},k)$ is arc-connected and (a), $\mathcal{GL}_{\mathrm{SO}(2)}(\mathbb{R}[k,m])$ is arc-connected.

(c) Fix $L \in \mathcal{GL}_{\mathrm{SO}(2)}(\mathbb{R}[k,m])$ and choose $Q \in GL(\mathbb{C},k)$ such that $J_{\mathbb{C}}(h(L)) = Qh(L)Q^{-1}$. Thus

$$h^{-1}(J_{\mathbb{C}}(h(L))) = h^{-1}(Q)Lh^{-1}(Q^{-1}) = h^{-1}(Q)L(h^{-1}(Q))^{-1} = J_{\mathbb{R}}(L),$$

which completes the proof of (c).

(d1) Fix $L \in \mathcal{GL}_{\mathrm{SO}(2)}(\mathbb{R}[k,m])$ and choose $Q \in GL(\mathbb{C},k)$ such that

$$Q^{-1}h(L)Q = J_{\mathbb{C}}(h(L)).$$

From (c), it follows that

$$J_{\mathbb{R}}(L) = h^{-1}(J_{\mathbb{C}}(h(L))) = h^{-1}(Q^{-1})Lh^{-1}(Q) = (h^{-1}(Q))^{-1}Lh^{-1}(Q).$$

Defining $P = h^{-1}(Q)$, we complete the proof of (d1).

(d2) Since $h(E) = m\sqrt{-1} \cdot \mathrm{id}_{\mathbb{C}^k}$, $Q^{-1}h(E)Q = h(E)$. Hence $P^{-1}EP = E$, which completes the proof. □

Lemma 2.8. *The following conditions are equivalent:*

(a) *$L \in \mathcal{TR}_{GL}(V)$,*
(b) *$L \in \mathcal{GL}_{\mathrm{SO}(2)}(V)$ and $\chi(L_{\perp}) \cap \sqrt{-1}\mathbb{R} = \emptyset$.*

Proof. Without loss of generality we assume that

$$V = \mathbb{R}[k,0] \oplus \mathbb{R}[k_1,m_1] \oplus \ldots \oplus \mathbb{R}[k_r,m_r].$$

For abbreviation, we write E_i instead of $E_{|\mathbb{R}[k_i,m_i]}$. Since

$$E_i = \operatorname{diag}\Big\{ \underbrace{\begin{pmatrix} 0 & -m_i \\ m_i & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -m_i \\ m_i & 0 \end{pmatrix}}_{k_i\text{-times}} \Big\},$$

we have $\chi(E_i) = \{\pm m_i\sqrt{-1}\}$. If $L \in \mathcal{GL}_{\mathrm{SO}(2)}(V)$, then, by the Schur lemma, $L(v_0,\ldots,v_r) = (L_0(v_0),\ldots,L_r(v_r))$. From Lemma 2.7 it follows that for any $i \in \{1,\ldots,r\}$ there is $P_i \in \mathcal{GL}_{\mathrm{SO}(2)}(\mathbb{R}[k_i,j_i])$ such that $P_i^{-1}L_iP_i = J_{\mathbb{R}}(L_i)$ and $P_i^{-1}E_iP_i = E_i$. Therefore, without restriction of generality we can assume that $J_{\mathbb{R}}(L_i) = L_i$ for $i = 1,\ldots,r$.

(a)$\Rightarrow$(b) Suppose contrary to our claim that $L \in \mathcal{TR}_{GL}(V)$ and that $\chi(L_\perp) \cap \sqrt{-1}\mathbb{R} \neq \emptyset$. This involves no loss of generality, if we assume that $\chi(L_1) \cap \sqrt{-1}\mathbb{R} \neq \emptyset$. What is left is to show that there is $v_1 \in \mathbb{R}[k_1,m_1] - \{0\}$ such that vectors $L_1(v_1)$ and $E_1(v_1)$ are linearly dependent. Let $\chi(L_1) = \{\pm\lambda_1 = \pm\beta\sqrt{-1}, \lambda_2,\ldots,\lambda_k\}$ and let $J(L_1,\lambda_i)$ is the generalized Jordan block corresponding to the eigenvalue λ_i. Since $J(L_1) = L_1$ we have

$$L_1 = (J(L_1,\pm\beta\sqrt{-1}), J(L_1,\lambda_2),\ldots,J(L_1,\lambda_k)).$$

Define $v_1 = (1,1,0,\ldots,0) \in \mathbb{R}[k_1,m_1]$ and notice that

$$E_1(v_1) = (-m_1,m_1,0,\ldots,0) \quad \text{and} \quad L_1(v_1) = (-\beta,\beta,0,\ldots,0) \tag{2.1}$$

Define $v = (0,v_1,0,\ldots,0) \in V = \mathbb{R}[k,0] \oplus \mathbb{R}[k_1,m_1] \oplus \ldots \oplus \mathbb{R}[k_r,m_r]$. By (2.1) we obtain $E(v) = (m_1/\beta)L(v)$, a contradiction.

(b)$\Rightarrow$(a) Suppose contrary to our claim that $L \in \mathcal{GL}_{\mathrm{SO}(2)}(V)$ and $\chi(L_\perp) \cap \sqrt{-1}\mathbb{R} = \emptyset$ and that $L \notin \mathcal{TR}_{GL}(V)$. Therefore, there is $v = (v_0,\ldots,v_r) \notin V^{\mathrm{SO}(2)}$ such that $E(v)$ and $L(v)$ are linearly dependent. There is no loss of generality in assuming that $v_1 \neq 0$. We claim that $E_1(v_1)$ and $L_1(v_1)$ are linearly independent. Indeed, let $v_1 = (\omega_1,\ldots,\omega_{2k_1})$. Define $j_0 = \max\{j : \omega_j \neq 0\}$ and notice that

$$E_1(v_1) = \begin{cases} (*,\ldots,*,-m_1\omega_{j_0}, m_1\omega_{j_0-1},0,\ldots,0) & \text{if } j_0 \text{ is even,} \\ (*,\ldots,*,0,m_1\omega_{j_0},0,\ldots,0) & \text{if } j_0 \text{ is odd,} \end{cases}$$

and that

$$L_1(v_1) = \begin{cases} (*,\ldots,*,\alpha\omega_{j_0-1} - \beta\omega_{j_0}, \beta\omega_{j_0-1} + \alpha\omega_{j_0},0,\ldots,0) & \text{if } j_0 \text{ is even,} \\ (*,\ldots,*,\alpha\omega_{j_0},\beta\omega_{j_0},0,\ldots,0) & \text{if } j_0 \text{ is odd.} \end{cases}$$

