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SOME NONLOCAL ELLIPTIC PROBLEM INVOLVING POSITIVE PARAMETER

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Abstract. We consider the following superlinear Kirchhoff type nonlocal problem:

$$\begin{cases} -\left(a+b\displaystyle\int_\Omega |\nabla u|^2\,dx\right)\Delta u = \lambda f(x,u) & \text{in } \Omega,\ a>0,\ b>0,\ \lambda>0,\\ u=0 & \text{on } \partial\Omega.\end{cases}$$

Here, $f(x,u)$ does not satisfy the usual superlinear condition, that is, for some $\theta>0$,

$$0\le F(x,u)\triangleq\int_0^u f(x,s)\,ds\le\frac{1}{2+\theta}f(x,u)\,u,\quad\text{for all } (x,u)\in\Omega\times\mathbb{R}^+$$

or the following variant

$$0\le F(x,u)\triangleq\int_0^u f(x,s)\,ds\le\frac{1}{4+\theta}f(x,u)\,u,\quad\text{for all } (x,u)\in\Omega\times\mathbb{R}^+$$

which is quiet important and natural. But this superlinear condition is very restrictive eliminating many nonlinearities. The aim of this paper is to discuss how to use the mountain pass theorem to show the existence of nontrivial solution to the present problem when we lose the above superlinear condition. To achieve the result, we first consider the existence of a solution for almost every positive parameter λ by varying the parameter λ. Then, it is considered the continuation of the solutions.

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1. Introduction

This paper considers a class of Kirchhoff type problem

$$
\begin{cases} -\Big(a + b\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = \lambda f(x,u) & \text{in } \Omega,\ \lambda > 0, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}
$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N = 1, 2$ or 3) and a, b are positive real numbers and $f(x,t)\colon \overline{\Omega} \times \mathbb{R}^1$ is a continuous real function and satisfies the subcritical condition

$$
|f(x,u)| \le C(|u|^{p-1} + 1) \quad \text{for some } 4 < p < 2^* = \begin{cases} 2N/(N-2), & N \ge 3, \\ +\infty, & N = 1, 2. \end{cases}
$$

Since the equation contains an integral over Ω, it is no longer a pointwise identity, and therefore is often called nonlocal problem.

In 1883, Kirchhoff in [10] proposed this type of problems as an extension of the classical D'Alembert wave equation for free vibrations of elastic strings. The model studies is

$$
\frac{\partial^2 u(x,t)}{\partial t^2} - \bigg\{m_0 + m_1 \int_0^l \Big(\frac{\partial u}{\partial x}\Big)^2 dx\bigg\}\frac{\partial^2 u(x,t)}{\partial x^2} = 0, \quad 0 < x < l,\ t > 0, \tag{1.2}
$$

where m_0 is connected with the initial tension, m_1 is dependent on the characteristic of the material of the string and $u(x,t)$ denote the vertical displacement of the point x of the string at time t. Some early classical studies of Kirchhoff equations were those of Bernstein [5] and Pohozaev [15]. However, (1.2) received great attention only after Lions [11] proposed an abstract framework for the problem. Some interesting results can be found in [4], [9], [6] and the references therein. In recent years, Alves et al. [1], Ma and Rivera [12] have obtained positive solutions of such problems by variational methods. Similar nonlocal problems also model several physical and biological systems where u describes a process which depends on the average of itself, for example the population density, see [3], [8].

Recently, some papers have been devoted to the study of boundary value problems of Kirchhoff type in the particular case of $a = 1$, $b = 0$, that is,

$$
\begin{cases} -\Delta u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}
$$

Ambrosetti and Rabinowitz in [2] established an existence of non-trivial solution result for problem (1.3), by assuming the following conditions:

(A_1) $f(x,0) = 0$, $\lim\limits_{s\to 0} \frac{f(x,s)}{s} = 0$, uniformly in almost every $x \in \Omega$.

(A_2) There exist positive constants a and b such that

$$|f(x,s)| \le a + b|s|^p, \quad 0 \le p < \frac{N+2}{N-2}, \quad \text{for all } s \in \mathbb{R}, \text{ for all } x \in \Omega.$$

(A_3) There are constants $\mu > 2$ and $s_0 > 0$ such that

$$0 < \mu F(x,s) \le s f(x,s), \quad |s| \ge s_0, \quad \text{for all } x \in \Omega,$$

where $F(x,s) = \int_0^s f(x,t)\,dt$.

It is easy to know that (A_3) implies another much weaker condition, which is a consequence of the superlinearity of f:

(A_3') $\lim\limits_{|s|\to\infty} \frac{F(x,s)}{s^2} = +\infty$, uniformly for almost every $x \in \Omega$.

