Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 29, 2007, 361–381

QUASILINEAR NON-UNIFORMLY PARABOLIC-ELLIPTIC SYSTEM MODELLING CHEMOTAXIS WITH VOLUME FILLING EFFECT. EXISTENCE AND UNIQUENESS OF GLOBAL-IN-TIME SOLUTIONS

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Abstract. A system of quasilinear non-uniformly parabolic-elliptic equations modelling chemotaxis and taking into account the volume filling effect is studied under no-flux boundary conditions. The proof of existence and uniqueness of a global-in-time weak solution is given. First the local solutions are constructed. This is done by the Schauder fixed point theorem. Uniqueness is proved with the use of the duality method. A priori estimates are stated either in the case when the Lyapunov functional is bounded from below or chemotactic forces are suitably weakened.

1. Introduction

Chemotaxis is a chemosensitive movement of cells which navigate towards the gradient of a chemical contained in the environment. Many kinds of such phenomenons include the situation when the chemical is produced by cells. In the present paper we are going to study the system that arises during the investigation of such a situation. The system is the mathematical description of

2000 *Mathematics Subject Classification.* 92C17, 35K60, 35K57.

Key words and phrases. Chemotaxis equations, global-in-time existence and uniqueness, quasilinear reaction-diffusion systems, prevention of blow-up, Schauder fixed point theorem.

C. Morales-Rodrigo gratefully acknowledges support from the EU Marie Curie Research Training Network Grant *Modelling, Mathematical Methods and Computer Simulations of Tumour Growth and Therapy*, contract number MCRTN-CT-2004-503661

chemotactic movement that takes into account the volume filling effect (cf. [12]). Previous models of chemotactic movement, so called minimal version of the classical Keller–Segel model (cf. [16], see also the survey [13] and the bibliography therein) did not consider this effect. It turns out that they predict a blow-up occurrence for dimensions $n \geq 2$ (see e.g. [15], [10]). In order to avoid this property there have been presented several models that were supposed to prevent blow-up which is interpreted as the overcrowding of cells. One of such attempts was presented in [12]. The main tool which was believed to stop blow-up mechanism was to consider when building the model the observation that the higher is the density at x, the lesser is the chance that another cell attains that position. This property was named the volume filling effect.

Let us call the density of cells by u and the density of a chemoattractant, the chemical which attracts cells, by v. By $\Omega \subset \mathbb{R}^n$ we denote the domain. Then the model derived in [12] reads as

$$\frac{\partial u}{\partial t} = \nabla\cdot[\alpha(u)\nabla u - u\beta(u)\nabla v] \quad \text{in } \Omega \times (0,T), \tag{1.1}$$

$$0 = \Delta v - v + u \quad \text{in } \Omega \times (0,T), \tag{1.2}$$

$$\nabla u \cdot \overrightarrow{n} = 0, \quad \nabla v \cdot \overrightarrow{n} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{1.3}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{1.4}$$

by $\overrightarrow{n}$ we denote the outer normal vector, the C^1 functions α and β are given in the following way:

$$\alpha(u) = q(u) - uq'(u), \tag{1.5}$$

$$\beta(u) = q(u), \tag{1.6}$$

where $q(u)$ reads as the probability that the particle attains a position (x,t) if the density of cells at this position equals u. Let us notice that assumptions $q(u) \geq 0$ for $0 \leq u < \infty$ and q is a decreasing function seem very natural. Notice that if $q(u) \equiv 1$ we arrive at the classical minimal version of Keller–Segel model.

In [11], assuming q vanishes for large cells densities, the authors proved the existence of global-in-time solutions to (1.1)–(1.4) with 0 replaced $\partial v/\partial t$ in (1.2). For the same system the global-in-time solutions are also proved to exist in the case when there is no value at which q vanishes in [5].

In the following paper we are going to investigate the solutions to (1.1)–(1.4) provided there is no value at which q vanishes. Since α is given by (1.5), when we consider the case $q(u) \to 0$ when $u \to +\infty$, we arrive at two possibilities:

$$uq'(u) \xrightarrow{u\to\infty} 0 \tag{1.7}$$

or not. When (1.7) holds we arrive at

$$\alpha(u) \xrightarrow{u\to\infty} 0, \tag{1.8}$$

a non-uniformly parabolic-elliptic system.

For dimension 2 (1.1)–(1.4), (1.8) was studied in [4]. The authors gave a priori estimates to the solution of the problem that prevent blow-up, but did not prove the existence of solutions. The purpose of this paper is to construct the solutions to (1.1)–(1.4), show their uniqueness and present the assumptions on α and β under which the blow-up of solutions to (1.1)–(1.4) is prevented also for dimension 3. We also show the easy fact that every system that arises as the volume-filling model of chemotaxis has global solution for dimension 1. Finally, with the use of techniques used in [5], we give the assumptions on α and β under which there is a blow-up prevention without the boundedness from below of the Lyapunov functional.

The problem is studied under the following hypotheses

(T) (a) $\beta \in C^1[0,\infty)$ is a positive bounded function.
(b) $\alpha \in C^1[0,\infty)$ is a positive bounded function.
(c) The boundary of Ω is sufficiently smooth.

Before we formulate our results in a precise way, let us present few facts about connections between α, β and q.

Notice that the functions α and β verify

$$\frac{\beta(\eta)}{\alpha(\eta)} \leq M \tag{1.9}$$

for $M = 1$ and every $\eta \in \mathbb{R}$.

Indeed, since q is decreasing $q(\eta) \leq (q(\eta) - \eta q'(\eta))$ and (1.9) follows.

This simple observation will be of importance in the further part of the paper. We will state the global existence results for (1.1)–(1.4) under hypothesis (1.9).

The examples of q that lead to the non-uniform parabolicity are $(1+u)^{-\lambda}$, $\lambda > 0$ or $e^{-\gamma u}$, $\gamma > 0$. The latter was mentioned in [12].

The paper is organized as follows. In Section 2 we give some preliminaries. In Section 3 we construct the weak solutions and present the prolongation principle. Then we show the uniqueness of such solutions in Section 4. Section 5 is devoted to proving L^∞ bounds of the solutions to (1.1)–(1.4). We show two mechanisms of preventing the blow-up. One of them, see Theorem 5.3, bases on a priori boundedness from below of the Lyapunov functional and the second uses the fact that the chemotactic forces are suitably weakened.

Throughout the paper C will denote the generic constant which value may vary from line to line.

In Sections 3 and 5 we shall multiply (1.1)–(1.4) by u^p for some p. Since the solution constructed in Section 3 is weak, the procedure mentioned above demands a justification. As it is standard and it is much more convenient not to remind it in every calculation, we only mention here that instead of multiplying the equation by u^p, one has to test the equation with the use of the proper cut-off functions $\psi_k(u)$. One can find such a choice of test functions in [3, p. 1196] and then pass with k to ∞.