Since $\chi(L_\perp) \cap \sqrt{-1}\mathbb{R} = \emptyset, \alpha \neq 0$. Hence vectors $L_1(v_1), E(v_1)$ are linearly independent, a contradiction. $\square$

3. Definition of degree for SO(2)-equivariant transversal maps

In this section we define degree theory for SO(2)-equivariant transversal maps.

Definition 3.1. Let $f_0, f_1 \in \mathcal{ORT}(\mathrm{cl}(\Omega), \partial\Omega)$ and let $h\colon \mathrm{cl}(\Omega) \times [0,1] \to V$ be a homotopy joining f_0 with f_1. Homotopy h is called an admissible orthogonal homotopy if for any $t \in [0,1]$ $h(\,\cdot\,,t) \in \mathcal{ORT}(\mathrm{cl}(\Omega), \partial\Omega)$. We say that maps f_0, f_1 are orthogonally homotopic.

Definition 3.2. Let $f_0, f_1 \in \mathcal{TR}(\mathrm{cl}(\Omega), \partial\Omega)$ and let $h\colon \mathrm{cl}(\Omega) \times [0,1] \to V$ be a homotopy joining f_0 with f_1. Homotopy h is called an admissible transversal homotopy if $h(\,\cdot\,,t) \in \mathcal{TR}(\mathrm{cl}(\Omega), \partial\Omega)$ for any $t \in [0,1]$. We say that maps f_0, f_1 are transversally homotopic.

Lemma 3.3. *Let $f \in \mathcal{TR}(\mathrm{cl}(\Omega), \partial\Omega)$. Then, there is an admissible transversal homotopy $h\colon \mathrm{cl}(\Omega) \times [0,1] \to V$ such that*

(a) $h(\,\cdot\,,0) = f(\,\cdot\,)$ *and* $h(\,\cdot\,,1) \in \mathcal{ORT}(\mathrm{cl}(\Omega), \partial\Omega)$,
(b) $h^{-1}(0) = f^{-1}(0) \times [0,1]$.

Proof. We first construct a homotopy h and next show that it is an admissible transversal homotopy. Let us define homotopy $h\colon \mathrm{cl}(\Omega) \times [0,1] \to V$ by the following formula

$$h(v,t) = \begin{cases} f(v) - \dfrac{t}{|E(v)|^2} \cdot \langle E(v), f(v)\rangle \cdot E(v) & \text{if } E(v) \neq 0, \\ f(v) & \text{if } E(v) = 0. \end{cases}$$

Taking into consideration decomposition $v = (v_1, v_2) \in V^{\mathrm{SO}(2)} \oplus \big(V^{\mathrm{SO}(2)}\big)^{\perp}$ we obtain the following

(a) $f(v_1, v_2) = (f_1(v_1, v_2), f_2(v_1, v_2))$,
(b) $f(v_1, 0) = (f_1(v_1, 0), f_2(v_1, 0)) = (f_1(v_1, 0), 0)$,
(c) $E(v_1, v_2) = (0, E_\perp(v_2))$.

Now homotopy h can be defined as follows

$$h(v_1, v_2, t) = \begin{cases} (f_1(v_1, v_2), f_2(v_1, v_2)) - \dfrac{t}{|E(v_2)|^2} & \\ \qquad\qquad \cdot\langle E(v_2), f_2(v_1, v_2)\rangle \cdot E(v_2) & \text{if } v_2 \neq 0, \\ (f_1(v_1, 0), 0), & \text{if } v_2 = 0. \end{cases}$$

It is clear that homotopy h is continuous at any point $(v_{1,0}, v_{2,0}, t_0)$ such that $v_{2,0} \neq 0$.

Fix a point $(v_{1,0}, 0, t_0)$ and choose sequence $(v_{1,n}, v_{2,n}, t_n)$ converging to $(v_{1,0}, 0, t_0)$ such that $v_{2,n} \neq 0$ for any $n \in \mathbb{N}$. What is left is to show that

$$\lim_{n\to\infty} h(v_{1,n}, v_{2,n}, t_n) = h(v_{1,0}, 0, t_0).$$

Since f_1 is continuous,

$$\lim_{n\to\infty} f_1(v_{1,n}, v_{2,n}) = f_1(v_{1,0}, 0).$$

It remains to prove that the second coordinate of $h(v_{1,n}, v_{2,n}, t_n)$ converges to $0 \in (V^{\mathrm{SO}(2)})^\perp$. Since $E_{|V^{\mathrm{SO}(2)}} = \Theta$, $E = (E_{\mathrm{SO}(2)}, E_\perp) = (\Theta, E_\perp)$. By the Schwartz inequality we obtain the following:

$$\begin{aligned}\bigg| f_2(v_{1,n}, v_{2,n}) &- \frac{t_n}{|E_\perp(v_{2,n})|^2} \cdot \langle E_\perp(v_{2,n}), f_2(v_{1,n}, v_{2,n})\rangle \cdot E_\perp(v_{2,n})\bigg| \\ &\leq |f_2(v_{1,n}, v_{2_n})| + \frac{t_n}{|E_\perp(v_{2,n})|} \cdot |\langle E_\perp(v_{2,n}), f_2(v_{1,n}, v_{2,n})\rangle| \\ &\leq |f_2(v_{1,n}, v_{2,n})| + t_n |f_2(v_{1,n}, v_{2,n})| \leq 2|f_2(v_{1,n}, v_{2,n})|.\end{aligned}$$

Since f_2 is continuous and $f_2(v_{1,0}, 0) = 0$,

$$\lim_{n\to\infty} h(v_{1,n}, v_{2,n}, t_n) = (f_1(v_{1,0}, 0), 0),$$

which completes the proof of the continuity of h. Since V is an orthogonal representation of the group $\mathrm{SO}(2)$, h is an $\mathrm{SO}(2)$-equivariant homotopy.