Recently M. Schechter and W. Zou in [16] proved that under hypotheses (A_1), (A_2) and

(A_4) Either $\lim\limits_{s\to-\infty} \frac{F(x,s)}{s^2} = +\infty$ or $\lim\limits_{s\to\infty} \frac{F(x,s)}{s^2} = +\infty$,

problem (1.3) has a non-trivial weak solution for almost every $\lambda > 0$.

M. Schechter and W. Zou also proved that, if addition we assume:

(A_5) $H(x,s)$ is convex in s, for all $x \in \Omega$, where $H(x,s) = sf(x,s) - 2F(x,s)$, for all $x \in \Omega$,

then there is a non-trivial solution for every positive λ.

More recently, Miyagaki and Souto in [14] pointed out that the convexity assumption on H in (A_5) is stronger than the following assumption:

(A_6) $\frac{f(x,s)}{s}$ is increasing in $s \ge s_0$ and decreasing in $s \le -s_0$, for all $x \in \Omega$.

Under the assumptions (A_1), (A_2), (A_3'), (A_6), the authors obtained the existence of a non-trivial weak solution of problem (1.3) for all $\lambda > 0$.

Different from the papers mentioned above, for $a > 0$, $b > 0$, the problem (1.1) has received growing attention in recent years. We list some recent results in this direction as follows. For $\lambda = 1$, Zhang–Perera in [17] and Mao–Zhang in [13] studied the existence of sign-changing solutions for problem (1.1) via sets of descent flow. In [17], the authors considered the 4-superlinear case:

$$\exists\, v > 4,\ s_0 > 0 \text{ s.t. } 0 < vF(x,s) \le sf(x,s),\ |s| \ge s_0,\ \forall\, x \in \Omega, \tag{1.4}$$

where $F(x,s) = \int_0^s f(x,t)\,dt$, which implies a weaker condition

$$F(x,s) \ge c_1|s|^v - c_2, \quad c_1, c_2 > 0,\ x \in \Omega,\ s \in \mathbb{R}.$$

The above condition implies another much weaker condition, which is a consequence of the superlinearity of f:

$$\lim_{|s|\to+\infty} \frac{F(x,s)}{s^4} = +\infty, \quad \text{uniformly for a.e. } x \in \Omega. \tag{1.5}$$

Note that Mao and Zhang in [13] studied the problem under the assumption of (1.5).

In another paper, B. Cheng and X. Wu in [7] also dealt with 4-superlinear problem where $\lambda = 1$ and $f(x,t)$ satisfies (1.5) and

(S_1) $f(x,s) \equiv 0$, for all $x \in \overline{\Omega}$.
(S_2) For almost every $x \in \Omega$, $\frac{f(x,s)}{s^3}$ is nondecreasing with respect to $s > 0$.

Motivated by the papers mentioned above, however, different from [17], [13], [7], we intend to study problem (1.1) with some new nonlinearities which satisfy weaker assumptions than (S_2). Via the mountain pass theorem, new existence result is given for problem (1.1) with $\lambda > 0$.

Next we give our main result.

Theorem 1.1. *Suppose*

(f_1) $|f(x,u)| \le C(|u|^{p-1} + 1)$ *for some* $4 < p < 2^* = 2N/(N-2)$, $N \ge 3$;
(f_2) $f(x,u) = o(u)$ *as* $|u| \to 0$ *uniformly in* $x \in \Omega$;
(f_3) $\frac{F(x,u)}{u^4} \to +\infty$ *as* $|u| \to +\infty$ *uniformly in* $x \in \Omega$;
(f_4) *Let* $H(x,s) := sf(x,s) - 4F(x,s)$, *there is* $C_* > 0$ *such that*

$$H(x,t) \le H(x,s) + C_*,$$

for all $0 < t < s$ *or* $s < t < 0$, *for all* $x \in \Omega$.

Then, problem (1.1) *has a non-trivial weak solution, for all* $\lambda > 0$.

Remark 1.2. Our argument is carried out with the Cerami sequence which is different from [17] because that our assumptions is weaker than (1.4).

Remark 1.3. Compared with [7], (f_4) is weaker than (S_2). Indeed, the condition (S_2) is equivalent to the condition

$$H(x,s) \text{ is increasing in } s > 0, \quad \text{for all } x \in \Omega.$$

Thus, it implies the condition (f_4). Observe that function $H(x,s)$ is a "*quasi-monotonic*" functions, and also if H is monotonic function in $s < 0$ and $s > 0$, or a convex function in $\mathbb{R}$, then it satisfies condition (f_4).

Remark 1.4. Since our hypothesis is weaker, the proof of compactness is very difficult. We overcome the difficulty by constructing a special flow.