Notation. The norm in the space $L^p(\Omega)$, $1 \le p \le \infty$ is denoted by $\|\cdot\|_p$. A classical Sobolev space will be denoted by $W^{1,p}(\Omega)$ for $1 \le p \le \infty$ and the associated norm will be denoted $\|\cdot\|_{1,p}$. The Hilbert space $W^{1,2}(\Omega)$ will be named $H^1(\Omega)$ and dual of this space by $(H^1)'(\Omega)$. Sometimes to shorten the notation we shall denote the vector valued function (u,v) by $\overline{u}$. We shall denote the Lebesgue measure of a set A by $|A|$. By f_-, f_+ we denote resp. the negative and the positive part of the function f. By Q_T we denote the set $\Omega \times (0,T)$. The norm of the space $L^4(0,T;L^2(\Omega))$ will be denoted by $|||\cdot|||$. The space of distributions on (0,T) with values in a Banach space X will be denoted by $\mathcal{D}'(0,T;X)$.

2. Preliminaries

In order to prove our results we need to use a theorem that let us handle the non-uniformly parabolic equations of the type (1.1) when ∇v is bounded. That theorem was proved in [6, Theorem 2.2] and independently (with the use of different methods) in [4]. For the reader's convenience we present it below. Let us first recall that a L^2-weak solution to a parabolic equation is a function w such that

$$w \in L^2(0,T;H^1(\Omega)), \quad w_t \in L^2(0,T;(H^1)'(\Omega)) \tag{2.1}$$

and w satisfies the equation in a weak sense. From (2.1) we see that $w \in C([0,T];L^2(\Omega))$.

Theorem 2.1. *Let $0 < T < \infty$, $\nabla v \in (L^\infty(Q_T))^n$, (1.9) and (T). Assume u to be a L^2-weak solution to*

$$\frac{\partial u}{\partial t} = \nabla\cdot[\alpha(u)\nabla u - u\beta(u)\nabla v] \quad \text{in } \Omega \times (0,T), \tag{2.2}$$

$$\nabla u \cdot \overrightarrow{n} = 0, \quad \nabla v \cdot \overrightarrow{n} = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{2.3}$$

$$u(x,0) = u_0(x) \quad \text{in } \Omega, \tag{2.4}$$

corresponding to $u_0 \in L^\infty(\Omega)$. Then

$$\sup_{[0,T_{max})\cap[0,T]} \|u(\cdot,t)\|_\infty < \infty,$$

where $T_{\max}$ is the maximal interval of solution's existence.

It is worth underlying that Theorem 2.1 says only that

$$u \in L^\infty_{\mathrm{loc}}((0,\infty), L^\infty(\Omega)).$$

We cannot infer the uniform in time boundedness of a solution from it. In several places we shall need the following Gagliardo–Nirenberg inequality

$$\|u\|_p \le C\|u\|_{1,2}^{\theta}\|u\|_r^{1-\theta} \quad \text{for all } u \in H^1(\Omega)$$

which holds for all $p \ge 1$ satisfying $p(n-2) < 2n$, $r \in (0,p)$ with

$$\theta = \frac{\dfrac{n}{r} - \dfrac{n}{p}}{1 - \dfrac{n}{2} + \dfrac{n}{r}}.$$

We define

$$\chi(s) := \int_0^s \int_1^\sigma \frac{\alpha(\tau)}{\tau\beta(\tau)}\, d\tau\, d\sigma, \quad s > 0.$$

Throughout this paper the functional

$$L(u,v) := \int_\Omega \chi(u) + \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega v^2 - \int_\Omega uv,$$

plays a fundamental role, because satisfies

$$\frac{d}{dt}L(u,v) \le 0. \tag{2.5}$$

The proof of this fact can be found in [5]. Let us point out that Lyapunov functionals of this form were introduced in [2], [4], [5] [9] and [14].

Moreover, assume the additional condition, there exist constants $D > 0$ and $\gamma < 2/n$, such that

$$\frac{\alpha(u)}{u\beta(u)} \ge Du^{-\gamma} \tag{2.6}$$

is satisfied, then the Lyapunov functional is bounded from below. The proof of this fact can be found in [14, Lemma 5.1 and the Remark after] because the Lyapunov functional L is of the form considered therein.

Moreover, we have the following proposition [14, Lemma 5.2].

Proposition 2.2. *Suppose $n \ge 2$ and*

$$\chi(s) \ge c_0 s^\alpha, \quad \textit{for all } s \ge 1$$

holds with some $c_0 > 0$ and some $\alpha > 2/n$. Then for any fixed $\lambda > 0$ there exist $\varepsilon_0 > 0$ and families $(u_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)} \subset W^{1,\infty}(\Omega)$ and $(v_\varepsilon)_{\varepsilon\in(0,\varepsilon_0)} \subset W^{1,\infty}(\Omega)$ such that $u_\varepsilon > 0$ and $v_\varepsilon > 0$ in $\overline{\Omega}$,

$$\int_\Omega u_\varepsilon = \lambda \quad \textit{for all } \varepsilon \in (0,\varepsilon_0)$$

but $L(u_\varepsilon, v_\varepsilon) \to -\infty$ as $\varepsilon \to 0$.

3. Existence of local solutions

In this section we are going to prove the existence of local-in-time solutions to (1.1)–(1.4). This will be done by Schauder's fixed point theorem. Since (1.1)–(1.4) is non-uniformly parabolic we first penalize it. Thus, the Schauder theorem can be applied. Also we present a prolongation rule.

For the needs of this and the next section we consider (1.1)–(1.4) with α and β prolonged also for negative values in such a way that they remain regular bounded and positive.

Definition 3.1. Fixed $u_0 \in L^\infty(\Omega)$. A weak solution to (1.1)–(1.4) is a couple (u,v) of functions such that, for each $T>0$,

$$\begin{gathered} u \in L^\infty(Q_T) \cap L^2(0,T;H^1(\Omega)), \quad u(0)=u_0, \quad \text{a.e. in } \Omega, \\ u_t \in L^2(0,T;(H^1)'(\Omega)), \quad v \in L^\infty(Q_T) \cap L^\infty(0,T;H^2(\Omega)), \end{gathered}$$

and (u,v) satisfy, for each $\varphi \in L^2(0,T;H^1(\Omega))$, the following

$$\begin{gathered} \int_0^T \langle u_t, \varphi\rangle \, dt + \int_0^T a\ (u,\varphi)\, dt - \int_0^T b(u,v,\varphi)\, dt = 0, \\ \int_\Omega \nabla v \cdot \nabla \varphi + \int_\Omega v\varphi = \int_\Omega u\varphi, \quad \text{a.e. } t \in (0,T), \end{gathered}$$

where

$$a(u,\varphi) = \int_\Omega \alpha(u) \nabla u \cdot \nabla \varphi \, dx, \qquad b(u,v,\varphi) = \int_\Omega u\beta(u)\nabla v \cdot \nabla \varphi\, dx$$

and $\langle \cdot\,,\,\cdot \rangle$ denotes the duality paring between $(H^1)'(\Omega)$ and $H^1(\Omega)$.