We claim that h is a family of transversal maps. Suppose, contrary to our claim, that there are $t_0 \in [0,1]$, $v_0 \in \Omega$ and $\lambda_0 \in \mathbb{R} \setminus \{0\}$ such that

(a) $h(v_0, t_0) \neq 0$,
(b) $E(v_0) \neq 0$,
(c) $h(v_0, t_0) = \lambda_0 E(v_0)$.

Since $h(v_0, t_0) \neq 0$, $f(v_0) \neq 0$. Moreover, from the above it follows that

$$\begin{aligned} h(v_0, t_0) &= f(v_0) - \frac{t_0}{|E(v_0)|^2} \cdot \langle E(v_0), f(v_0)\rangle \cdot E(v_0) = \lambda_0 E(v_0), \\ f(v_0) &= \bigg(\frac{t_0}{|E(v_0)|^2} \cdot \langle E(v_0), f(v_0)\rangle + \lambda_0\bigg) \cdot E(v_0), \end{aligned}$$

which means that $f \notin \mathcal{TR}(\mathrm{cl}(\Omega), \partial\Omega)$, a contradiction.

Fix $t_0 \in [0,1]$ and $v_0 \in \Omega \subset V^{\mathrm{SO}(2)} \oplus (V^{\mathrm{SO}(2)})^\perp$. If $f(v_0) = 0$ then $h(v_0, t_0) = 0$. In other words $f^{-1}(0) \subset h(\,\cdot\,, t_0)^{-1}(0)$. What is left is to show that $h(\,\cdot\,, t_0)^{-1}(0) \subset f^{-1}(0)$. If $h(v_0, t_0) = 0$, then $f_1(v_0) = 0$. If $E(v_0) = 0$, then $v_0 \in V^{\mathrm{SO}(2)}$ and $f_2(v_0) = 0$. Suppose now that $E(v_0) \neq 0$ and notice that

$$(0, f_2(v_0)) - \frac{t_0}{|E(v_0)|^2} \cdot \langle E(v_0), f(v_0)\rangle \cdot (0, E_\perp(v_0)) = 0. \tag{3.1}$$

Suppose, contrary to our claim, that $f_2(v_0) \neq 0$. Since $f \in \mathcal{TR}(\mathrm{cl}(\Omega), \partial\Omega)$ vectors $(0, f_2(v_0))$ and $(0, E_\perp(v_0))$ are linearly independent, which contradicts (3.1). In other words we have just shown that $h(\,\cdot\,, t_0)^{-1}(0) \subset f^{-1}(0)$. Summing up, we have shown that for any $t \in [0,1]$ $h(\cdot,t)^{-1}(0) = f^{-1}(0)$. Finally notice that

$$\langle h(v,1), E(v)\rangle = \langle f(v) - \frac{1}{|E(v)|^2} \cdot \langle E(v), f(v)\rangle \cdot E(v), E(v)\rangle = 0$$

for any $v \in \mathrm{cl}(\Omega)$ such that $E(v) \neq 0$, which completes the proof. $\square$

Lemma 3.4. *If $f_0, f_1 \in \mathcal{ORT}(\mathrm{cl}(\Omega), \partial\Omega)$ are transversally homotopic maps, then they are orthogonally homotopic.*

Proof. Let h be an admissible transversal homotopy joining f_0 with f_1. Define homotopy $g\colon \mathrm{cl}(\Omega) \times [0,1] \to V$ as follows:

$$g(v,t) = \begin{cases} h(v,t) - \dfrac{1}{|E(v)|^2}\langle E(v), h(v,t)\rangle \cdot E(v) & \text{if } E(v) \neq 0, \\ h(v,t) & \text{if } E(v) = 0. \end{cases}$$

Repeating the reasoning from the proof of Lemma 3.3 we show that g is an admissible orthogonal homotopy joining $g(\,\cdot\,,i) = f_i(\,\cdot\,)$, $i = 0,1$. $\square$

We are now in a position to define degree theory for SO(2)-equivariant transversal maps.

Definition 3.5. The degree for SO(2)-equivariant transversal maps is an element of the group $\mathbb{Z} \oplus (\bigoplus_{i=1}^{\infty} \mathbb{Z})$ defined as follows

$$\mathcal{DEG}(f,\Omega) := DEG(f_0,\Omega) \in \mathbb{Z} \oplus \bigg(\bigoplus_{i=1}^{\infty} \mathbb{Z}\bigg),$$

where $f \in \mathcal{TR}(\mathrm{cl}(\Omega), \partial\Omega)$, $f_0 \in \mathcal{ORT}(\mathrm{cl}(\Omega), \partial\Omega)$ is chosen as in Lemma 3.3 and $DEG(\,\cdot\,,\,\cdot\,)$ denotes the degree for SO(2)-equivariant orthogonal maps defined in [20].

First of all we will show that the above definition does not depend on the choice of the map f_0. Let $f_1 \in \mathcal{ORT}(\mathrm{cl}(\Omega), \partial\Omega)$ be a map transversally homotopic with f such that $f_0 \neq f_1$. Notice that $f_0, f_1 \in \mathcal{ORT}(\mathrm{cl}(\Omega), \partial\Omega)$ and these maps are transversally homotopic. By Lemma 3.4 f_0 and f_1 are orthogonally homotopic. Finally, by the homotopy invariance of degree theory for SO(2)-equivariant orthogonal maps, see Theorem 3.9(d) in [20], we obtain $DEG(f_0,\Omega) = DEG(f_1,\Omega)$. In other words the definition of degree for SO(2)-equivariant transversal maps does not depend on the choice of f_0.