In order to achieve the result, we first consider the existence of a solution for almost every positive parameter λ by varying the parameter λ. Then, it is considered the continuation of the solutions. The proof here is made by adapting some arguments used by Miyagaki and Souto in [14].

2. Preliminary results

Let $H := H_0^1(\Omega)$ be the Sobolev space equipped with the inner product and the norm

$$(u,v) = \int_\Omega \nabla u \nabla v \, dx, \quad \|u\| = (u,u)^{1/2}.$$

$B_\delta := \{u \in H : \|u\| < \delta\}$, $S_\delta := \partial B_\delta$, $\delta \in (0,\ +\infty)$. And we denote the usual L^p-norm:

$$|u|_p = \bigg(\int_{R^N} |u(x)|^p \, dx \bigg)^{1/p}.$$

Denote by $0 < \lambda_1 < \lambda_2 < \ldots$ the distinct Dirichlet eigenvalues of $-\Delta$ on Ω, and by $\varphi_1, \varphi_2, \ldots$ the eigenfunctions corresponding to the eigenvalues, then

$$\lambda_1(\Omega) = \inf_{u \in H,\, |u|_2 = 1} |\nabla u|_2^2,$$

is achieved by $\varphi_1 > 0$. Let

$$E_j := \bigoplus_{i \leq j} \operatorname{Ker}(-\Delta - \lambda_i), \quad \text{for } j \in \mathbb{N}.$$

Recall that a function $u \in H$ is called a weak solution of (1.1) if

$$(a + b \|u\|^2) \int_\Omega \nabla u \cdot \nabla v = \lambda \int_\Omega f(x,u) v, \quad \text{for all } v \in h.$$

Weak solutions are the critical points of the C^1 functional

$$\Phi_\lambda(u) = \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \lambda \int_\Omega F(x,u) \, dx, \quad \text{for all } u \in H.$$

Then

$$\Phi'_\lambda(u) v = (a + b \|u\|^2) \int_\Omega \nabla u \nabla v \, dx - \lambda \int_\Omega f(x,u) v \, dx, \quad \text{for all } u, v \in H.$$

They are also classical solutions if $f(x,u)$ is locally Lipschitz on $\overline{\Omega} \times \mathbb{R}^1$.

In what follows we will give some lemmas concerning the energy function Φ. First of all, we discuss the mountain pass structure of the functional Φ.

Lemma 2.1.

(a) Φ_λ *is unbounded from below.*
(b) $u = 0$ *is a strict local minimum for* Φ_λ.

Proof. (a) It follows from (f_3) that, for all $M > 0$, there exists $C_M > 0$ such that

$$F(x,u) \geq M|u|^4 - C_M, \quad \text{for all } (x,u) \in \Omega \times \mathbb{R},\ p > 4. \tag{2.1}$$

Since the space is finite dimensional, then for for all $u \in E_j = \operatorname{span}\{\varphi_1, \ldots, \varphi_j\}$, tehre exists $\rho_{j,s} > 0$ such that $|u|_s \geq \rho_{j,s}\|u\|$, we infer that

$$\Phi_\lambda(u) \leq \frac{a}{2}\|u\|^2 + \left(\frac{b}{4} - \lambda M \rho_{j,4}\right)\|u\|^4 + \lambda C_M|\Omega|, \tag{2.2}$$

then $\Phi_\lambda(u) \to -\infty$, as $\|u\| \to +\infty$, for M sufficiently large.

(b) By (f_1) and (f_2), for any given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$F(x,u) \leq \frac{a\varepsilon}{2}|u|^2 + C_\varepsilon|u|^p, \quad \text{for all } (x,u) \in \Omega \times \mathbb{R},\ p > 4, \tag{2.3}$$

which implies that

$$\begin{aligned} \Phi_\lambda(u) &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda \int_\Omega \frac{a\varepsilon}{2}u^2 + C_\varepsilon|u|^p\,dx \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda\frac{a\varepsilon}{2}\frac{\|u\|^2}{\lambda_1} - \lambda C_1\|u\|^p \\ &\geq \frac{a}{2}\|u\|^2\left(1 - \frac{\lambda}{\lambda_1}\varepsilon\right) + \frac{b}{4}\|u\|^4 - \lambda C_1\|u\|^p \\ &\geq \frac{b}{4}\|u\|^4 - \lambda C_1\|u\|^p \geq \frac{b}{8}\|u\|^4, \end{aligned} \tag{2.4}$$

therefore, 0 is local minimum of Φ_λ.

