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PERTURBING FULLY NONLINEAR SECOND ORDER ELLIPTIC EQUATIONS

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Abstract. We present two types of perturbations with reverse effects on some scalar fully nonlinear second order elliptic differential operators: on the other hand, first order perturbations which destroy the global solvability of the Dirichlet problem, in smooth bounded domains of $\mathbb{R}^n$; on the other hand, an integral perturbation which restore the local solvability, on compact connected manifolds without boundary.

Introduction

Perturbing scalar second order elliptic equations can bring both bad news and good news. The bad news (Section 1) is that *positivity*, hence in some cases ellipticity, can be destroyed by a first order perturbation. Let us illustrate this phenomenon with an example. Denote by $B(0,1)$ the open unit ball centered at the origin in $\mathbb{R}^2$; there exists a smooth (in fact radial) solution of the Dirichlet problem: $u_{xx}u_{yy} - u_{xy}^2 = 1$ in $B(0,1)$, $u = 0$ on $\partial B(0,1)$. By Theorem 2 below, for any small enough real $\varepsilon \neq 0$, there exists a smooth function f positive on $\overline{B}(0,1)$ with $f \equiv 1 + \varepsilon u_x$ outside an arbitrarily small ball in $B(0,1)$, such that the perturbed problem: $z_{xx}z_{yy} - z_{xy}^2 + \varepsilon z_x = f$ in $B(0,1)$, $z = 0$ on $\partial B(0,1)$, admits no smooth solution in the connected component of $\{z_{xx}z_{yy} - z_{xy}^2 > 0\}$

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where u lies. A similar result holds *e.g.* with the laplacian $u_{xx} + u_{yy}$ instead of the Monge–Ampère operator, but it does not affect the ellipticity of the solution (just the positivity of the laplacian). The idea of the proof first arose in [8] in connection with a particular geometric equation in dimension 4.

The good news (Section 2) concern the local solvability of a generic (scalar second order elliptic) fully nonlinear equation without zeroth-order term posed on a compact manifold. Here the difficulty lies in the fact that the local image of the differential operator is expected to have codimension 1, but no equation is known for it. We provide an integral perturbation device, first used in [5], to cope with this situation. We treat also zeroth-order perturbations regardless of monotonicity.

1. Non-existence *via* a first order perturbation

1.1. Assumptions. Let $\mathcal{D}$ be a domain of $\mathbb{R}^n$, $n > 1$. On the second jet-bundle $J^2\mathcal{D} \to \mathcal{D}$ we are given a smooth real function f positive on a strict subset $\mathcal{P}(f)$ of $J^2\mathcal{D}$ which still projects onto $\mathcal{D}$, with f vanishing on the boundary of $\mathcal{P}(f)$. We assume that the zero section lies in the boundary of $\mathcal{P}(f)$ and that, for any $X \in \mathcal{P}(f)$, there exists a point in the kernel through X of the natural projection $J^2\mathcal{D} \to J^1\mathcal{D}$ which lies in the boundary of $\mathcal{P}(f)$. In other words, if r denotes the variable of $\ker(J^2\mathcal{D} \to J^1\mathcal{D})$ and (x, z, p) the $J^1\mathcal{D}$ variables (with $x \in \mathcal{D}$, $z \in \mathbb{R}$ and $p \in T_x^*\mathcal{D}$), then we have: $f(x, 0, 0, 0) = 0$ and

$$\forall (x, z, p, r) \in \mathcal{P}(f),\ \exists r', f(x, z, p, r + r') = 0. \tag{1}$$

Let Ω be a smooth bounded domain of $\mathbb{R}^n$ with closure contained in $\mathcal{D}$, F, the differential operator associated to f on Ω and $P(F)$, a connected component of the counterset of $\mathcal{P}(f)$ by the second-jet map $u \in C^\infty(\overline{\Omega}) \mapsto j^2u \in J^2\overline{\Omega}$. We assume that $P(F)$ is convex, the operator F, elliptic on $P(F)$ and that, for any $z \in P(F)$, if the principal symbol of $dF[z]$ is positive (resp. negative) definite, then its zeroth-order coefficient $dF[z](1)$ is non-positive (resp. non-negative) in other words $\partial f/\partial z \le 0$ (resp. $\partial f/\partial z \ge 0$). In particular then, the maximum principle ([10]) implies that $dF[z]$ is one-to-one whenever $z \in P(F)$.

The preceding set of assumptions is typically fulfilled for a k-hessian operator $F[z] = \sigma_k[\lambda(Ddz)]$ (D stands for the canonical flat connection of $\mathbb{R}^n$; see [4]).

