Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 37, 2011, 357–376

ROBUSTNESS OF NONUNIFORM POLYNOMIAL DICHOTOMIES FOR DIFFERENCE EQUATIONS

Luis Barreira — Meng Fan — Claudia Valls — Jimin Zhang

Abstract. For a nonautonomous dynamics with discrete time defined by a sequence of linear operators in a Banach space, we establish the robustness of polynomial contractions and of polynomial dichotomies under sufficiently small linear perturbations. In addition, we consider the general case of nonuniform polynomial behavior.

1. Introduction

We consider in this paper the robustness problem for difference equations defined by a sequence of linear operators in a Banach space, or equivalently for a nonautonomous dynamics with discrete time. In loose terms, the problem asks whether the behavior of a dichotomy does not change much under sufficiently small linear perturbations. Our main aim is to show that a relatively weak form of dichotomy, which we call polynomial dichotomy, persists under sufficiently small perturbations of the original dynamics. We also consider the general case of nonuniform polynomial behavior.

The notion of exponential dichotomy, essentially introduced in seminal work of O. Perron [13], plays a central role in a substantial part of the theory of

2010 *Mathematics Subject Classification.* 34D09, 34D10, 37D25.
Key words and phrases. Nonuniform polynomial dichotomy, robustness.
L. Barreira and C. Valls were partially supported by FCT through CAMGSD, Lisbon. M. Fan and J. Zhang were supported by the NSFC and NCET-08-0755.

differential equations and dynamical systems, particularly in what concerns the study of stable and unstable invariant manifolds. Although strictly speaking the notion is not introduced in [13], together with Hadamard's work on the geodesic flow on surfaces of negative curvature, the paper can be considered one of main original sources for the study of hyperbolicity. In particular, it may be considered the first source for the study of robustness, via the the notion of admissibility. Due to the role played by the notion of exponential dichotomy, it is not surprising that the study of robustness has a long history. In particular, the problem was discussed by J. Massera and J. Schäffer [9] (see also [10]), W. Coppel [7], and in the case of Banach spaces by Ju. Dalec'kiĭ and M. Kreĭn [8]. For more recent works we refer to [2], [6], [11], [15], [16] and the references therein.

In this paper we consider a notion of polynomial dichotomy mimicking a corresponding notion of contraction introduced in [4], now with rates of expansion and contraction varying polynomially instead of exponentially. We note that it follows from results in that paper that the notion of nonuniform polynomial dichotomy occurs naturally, in fact being related to the nonvanishing of a certain Lyapunov exponent. To formulate a rigorous statement we first introduce the notion of polynomial dichotomy in a particular case.

Let $B(X)$ be the space of bounded linear operators in a Banach space X. For simplicity of the exposition, we assume that there is a decomposition $X = E \oplus F$. Moreover, given a sequence $(A_m)_{m\in\mathbb{N}} \subset B(X)$ of invertible operators we assume that

$$A_m = \begin{pmatrix} B_m & 0 \\ 0 & C_m \end{pmatrix}$$

with respect to the above decomposition. We say that the sequence $(A_m)_{m\in\mathbb{N}}$ admits a *nonuniform polynomial dichotomy* if there exist constants $a < 0$, $\varepsilon \geq 0$ and $K > 0$ such that

$$\begin{aligned} \|B_{m-1} \dots B_n\| &\leq K(m/n)^a n^\varepsilon, \quad m > n, \\ \|C_m^{-1} \dots C_{n-1}^{-1}\| &\leq K(n/m)^a n^\varepsilon, \quad m < n. \end{aligned} \tag{1.1}$$

We also introduce a natural notion of Lyapunov exponent in the present context. Namely, the *polynomial Lyapunov exponent* of a vector $v \in X$ (with respect to the sequence $(A_m)_{m\in\mathbb{N}}$) is defined by

$$\lambda(v) = \limsup_{n\to\infty} \frac{\log \|A_n \dots A_1 v\|}{\log n}.$$

One can easily verify that if the sequence $(A_m)_{m\in\mathbb{N}}$ admits a nonuniform polynomial dichotomy, then

$$\lambda|(E \setminus \{0\}) < 0 \quad \text{and} \quad \lambda|(F \setminus \{0\}) > 0. \tag{1.2}$$

On the other hand, it follows from results in [4] that if condition (1.2) holds, and the Lyapunov exponents of the sequences A_m and $(A_m^*)^{-1}$ are finite for nonzero vectors, then the sequence $(A_m)_{m\in\mathbb{N}}$ admits a nonuniform polynomial dichotomy. In a certain sense, this shows that the notion of nonuniform polynomial dichotomy occurs naturally, in the sense that it can be deduced from the nonvanishing of an appropriate Lyapunov exponent.

We emphasize that we also consider the general case of nonuniform polynomial behavior. Indeed, the constant ε in (1.1) may be positive. It turns out that the classical notion of (uniform) exponential dichotomy is very stringent for the dynamics and it is of interest to look for more general types of hyperbolic behavior. This is precisely what happens with the notion of nonuniform exponential behavior. We refer to [1] for a detailed exposition of the theory, which goes back to the landmark works of V. Oseledets [12] and Pesin [14]. In particular, the notion of nonuniform hyperbolicity plays an important role in the construction of stable and unstable invariant manifolds (see [14], [17], [18]). We refer to [1], [3] for related discussions.