Set $0 < \alpha < \beta$. We can show that mountain pass geometry on Φ_λ works uniformly on $[\alpha,\beta]$. In fact, for each $\alpha \leq \lambda \leq \beta$, take $0 < \varepsilon < \lambda_1/(2\beta)$, such that $1 - \beta\varepsilon/\lambda_1 \geq 1/2$, for each $u \in H$,

$$\Phi_\lambda(u) \geq \frac{a}{4}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda C_1\|u\|^p. \tag{2.5}$$

Define $Y = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0 \text{ and } \gamma(1) = e\}$, for $\alpha \leq \lambda \leq \beta$, let

$$c_\lambda = \inf_{\gamma\in Y}\max_{t\in[0,1]} \Phi_\lambda(\gamma(t)). \tag{2.6}$$

Let $c\colon [\alpha,\beta] \to \mathbb{R}_+$ given by $c(\lambda) = c_\lambda > 0$. Next we will show (i) c_λ/λ is decreasing, (ii) c_λ and c_λ/λ is left semi-continuous.

(i) Choosing $e \in H$ ($\|e\|$ sufficiently large), such that $\Phi_\alpha(e) < 0$, it is easy to check

$$\frac{\Phi_\lambda(e)}{\lambda} \leq \frac{\Phi_\alpha(e)}{\alpha} < 0, \quad \alpha \leq \lambda \leq \beta, \tag{2.7}$$

and

$$\frac{\Phi_\lambda(u)}{\lambda} \leq \frac{\Phi_\mu(u)}{\mu}, \quad \text{for all } u \in H,\ 0 < \mu < \lambda. \tag{2.8}$$

(ii) Fix $\mu \in [\alpha,\beta]$ and $\varepsilon > 0$, then fix $\gamma \in Y$ which satisfies

$$c_\mu \leq \max_{t\in[0,1]} \Phi_\mu(\gamma(t)) \leq c_\mu + \frac{\varepsilon\mu}{8}. \tag{2.9}$$

Let

$$Q = \max_{t\in[0,1]} \int_\Omega F(x,\gamma(t))\,dx,$$

then, for any $t \in [0,1]$ and $\lambda > \mu/4$, such that

$$\frac{1}{\lambda} < \frac{1}{\mu} + \frac{\varepsilon}{4\mu},$$

which implies that

$$\Phi_\lambda(\gamma(t)) \leq c_\mu + \frac{\varepsilon\mu}{8} + Q|\lambda-\mu|,$$

when $|\mu-\lambda| < \varepsilon\mu/(8Q)$, we have

$$c_\lambda \leq c_\mu + \frac{\varepsilon\mu}{8} + Q\frac{\varepsilon\mu}{8Q} = c_\mu + \frac{\varepsilon\mu}{4}. \tag{2.10}$$

Hence, if $\mu > \lambda$,

$$\frac{c_\mu}{\mu} - \left(1+\frac{c_\mu}{4\mu}\right)\varepsilon \leq \frac{c_\mu}{\mu} \leq \frac{c_\lambda}{\lambda} < \frac{c_\mu}{\mu} + \left(1+\frac{c_\mu}{4\mu}\right)\varepsilon, \tag{2.11}$$

and

$$c_\mu - \frac{\varepsilon\mu}{4} < c_\mu < c_\lambda \leq c_\mu + \frac{\varepsilon\mu}{4}. \tag{2.12}$$

This proves the left semi-continuity of c_λ/λ and c_λ. □

The next lemma estimates the dependence of the parameter λ of the derivative Φ'_λ in the $H^{-1,2}(\Omega)$-norm $\|\cdot\|_*$, and the proof is similar to the Lemma 2.2 in [14].

Lemma 2.2. *There exists $C > 0$, such that*

$$\|\Phi'_\mu(u) - \Phi'_\lambda(u)\|_* \leq C(1+\|u\|^{p-1})|\lambda-\mu|, \quad \textit{for all } \lambda,\ \mu > 0.$$

Remark 2.3. The map c_λ and $g(\lambda) = c_\lambda/\lambda$ are monotone decreasing. Thus, c_λ and $g(\lambda)$ are differentiable at almost all values $\lambda \in [\alpha,\beta]$.

Next we deal with the Cerami sequence ((C) sequence for short). Recall that a sequence $\{u_n\} \subset H$ is said to be a Cerami sequence of Φ provided that $\Phi(u_n) \to c$, $(1+\|u_n\|)\Phi'(u_n) \to 0$ as $n \to +\infty$.