Theorem 3.6. *Assume that $\Omega \subset V$ is an open, bounded and $\mathrm{SO}(2)$-invariant subset of a finite-dimensional, real, orthogonal, representation V of $\mathrm{SO}(2)$ and that $f \in \mathcal{TR}(\mathrm{cl}(\Omega), \partial\Omega)$. Then*

(a) *if $\mathcal{DEG}_K(f,\Omega) \neq 0$, then $f^{-1}(0) \cap \Omega^K \neq \emptyset$,*

(b) *if* $\Omega_0 \subset \Omega$ *is an open* SO(2)*-invariant subset such that* $f^{-1}(0) \cap \Omega \subset \Omega_0$, *then*

$$\mathcal{DEG}(f,\Omega) = \mathcal{DEG}(f,\Omega_0),$$

(c) *if* $\Omega_1, \Omega_2 \subset \Omega$ *are open, disjoint,* SO(2)*-invariant subsets such that* $f^{-1}(0) \subset \Omega_1 \cup \Omega_2$, *then*

$$\mathcal{DEG}(f,\Omega) = \mathcal{DEG}(f,\Omega_1) + \mathcal{DEG}(f,\Omega_2),$$

(d) *if* $h\colon (\mathrm{cl}(\Omega)\times[0,1], \partial\Omega\times[0,1]) \to (V, V-\{0\})$, *is an admissible transversal homotopy, then*

$$\mathcal{DEG}(h(\,\cdot\,,0),\Omega) = \mathcal{DEG}(h(\,\cdot\,,1),\Omega),$$

(e) *if* $\mathcal{U} \subset W$ *is an open, bounded and an* SO(2)*-invariant neighbourhood of* $0 \in W$, *then if a map* $F\colon (\mathrm{cl}(\mathcal{U}\times\Omega), \partial(\mathcal{U}\times\Omega)) \to (W\oplus V, W\oplus V-\{(0,0)\})$ *is given by the formula* $F(w,v) = (w, f(v))$, *then* $\mathcal{DEG}(F,\mathcal{U}\times\Omega) = \mathcal{DEG}(f,\Omega)$.

Proof. (1) Choose $f_0 \in \mathcal{ORT}(\mathrm{cl}(\Omega),\partial\Omega)$ as in Lemma 3.3. Therefore by Definition 3.5 we have

$$\mathcal{DEG}_K(f,\Omega) = DEG_K(f_0,\Omega) \neq 0.$$

From Theorem 3.9(a) in [20] we obtain $f_0^{-1}(0)\cap\Omega^K \neq \emptyset$. Since $f^{-1}(0) = f_0^{-1}(0)$ we have $f^{-1}(0)\cap\Omega^K \neq \emptyset$.

(2) Choose $f_0 \in \mathcal{ORT}(\mathrm{cl}(\Omega),\partial\Omega)$ as in Lemma 3.3. By Definition 3.5 we have $\mathcal{DEG}(f,\Omega) = DEG(f_0,\Omega)$. Since $f^{-1}(0) = f_0^{-1}(0)$, $f_0^{-1}(0)\cap\Omega \subset \Omega_0$. That is why from Theorem 3.9(b) of [20] it follows that $DEG(f_0,\Omega) = DEG(f_0,\Omega_0)$. Consequently by Lemma 3.3 and Definition 3.5 we have $\mathcal{DEG}(f,\Omega_0) = DEG(f_0,\Omega_0)$.

(3) Choose $f_0 \in \mathcal{ORT}(\mathrm{cl}(\Omega),\partial\Omega)$ as in Lemma 3.3. Therefore by Definition 3.5 we have $\mathcal{DEG}(f,\Omega) = DEG(f_0,\Omega)$. Since $f^{-1}(0) = f_0^{-1}(0)$ and Theorem 3.9(c) of [20], $DEG(f_0,\Omega) = DEG(f_0,\Omega_1) + DEG(f_0,\Omega_2)$. Consequently by Lemma 3.3 and Definition 3.5 we have $\mathcal{DEG}(f,\Omega_1) = DEG(f_0,\Omega_1)$ and $\mathcal{DEG}(f,\Omega_2) = DEG(f_0,\Omega_2)$.

(4) By Lemma 3.3 $h(\,\cdot\,,0)$, $h(\,\cdot\,,1)$ are transversally homotopic with orthogonal maps $g_0, g_1 \in \mathcal{ORT}(\mathrm{cl}(\Omega),\partial\Omega)$, respectively. Since h is an admissible transversal homotopy, orthogonal maps g_0 and g_1 are transversally homotopic. From Lemma 3.4 it follows that maps g_0 and g_1 are orthogonally homotopic. Summing up, by Theorem 3.9(d) of [20] we obtain:

$$\mathcal{DEG}(h(\,\cdot\,,0),\Omega) = DEG(g_0,\Omega) = DEG(g_1,\Omega) = \mathcal{DEG}(h(\,\cdot\,,1),\Omega).$$

(5) Choose $f_0 \in \mathcal{ORT}(\mathrm{cl}(\Omega),\partial\Omega)$ as in Lemma 3.3 and notice that by Definition 3.5 and Theorem 3.9(e) of [20] we obtain:

$$\mathcal{DEG}(F,\mathcal{U}\times\Omega) = DEG(\mathrm{id}\times f_0,\mathcal{U}\times\Omega) = DEG(f_0,\Omega) = \mathcal{DEG}(f,\Omega). \qquad \square$$

4. Computations of degree for SO(2)-equivariant transversal maps

The aim of this section is to compute index of an isolated zero of an SO(2)-equivariant transversal map. In the first step we reduce our computation to computation of degree of a map from $\mathcal{TR}_{GL}(V)$. In the second step we prove formula for degree of SO(2)-equivariant transversal isomorphism.

The principal significance of the following lemma is that it allows one to reduce the computation of a local index of f at $0 \in V$ to computation of a local index of its linearization $Df(0)$. This lemma will prove extremely useful in applications.