1.2. A non-existence theorem. Under the preceding assumptions, we shall prove the following result:

THEOREM 1. *Let G be a first order differential operator on $\mathcal{D}$ and $u \in P(F)$. Assume there exists $x_0 \in \Omega$ such that $G[u](x_0) > 0$. Then, for any compact subset K of $P(F)$, there exists a real $\varepsilon > 0$ such that, for any $s \in (0, \varepsilon)$, setting $F_s := F + sG$, there exists a function $\psi \in C^\infty(\overline{\Omega})$ positive on $\overline{\Omega}$ such that*

$\psi \equiv F_s[u]$ outside an arbitrarily small ball centered at x_0 and that the Dirichlet problem: $F_s[z] = \psi$ in Ω, $z = u$ on $\partial\Omega$, admits no solution in K.

The sign of s is of course essential in this statement. Let us differ the proof to the next section and concentrate on the basic case when the first order operator G is a fixed directional derivative.

PROPOSITION 1. *Let $u \in P(F)$ be non-constant. Then there exists a unit vector $\xi \in \mathbb{R}^n$ such that Theorem 1 holds with $G[z] = dz(\xi)$.*

PROOF. Let $u \in P(F)$ be non-constant. Since $F[0] = 0$, the function u satisfies in Ω the second order linear equation $Lu = v$ where $v = F[u]$ and $L = \int_0^1 dF[tu]\,dt$. But $0 \in \partial P(F)$ and $P(F)$ is convex, so L is elliptic. Let $y_0 \in \partial\Omega$ be such that $u(y_0) = \max_{\partial\Omega} u$. Take for ξ the outward unit normal to $\partial\Omega$ at y_0. Since $v > 0$ and u is non-constant, Hopf–Oleinik's lemma (see [10]) implies $du(\xi)(y_0) > 0$. Taking $x_0 \in \Omega$ close enough to y_0 proves the proposition.$\square$

From this proof, one readily infers the

COROLLARY 1. *Let $u \in P(F)$ be non-constant, and constant on $\partial\Omega$. Then, for any unit vector $\xi \in \mathbb{R}^n$, Proposition 1 holds.*

Under the additional assumption

$$\frac{\partial f}{\partial z} \equiv 0,$$

one can strengthen the preceding results as follows:

THEOREM 2. *Let $u \in P(F)$, then there exists a unit vector $\xi \in \mathbb{R}^n$ and a real number $\varepsilon > 0$ such that, for any $s \in (0, \varepsilon)$, there exists a function $\psi \in C^\infty(\overline{\Omega})$ positive on $\overline{\Omega}$ with $\psi = F[u] + s\,du(\xi)$ outside an arbitrarily small ball centered at x_0, such that the Dirichlet problem: $F[z] + s\,dz(\xi) = \psi$ in Ω, $z = u$ on $\partial\Omega$, admits no solution in $P(F)$. Furthermore, if u is constant on $\partial\Omega$, then the preceding statement holds with the unit vector $\xi \in \mathbb{R}^n$ arbitrary and with $s \in (-\varepsilon, \varepsilon)$.*

Theorem 2, whose proof follows closely that of Theorem 1 (see below), takes a considerable strength when the Dirichlet map associated to F, sends $P(F)$ onto $\{\psi \in C^\infty(\overline{\Omega}),\ \psi > 0\} \times C^\infty(\partial\Omega)$ and the ellipticity of F may fail on $\partial P(F)$, as it is the case for k-hessian operators when Ω is a $(k-1)$-convex domain, $k > 1$ (see [4]), in particular, for the example given in the introduction.

Proof of Theorem 1. We need a few auxiliary lemmas.

LEMMA 1. *Let $u \in P(F)$ and $x_0 \in \Omega$. For any small real $\rho > 0$, there exists a function $u_0 \in \partial P(F)$ with the following properties:*

(i) *u_0 coincides with u outside the euclidean ball $B(x_0, \rho)$,*
(ii) *the $C^1[\overline{B(x_0, \rho)}]$ norm of $(u - u_0)$ is $O(\rho)$.*

Proof. Fix $r > 0$ such that $B(x_0, r) \subset \Omega$ and let ϕ be a smooth cut-off function satisfying: $\phi = 1$ in $B(x_0, r/2)$, $\phi = 0$ outside $B(x_0, r)$. By (1) we can find a quadratic polynomial q_0 satisfying: $q_0(x_0) = 0$, $dq_0(x_0) = 0$ and

$$f[x_0, du(x_0), Dd(u + q_0)(x_0)] = 0.$$

Setting $y = x - x_0$, let us define:

$$w(y) = \phi(x)q_0(x)$$

and, for any real $R > 1$,

$$z_R(x) = R^{-2}w(Ry).$$

The function z_R belongs to $C^\infty(\overline{\Omega})$ and it is supported in $B(x_0, r/R)$. Furthermore, since

$$dz_R(x) = R^{-1}dw(Ry),$$

the $C^1[\overline{B(x_0, r/R)}]$ norm of z_R is $O(R^{-1})$. However, at x_0,

$$Ddz_R(x_0) \equiv Ddq_0(x_0),$$

therefore the smaller positive real a_0 such that the function $(u + a_0 z_R) =: u_0$ belongs to $\partial P(F)$ is well-defined and satisfies $a_0 \leq 1$. For R large enough (depending on ρ) the function u_0 fulfills all the requirements of Lemma 1. $\square$