Remark 3.2. In the definition, since $u \in L^2(0,T;H^1(\Omega))$ and $u_t \in L^2(0,T;(H^1)'(\Omega))$ then $u \in C([0,T];L^2(\Omega))$. This gives a sense to the condition $u(0) = u_0$ a.e. in Ω.

Proposition 3.3. *If the initial data u_0 is nonnegative then $u(x,t) \geq 0$ a.e. in Q_T and $v(x,t) \geq 0$ in $\overline{Q}_T$. Moreover, we have $\|u_0\|_1 = \|u\|_1 = \|v\|_1$.*

Proof. Mass conservation follows easily by integrating (1.1)–(1.2) over Ω. For the nonnegativity of solutions we apply a standard argument. Consider the equation

$$u_t = \nabla \cdot (\alpha(u)\nabla u - u_+ \beta(u) \nabla v) \tag{3.1}$$

under no-flux boundary conditions. Multiplying (3.1) by u_- and integrating over Ω, we obtain

$$\frac{d}{2dt} \int_\Omega (u_-)^2 \, dx = - \int_\Omega \alpha(u) |\nabla u_-|^2 \, dx. \tag{3.2}$$

This results in

$$\int_\Omega (u_-(x,t))^2\,dx \le \int_\Omega (u_-(x,0))^2\,dx = 0, \quad t > 0. \tag{3.3}$$

This implies that $u_- = 0$ a.e. Now, $v(x,t) \ge 0$ in $\overline{Q}_T$ by the maximum principle. $\square$

THEOREM 3.4. *Suppose that Ω is a bounded and regular domain in $\mathbb{R}^n$, $n \le 3$. Let* (T) *be satisfied and $0 \le u_0 \in L^\infty(\Omega)$ then there exists a local weak solution (u,v) to* (1.1)–(1.4) *in the sense of Definition* 3.1.

PROOF. For $T > 0$ we consider the spaces of functions

$$X = L^4(0,T;L^2(\Omega)), \qquad Y = L^4(0,T;H^2(\Omega)).$$

We define the linear operator $F\colon X \to Y$ such that for each $\psi \in X$, $F(\psi) = v$ is the unique solution of the linear equation

$$\begin{cases} -\Delta v + v = \psi, \\ \nabla v \cdot \overrightarrow{n} = 0. \end{cases}$$

Since $\psi(\,\cdot\,,t) \in L^2(\Omega)$ a.e. in $t \in [0,T]$ we obtain, by elliptic regularity, that $\nabla v(\,\cdot\,,t) \in H^1(\Omega)$ a.e. in $t \in [0,T]$. In order to simplify the notation, from now on we do not write explicitly the dependence of the functions on t. We know, by Sobolev imbedding, that

$$\|\nabla v\|_q \le C\|\psi\|_2$$

where $q = \infty$ if $n = 1$, $q < \infty$ if $n = 2$ and $q = 6$ if $n = 3$.

Now, we fix $k > \|u_0\|_\infty + 1$ and define the function α_k

$$\alpha_k(t) = \begin{cases} \alpha(t) & \text{if } t < k, \\ \phi_k(t) & \text{if } k \le t < k+1, \\ \alpha(k+1) & \text{if } t \ge k+1, \end{cases}$$

where ϕ_k is chosen strictly positive function such that α_k is of the same regularity as α. We define

$$0 < m_k = \inf_{s \ge 0} \alpha_k(s). \tag{3.4}$$

The problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \nabla \cdot (\alpha_k(\psi)\nabla u - u\beta(\psi)\nabla v) & \text{in } \Omega \times (0,T), \\ \nabla u \cdot \overrightarrow{n} = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{3.5}$$

has, for $n \le 3$ a unique solution in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ (see [17, Chapter 3, Theorem 5.1]). We define the nonlinear operator $G\colon Y \to X$, such that, for each v, $G(v) = u$ where u is the unique solution of (3.5). Finally, it is clear that if u is a fixed point of $H = G \circ F\colon X \to X$ and $\|u\|_\infty < k$ then u is a solution to (1.1)–(1.4). In what follows we will apply Schauder's fixed point theorem what requires some steps.

Step 1. There exists a ball B_R in X invariant under H. Taking $L^2(\Omega)$ inner product of (3.5) with u we obtain

$$\frac{d}{2dt}\|u\|_2^2 + \int_\Omega \alpha_k(\psi)|\nabla u|^2 = \int_\Omega u\beta(\psi)\nabla u \cdot \nabla v. \tag{3.6}$$

We can estimate the right hand side of (3.6) using the Hölder inequality, interpolation in L^p spaces, the Sobolev imbedding and Young's inequality.

$$\begin{aligned}\bigg|\int_\Omega u\beta(\psi)\nabla u \cdot \nabla v\bigg| &\leq C\|u\|_3\|\nabla u\|_2\|\nabla v\|_6 \leq C\|u\|_{1,2}^{1/2}\|u\|_2^{1/2}\|\nabla u\|_2\|\psi\|_2 \\ &\leq C\|u\|_{1,2}^{3/2}\|u\|_2^{1/2}\|\psi\|_2 \leq \frac{m_k}{2}\|u\|_{1,2}^2 + C\|u\|_2^2\|\psi\|_2^4,\end{aligned}$$

where $m_k > 0$ is the constant defined in (3.4). Hence we have

$$\frac{d}{dt}\|u\|_2^2 + m_k\|\nabla u\|_2^2 \leq (C_k\|\psi\|_2^4 + 1)\|u\|_2^2 := \gamma(t)\|u\|_2^2. \tag{3.7}$$

Since $\psi \in L^4(0,T;L^2(\Omega))$ the coefficient $\gamma(t)$ is integrable on $[0,T]$. We infer from (3.7)

$$\|u(t)\|_2^2 \leq \|u_0\|_2^2 \exp\bigg(\int_0^T \gamma(s)\,ds\bigg) < \infty. \tag{3.8}$$

Thanks to (3.8) we have

$$|||u||| \leq T^{1/4}\|u_0\|_2 \exp\bigg(\frac{1}{2}\int_0^T \gamma(s)\,ds\bigg).$$

Now we consider, for a fixed $R > 0$, the ball $B_R = \{\phi \in L^4(0,T;L^2(\Omega)) : |||\phi||| \leq R\}$. Let $\psi \in B_R$. From (3.7) and (3.8) we obtain

$$|||u = H(\psi)||| < T^{1/4}\|u_0\|_2 \exp((1/2)C_kR^4 + T/2). \tag{3.16}$$

In order to have $H(B_R) \subset B_R$ we choose T such that $|||H(\psi)||| < R$. For that, let $f(t) = t^{1/4}\exp(t/2)$. Since $f(0) = 0$ and f is an increasing function, then there exists T such that

$$f(t) < \frac{R}{\exp(C_kR^4/2)\|u_0\|_2} \quad \text{for all } t \in [0,T].$$

Step 2. The closure of H(B_R) is compact in X. Thanks to (3.7) and (3.8) we have $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$. Moreover,

$$u_t = \nabla \cdot (\alpha_k(\psi)\nabla u - u\beta(\psi)\nabla v) \in L^2(0,T;(H^1)'(\Omega)). \tag{3.9}$$