Lemma 4.1. *Let $\Omega \subset V$ be an open, bounded $\mathrm{SO}(2)$-invariant neighbourhood of $0 \in V$ and let $f \in \mathcal{TR}(\mathrm{cl}(\Omega), \partial\Omega)$ be a C^1-map such that*

(a) $f(0) = 0$,
(b) $Df(0) \in \mathcal{TR}_{GL}(V)$.

Then, there is $\gamma_0 > 0$ such that $\mathcal{DEG}(f, D_\gamma(V)) = \mathcal{DEG}(Df(0), D_\gamma(V))$, for any $\gamma < \gamma_0$.

Proof. We claim that there is $\gamma_0 > 0$ such that for any $\gamma < \gamma_0$ a homotopy

$$h\colon (cl(D_\gamma(V)) \times [0,1], \partial D_\gamma(V) \times [0,1]) \to (V, V - \{0\})$$

defined by

$$h(v,t) = Df(0)(v) + t \cdot (f(v) - Df(0)(v)) = Df(0)(v) + t \cdot \varphi(v) \tag{4.1}$$

is well defined transversal homotopy. We first prove that $h^{-1}(0) \cap (\partial D_\gamma(V) \times [0,1]) = \emptyset$ for sufficiently small γ. It is clear that

$$\text{for any } \varepsilon > 0 \text{ there is } \beta > 0 \text{ such that if } |v| < \beta, \text{ then } |\varphi(v)| < \varepsilon|v| \tag{4.2}$$

Fix $\varepsilon < |Df(0)^{-1}(0)|^{-1}/2$ and choose $\gamma_1 = \beta$ as in (4.2). For any $\gamma < \gamma_1$ and $v \in \partial D_\gamma(V)$ we have

$$\begin{aligned} |h(v,t)| &= |Df(0)(v) + t \cdot \varphi(v)| \geq |Df(0)(v)| - t \cdot |\varphi(v)| \\ &\geq |Df(0)^{-1}|^{-1}|v| - t \cdot |\varphi(v)| \geq |Df(0)^{-1}|^{-1}|v| - t \cdot \varepsilon \cdot |v| \\ &\geq \left(1 - \frac{t}{2}\right) \cdot |Df(0)^{-1}|^{-1}|v| \geq \frac{1}{2} \cdot |Df(0)^{-1}|^{-1}|v| > 0 \end{aligned} \tag{4.3}$$

Suppose that $v_2 \in (V^{\mathrm{SO}(2)})^\perp - \{0\}$. Since $Df(0) \in \mathcal{TR}_{GL}(V)$, $E(v_2)$ and $Df(0)(v_2)$ are linearly independent. Since $\partial D_\gamma((V^{\mathrm{SO}(2)})^\perp)$ is compact, there is $\delta_\gamma > 0$ such that

$$\sup_{|v_2|=\gamma} \frac{|\langle E(v_2), Df(0)(v_2)\rangle|}{|E(v_2)| \cdot |Df(0)(v_2)|} < \delta_\gamma < 1 \tag{4.4}$$

It is easy to check that constant δ_γ does not depend on γ. Therefore for simplicity of notation, we write δ instead of δ_γ. Fix $v = (v_1, v_2) \in V^{\mathrm{SO}(2)} \oplus (V^{\mathrm{SO}(2)})^\perp$ such that $v_2 \neq 0$. The orthogonal projection of $Df(0)(v)$ on $E(v)$ is given by

$$\frac{1}{|E(v)|^2} \cdot \langle E(v), Df(0)(v)\rangle \cdot E(v) = \frac{1}{|E(v_2)|^2} \cdot \langle E(v_2), Df(0)(v_2)\rangle \cdot E(v_2).$$

That is why the distance from $Df(0)(v)$ to $\operatorname{span}\{E(v)\}$ is given by

$$\begin{aligned}\bigg|Df(0)(v) - &\frac{1}{|E(v)|^2} \cdot \langle E(v), Df(0)(v)\rangle \cdot E(v)\bigg| \\ &= \bigg|Df(0)(v) - \frac{1}{|E(v_2)|^2} \cdot \langle E(v_2), Df(0)(v_2)\rangle \cdot E(v_2)\bigg|.\end{aligned}$$

By (4.4) we have

$$\begin{aligned}\bigg|Df(0)(v) \; - \; &\frac{1}{|E(v_2)|^2} \cdot \langle E(v_2), Df(0)(v_2)\rangle \cdot E(v_2)\bigg| \\ &\geq |Df(0)(v)| - \frac{1}{|E(v_2)|} \cdot |\langle E(v_2), Df(0)(v_2)\rangle| \\ &\geq |Df(0)(v)| - \delta \cdot |Df(0)(v_2)| \geq (1-\delta) \cdot |Df(0)(v)| \\ &\geq (1-\delta) \cdot |Df(0)^{-1}|^{-1} \cdot |v|.\end{aligned} \tag{4.5}$$

Fix $\varepsilon < (1-\delta)|Df(0)^{-1}(0)|^{-1}/2$ and choose $\gamma_2 = \beta$ as in (4.2). For any $\gamma < \gamma_2$ and $v \in \partial D_\gamma(V)$ by (4.5) we have

$$\bigg|Df(0)(v) - \frac{1}{|E(v)|^2} \cdot \langle E(v), Df(0)(v)\rangle \cdot E(v)\bigg| > \varphi(v). \tag{4.6}$$

Define $\gamma_0 = \min\{\gamma_1, \gamma_2\}$. Combining (4.3) with (4.6) we deduce that for any $\gamma < \gamma_0$ homotopy (4.1) is well defined admissible transversal homotopy, which completes the proof. $\square$

In the following example we show that the assumptions in the above lemma cannot be relaxed. Namely, it can happen that f is transversal and $Df(0)$ is not transversal.