Lemma 2. *Let $u \in P(F)$ and u_0 be as in Lemma 1, with x_0 as in Theorem 1. There exists a real number $\varepsilon_0 > 0$ such that, for any $s \in (0, \varepsilon_0)$ and any small enough $\rho > 0$ (as in Lemma 1), the function $\psi = F_s[u_0]$ is positive on $\overline{\Omega}$.*

Proof. For $\rho > 0$ small enough, setting $2\delta = G[u](x_0)$, we have $G[u_0] \geq \delta$ on $\overline{B(x_0, \rho)}$ by Lemma 1(ii). Moreover, $F[u_0] \geq 0$ because $u_0 \in \partial P(F)$. Therefore

$$F_s[u_0] \geq s\delta > 0 \text{ on } \overline{B(x_0, \rho)}.$$

Outside $B(x_0, \rho)$ we have $F_s[u_0] = F_s[u]$ by Lemma 1(i). So there exists $\varepsilon_0 > 0$ such that, for any $s \in (0, \varepsilon_0)$, the function $F_s[u_0]$ is positive outside $B(x_0, \rho)$. Altogether, the function $\psi = F_s[u_0]$ is positive on $\overline{\Omega}$ as claimed. $\square$

Lemma 3. *For any $u_0 \in \partial P(F)$ and any compact subset $K \subset P(F)$, there exists a real $\varepsilon_1 > 0$ such that, for any $s \in (-\varepsilon_1, \varepsilon_1)$ and any $u \in K$, setting $u_t = tu + (1-t)u_0$ for $t \in [0, 1]$, the Dirichlet map:*

$$z \in C^\infty(\overline{\Omega}) \mapsto (L_{u,s}[z], z|_{\partial\Omega}) \in C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega)$$

associated to the linear operator $L_{u,s} = \int_0^1 dF_s[u_t]\, dt$ is an isomorphism.

Proof. Since $P(F)$ is convex, the function u_t lies in $P(F)$ for $t > 0$, so the operator $L_{u,s}$ is *elliptic*.

For each $z \in P(F)$, the Dirichlet map associated to the linear operator $dF[z]$ is an *isomorphism*. Indeed, by ellipticity it is Fredholm and it can readily be deformed continuously into an isomorphism, so it has zero index (e.g. by [11, Theorem IV, 5.17]). By the maximum principle [10] it is one-to-one (recalling our sign assumption on $\partial f/\partial z$), it is thus also onto, by the Fredholm alternative theory (e.g. [3, p. 464]), hence an isomorphism, by the open mapping theorem (e.g. [12, Chapter 1]).

Let us consider the Fréchet space $\mathcal{L}_2^\infty$ of linear maps of second order L from

$$C_0^\infty := \{z \in C^\infty(\overline{\Omega}),\ z|_{\partial\Omega} = 0\}$$

to $C^\infty(\overline{\Omega})$ such that, for each integer j and, for some fixed $\alpha \in (0,1)$, the norm

$$\|L\|_j = \sup\{|Lz|_{C^{j,\alpha}(\overline{\Omega})},\ z \in C_0^\infty, |z|_{C^{j+2,\alpha}(\overline{\Omega})} = 1\}$$

is finite. Recall $\mathcal{L}_2^\infty$ can be endowed with the metric (e.g. [12, Chapter 1]):

$$d(L, L') := \sum_{j=0}^{\infty} 2^{-j} \frac{\|L - L'\|_j}{1 + \|L - L'\|_j}.$$

Let $\mathcal{L}_2^0$ be the completion of $\mathcal{L}_2^\infty$ for the norm $\|\cdot\|_0$. The canonical imbedding $\mathcal{J}_0 \colon \mathcal{L}_2^\infty \to \mathcal{L}_2^0$ is continuous and the set Isom_2^0 of isomorphisms in $\mathcal{L}_2^0$ is *open* (e.g. by [11, Theorem IV, 1.16]), hence so is $\mathrm{Isom}_2^\infty = \mathcal{J}_0^{-1}\{\mathrm{Isom}_2^0\}$. Moreover, given any small real $\delta > 0$, the map

$$(u, s) \in P(F) \times [-\delta, \delta] \mapsto L_{u,s} \in \mathcal{L}_2^\infty$$

is continuous, hence uniformly continuous on $K \times [-\delta, \delta]$, and

$$\widetilde{K} = \{L_{u,0} \mid u \in K\}$$

is a compact subset of Isom_2^∞. Therefore, on the one hand, there exists a tubular neighbourhood $\mathcal{V}$ (for the metric d) of the compact $\widetilde{K}$, contained in Isom_2^∞, on the other hand, given this neighbourhood $\mathcal{V}$, there exists $\varepsilon_1 \in (0, \delta)$ such that:

$$L_{u,s} \in \mathcal{V} \quad \text{for all } s \in (-\varepsilon_1, \varepsilon_1) \text{ and all } u \in K.$$