Indeed, for any test function $\varphi \in H^1(\Omega)$ we have

$$|(\nabla \cdot (\alpha_k(\psi)\nabla u), \varphi)| \leq m_k|(\nabla u, \nabla\varphi)| \leq C_k\|\nabla u\|_2\|\varphi\|_{1,2}$$

and

$$|(\nabla \cdot (u\beta(\psi)\nabla v), \varphi)| \leq C|(u\nabla v, \nabla\varphi)| \leq C\|u\|_3\|\nabla v\|_6\|\varphi\|_{1,2}$$
$$\leq C\|u\|_{1,2}^{1/2}\|u\|_2^{1/2}\|\psi\|_2\|\varphi\|_{1,2} \leq C(\|\nabla u\|_2^{1/2} + 1)\|\psi\|_2\|\varphi\|_{1,2}.$$

The last inequality is due to the fact that $u \in L^\infty(0,T;L^2(\Omega))$. Since

$$(\|\nabla u(t)\|_2^{1/2} + 1)\|\psi(t)\|_2 \in L^2(0,T),$$

we obtain (3.9). From [19, Chapter 1, Section 5.2] it follows that the set $H(B_R)$ is relatively compact in $L^2(Q_T)$. As we also have the boundedness of u in $L^\infty(0,T;L^2(\Omega))$ we can infer the compactness in X.

Step 3. The operator $H\colon B_R \to B_R$ is continuous. Let $v_n \in B_R$ such that $v_n \to v$ in X. Because $\|v_n\|_X \leq R$ and $H(B_R)$ is precompact in X there exists a subsequence (that we will denote again with the same index) $y_n = H(v_n) \to y$. By elliptic regularity, we know that $F\colon X \to Y$ is a continuous operator. For that, if $v_n \in B_R$ converges to v then $z_n = F(v_n) \to F(v) = z$ in Y. In order to conclude this step we have to prove that $y_n = G(z_n) \to G(z)$.

Thanks to (3.7)–(3.9) we see that

$$y_n \text{ is bounded in } L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \tag{3.10}$$

and

$$(y_n)_t \text{ is bounded in } L^2(0,T;(H^1)'(\Omega)). \tag{3.11}$$

From (3.10) we get

$$y_n \rightharpoonup y \quad \text{weak in } L^2(0,T;H^1(\Omega)), \tag{3.12}$$

this implies that $y_n \to y$ in $\mathcal{D}'(0,T;(H^1)'(\Omega))$, so thanks to $(y_n)_t \to y_t$ in $\mathcal{D}'(0,T;(H^1)'(\Omega))$ and (3.11) we infer

$$(y_n)_t \rightharpoonup y_t \quad \text{weak in } L^2(0,T;(H^1)'(\Omega)). \tag{3.13}$$

We have $y_n \to y$ in X and a.e.

Let $\varphi \in L^2(0,T;H^1(\Omega))$, thanks to (3.13) we obtain

$$\int_0^T \langle (y_n)_t, \varphi\rangle \to \int_0^T \langle y_t, \varphi\rangle.$$

The sequence

$$\alpha_k(v_n) \text{ is bounded in } L^\infty(Q_T), \tag{3.14}$$

$v_n \to v$ in X together with α_k continuous implies $\alpha_k(v_n) \to \alpha_k(v)$ a.e. in Q_T. Therefore,

$$\alpha_k(v_n) \to \alpha_k(v) \quad \text{in } L^p(Q_T) \tag{3.15}$$

for every $p < \infty$ by the Lebesgue majorised theorem.

Thanks to (3.14) and (3.10) we conclude that $\alpha_k(v_n)\nabla y_n$ is bounded in $L^2(Q_T)$, in particular

$$\alpha_k(v_n)\nabla y_n \rightharpoonup \xi \quad \text{in } L^2(Q_T).$$

We observe that for $\varphi \in L^2(0,T;W^{1,3}(\Omega))$ by (3.15) and (3.12)

$$\int_0^T \int_\Omega \alpha_k(v_n)\nabla y_n \cdot \nabla\varphi \to \int_0^T \int_\Omega \alpha_k(v)\nabla y \cdot \nabla\varphi.$$

Since $L^2(0,T;W^{1,3}(\Omega))$ is dense in $L^2(0,T;H^1(\Omega))$ $\xi = \alpha_k(v)\nabla y$.

As $z_n \to z$ in Y, then $\nabla z_n \to \nabla z$ in $L^2(0,T;L^6(\Omega))$, in particular also a.e. Reasoning as before, we also have $\beta(v_n) \to \beta(v)$ a.e. Finally, from Aubin–Lions's lemma we infer $y_n \to y$ in $L^2(0,T;L^3(\Omega))$. Now, it follows by the Lebesgue majorised theorem

$$\int_0^T \int_\Omega y_n\beta(v_n)\nabla z_n \cdot \nabla\varphi \to \int_0^T \int_\Omega y\beta(v)\nabla z \cdot \nabla\varphi.$$

Then we have proved that $y_n \to y$, where y is the unique weak solution to

$$\begin{cases} \dfrac{\partial y}{\partial t} = \nabla\cdot(\alpha_k(v)\nabla y - y\beta(v)\nabla z) & \text{in } \Omega\times(0,T), \\ \nabla y\cdot\overrightarrow{n} = 0 & \text{on } \partial\Omega\times(0,T), \\ y(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{3.16}$$

This implies that $y = G(z)$. Moreover, thanks to the uniqueness of solutions to (3.16) we can infer that all the original sequence (y_n) tends to $G(z)$.

Step 4. L^∞ bounds. Multiplying (3.5) by pu^{p-1}, where $p \geq 2$, we see that

$$\frac{d}{dt}\|u\|_p^p + \frac{4(p-1)}{p}\int_\Omega \alpha_k(u)|\nabla(u^{p/2})|^2\,dx = 2(p-1)\int_\Omega u^{p/2}\beta(u)\nabla(u^{p/2})\cdot\nabla v\,dx$$

Doing the same calculus as in Step 1 we obtain,

$$\frac{d}{dt}\|u\|_p^p + m_k\|\nabla(u^{p/2})\| \leq C_k p^4\|u\|_p^p. \tag{3.17}$$

Owing to (3.17) we infer that

$$u \in L^\infty(0,T;L^p(\Omega)) \quad \text{and} \quad u^{p/2} \in L^2(0,T;H^1(\Omega)) \quad \text{for all } p \text{ finite.} \tag{3.18}$$

Now, we can prove that $u \in L^\infty(Q_T)$. This proof is based on De Giorgi L^∞ technique. In the proof we will require the following Lemma [8, Lemma 4.1].