Lemma 2.4. *Suppose the map $c\colon [\alpha,\beta] \to \mathbb{R}_+$, given by $c(\lambda) = c_\lambda$, is differentiable in λ at $\lambda = \mu$, then there exists a bounded* (C) *sequence $\{u_n\} \in H$, that is,*

$$\Phi_\mu(u_n) \to c_\mu, \quad (1+\|u_n\|)\Phi'_\mu(u_n) \to 0 \quad \textit{and} \quad \|u_n\|^2 \leq K_1,$$

as $n \to \infty$, and actually

$$K_1 = \frac{c_\mu + \mu(2-c'(u)) + 1)^{1/2}}{d^{1/2}}, \quad d = \frac{b}{4}.$$

Proof. Arguing by contradiction, suppose for that K_1 there exists $\delta > 0$ such that

$$(1+\|u_n\|)\Phi'_\mu(u_n) \geq 2\delta, \quad \text{for any } u\in N_\delta=\{v\in H : \|v\|^2\leq K_1,\ |\Phi_\mu(v)-c_\mu|\leq\delta\}.$$

Let $I\colon N_\delta \to H$ be the pseudo-gradient vector field for Φ_μ in N_δ, that is, I is locally Lipschitz, $\|I\| \leq 1$, and

$$\Phi'_\mu(u)(I(u)) \geq \|\Phi'_\mu(u)\|^2 \geq \frac{4\delta^2}{(1+\sqrt{K_1})^2}, \quad \text{for any } u \in N_\delta. \tag{2.13}$$

Let $\{\lambda_n\}$be such a sequence in (α,β) that $\mu < \lambda_{n+1} < \lambda_n$, converging to μ, $|\lambda_n - \mu| \leq \delta/8$ and $|c_\mu - c_{\lambda_n}| \leq \delta/8$. For each n, we can take $\gamma_n \in Y$ such that

$$\max_{t\in[0,1]} \Phi_\mu(\gamma_n(t)) \leq c_\mu + (\lambda_n - \mu). \tag{2.14}$$

Consider the open set

$$E_n = \{t \in [0,1] : \Phi_{\lambda_n}(\gamma_n(t)) > c_{\lambda_n} - (\lambda_n - \mu)\},$$

by the definition of c_{λ_n}, E_n is nonempty set. Moreover, if $\nu \in \gamma_n(E_n)$, we show that $\gamma_n(E_n) \subset N_{\delta/2}$. Indeed, from (2.14) we obtain

$$\int_\Omega F(x,u)\,dx = \frac{\Phi_\mu(\nu)-\Phi_{\lambda_n}(\nu)}{\lambda_n-\mu} \leq \frac{c_\mu - c_{\lambda_n}}{\lambda_n-\mu} + 2 = -c'(\mu) + 2 + o_n(1),$$

where $c_\mu - c_{\lambda_n} = (c'(\mu)+o_n(1))(\mu-\lambda_n)$, then

$$\frac{a}{2}\|\nu\|^2 + \frac{b}{4}\|\nu\|^4 < c_\mu + \mu(2-c'(\mu)) + 1. \tag{2.15}$$

Let $d = b/4$ be such that

$$\frac{a}{2}\|\nu\|^2 + \frac{b}{4}\|\nu\|^4 \geq d\|\nu\|^4,$$

for n large, which implies that

$$\|\nu\|^2 \leq \left(\frac{c_\mu + \mu(2-c'(\mu))+1}{d}\right)^{1/2} = K_1. \tag{2.16}$$

On the other hand, we have

$$|\Phi_{\lambda_n}(\nu) - \Phi_\mu(\nu)| = (\lambda_n-\mu)\left|\int_\Omega F(x,\nu)\,dx\right| \leq K_2|\lambda_n-\mu||\Omega|, \tag{2.17}$$

where $K_2 = \max\limits_{\nu\in\gamma_n(E_n)} F(x,\nu)$.

By the definition of E_n, $\Phi_{\lambda_n}(\nu) \geq c_{\lambda_n} - (\lambda_n-\mu)$, and

$$\Phi_\mu(\nu) \leq c_\mu + (\lambda_n - \mu), \tag{2.18}$$

we have

$$c_{\lambda_n} - (K_2|\Omega|+1)(\lambda_n-\mu) \leq \Phi_\mu(\nu) \leq c_\mu + (\lambda_n-\mu),$$

then, for $|c_\mu - c_{\lambda_n}| \le \delta/8$,

$$c_\mu - \frac{\delta}{8} - (K_2|\Omega| + 1)(\lambda_n - \mu) \le \Phi_\mu(\nu) \le c_\mu + (\lambda_n - \mu),$$

so, for n large,

$$c_\mu - \frac{\delta}{2} < \Phi_\mu(\nu) < c_\mu + \frac{\delta}{2}, \quad \text{for all } \nu \in \gamma_n(E_n). \tag{2.19}$$

Now, consider a Lipschitz continuous cut-off function ω, such that $0 \le \omega \le 1$,

$$\omega(u) = \begin{cases} 0 & \text{for } u \notin N_\delta, \\ 1 & \text{for } u \in N_{\delta/2}. \end{cases}$$