Lemma 3 is proved. □

Proof of Theorem 1. We are given u, x_0 and K. Let u_0 be as in Lemma 1 and ε_1, as in Lemma 3. Take ε_0 and ψ as in Lemma 2, with $\varepsilon_0 \le \varepsilon_1$. Let us argue by contradiction and assume the existence of $u_1 \in K$ satisfying: $F_s[u_1] = \psi$ in Ω, $u_1 = u_0$ on $\partial\Omega$. For $t \in [0,1]$, set $u_t = tu_1 + (1-t)u_0$. The function $u = u_1 - u_0$ satisfies $L[u] = 0$ in Ω, $u = 0$ on $\partial\Omega$, with $L = \int_0^1 dF_s[u_t]\,dt$. By Lemma 3, it implies $u \equiv 0$, which is absurd since $u_0 \in \partial P(F)$. So Theorem 1 holds. □

Proof of Theorem 2. First of all, when $\partial f/\partial z = 0$ necessarily $F[z] = 0$ if z is constant; so $u \in P(F)$ cannot be constant. Given u and x_0, take u_0 as in Lemma 1, ε_0 and ψ as in Lemma 2, and argue again by contradiction, now with an arbitrary function u_1 fixed in $P(F)$. Since $\partial f/\partial z = 0$ and $G[z] = dz(\xi)$, the operator $L = \int_0^1 dF_s[u_t]\,dt$ has no zeroth-order term. Moreover, it is elliptic by the convexity of $P(F)$. So L is one-to-one, by the maximum principle (see [10]), which is enough to conclude as above. $\square$

2. Local existence *via* integral perturbation

In this section, we are given a second order differential operator F_0 on a compact connected manifold M of dimension n (without boundary), satisfying:

$$F_0[u + \text{constant}] \equiv F_0[u]. \tag{2}$$

Let $u_0 \in C^\infty(M)$ be a smooth real function on M at which F_0 is elliptic. Given $\psi \in C^\infty(M)$ close to $\psi_0 = F_0[u_0]$, we want to solve the equation $F_0[u] = \psi$ with $u \in C^\infty(M)$ close to u_0.

2.1. The local image problem. Let us start with a couple of elementary observations.

LEMMA 4. *Given any neighbourhood $\mathcal{U}$ of ψ_0 in $C^\infty(M)$, there is no neighbourhood of $u_0 \in C^\infty(M)$ mapped onto $\mathcal{U}$ by F_0.*

PROOF. Let us argue by contradiction and assume the existence of a *nonzero* real number c, arbitrarily small, such that the equation $F_0[u] = \psi_0 + c$ admits a solution $u_1 \in C^\infty(M)$ close to u_0. Setting $u_t = tu_1 + (1-t)u_0$ for $t \in [0,1]$, we infer that $v = u_1 - u_0$ satisfies on M the second order linear equation $Lv = c$, where $L = \int_0^1 dF_0[u_t]\,dt$. For u_1 close to u_0, this equation is elliptic; moreover, condition (2) readily implies that L has no zeroth-order term. The maximum principle [10] thus implies that v is constant, contradicting $c \neq 0$. $\square$

LEMMA 5. *For any $u \in C^\infty(M)$ close enough to u_0, the kernel of $dF_0[u]$ coincides with the functions on M which are* constant*: $\ker dF_0[u] = \mathbb{R}$.*

PROOF. For any $u \in C^\infty(M)$, the kernel of $dF_0[u]$ certainly contains the constant functions, due to (2). Conversely, if $dF_0[u]$ is elliptic (as it is the case for u close to u_0), then $\ker dF_0[u] \subset \mathbb{R}$ due to the maximum principle. $\square$

Fixing an auxiliary Lebesgue measure $d\lambda$ on M, it is easy to see that the restriction of $dF_0[u]$ to the subspace

$$\mathbb{R}^\perp_{d\lambda} = \{v \in C^\infty(M),\ \langle v\rangle = 0\}$$

(where $\langle v\rangle$ stands for the $d\lambda$-average of v over M), is *one-to-one* when u is close to u_0 in $C^\infty(M)$. Therefore the restriction of F_0 to the affine subspace $u_0 + \mathbb{R}^\perp_{d\lambda}$ is an *immersion* near u_0 into $C^\infty(M)$. Moreover, the local image of that immersion coincides with that of F_0 due to condition (2).