Lemma 3.5. *Let $\psi(s)$ a nonnegative, non-increasing function on $[s_1,s_2)$ where $s_1 < s_2 \leq \infty$. Suppose that there are positive constants M, γ, β such that*

$$\psi(\widehat{s}) \leq M(\widehat{s}-s)^{-\gamma}\psi(s)^{1+\beta}$$

for all $s_1 < s < \widehat{s} < s_2$. If

$$s_0 = s_1 + 2^{(1+\beta)/\beta}M^{1/\gamma}\psi(s_1)^{\beta/\gamma} \in (s_1,s_2)$$

then $\psi(s) = 0$ *on* $[s_0, s_2)$.

Let l a fixed constant and consider the function $u_l = (u-l)_+$. We denote by Ω_l the sets

$$\Omega_l = \Omega_l(t) = \{x \in \Omega : u(x,t) > l,\ a.e.\}.$$

If we multiply (3.5) by u_l we estimate

$$\frac{d}{2dt}\|u_l\|_2^2 + \frac{m_k}{2}\|\nabla u_l\|_2^2 \le M\int_\Omega (u_l + l)\nabla v \cdot \nabla u_l. \tag{3.19}$$

On the other hand multiplying (1.2) by u_l^2 and applying integration by parts we have

$$\frac{1}{2}\int_\Omega \nabla v \cdot \nabla u_l^2 = \frac{1}{2}\int_\Omega u_l^2 u - v u_l^2 = \frac{1}{2}\int_\Omega u_l^3 + k u_l^2 - v u_l^2. \tag{3.20}$$

Analogously

$$\int_\Omega \nabla v \cdot \nabla u_l = \int_\Omega u_l^2 + k u_l - v u_l. \tag{3.21}$$

Using the estimates (3.20) and (3.21) in (3.19) we obtain

$$\frac{d}{2dt}\|u_l\|_2^2 + \frac{m_k}{2}\|\nabla u_l\|_2^2 \le C(\|u_l\|_3^3 + \|u_l\|_2^2 + \|u_l\|_1). \tag{3.22}$$

Combining Hölder's inequality and (3.18) we find

$$\int_{\Omega_l} u_l^3 \le \|u_l\|_6^3 |\Omega_l|^{1/2} \le C\|u_l\|_6^{4/3}|\Omega_l|^{1/2} \le \varepsilon\|u_l\|_6^2 + C(\varepsilon)|\Omega_l|^{3/2}.$$

Finally, we can estimate $\|u_l\|_6$ using Sobolev's imbedding

$$\int_{\Omega_l} u_l^3 \le \varepsilon C\|u_l\|_{1,2}^2 + C(\varepsilon)|\Omega_l|^{3/2}. \tag{3.23}$$

In a similar way we can prove the following estimate

$$\int_{\Omega_l} u_l^2 \le \varepsilon C\|u_l\|_{1,2}^2 + C(\varepsilon)|\Omega_l|^{3/2}. \tag{3.24}$$

In addition we have

$$\int_{\Omega_l} u_l \le \|u_l\|_4 |\Omega_l|^{3/4} \le \varepsilon C\|u_l\|_{1,2}^2 + C(\varepsilon)|\Omega_l|^{3/2}.$$

Let $\delta > 0$ a positive constant. If we add $\delta\|u_l\|_2^2$ in both sides of (3.22), thanks to (3.23), (3.24) and choosing $\varepsilon > 0$ properly we get

$$\frac{d}{dt}\|u_l\|_2^2 + \delta\|u_l\|_2^2 \le C(\varepsilon)|\Omega_l|^{3/2}. \tag{3.25}$$

Choosing $l \ge \|u_0\|_\infty = l_0$, we have $\|u_l(0)\|_2 = 0$, so from (3.25) we infer

$$\|u_l(t)\|_2^2 \le \frac{C(\varepsilon)}{\delta}\left(1 - e^{-\delta t}\right) \sup_{s\in[0,t)} |\Omega_l(s)|^{3/2}. \tag{3.26}$$

On the other hand, fixing $j > l$, since $\Omega_j \subset \Omega_l$ and $u > j$ in Ω_j then

$$\|u_l(t)\|_2^2 \geq \int_{\Omega_j} (u-l)_+^2 \geq (j-l)^2 |\Omega_j(t)|. \tag{3.27}$$

Putting together the inequalities (3.26) and (3.27)

$$(j-l)^2 |\Omega_j(t)| \leq \frac{C(\varepsilon)}{\delta}(1-e^{-\delta t}) \sup_{s\in[0,t)} |\Omega_l(s)|^{3/2}. \tag{3.28}$$

Let $t < z < T$, then taking $\sup_{t\in[0,z]}$ on both sides of (3.28) and thanks to the fact that $\sup(AB) \leq \sup A \sup B$,

$$(j-l)^2 \sup_{t\in[0,z]} |\Omega_j(t)| \leq \frac{C(\varepsilon)}{\delta}\left(1-e^{-\delta z}\right) \sup_{t\in[0,z]} |\Omega_l(t)|^{3/2}. \tag{3.29}$$

Denoting $\psi_z(j) = \sup_{t\in[0,z]} |\Omega_j(t)|$, we rewrite (3.29) in the following form

$$\psi_z(j) \leq M_z (j-l)^{-2} \psi_z(l)^{3/2},$$

where $M_z = (C(\varepsilon)/\delta)(1-e^{-\delta z})$. Applying Lemma 3.5 on the interval $[\|u_0\|_\infty, \infty)$ we obtain, $\psi_z(s) = 0$ on $[s_0^z, \infty)$ with

$$s_0^z = \|u_0\|_\infty + 8M_z^{1/2}\psi_z(\|u_0\|_\infty)^{1/4} < \|u_0\|_\infty + 8|\Omega|^{1/4} M_z^{1/2}.$$

The function $M_z \to 0$ when $z \to 0$. Thus we can choose T_0 such that $s_0^z < 1 + \|u_0\|_\infty < k$ for all $z < T_0$. Hence $u < k$ a.e. in Q_{T_0}, and this implies $\alpha_k(u) = \alpha(u)$. Thus, the fixed point of H is a solution to (1.1)–(1.4). $\square$

Lemma 3.6. *For arbitrary $0 < T < \infty$, provided there exists $k(T)$ such that $\|u\|_{\infty,Q_T} \leq k(T)$, the local solutions constructed in Theorem 3.4 can be continued until T.*

Proof. We have $\|u\|_{\infty,Q_T} < k(T)$. Given u_0 we obtain a solution defined until time t_1. Then, choosing as initial data $u_1 = u(\,\cdot\,, t_1)$ we have a solution defined until t_2. Applying this procedure we have a sequence of times $\{t_l\}$ and initial data $\{u_l\}$. We shall prove that there exists $\overline{t} > 0$, such that $t_l = \overline{t}$. For that fix $k > k(T) + 1$, then in every step the function $\alpha_k(u)$ will be the same. We fix $R > 0$. Since,

$$\begin{aligned} f(t)\exp\left(\frac{CR^4}{2}\right)\|u_l\|_2 &< f(t)\exp\left(\frac{CR^4}{2}\right)|\Omega|\|u_l\|_\infty \\ &< f(t)\exp\left(\frac{CR^4}{2}\right)|\Omega| k(T) \end{aligned}$$

then, if we choose $\overline{t}$ such that

$$f(t) < \frac{R}{\exp(CR^4/2)|\Omega| k(T)}, \quad t \in (0, \overline{t}]$$

and in view of Step 1 lemma follows. $\square$

THEOREM 3.7. *The local solution constructed in Theorem* 3.4 *are global-in-time provided for arbitrary* $T < \infty$ *there exists* $k(T)$ *such that* $\|u\|_{\infty,Q_T} \leq k(T)$.