Let η be the flow generated by ωI, that is,

$$\frac{\partial \eta}{\partial r}(u,r) = -\omega(\eta(u,r)) I(\eta(u,r)), \quad \text{for any } r, \text{ and } \eta(u,0) = u.$$

By the existence and uniqueness theorem for ordinary differential equation, we have:

(1) if $u \notin N_\delta$, then $\eta(u,r) = u$, for all $r \ge 0$;

(2) if $u \in N_\delta$, then $\eta(u,r) \in N_\delta$, for all $r \ge 0$;

(3) if $u \in H_0^1(\Omega)$, then

$$\Phi'_{\lambda_n}(\eta(u,r))\left(\frac{\partial \eta}{\partial r}(u,r)\right) \le 0, \quad \text{for all } r \ge 0.$$

Indeed, for

$$\langle \Phi'_{\lambda_n}(\eta(u,r)), I(\eta(u,r))\rangle > \frac{\|\Phi'_{\lambda_n}(\eta(u,r))\|^2}{2} \ge \frac{2\delta^2}{1 + \|\eta\|^2} \ge \frac{2\delta^2}{(1+\sqrt{K_1})^2},$$

then

$$\begin{aligned} \frac{\partial \Phi_{\lambda_n}(\eta(u,r))}{\partial r} &= \langle -\Phi'_{\lambda_n}(\eta(u,r)), \omega\,(\eta(u,r)) I(\eta(u,r))\rangle \\ &\le -\frac{2\delta^2}{(1+\sqrt{K_1})^2}\,\omega(\eta(u,r)) \le 0. \end{aligned}$$

(4) if $\eta(u,r) \in N_{\delta/2}$, for all $r < \infty$, then

$$\Phi_{\lambda_n}(\eta(u,r)) \le \Phi_{\lambda_n}(u) - \frac{2\delta^2 r}{(1+\sqrt{K_1})^2}.$$

In fact, if $\eta(u,r) \in N_{\delta/2}$, then $\omega(\eta(u,r)) = 1$,

$$\int_0^r \frac{\partial \Phi_{\lambda_n}(\eta(u,s))}{\partial s}\, ds = \Phi_{\lambda_n}(\eta(u,r)) - \Phi_{\lambda_n}(u),$$

and, for all $r < \infty$, we have

$$\begin{aligned}\int_0^r \frac{\partial \Phi_{\lambda_n}(\eta(u,s))}{\partial s}\,ds &= \int_0^r \Phi'_{\lambda_n}(\eta(u,s))\frac{\partial \eta(u,s)}{\partial s}\,ds \\ &= -\int_0^r \Phi'_{\lambda_n}(\eta(u,s))I(\eta(u,s))\,ds \le -\int_0^r \frac{2\delta^2}{(1+\sqrt{K_1})^2}\,ds \\ &= -\frac{2\delta^2}{(1+\sqrt{K_1})^2} r,\end{aligned}$$

Therefore,

$$\Phi_{\lambda_n}(\eta(u,1)) \le \Phi_{\lambda_n}(u) - \frac{2\delta^2}{(1+\sqrt{K_1})^2}, \quad \text{for all } u \in N_{\delta/2}. \tag{2.20}$$

Take $\|e\|$ sufficiently large, such that $\|e\| > K_1$, then $e \notin N_\delta$, so by (1) we have

$$\eta(e,r) = e, \quad \text{for all } r \ge 0,$$

also take δ sufficiently small, such that $0 \notin N_\delta$, then

$$\eta(0,r) = 0, \quad \text{for all } r \ge 0,$$

thus, for any $r \ge 0$ and $\gamma \in Y$,

$$\eta(\gamma(0),0) = 0, \qquad \eta(\gamma(1),1) = e,$$

as ω and I is continuous, then η is continuous. Therefore we have $\eta(\gamma, r) \in Y$.

Let $h_n(t) = \eta(\gamma_n(t),1)$ is a continuous path in Y such that

$$\Phi_{\lambda_n}(\eta(\gamma_n(t),1)) = \Phi_{\lambda_n}(h_n(t)) \le \Phi_{\lambda_n}(\gamma_n(t)), \quad \text{for all } t \in [0,1],$$

and for its maximum point $t_n \in [0,1]$, it is easy to check that $t_n \in E_n$ and

$$\begin{aligned}c_\mu - o_n(1) = c_{\lambda_n} &\le \max_{t\in[0,1]} \Phi_{\lambda_n}(h_n(t)) \\ &= \Phi_{\lambda_n}(\eta(\gamma_n(t_n)),1)) \le \Phi_{\lambda_n}(\gamma_n(t_n)) - \frac{2\delta^2}{(1+\sqrt{K_1})^2}.\end{aligned}$$