The problem which we are now facing consists in identifying an *equation* for the local image of F_0. In other words, in order to solve locally the equation

$F_0[u] = \psi$, we look for an *a priori* constraint on $F_0[u]$, for u near u_0, telling us where ψ should lie (near ψ_0) for the equation to be solvable.

2.2. A self-adjointness ansatz? Whenever F_0 is *linear*, the Fredholm alternative theory (cf. e.g. [3, p. 464]) solves the problem. Specifically then, there exists a riemannian metric g and a vector field ξ on M such that, up to sign:

$$F_0[z] = \Delta z + dz(\xi),$$

where Δ stands for the (positive) laplacian of g. Now ψ lies in the image of F_0 if and only if it is L^2 orthogonal to the 1-dimensional subspace:

$$\operatorname{coker} F_0 = \{v \in C^\infty(M),\ \Delta v + \operatorname{div}(v\xi) = 0\},$$

where L^2 and div are both relative to (the Lebesgue measure of) g.

In the *fully nonlinear* case, which we are considering here for F_0, we can first complement Lemma 5 with

LEMMA 6. *For any u close to u_0 in $C^\infty(M)$, the image of $dF_0[u]$ has codimension* 1.

PROOF. We can speak of the (formal) *adjoint* of $dF_0[u]$ in $L^2(M, d\lambda)$. The elliptic operator $dF_0[u]$ is Fredholm, of *index zero* because it can be deformed continuously into a (second-order elliptic) *self-adjoint* operator. So it has a 1-dimensional cokernel, whose $L^2(M, d\lambda)$-orthogonal coincides with the image of $dF_0[u]$ according to Fredholm theory. The lemma is proved. $\square$

The local image problem thus amounts to integrating near ψ_0 in $C^\infty(M)$ the codimension 1 distribution $(\operatorname{coker} dF_0[u])^\perp$. The simplest way to do it is to find a Lebesgue measure $d\lambda$ with respect to which $dF_0[u]$ is identically *self-adjoint*. Indeed then, we have the following result (pointed out to us by Pengfei Guan):

PROPOSITION 2. *Let F_0 be as above, satisfying* (2), *and $d\lambda$ be a Lebesgue measure on M. If $dF_0[u]$ is* self-adjoint *in $L^2(M, d\lambda)$ for all u close to u_0 in $C^\infty(M)$, then the local image of F_0 near ψ_0 consists of the codimension* 1 *affine submanifold:*

$$\Sigma_0 = \left\{\psi \in C^\infty(M),\ \psi \text{ close to } \psi_0,\ \int_M \psi\, d\lambda = \int_M \psi_0\, d\lambda\right\}.$$

PROOF. Near u_0 in $C^\infty(M)$, consider the map $u \mapsto \int_M F_0[u]\, d\lambda$. Under the self-adjointness assumption, recalling Lemma 5, we see that its derivative at u, given by $v \mapsto \int_M dF_0[u](v)\, d\lambda$, vanishes identically. So the map is constant, proving that the local image of F_0 lies in Σ_0.

It remains to prove that F_0 is *onto* Σ_0 near u_0. To do so, we use the elliptic inverse function theorem with constraints of [7, Theorem 2, p. 686] applied at u_0 to the map (for u close to u_0):

$$u \in (u_0 + \mathbb{R}^{\perp}_{d\lambda}) \mapsto F_0[u] \in \Sigma_0.$$

Under the self-adjointness assumption, the derivative of this map at u_0 is readily seen to be an automorphism of $\mathbb{R}^{\perp}_{d\lambda}$ by the Fredholm alternative theory. So [7, Theorem 2] implies:

$$\forall \psi \in \Sigma_0 \text{ near } \psi_0, \exists u \in (u_0 + \mathbb{R}^{\perp}_{d\lambda}) \text{ near } u_0, F_0[u] = \psi.$$

The proof is complete. □

Nontrivial examples for Proposition 2 are provided by the Calabi–Yau operator on compact Kähler manifolds, $d\lambda$ being the riemannian measure (cf. e.g. [2]), and by the almost-Kähler version of it (as easily verified) ([8]).

Can Proposition 2 serve as an *ansatz* to solve our image problem? In other words, given (F_0, u_0), can one *always* find a Lebesgue measure $d\lambda$ such that Proposition 2 holds? The answer is *no*, as shown by the following counterexample (Proposition 3 below).