PROOF. We can choose T exactly as in the Lemma 3.6 and continue successively the solution to (1.1)–(1.4) with the step of the length T. $\square$

4. Uniqueness

In this section we prove that solutions constructed in the previous section are unique. We present two theorems on uniqueness. In the first theorem we prove the uniqueness without additional hypotheses on α and β in the class of spaces where we defined the weak solutions. In the second the uniqueness is proved in less restrictive spaces but we must assume more on β. Both proofs rely on a classical technique, so called duality method i.e. choosing suitable test functions in the weak formulation. In particular, we use the techniques introduced in [7] and [18]. In order to find out more details about the method we refer the reader to [7] where the authors proved, basing on the duality methods, three theorems on uniqueness for a model arising in semiconductors theory.

THEOREM 4.1. *Let* $T > 0$*, assume* (T) *and let* u *a weak solution to* (1.1)–(1.4) *in the sense of Definition* 3.1 *then* u *is unique in* Q_T.

PROOF. Let (u, v) and (u_1, v_1) be two solutions to (1.1)–(1.4). We fix $T > 0$. We put

$$U(t,x) = u(t,x) - u_1(t,x), \quad V(t,x) = v(t,x) - v_1(t,x),$$

We introduce the subset $L^2_0(\Omega)$ of $L^2(\Omega)$ defined by

$$L^2_0(\Omega) := \left\{ w \in L^2(\Omega), \ \int_\Omega w(x)\,dx = 0 \right\}.$$

Then, by Ψ we denote the unique solution in $L^2_0(\Omega) \cap H^1(\Omega)$ to

$$-\Delta\Psi = U \quad \text{in } \Omega, \tag{4.1}$$

$$\nabla\Psi \cdot \overrightarrow{n} = 0 \quad \text{on } \partial\Omega. \tag{4.2}$$

for $(x,t) \in Q_T$. Thus

$$\begin{aligned}\int_0^t \langle \partial_t U, \psi\rangle\,ds = -\int_0^t \int_\Omega [\nabla(A(u) - A(u_1)) - (u\beta(u) \\ - u_1\beta(u_1))\nabla v - u_1\beta(u_1)\nabla V] \cdot \nabla\psi\,dx\,ds,\end{aligned} \tag{4.3}$$

for each $\psi \in L^2((0,t); H^1(\Omega))$, where $\langle \cdot\,,\cdot \rangle$ denotes the duality pairing between $H^1(\Omega)$ and its dual.

$$A \text{ is a primitive of } \alpha, \ \text{i.e. } \frac{d}{ds}A(s) = \alpha(s), A(0) = 0.$$

In (4.3) we set $\Psi = \psi$. Thus, we have

$$\begin{aligned}
\int_\Omega |\nabla\Psi(t)|^2\,dx = \int_\Omega |\nabla\Psi(0)|^2\,dx + 2\int_0^t \langle \partial_t U, \Psi\rangle\,ds \\
= 2\int_0^t\int_\Omega \Delta\Psi(A(u)-A(u_1))\,dx\,ds \\
+ 2\int_0^t\int_\Omega (u\beta(u)-u_1\beta(u_1))\nabla\Psi\cdot\nabla v\,dx\,ds \\
+ 2\int_0^t\int_\Omega u_1\beta(u_1)\nabla\Psi\cdot\nabla V\,dx\,ds.
\end{aligned}\tag{4.4}$$

Since $u \in L^\infty(Q_T)$ classical elliptic regularity assures that $\nabla v \in (L^\infty(Q_T))^n$. For that, we can infer from (4.4) after applying Hölder's inequality

$$\begin{aligned}
\int_\Omega |\nabla\Psi(t)|^2\,dx \le -2\int_0^t\int_\Omega (u-u_1)(A(u)-A(u_1))\,dx\,ds \\
+ 2\int_0^t \|\nabla v\|_\infty \|u\beta(u)-u_1\beta(u_1)\|_2\|\nabla\Psi\|_2\,ds \\
+ 2\|u\beta(u)\|_\infty \int_0^t \|\nabla\Psi\|_2\|\nabla V\|_2\,ds.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_\Omega |\nabla\Psi(t)|^2\,dx \le \int_0^t\int_\Omega F(u,u_1)\,dx\,ds \\
+ C(\varepsilon)\int_0^t \|\nabla\Psi\|_2^2\,ds + \int_0^t \|\nabla V\|_2^2\,ds,
\end{aligned}\tag{4.5}$$

where $F(u,u_1) = \varepsilon(u\beta(u)-u_1\beta(u_1))^2 - 2(u-u_1)(A(u)-A(u_1))$ and $\varepsilon \in (0,1)$.

On the other hand by (1.2) and taking into account the fact that U satisfies (4.1)–(4.2) we obtain after integrating by parts that

$$\int_\Omega |\nabla V(t)|^2 dx + \int_\Omega V(t)^2 dx = \int_\Omega \nabla\Psi(t)\cdot\nabla V(t) dx.\tag{4.6}$$

After applying Young's inequality in (4.6) we obtain

$$\int_\Omega |\nabla V(t)|^2\,dx \le \int_\Omega |\nabla\Psi(t)|^2\,dx.\tag{4.7}$$

Thus, it follows from (4.5) and (4.7) that

$$\int_\Omega |\nabla\Psi(t)|^2\,dx \le \int_0^t\int_\Omega F(u,u_1)\,dx\,ds + (C(\varepsilon)+1)\int_0^t \|\nabla\Psi\|_2^2\,ds.$$

If $F(u,u_1) \le 0$ in Q_T then from Gronwall's lemma we see $\nabla\Psi = 0$ a.e. in Q_T, therefore, since Ω is connected $\Psi = k$ in Q_T. Thus by (4.1) $U = 0$ a.e. in Q_T. Let us prove that $F(u,u_1) \le 0$. Observe that

$$(A(r)-A(s))(r-s) = \alpha(\xi)(r-s)^2 > m(r-s)^2,$$

where $m > 0$. On the other hand,

$$\varepsilon(r\beta(r) - s\beta(s)) = \varepsilon(\beta(\xi') + \xi\beta'(\xi'))(r-s)^2 < \varepsilon M(r-s)^2.$$

Now, choosing ε such that $\varepsilon M < m$ we conclude the proof. □

Under an additional condition we prove the uniqueness without assuming $u \in L^\infty(Q_T)$.

THEOREM 4.2. *Assume* (T). *If there exists a positive constant M such that $\beta(u) \le M/u$ then solutions to* (1.1)–(1.4) *are unique in* $[L^4(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))] \times L^4(0,T;H^2(\Omega))$.