On the other hand, from (2.17) and (2.18), we have $\Phi_{\lambda_n}(\gamma_n(t_n)) \to c_\mu$, which contradicts $\Phi_{\lambda_n}(\gamma_n(t_n)) \ge c_\mu - o_n(1) + 2\delta^2/(1+\sqrt{K_1})^2$. □

Lemma 2.5. *For almost all $\lambda > 0$, c_λ is a critical value of Φ_λ.*

3. Proof of main theorem

Since c_λ is left semi-continuous, from Lemma 2.5, for each $\mu > 0$, we can fix a sequence $\{u_n\}$ in H, and $\{\lambda_n\} \subset \mathbb{R}$ such that $\lambda_n \to \mu$, $c_{\lambda_n} \to c_\mu$, as $n \to \infty$,

$$\Phi_{\lambda_n}(u_n) = c_{\lambda_n} \quad \text{and} \quad \Phi'_{\lambda_n}(u_n) = 0. \tag{3.1}$$

Firstly, we claim that $\{u_n\}$ is bounded. Assume as a contradiction that $\|u_n\| \to \infty$. Let $\xi_n = u_n/\|u_n\|$, as in [16] we will show that $\xi_n \to 0$ in $L^p(\Omega)$,

$n \to \infty$. Without loss of generality, we suppose that there is $\xi \in H$, $h \in L^p(\Omega)$ such that $|\xi_n(x)| \leq h(x)$ almost everywhere in Ω, $\xi_n \to \xi$ in $L^p(\Omega)$ and almost everywhere in Ω.

Let $\Omega_0 = \{x \in \Omega : \xi(x) \neq 0\}$, suppose $|\Omega_0| > 0$, if $x \in \Omega_0$, then

$$\lim_{n\to\infty} \frac{F(x, u_n(x))}{u_n^4(x)}\, \xi_n^4(x) = \infty,$$

for any $M > 0$, we have

$$\int_\Omega \lim_{n\to\infty} \frac{F(x, u_n(x))}{u_n^4}\, \xi_n^4\, dx \geq M|\Omega_0|. \tag{3.2}$$

Since

$$\frac{\Phi_\mu(u_n)}{\mu\|u_n\|^4} = \frac{a}{2\mu\|u_n\|^4} + \frac{b}{4\mu} - \int_\Omega \frac{F(x, u_n)}{\|u_n\|^4}\, dx, \tag{3.3}$$

and

$$\lim_{n\to\infty} \frac{\Phi_\mu(u_n)}{\|u_n\|^4} = 0,$$

then

$$\lim_{n\to\infty} \int_\Omega \frac{F(x, u_n)}{u_n^4}\, \xi_n^4\, dx = \lim_{n\to\infty} \int_\Omega \frac{F(x, u_n)}{\|u_n\|^4}\, dx = \frac{b}{4\mu}. \tag{3.4}$$

Applying the Fatou Lemma, for any $M > 0$, we know that

$$\frac{b}{4\mu} = \lim_{n\to\infty} \int_\Omega \frac{F(x, u_n)}{u_n^4}\, \xi_n^4\, dx \geq \int_\Omega \lim_{n\to\infty} \frac{F(x, u_n)}{u_n^4}\, \xi_n^4\, dx \geq M|\Omega_0|,$$

which is a contradiction. We conclude that Ω_0 has zero measure and $\xi = 0$ almost everywhere in Ω.

Let $t_n \in [0, 1]$ such that

$$\Phi_{\lambda_n}(t_n u_n) = \max_{t\in(0,1]} \Phi_{\lambda_n}(t u_n).$$

Since $\Phi'_{\lambda_n}(t_n u_n)(t_n u_n) = 0$, from ($\mathrm{f}_4$), for any $t \in [0, 1]$, we have

$$\begin{aligned} \Phi_{\lambda_n}(t u_n) &\leq \Phi_{\lambda_n}(t_n u_n) - \frac{1}{4}\, \Phi'_{\lambda_n}(t_n u_n)(t_n u_n) \\ &\leq \frac{a}{4}\|u_n\|^2 + \frac{\lambda_n}{4} \int_\Omega u_n f(x, u_n) - 4F(x, u_n) + C_*\, dx \\ &= \frac{a}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 - \lambda_n \int_\Omega F(x, u_n)\, dx + C_* \lambda_n |\Omega| \\ &= \Phi_{\lambda_n}(u_n) + C_* \lambda_n |\Omega| = c_{\lambda_n} + C_* \lambda_n |\Omega|. \end{aligned} \tag{3.5}$$

On the other hand, for any $R > 0$,

$$\begin{aligned}\Phi_{\lambda_n}(R\xi_n) &= \frac{a\|R\xi_n\|^2}{2} + \frac{b\|R\xi_n\|^4}{4} - \lambda_n \int_\Omega F(x.R\xi_n)\,dx \\ &= \frac{aR^2}{2} + \frac{bR^4}{4} - \lambda_n \int_\Omega F(x.R\xi_n)\,dx \\ &= \frac{aR^2}{2} + \frac{bR^4}{4} + o_n(1),\end{aligned} \tag{3.6}$$

which contradicts $\Phi_{\lambda_n}(R\xi_n) \le c_{\lambda_n} + C_*\lambda_n|\Omega|$, for n large.