Pick a riemannian metric g on M, with Levi–Civita connection ∇, and take for $F_0[z]$ the second elementary symmetric function $\widetilde{\sigma_2}[\lambda(z)]$ of the eigenvalues with respect to g of $(g + \nabla dz)$, with $\widetilde{\sigma_2}$ normalized by $F_0[0] = \widetilde{\sigma_2}(1, \ldots, 1) = 1$. This is indeed a second order fully nonlinear operator satisfying (2). Moreover (see [9]), it is elliptic on the open convex set

$$P(F_0) = \{z \in C^\infty(M),\ F_0[z] > 0\}.$$

If $|\cdot|$ stands for the g-norm and Δ, for the (positive) g-laplacian, we have:

$$F_0[z] = 1 - \frac{2}{n}\Delta z + \frac{1}{n(n-1)}[(\Delta z)^2 - |\nabla dz|^2],$$

and a routine computation yields the identity:

$$\int_M (F_0[z] - 1)\, d\mu \equiv \frac{1}{n(n-1)} \int_M \mathrm{Ricci}(\overrightarrow{\nabla} z, \overrightarrow{\nabla} z)\, d\mu, \tag{3}$$

where $d\mu$ (resp. Ricci, $\overrightarrow{\nabla}$) denotes the Lebesgue measure of g (resp. its Ricci tensor, its gradient operator). Therefore, whenever g is Ricci-flat, the image of F_0 lies *a priori* in the following smooth codimension 1 submanifold:

$$\left\{\psi \in C^\infty(M),\ \int_M (\psi - 1) d\mu = 0\right\}.$$

Actually then, F_0 and $d\mu$ also fulfill the assumptions of Proposition 2 near $u_0 = 0$ (routine exercise).

Proposition 3. *If g is not Ricci-flat, there exists no Lebesgue measure $d\lambda$ on M such that, for any u close enough to $u_0 = 0$ in $C^\infty(M)$, the operator $dF_0[u]$ is formally self-adjoint with respect to $d\lambda$.*

Proof. Let us argue by contradiction and pick a Lebesgue measure $d\lambda$ for which $dF_0[u]$ is self-adjoint at each u close enough to $u_0 = 0$ in $C^\infty(M)$. By Proposition 2, the functional

$$\phi(u) = \int_M (F_0[u] - 1)\, d\lambda$$

vanishes identically in $C^\infty(M)$ near $u_0 = 0$. Therefore $d\phi(0)(v) = 0$ for all $v \in C^\infty(M)$, which reads

$$\int_M dF_0[0](v)\, d\lambda = 0$$

or else,

$$\int_M \Delta v\, d\lambda = 0.$$

In other words, the Radon–Nikodym derivative ρ of $d\lambda$ with respect to $d\mu$, satisfies $\Delta\rho = 0$ in the distribution sense on M. Since Δ is elliptic, ρ must be smooth [2, p. 85] and the maximum principle [10] implies that ρ is *constant.* Recalling $\phi(u) \equiv 0$ and (3), we reach a contradiction unless g is Ricci-flat. □

From Proposition 3 we conclude that the image problem remains open for a *generic* fully nonlinear second-order differential operator F_0 satisfying (2) on M compact.

2.3. An integral perturbation device. To cope with the preceding situation and restore a local solvability, the idea is to break the invariance of F_0 expressed by (2), at a somewhat lower cost (loosing the locality of the operator). Let $\langle z\rangle$ still denote the *average* on M of a function z with respect to a fixed Lebesgue measure $d\lambda$.

Theorem 3. *Let (F_0, u_0, ψ_0) be as above, with F_0 satisfying* (2). *Without loss of generality, assume: $\langle u_0\rangle = 0$. Then, given any* nonzero *real number s, the perturbed operator*

$$F_s[z] := F_0[z] + s\langle z\rangle$$

is a smooth diffeomorphism from a neighbourhood of u_0 in $C^\infty(M)$ onto *a neighbourhood of ψ_0 in $C^\infty(M)$.*

Remark 1. If $F_s[z_s] = F_t[z_t]$ with $st \neq 0$ and both z_s and z_t close enough to u_0 in $C^\infty(M)$, then Theorem 3 implies:

$$z_t = z_s + \frac{1}{t}(s-t)\langle z_s\rangle.$$

One may thus use the normalization $s = 1$ without loss of generality.

REMARK 2. The idea of adding an average term goes back to [5, Theorem 1] (see also [6, p. 426]) where it is used to invert (globally) in $C^\infty(M)$ the elliptic riemannian Monge–Ampère operator:

$$u \mapsto F_0[u] = \log\left[\frac{\det(g + \nabla du)}{\det(g)}\right],$$

g standing for a smooth riemannian metric and ∇, for its Levi–Civita connection. Another global application is drawn in [9].