PROOF. We keep the notation of Theorem 4.1. Thus, by (4.4) we have

$$\begin{aligned}\int_\Omega |\nabla\Psi(t)|^2\,dx \le &-2\int_0^t\int_\Omega (u-u_1)(A(u)-A(u_1))\,dx\,ds \\ &+2\int_0^t \|\nabla v\|_2 \|u\beta(u) - u_1\beta(u_1)\|_\infty \|\nabla\Psi\|_2\,ds \\ &+2\|u\beta(u)\|_\infty \int_0^t \|\nabla\Psi\|_2\|\nabla V\|_2\,ds.\end{aligned}$$

Notice that A is increasing as a primitive of positive function. Thus, first term of the right-hand side of the last inequality is negative. Using the facts that $\beta(u) \le M/u$, $\|\nabla v\|_{L^\infty(0,T;L^2(\Omega))} \le C$ and (4.7) we arrive at

$$\int_\Omega |\nabla\Psi(t)|^2\,dx \le C\int_0^t \|\nabla\Psi\|_2\,ds.$$

Then we see that $u = u_1$ with the use of Gronwall's lemma. Thus, from the standard arguments for elliptic equations we infer $v = v_1$. The proof is finished.□

5. Energy estimates

In this section we present three theorems. First of them is going to state the result for dimension 1. Theorem 5.2 says that we have the global-in-time existence of solutions for dimensions 2 and 3 provided β goes to 0 fast enough. It is of importance that Theorem 5.2 specifies the assumptions on α and β under which we still have the prevention of blow-up and the Lyapunov functional could not be bounded from below. Next we shall give the assumptions on α and β under which we have the global existence of solutions for dimension 3. In [4, Theorem 6.18] the authors proved for dimension 2 the result that seems to be optimal from the point of view of the prevention of blow-up.

Let us underline that from Theorem 3.7 we know that in order to prolong the solution to (1.1)–(1.4) it is enough to estimate their L^∞ norms. We also see that if we have L^∞ bounds on u we immediately get ones on v.

Proposition 5.1. *Assume $n = 1$ and $u_0 \in L^\infty(\Omega)$ are nonnegative functions and hypotheses* (T) *and* (1.9) *hold. Then the solution to* (1.1)–(1.4) *is global-in-time.*

Proof. As we noticed in the foreword of Section 5 it is enough to prove L^∞ bounds on u. To this end (see Theorem 2.1) we need L^∞ estimate of ∇v. $\Omega = [a, b]$. But from (1.2) after integration from a to $x_0 \in \Omega$ we see that

$$|v_x(x_0)| \leq 2M$$

thanks to the boundary conditions (1.3) and Proposition 3.3. □

Let us underline that the following result is an application of the result of [5] to the parabolic-elliptic case.

Theorem 5.2. *Assume $n = 2, 3$. Let the nonnegative functions $u_0 \in L^\infty(\Omega)$ and hypotheses* (T) *and* (1.9) *hold. Then the solution to* (1.1)–(1.4) *exists globally in time provided there exist a positive constant M_1 such that*

$$\beta(u) \leq M_1 u^{-\gamma_1}, \tag{5.1}$$

where $\gamma_1 > 1$ for $n = 2$ or $\gamma_1 > 2$ for $n = 3$.

Proof. This proof follows the steps of the proof presented in [5] for fully parabolic system. It differs only in few places. For completeness we present it also below. On multiplying (1.1) by u^{p-1}, $p > n$ and (1.2) by Δv we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\,dx + (p-1)\int_\Omega \alpha(u)|\nabla u|^2 u^{p-2}\,dx \\ = (p-1)\int_\Omega u^{p-1}\beta(u)\nabla v\cdot\nabla u\,dx\end{aligned} \tag{5.2}$$

and

$$\int_\Omega |\Delta v|^2\,dx + \int_\Omega |\nabla v|^2\,dx \leq C\int_\Omega u^2\,dx. \tag{5.3}$$

Owing to the equality

$$u^{p-1}\beta(u) = u^{(p-2)/2}u^{p/2}\sqrt{\beta(u)}\sqrt{\beta(u)}$$

and since (1.9) holds, thanks to the Cauchy–Schwarz inequality we obtain from (5.2)

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p\,dx + \frac{p-1}{2}\int_\Omega \alpha(u)|\nabla u|^2 u^{p-2}\,dx \leq C\int_\Omega u^p\beta(u)|\nabla v|^2\,dx. \tag{5.4}$$

Now adding (5.4) and (5.3) and applying the Hölder inequality to the right-hand-side of (5.3) we obtain ($p > 2$)

$$\frac{d}{dt}\int_\Omega u^p\,dx + \frac{1}{2}\int_\Omega |\Delta v|^2\,dx \leq C\bigg(\int_\Omega u^p\,dx\bigg)^{2/p} + C\int_\Omega u^p\beta(u)|\nabla v|^2\,dx. \tag{5.5}$$

Thanks to (5.1) $u^p\beta(u) \le M_1 u^{p-\gamma_1}$. Next from Hölder's inequality we see that

$$\int_\Omega u^p\beta(u)|\nabla v|^2\,dx \le C\|u^{p-\gamma_1}\|_l\|\nabla v\|^2_{2l'}, \tag{5.6}$$

where l, l' are positive constants such that $1/l + 1/l' = 1$. Now we focus on the proper choice of the constants l' and l. To this end we notice by the standard results in the regularity theory for elliptic equations that from (1.2) and the estimate of L^1 norm of u we have, for $q = n/(n-1)$,

$$\sup_{t\in(0,T]}\|\nabla v(\,\cdot\,,t)\|_q \le C\sup_{t\in(0,T]}\|u(\,\cdot\,,t)\|_1 = C. \tag{5.7}$$

By the Gagliardo–Nirenberg inequality we have (the terms of lower order are absorbed, for the details see [14, (27)])

$$\|\nabla v(\,\cdot\,,t)\|^2_{2l'} \le C\|\nabla v(\,\cdot\,,t)\|^2_q\|\Delta v(\,\cdot\,,t)\|_2^{2b'}, \tag{5.8}$$

where C is a positive constant, $l' < n/(n-2)$ and

$$b' = \frac{\dfrac{n}{q} - \dfrac{n}{2l'}}{1 - \dfrac{n}{2} + \dfrac{n}{q}}. \tag{5.9}$$

Choosing $l' = q = 2$ for dimension 2 and $l' = q = 3/2$ for dimension 3 with the use of the Young inequality to the right-hand side of (5.6) and applying (5.8), where $\|\nabla v(\,\cdot\,,t)\|_q$ is estimated using (5.7) for $\delta = 1$ we arrive at

$$\|u^{p-\gamma}\|_l\|\nabla v\|^2_{2l'} \le C\int_U u^p dx + C\|\Delta v\|_2^{2b'l'}. \tag{5.10}$$

Estimating the second term of the right-hand side of (5.5) by (5.10) and then using the Gronwall lemma we estimate $\|u(\,\cdot\,,t)\|_p$ on finite time intervals.