Example 4.2. Consider an SO(2)-equivariant map $f\colon \mathbb{R}[1,1] \to \mathbb{R}[1,1]$ of the form

$$f(x,y) = Df(0,0)(x,y) + \nabla\varphi(x,y) = (-y,x) + (x(x^2+y^2), y(x^2+y^2)).$$

It is clear that

(a) $Df(0,0) \notin \mathcal{TR}_{GL}(\mathbb{R}[1,1])$ since $\chi(Df(0,0)) = \{\pm\sqrt{-1}\}$ and Lemma 2.8,
(b) $Df(0,0) \perp_{\mathbb{R}[1,1]} \nabla\varphi$ since $\nabla\varphi \in \mathcal{GRAD}(\mathbb{R}[1,1], \emptyset)$ and $Df(0,0) = E$,
(c) $f \in \mathcal{TR}(\mathbb{R}[1,1], \emptyset)$ since $\varphi^{-1}(0,0) = \{(0,0)\}$.

The remainder of this section will be devoted to the computation of degree of a linear SO(2)-equivariant transversal isomorphism. The point of the following lemma is that it allows one to compute degree of elements of $\mathcal{TR}_{GL}(V)$ in terms of spectrum of such isomorphisms. Let $L \in \mathcal{TR}_{GL}(V)$, $\chi_-(L) = \{\lambda \in \chi(L) : \operatorname{Re}\lambda < 0\}$ and let $\mu(\lambda)$ denotes the algebraic multiplicity of $\lambda \in \chi(L)$. Define

$$n^-(L) := \sum_{\lambda \in \chi_-(L)} \mu(\lambda).$$

In other words $n^-(L)$ is the sum of algebraic multiplicities of eigenvalues of matrix L with negative real parts.

Theorem 4.3. *Let $L \in \mathcal{TR}_{GL}(V)$ and $V = \mathbb{R}[k,0] \oplus \mathbb{R}[k_1,m_1] \oplus \ldots \oplus \mathbb{R}[k_r,m_r]$. Then we have $L = \operatorname{diag}(L_0,\ldots,L_r)$ and for any $\alpha > 0$*

$$\mathcal{DEG}_{\mathbb{Q}}(L, D_\alpha(V)) = \begin{cases} \operatorname{sign}(\det L_0) & \text{if } \mathbb{Q} = \mathrm{SO}(2), \\ \operatorname{sign}(\det L_0)\cdot n^-(L_i)/2 & \text{if } \mathbb{Q} = \mathbb{Z}_{m_i},\ i = 1,\ldots,r, \\ 0 & \text{otherwise}, \end{cases}$$

where it is understood that if $k = 0$ then $\operatorname{sign}(\det L_0) = 1$.

Proof. We first prove that L is transversally homotopic with an SO(2)-equivariant orthogonal isomorphism T. For abbreviation, we write E_i instead of $E_{|\mathbb{R}[k_i,m_i]}$.

From Lemma 2.7 it follows that there is $P_i \in \mathcal{GL}_{\mathrm{SO}(2)}(\mathbb{R}[k_i,j_i])$ for any $i = 1,\ldots,r$ such that $P_i^{-1}L_iP_i = J_{\mathbb{R}}(L_i)$.

Define paths $\gamma_i\colon [0,1] \to \mathcal{GL}_{\mathrm{SO}(2)}(\mathbb{R}[k_i,m_i])$, for $i = 1,\ldots,r$, as follows

$$\gamma_i(t) = P_i(\operatorname{diag}(J_{\mathbb{R}}(L_i)) + (1-t)(J_{\mathbb{R}}(L_i) - \operatorname{diag}(J_{\mathbb{R}}(L_i))))P_i^{-1}, \quad t \in [0,1]$$

and homotopy $h^0\colon V \times [0,1] \to V$ in the following way

$$h^0(v_0,\ldots,v_r,t) = (L_0v_0, \gamma_1(t)v_1,\ldots,\gamma_r(t)v_r).$$

Since $L \in \mathcal{TR}_{GL}(V)$ and Lemma 2.8, $\chi(L_i) \cap \sqrt{-1}\mathbb{R} = \gamma_i(t) \cap \sqrt{-1}\mathbb{R} = \emptyset$ for any $i = 1,\ldots,r$ and $t \in [0,1]$. Therefore $h^0(\,\cdot\,,t) \in \mathcal{TR}_{GL}(V)$ for any $t \in [0,1]$. Thus

$$\begin{aligned} h^0(\,\cdot\,,1) &= (L_0, \gamma_1(1),\ldots,\gamma_r(1)) \\ &= (L_0, P_1\operatorname{diag}J_{\mathbb{R}}(L_1)P_1^{-1},\ldots,P_r\operatorname{diag}J_{\mathbb{R}}(L_r)P_r^{-1}) \end{aligned}$$

and $h^0(\,\cdot\,,1) \in \mathcal{TR}_{GL}(V)$. From Lemma 2.7(b) it follows that there are paths $\xi_i\colon [0,1] \to \mathcal{GL}_{\mathrm{SO}(2)}(\mathbb{R}[k_i,m_i])$, $i = 1,\ldots,r$, such that $\xi_i(0) = P_i$ and $\xi_i(1) = \mathrm{id}_{\mathbb{R}[k_i,m_i]}$. Define homotopy $h^1\colon V \times [0,1] \to V$ as follows

$$h^1(\,\cdot\,,t) = (L_0, \xi_1(t)\operatorname{diag}J_{\mathbb{R}}(L_1)\xi_1^{-1}(t),\ldots,\xi_r(t)\operatorname{diag}J_{\mathbb{R}}(L_r)\xi_r^{-1}(t)).$$

Since $L \in \mathcal{TR}_{GL}(V)$, $h^1(\,\cdot\,,t) \in \mathcal{TR}_{GL}(V)$ for any $t \in [0,1]$. Define transversal homotopy $h\colon V \times [0,1] \to V$ in the following way

$$h(v,t) = \begin{cases} h^0(v,2t) & \text{for } t \in [0,1/2], \\ h^1(v,2t-1) & \text{for } t \in [1/2,1]. \end{cases}$$