REMARK 3. Fix $s \neq 0$ and set: $\psi \mapsto S(\psi)$ for the local solution map defined by Theorem 3, G_s for the open subset of functions ψ_1 close to ψ_0 in $C^\infty(M)$ satisfying $\langle S(\psi_1)\rangle \neq 0$. Granted the next corollary, we now know an equation for the image by F_0 of a neighbourhood of u_0 in $C^\infty(M)$, namely: $\langle S(\psi)\rangle = 0$.

The proof of Theorem 3, given below, relies on an inverse function theorem argument. With Theorem 3 at hand, we can characterize the *ill-posedness* of the original equation $F_0[z] = \psi$ with ψ near ψ_0 as follows:

COROLLARY 2. *Let s, F_0, u_0 (and ψ_0) be as in Theorem 3, and let ψ_1 be given in G_s (cf. Remark 3). Then the equation $F_0[u_1] = \psi_1$ admits* no *solution $u_1 \in C^\infty(M)$ such that the operator F_0 remains elliptic along the path $t \in [0,1] \mapsto u_t := tu_1 + (1-t)u_0$.*

PROOF. Let us argue by contradiction and take $\psi_1 \in G_s$ and u_1 as stated. By (2), we may assume $\langle u_1\rangle = 0$. So u_1 solves $F_s[u_1] = \psi_1$ as well. If u_1 is close enough to u_0 we reach a contradiction by the very definition of G_s; if not, we need to argue further. Since ψ_1 is close to ψ_0, Theorem 3 provides a solution z_1 of $F_s[z_1] = \psi_1$ close to u_0 in $C^\infty(M)$. It follows that F_0 remains elliptic along the path $t \in [0,1] \mapsto v_t = tu_1 + (1-t)z_1$. Now $v = u_1 - z_1$ satisfies on M the linear elliptic equation $L[v] = 0$, where $L = \int_0^1 dF_s[v_t]\,dt$. Noting that $L[v + \text{constant}] \equiv L[v]$, we conclude from the maximum principle that $v = 0$, which is absurd since $\langle v\rangle = -\langle z_1\rangle \neq 0$. □

Corollary 2 yields the *global* ill-posedness of elementary hessian equations $\widetilde{\sigma_k}[\lambda(z)] = \psi > 0$ (with notations used for Proposition 3 above) on a compact riemannian manifold, since these equations are *a priori* elliptic and their ellipticity set is *convex* (cf. [9, Section 1]).

2.4. Proof of Theorem 3. For any fixed nonzero real s, the operator F_s is an *elliptic map* in the sense of [7] from a neighbourhood of u_0 in $C^\infty(M)$, to $C^\infty(M)$. Theorem 3 thus follows from the elliptic inverse function Theorem [7, Theorem 2] provided we can prove the following linear result:

THEOREM 4. *For $s \neq 0$, the linear operator $dF_s[u_0]$ is an automorphism of $C^\infty(M)$.*

PROOF. Since F_0 is elliptic at u_0, satisfying (2), there exist a riemannian metric g and a vector field ξ, both smooth on M, such that the linear operator $L_s = dF_s[u_0]$ reads, up to sign:

$$L_s[z] = \Delta z + dz(\xi) \pm s\langle z\rangle, \tag{4}$$

where Δ stands for the (positive) laplacian of g. Without loss of generality, we may take $+s\langle z\rangle$ in the right-hand side (the sign of s here is unimportant).

Clearly, L_s is a continuous linear map from $C^\infty(M)$ to itself; it is one-to-one by the maximum principle (easy check). According to the open mapping Theorem [12, Chapter 2], it remains only to show that L_s is *onto*, which we now do with an argument inspired from [5, Lemma 2, p. 346]. Let $d\mu_g$ be the canonical Lebesgue measure of the metric g, and $d\lambda$, the Lebesgue measure used to define the average $\langle\,\cdot\,\rangle$ in Theorem 3; let $\rho \in L^1(M, d\mu_g)$ be the density of $d\lambda$ with respect to $d\mu_g$. Set:

$$V = \int_M d\lambda \equiv \int_M \rho\, d\mu_g, \quad V_g = \int_M d\mu_g, \quad \langle z\rangle_g = \frac{1}{V_g}\int_M z\, d\mu_g$$

for a generic real function $z \in L^1(M, d\mu_g)$.