Since by standard regularity estimates for elliptic equations we know

$$\sup_{t\in(0,T]}\|\nabla v(\,\cdot\,,t)\|_\infty \le \sup_{t\in(0,T]}\|u(\,\cdot\,,t)\|_p, \tag{5.11}$$

$p > n$, we see that $\sup_{t\in(0,T]}\|\nabla v(\,\cdot\,,t)\|_\infty < \infty$ on finite time intervals. Now we finish the proof applying Theorem 2.1. $\square$

THEOREM 5.3. *Assume $n = 3$. Let the nonnegative functions $u_0 \in L^\infty(\Omega)$ and hypotheses* (T) *and* (1.9) *hold. Then there exists a global in time nonnegative weak solution to* (1.1)–(1.4) *provided* (2.6) *holds.*

REMARK 5.4. The assumptions of Theorem 5.3 are optimal in a sense that they cause the boundedness from below of the Lyapunov functional. If they are not satisfied then the Lyapunov functional is no more bounded from below (see Proposition 2.2).

Proof. Fix $T > 0$. First of all let us point out the following fact. By (2.6) we see that the Lyapunov functional L is bounded from below. Moreover [14, Remark after Lemma 5.1] says that also there exists $N > 0$ such that for $\kappa = 1 - \gamma$ holds

$$\sup_{t\in[0,T]} \|u\|_{1+\kappa} < N. \tag{5.12}$$

Following [6] and [4] we introduce the nonlinear convex functional $\phi_p(u)$ satisfying both $\phi_p(0) = 0$ and $\phi_p'(0) = 0$ by

$$p(p-1)\eta^{p-2} = \phi_p''(\eta)\alpha(\eta). \tag{5.13}$$

for every $\eta > 0$. Notice that

$$\int_\Omega \eta^p\,dx \le C\int_\Omega \phi_p(\eta)\,dx. \tag{5.14}$$

The inequality can be easily derived integrating twice (5.13).

If we had the estimates for

$$\sup_{t\in(0,T]} \|u(\,\cdot\,,t)\|_p, \tag{5.15}$$

$p > n$ then by (5.11) we would have

$$\sup_{t\in(0,T]} \|\nabla v(\,\cdot\,,t)\|_\infty < \infty.$$

Then application of Theorem 2.1 finishes the proof. □

In order to obtain (5.15) we first have to prove the auxiliary lemma.

Lemma 5.5. *Let the assumptions of Theorem* 5.3 *be fulfilled. Moreover, assume* (5.12) *then* (5.15) *holds.*

Proof. We shall prove the lemma using a technique introduced in [15] and applied for the quasilinear non-uniformly parabolic case in [4]. Multiplying (1.1) by $\phi_p'(u)$ and then integrating we arrive at

$$\frac{d}{dt}\int_\Omega \phi_p(u)\,dx = -\int_\Omega \phi_p''(u)\alpha(u)|\nabla u^{p/2}|^2\,dx + \int_\Omega \phi_p''(u)u\beta(u)\nabla u\cdot\nabla v\,dx \tag{5.16}$$

Owing to (2.6) and (5.13) we derive from (5.16) the following inequality

$$\frac{d}{dt}\int_\Omega \phi_p(u)\,dx \le -C\int_\Omega |\nabla u^{p/2}|^2\,dx + C\int_\Omega u^{p-2+\gamma}\nabla u\cdot\nabla v\,dx.$$

Next, multiplying (1.2) by $u^{p+\gamma-1}$ we obtain from (5.16)

$$\begin{aligned}\frac{d}{dt}\int_\Omega \phi_p(u)\,dx &\le -C\int_\Omega |\nabla u^{p/2}|^2\,dx + C\int_\Omega u^{p+\gamma}\,dx \\ &= -C\|\nabla u^{p/2}\|_2^2 + C\|u^{p/2}\|_{2(p+\gamma)/p}^{2(p+\gamma)/p}.\end{aligned}$$

Next we estimate $\|u^{p/2}\|_{2(p+\gamma)/p}^{2(p+\gamma)/p}$ with the use of the Gagliardo–Nirenberg inequality

$$\|u^{p/2}\|_{2(p+\gamma)/p}^{2(p+\gamma)/p} \le C(\|\nabla u^{p/2}\|_2^{2\theta(p+\gamma)/p} + \|u^{p/2}\|_2^{2\theta(p+\gamma)/p})\|u^{p/2}\|_d^{2(1-\theta)(p+\gamma)/p}, \tag{5.17}$$

where

$$d = \frac{2c}{p} \quad \text{and} \quad \theta = \frac{\dfrac{3p}{2c} - \dfrac{3p}{2(p+\gamma)}}{\dfrac{3p}{2c} - \dfrac{1}{2}},$$

we need to chose c in such a way that

$$c < 1 + \kappa \tag{5.18}$$

and

$$\theta\,\frac{p+\gamma}{p} < 1. \tag{5.19}$$

Then from (5.17) we obtain

$$\frac{d}{dt}\int_\Omega \phi_p(u) \le -C\|\nabla u^{p/2}\|_2^2 + C\|u^{p/2}\|_2^2 + C, \tag{5.20}$$

where the last inequality was derived thanks to on the one hand (5.12) and (5.18) and on the other hand (5.19) and the Young inequality. Then to finish the proof of Lemma 5.5 we proceed in the following way. Integrating in time (5.20), with the use of (5.14) and Gronwall's lemma, we obtain (5.15).

Thus, in order to complete the proof of Lemma 5.5 we have to show that we can chose c such that (5.18) and (5.19) hold. But,

$$\theta\,\frac{p+\gamma}{p} = \frac{3(p+\gamma-c)}{3p-c}.$$

Thus (5.19) is satisfied whenever $3\gamma < 2c$ and this is satisfied for $1 < c < 1+\kappa$. $\square$

Remark 5.6. One sees that conditions (5.18) and (5.19) are satisfied whenever $\gamma < 4/5$. One could then expect that our proof of Theorem 5.3 acts also for this range of parameters. It is false. Notice that in order to obtain (5.20) from (5.17) we heavily based on (5.12) which is a consequence of the boundedness from below of the Lyapunov functional which demands $\gamma < 2/3$, see (2.6) and Proposition 2.2.

Conclusion. Beside constructing the solution we showed two mechanisms of preventing the blow-up. They are independent in the way that they cover different assumptions on α and β, in order to prove the prevention of blow-up. The differences between them are also important from the applications point of view. For example in Hillen–Painter models one can show the boundedness of a solution on finite time intervals for $q(u) = (1+u)^{-\gamma}$, $\gamma > 0$. Then α and β are given by (1.5) and (1.6). For dimension 3 thanks to Theorem 5.2 we have the prevention of blow-up for $\gamma > 2$ and for dimension 2 for $\gamma > 1$. Theorem 5.3 is not applicable for such a choice of q for any γ neither in dimension 2 nor 3. It is worth to underline that for dimension 2 the theorem proved in [4] covers such a choice of q for every $\gamma > 0$.

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Manuscript received April 4, 2006

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