It is clear that $h(\,\cdot\,,0) = L$ and that $h(\,\cdot\,,1) \in \mathcal{GRAD}_{GL}(V) \subset \mathcal{ORT}_{GL}(V)$. Define $T = h(\,\cdot\,,1)$. By Theorem 3.6(d) we obtain that

$$\mathcal{DEG}(L, D_\alpha(V)) = \mathcal{DEG}(T, D_\alpha(V)).$$

Moreover, since $T \in \mathcal{GRAD}_{GL}(V)$, $\mathcal{DEG}(T, D_\alpha(V)) = DEG(T, D_\alpha(V))$. Consequently, by Corollary 4.3 in [20] we obtain:

$$DEG_{\mathbb{Q}}(T, D_\alpha(V)) = \begin{cases} \operatorname{sign}(\det T_0) & \text{if } \mathbb{Q} = \mathrm{SO}(2), \\ \operatorname{sign}(\det T_0) \cdot m^-(T_i)/2 & \text{if } \mathbb{Q} = \mathbb{Z}_{m_i},\ i = 1,\dots,r, \\ 0 & \text{otherwise}, \end{cases}$$

where $m^-(T_i)$ denotes the Morse index of $T_i \in \mathcal{GRAD}_{GL}(\mathbb{R}([k_i,m_i]))$, $i = 1,\dots,r$. Directly from the definition of homotopy h we obtain that

(a) $T_0 = L_0$,
(b) $n^-(L_i) = m^-(T_i)$,

which completes the proof. □

Remark 4.4. Computing the Brouwer degree of $L \in GL(n,\mathbb{R})$ on a disc centered at the origin we take into account the multiplicities of negative eigenvalues of L. If $L \in \mathcal{ORT}_{GL}(V)$ ($L \in \mathcal{GRAD}_{GL}(V)$) then from the Schur Lemma and Lemma 2.5 it follows that isomorphism $L_\perp\colon (V^{\mathrm{SO}(2)})^\perp \to (V^{\mathrm{SO}(2)})^\perp$ is self-adjoint and therefore $\sigma(L_\perp) \subset \mathbb{R}$. Moreover, computing degree for SO(2)-equivariant orthogonal (gradient) maps we take into account the multiplicities of negative eigenvalues of L and the structure of representation V of the group SO(2). Finally, if $L \in \mathcal{TR}_{GL}(V)$ then from the Schur Lemma and Lemma 2.8 it follows that isomorphism $L_\perp\colon (V^{\mathrm{SO}(2)})^\perp \to (V^{\mathrm{SO}(2)})^\perp$ is such that $\chi(L_\perp) \cap \sqrt{-1}\mathbb{R} = \emptyset$. Moreover, computing degree for SO(2)-equivariant transversal maps we take into account the multiplicities of elements of $\chi_-(L)$, i.e. eigenvalues of L^c with negative real part, and the structure of representation V of the group SO(2).

5. Final remarks

In the forthcoming articles we will apply degree theory constructed in this article to the study of the existence, multiplicity, continuation and global bifurcation of solutions of SO(2)-symmetric differential equations without variational structure. The simplest example of this is furnished by perturbation of autonomous Hamiltonian systems.

In the rest of this section we will discuss bifurcations of zeros of the family of smooth SO(2)-equivariant transversal maps forced by the change of degree for SO(2)-equivariant transversal maps.

Let $f\colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a continuous map such that $f(0,\lambda) = 0$ for all $\lambda \in \mathbb{R}$. Set $\{0\} \times \mathbb{R}$ is said to be the set of trivial solutions of the equation $f(x,\lambda) = 0$. Let $\Sigma \subset \mathbb{R}^n \times \mathbb{R}$ denote the closure of the set of non-trivial solutions of equation $f(x,\lambda) = 0$. A point $(0,\lambda_0) \in \mathbb{R}^n \times \mathbb{R}$ is called a bifurcation point of solutions of equation $f(x,\lambda) = 0$ if $(0,\lambda_0) \in \Sigma$. If Σ contains a connected set S such that $(0,\lambda_0) \in S$ and $S \setminus \{(0,\lambda_0)\} \neq \emptyset$, we will say that $(0,\lambda_0)$ is a branching point of solutions of the equation $f(x,\lambda) = 0$.

Let $\varphi_i\colon \mathbb{R}[1,1] \to \mathbb{R}[1,0]$, $i = 1,2$, be SO(2)-equivariant maps defined as follows

$$\varphi_i(x,y) = \frac{1}{2}(x^2+y^2)^i \quad \text{and} \quad A(\mu,\nu) = \begin{bmatrix} \mu & -\nu \\ \nu & \mu \end{bmatrix}$$

for $\mu,\nu \in \mathbb{R}$. Fix $\alpha,\beta \in \mathbb{R}$ and define family of smooth SO(2)-equivariant maps $f_{\alpha,\beta}\colon \mathbb{R}[1,1] \times \mathbb{R} \to \mathbb{R}[1,1]$, as follows

$$\begin{aligned} f_{\alpha,\beta}(x,y,\lambda) =& A(\lambda,\lambda)\nabla\varphi_1(x,y) + A(\alpha,\beta)\nabla\varphi_2(x,y) \\ =& \begin{bmatrix} \lambda & -\lambda \\ \lambda & \lambda \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \begin{pmatrix} x(x^2+y^2) \\ y(x^2+y^2) \end{pmatrix} \\ =& (\lambda + \alpha(x^2+y^2)) \begin{pmatrix} x \\ y \end{pmatrix} + (\lambda + \beta(x^2+y^2)) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ =& (\lambda + \alpha(x^2+y^2)) \begin{pmatrix} x \\ y \end{pmatrix} + (\lambda + \beta(x^2+y^2)) E \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

Notice that $\{0\} \times \mathbb{R} \subset f_{\alpha,\beta}^{-1}(0)$. Since $\det(Df_{\alpha,\beta}(0,0,\lambda)) = 2\lambda^2$, $(0,0,0) \in \mathbb{R}[1,1] \times \mathbb{R}$ is the only possible bifurcation point of solutions of the equation $f_{\alpha,\beta}(x,y,\lambda) = 0$. We will discuss the existence of bifurcation point of solutions of equation $f_{\alpha,\beta}(x,y,\lambda) = 0$ with respect to α and β.