LEMMA 7. *The formal $L^2(M, d\mu_g)$ adjoint of the operator L_s is given by*

$$L_s^*[z] = \Delta z + \operatorname{div}(z\xi) + s\rho\frac{V_g}{V}\langle z\rangle_g. \tag{5}$$

In particular, the adjoint of the differential *operator L_0 is given by*

$$L_0^* z = \Delta z + \operatorname{div}(z\xi).$$

The latter has a 1-dimensional null space and, if $w \in \ker L_0^*$, then

$$w \neq 0 \Rightarrow \langle w\rangle_g \neq 0.$$

Formula (5) is routinely obtained, integrating by parts on M (compact without boundary) with the measure $d\mu_g$. The assertion on the dimension was proved in Lemma 6; for the last assertion, we argue by contradiction: if $w \neq 0$ satisfies $\langle w\rangle_g = 0$ and spans $\ker L_0^*$ then, according to Fredholm Theorem [3, p. 464], one can solve on M the equation

$$\Delta z + dz(\xi) = 1,$$

because its right-hand side is orthogonal to w in $L^2(M, d\mu_g)$. But the maximum principle ([10]) implies that the solution z must be constant, which is absurd.

With Lemma 7 at hand, we can complete the proof of Theorem 4 as follows. Given any $\psi \in C^\infty(M)$ and $s \neq 0$, consider the ratio

$$c(s,\psi) := \left(\int_M w\psi\, d\mu_g\right)\left(s\int_M w\, d\mu_g\right)^{-1}$$

where w stands for a nonzero element of $\ker L_0^*$; clearly, $c(s,\psi)$ does not depend on a particular choice of such a w. The function $[\psi - s\,c(s,\psi)]$ is orthogonal to $\ker L_0^*$ in $L^2(M, d\mu_g)$, so by the Fredholm Theorem [3, p. 464] one can solve on M the equation $L_0 z = \psi - s\,c(s,\psi)$. Moreover, since the solution z is defined up to an additive constant (by Lemma 5), we can define z by imposing:

$$\langle z \rangle = c(s,\psi).$$

Now $z \in C^\infty(M)$ satisfies $L_s z = \psi$ as required, so L_s is indeed onto. □

REMARK 4. When the density ρ lies only in $L^1(M, d\mu_g)$, the operator L_s provides an example of an automorphism of $C^\infty(M)$ whose formal $L^2(M, d\mu_g)$-adjoint, given by (5), maps $C^\infty(M)$ to $L^1(M, d\mu_g)$ only.

2.5. Zeroth-order perturbation. Let F_0, u_0 and ψ_0 be as in Theorem 3. We wish to deduce from Theorem 4 a local existence result for the equation:

$$F_0[z] = \psi + sz \tag{6}$$

near u_0, where $\psi \in C^\infty(M)$ is close to ψ_0 and $s \neq 0$ is a small real parameter. Although the *sign* of s is irrelevant for our result, let us stress that it can be on the resonant side of zero, where s could interfere with the spectrum of $dF_0[z]$ (whereas for s on the other side of zero, equation (6) is *a priori* locally invertible).

THEOREM 5. *Let F_0, u_0 and ψ_0 be as in Theorem 3. Then there exists a neighbourhood $\mathcal{V}$ of ψ_0 in $C^\infty(M)$ and a real number $s_0 > 0$ such that, for any $\psi \in \mathcal{V}$ and any nonzero $s \in (-s_0, s_0)$, equation (6) admits a unique solution u close to u_0 in $C^\infty(M)$. Moreover, this solution depends smoothly on the data (ψ, s), for $s \neq 0$.*

PROOF. We first consider, near the origin in $\mathbb{R} \times C^\infty(M) \times C^\infty(M)$, the smooth map $(s,\phi,v) \mapsto \mathcal{G}(s,\phi,v) \in C^\infty(M)$ defined by

$$\mathcal{G}(s,\phi,v) = F_0[u_0+v] + \langle v \rangle - (\psi_0+\phi) - s(u_0+v).$$

It satisfies $\mathcal{G}(0,0,0) = 0$ and, by Theorem 4,

$$\frac{\partial \mathcal{G}}{\partial v}(0,0,0) \equiv dF_1[u_0]$$

is an *automorphism* of $C^\infty(M)$. An implicit function theorem argument, using [7, Theorem 2], yields near zero the existence of a smooth solution-map $(s,\phi) \mapsto v = \mathcal{S}(s,\phi)$ such that: $\mathcal{G}[s,\phi,\mathcal{S}(s,\phi)] = 0$. The theorem follows for $s \neq 0$, by letting $\psi = \psi_0 + \phi$ and

$$u = u_0 + \mathcal{S}(s,\phi) - \frac{1}{s}\langle \mathcal{S}(s,\phi) \rangle. \qquad \square$$

REMARK 5. The idea of pushing the parameter s toward the resonant side accross the value $s = 0$ by means of the equation $\mathcal{G}(s,\phi,v) = 0$ goes back to [5, Theorem 2] (where an existence result is proved for *any* value of $s \neq 0$, for the riemannian Monge–Ampère equation). Using it for the complex Monge–Ampère equation, Th. Aubin could take up the $c_1 > 0$ case of the Calabi's conjecture (cf. [1, footnote p. 148]).

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