Next, we show that $\{u_n\}$ has a convergent subsequence. There exists $\{u_n\}$ such that

$$\Phi_{\lambda_n}(u_n) = c_{\lambda_n} \quad \text{and} \quad (1 + \|u_n\|)\Phi'_{\lambda_n}(u_n) = 0.$$

Since $\|u_n\|$ is bounded, for a subsequence, u_n weakly converges to u in H, strongly in $L^p(\Omega)$, $4 < p < 2^*$, and almost everywhere in Ω. By the definition of Φ', we have

$$\begin{aligned}&\Phi'_{\lambda_n}(u_n)(u - u_n) \\ &\qquad = (a + b\|u_n\|^2)[(u_n, u) - (u_n, u_n)] - \lambda_n \int_\Omega f(x, u_n)(u - u_n)\,dx.\end{aligned} \tag{3.7}$$

Noticing that $q = p/(p - 1)$, by (f_1), there exist $C_4, C_5 > 0$, such that

$$\begin{aligned}\bigg(\int_\Omega |f(x, u_n)|^q\,dx\bigg)^{1/q} &\le C\bigg(\int_\Omega (1 + |u_n|^{p-1})^q\,dx\bigg)^{1/q} \\ &\le C\bigg[\int_\Omega 2^q(1 + |u_n|^{q(p-1)})\,dx\bigg]^{1/q} \le 4C(|\Omega|^{1/q} + C_4\|u_n\|^{p/q}) = C_5,\end{aligned}$$

which implies that

$$\int_\Omega |f(x, u_n)(u - u_n)|\,dx \to 0, \tag{3.8}$$

as $n \to +\infty$. Since $(1 + \|u_n\|)\Phi'_{\lambda_n}(u_n) = 0$ and $\lambda_n \to \mu$, as $n \to +\infty$, by (3.7) and (3.8), we obtain $(u_n, u_n) \to (u, u)$, as $n \to +\infty$, thus, $u_n \to u$ in H.

Finally, we prove

$$\Phi_\mu(u_n) \to c_\mu, \quad (1 + \|u_n\|)\Phi'_\mu(u_n) \to 0.$$

Since $\lambda_n \to \mu$, $c_{\lambda_n} \to c_\mu$, then we have

$$\begin{aligned}|\Phi_\mu(u_n) - c_\mu| &= |\Phi_\mu(u_n) - \Phi_{\lambda_n}(u_n) + c_{\lambda_n} - c_\mu| \\ &\le |\Phi_\mu(u_n) - \Phi_{\lambda_n}(u_n)| + |c_{\lambda_n} - c_\mu| \\ &\le |\lambda_n - \mu|\bigg|\int_\Omega F(x, u_n)\,dx\bigg| + |c_{\lambda_n} - c_\mu| \to 0.\end{aligned}$$

On the other hand,

$$\begin{aligned}(3.9)\quad \|\Phi'_{\lambda_n}(u_n)-\Phi'_\mu(u_n)\| &= \sup_{\|v\|\le 1}|(\Phi'_{\lambda_n}(u_n)-\Phi'_\mu(u_n),v)| \\ &= \sup_{\|v\|\le 1}|(\Phi'_{\lambda_n}(u_n),v)-(\Phi'_\mu(u_n),v)| \le \sup_{\|v\|\le 1}|\lambda_n-\mu|\bigg|\int_\Omega f(x,u_n)v\,dx\bigg|,\end{aligned}$$

and

$$\begin{aligned}(3.10)\quad \bigg|\int_\Omega f(x,u_n)v\,dx\bigg| &\le \int_\Omega C(1+|u_n|^{p-1})|v|\,dx \\ &\le C_1\|v\|+C\bigg(\int_\Omega |u_n|^{(p-1)q}\,dx\bigg)^{1/q}\bigg(\int_\Omega |v|^p\,dx)^{1/p}\le C_3,\end{aligned}$$

by (3.9) and (3.10), we have $(1+\|u_n\|)\Phi'_\mu(u_n)\to 0$. The proof is done. □

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