Case $\alpha \neq \beta$. It is easy to verify that $(0,0,0) \in \mathbb{R}[1,1] \times \mathbb{R}$ is not a bifurcation point of solutions of the equation $f_{\alpha,\beta}(x,y,\lambda) = 0$. Fix $(x_0,y_0,\lambda_0) \in (\mathbb{R}[1,1] \setminus \{0\}) \times \mathbb{R}$ such that $\lambda_0 + \alpha(x_0^2+y_0^2) = 0$. Since $\alpha \neq \beta$, $\lambda_0 + \beta(x_0^2+y_0^2) \neq 0$ and

$$f_{\alpha,\beta}(x_0,y_0,\lambda_0) = (\lambda_0 + \beta(x_0^2+y_0^2)) E \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \neq 0.$$

Summing up, we have shown that $f_{\alpha,\beta}(\,\cdot\,,\,\cdot\,,\lambda_0) \notin \mathcal{TR}(\mathbb{R}[1,1],\emptyset)$.

Case $\alpha = \beta$. First of all notice that $f_{\alpha,\alpha}(\,\cdot\,,\,\cdot\,,\lambda) \in \mathcal{TR}(\mathbb{R}[1,1],\emptyset)$ for all $\lambda \in \mathbb{R}$. Put $\lambda_\pm = \pm 1$. Since $\mathbb{R}[1,1]^{\mathrm{SO}(2)} = \{0\}$, $\chi(Df(0,0,\lambda_+)) = \{1 \pm \sqrt{-1}\}$ and $\chi(Df(0,0,\lambda_-)) = \{-1 \pm \sqrt{-1}\}$, from Lemma 2.8 it follows that $Df(0,0,\lambda_\pm) \in \mathcal{TR}_{GL}(\mathbb{R}[1,1])$. Hence from Lemma 4.1 it follows that there is $\gamma_0 > 0$ such that for any $\gamma < \gamma_0$

$$\mathcal{DEG}(f_{\alpha,\alpha}(\,\cdot\,,\,\cdot\,,\lambda_\pm), D_\gamma(\mathbb{R}[1,1])) = \mathcal{DEG}(Df_{\alpha,\alpha}(0,0,\lambda_\pm), D_\gamma(\mathbb{R}[1,1])).$$

On the other hand from Lemma 4.3 it follows that

$$\mathcal{DEG}_{\mathbb{Q}}(Df_{\alpha,\alpha}(0,0,\lambda_+), D_\gamma(\mathbb{R}[1,1])) = \begin{cases} 1 & \text{if } \mathbb{Q} = \mathrm{SO}(2), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathcal{DEG}_{\mathbb{Q}}(Df_{\alpha,\alpha}(0,0,\lambda_-), D_\gamma(\mathbb{R}[1,1])) = \begin{cases} 1 & \text{if } \mathbb{Q} = \mathrm{SO}(2), \\ 1 & \text{if } \mathbb{Q} = \mathbb{Z}_1, \\ 0 & \text{otherwise.} \end{cases}$$

In other words we have shown that bifurcation index $\eta(0,0,0) \in \mathbb{Z} \oplus (\bigoplus_{i=1}^\infty \mathbb{Z})$ computed at the point $(0,0,0) \in \mathbb{R}[1,1] \times \mathbb{R}$ defined by

$$\eta(0,0,0) = \mathcal{DEG}(f_{\alpha,\alpha}(\,\cdot\,,\,\cdot\,,\lambda_+), D_\gamma(\mathbb{R}[1,1])) - \mathcal{DEG}(f_{\alpha,\alpha}(\,\cdot\,,\,\cdot\,,\lambda_-), D_\gamma(\mathbb{R}[1,1]))$$

is nontrivial in $\mathbb{Z} \oplus (\bigoplus_{i=1}^\infty \mathbb{Z})$. Therefore, $(0,0,0) \in \mathbb{R}[1,1] \times \mathbb{R}$ is a bifurcation point of solutions of the equation $f_{\alpha,\alpha}(x,y,\lambda) = 0$.

It is known that non-triviality of bifurcation index computed in terms of any reasonable topological degree implies the global bifurcation (in the sense of Rabinowitz) of solutions of a suitable equation. Since the bifurcation index $\eta(0,0,0)$ is nontrivial and $(0,0,0) \in \mathbb{R}[1,1] \times \mathbb{R}$ is the only possible bifurcation point of solutions of equation $f_{\alpha,\alpha}(x,y,\lambda) = 0$, it follows that $(0,0,0) \in \mathbb{R}[1,1] \times \mathbb{R}$ is a branching point. Moreover, it is possible to show that connected set of nontrivial solutions bifurcating from $(0,0,0) \in \mathbb{R}[1,1] \times \mathbb{R}$ is unbounded in $\mathbb{R}[1,1] \times \mathbb{R}$. In fact, it is easy to show that, $\{(x,y,\lambda) \in \mathbb{R}[1,1] \times \mathbb{R} : \lambda = 0\} \subset f_{0,0}^{-1}(0)\}$ and if $\alpha \neq 0$ then $\{(x,y,\lambda) \in \mathbb{R}[1,1] \times \mathbb{R} : x^2 + y^2 = -\lambda/\alpha\} \subset f_{\alpha,\alpha}^{-1}(0)$.

Finally notice that map $f_{\alpha,\alpha}(x,y,\lambda)$ is not a gradient, because matrix

$$Df_{\alpha,\alpha}(0,0,\lambda) = A(\lambda,\lambda)$$

is not symmetric. Therefore we cannot apply critical point theory to study of solutions of the equation $f_{\alpha,\alpha}(x,y,\lambda) = 0$. Moreover, since

$$\det Df_{\alpha,\alpha}(0,0,\lambda_\pm) = 2,$$

bifurcation index, computed in terms of the Brouwer degree, is trivial. Therefore we are not able to force the bifurcation of solutions of equation $f_{\alpha,\alpha}(x,y,\lambda) = 0$ using the Brouwer degree.

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