We note that a different notion of nonuniform polynomial dichotomy was introduced in [5]. It corresponds to replace the terms $(m/n)^a$ and $(n/m)^a$ in (1.1), respectively by $(1+m-n)^a$ and $(1+n-m)^a$. However, since

$$m/n = 1 + (m-n)/n \le 1 + m - n \quad \text{for } m \ge n,$$

and

$$n/m \le 1 + (n-m)/m \le 1 + n - m \quad \text{for } m \le n,$$

our notion is less restrictive (recall that $a < 0$). Moreover, as explained above, the inequalities in (1.1) occur naturally, in the sense that they can be derived from the nonvanishing of a Lyapunov exponent. To the best of our understanding the corresponding notion of dichotomy in [5] has no corresponding motivation.

2. Robustness of polynomial contractions

We consider in this section the simpler problem of the robustness of nonuniform polynomial contractions, asking whether a nonuniform polynomial contraction persists under sufficiently small linear perturbations.

Let again $B(X)$ be the space of bounded linear operators in a Banach space X. Given a sequence $(A_m)_{m\in\mathbb{N}} \subset B(X)$, we define

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\dots A_n & \text{if } m > n, \\ \mathrm{id} & \text{if } m = n. \end{cases}$$

Following [4], the sequence $(A_m)_{m\in\mathbb{N}}$ is said to admit a *nonuniform polynomial contraction* if there exist constants $a < 0$, $\varepsilon \ge 0$ and $K > 0$ such that

$$\|\mathcal{A}(m,n)\| \le K(m/n)^a n^\varepsilon, \quad m \ge n. \tag{2.1}$$

We also consider the perturbed dynamics

$$x_{m+1} = (A_m + B_m)x_m \quad m \in \mathbb{N}, \tag{2.2}$$

and we define

$$\widetilde{\mathcal{A}}(m,n) = \begin{cases} (A_{m-1}+B_{m-1})\dots(A_n+B_n) & \text{if } m > n, \\ \text{id}, & \text{if } m = n. \end{cases}$$

The following is our robustness result for contractions.

Theorem 2.1. *Assume that $(A_m)_{m\in\mathbb{N}}$ admits a nonuniform polynomial contraction and that there exist constants $\eta, \rho > 0$ such that $\|B_m\| \le \eta m^{-\rho}$ for $m \in \mathbb{N}$. If $\rho > \varepsilon + 1$ and η is sufficiently small, then $(A_m + B_m)_{m\in\mathbb{N}}$ admits a nonuniform polynomial contraction with*

$$\|\widetilde{\mathcal{A}}(m,n)\| \le \frac{K}{1 - K\eta 2^{\varepsilon - a}\zeta(\rho-\varepsilon)}(m/n)^a n^\varepsilon, \quad m \ge n, \tag{2.3}$$

where ζ is the zeta function.

Proof. For each $n \in \mathbb{N}$ we consider the space

$$\Omega_0 := \{\widetilde{\mathcal{A}} = (\widetilde{\mathcal{A}}(m,n))_{m\ge n} : \|\widetilde{\mathcal{A}}\|_0 < \infty\},$$

with the norm

$$\|\widetilde{\mathcal{A}}\|_0 := \sup\bigg\{\frac{\|\widetilde{\mathcal{A}}(m,n)\|}{(m/n)^a n^\varepsilon} : m \ge n\bigg\}.$$

It is not difficult to verify that Ω_0 is a Banach space. We define an operator T_0 in Ω_0 by

$$(T_0\widetilde{\mathcal{A}})(m,n) = \mathcal{A}(m,n) + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1)B_k\widetilde{\mathcal{A}}(k,n).$$

It follows from (2.1) that

$$\begin{aligned} \|(T_0\widetilde{\mathcal{A}})(m,n)\| &\le \|\mathcal{A}(m,n)\| + \sum_{k=n}^{m-1} \|\mathcal{A}(m,k+1)\|\cdot\|B_k\|\cdot\|\widetilde{\mathcal{A}}(k,n)\| \\ &\le K(m/n)^a n^\varepsilon + K\eta\sum_{k=n}^{m-1}(m/(k+1))^a(k+1)^\varepsilon k^{-\rho}(k/n)^a n^\varepsilon\|\widetilde{\mathcal{A}}\|_0 \\ &\le K(m/n)^a n^\varepsilon + K\eta(m/n)^a n^\varepsilon\sum_{k=n}^{m-1} k^{a-\rho}(k+1)^{\varepsilon-a}\|\widetilde{\mathcal{A}}\|_0 \\ &\le K(m/n)^a n^\varepsilon + K\eta 2^{\varepsilon-a}(m/n)^a n^\varepsilon\sum_{k=n}^{m-1} k^{\varepsilon-\rho}\|\widetilde{\mathcal{A}}\|_0, \end{aligned}$$

and hence

$$\|T_0\widetilde{\mathcal{A}}\|_0 \le K + K\eta 2^{\varepsilon-a}\zeta(\rho-\varepsilon)\|\widetilde{\mathcal{A}}\|_0 < \infty. \tag{2.4}$$

This shows that the operator $T_0\colon \Omega_0 \to \Omega_0$ is well-defined. Moreover, for each $\widetilde{\mathcal{A}}_1, \widetilde{\mathcal{A}}_2 \in \Omega_0$ and $m \geq n$, we have

$$\begin{aligned}
&\|(T_0\widetilde{\mathcal{A}}_1)(m,n) - (T_0\widetilde{\mathcal{A}}_2)(m,n)\| \\
&\qquad \leq \sum_{k=n}^{m-1} \|\mathcal{A}(m,k+1)\| \cdot \|B_k\| \cdot \|\widetilde{\mathcal{A}}_1(k,n) - \widetilde{\mathcal{A}}_2(k,n)\| \\
&\qquad \leq K\eta \sum_{k=n}^{m-1} (m/(k+1))^a (k+1)^\varepsilon k^{-\rho} (k/n)^a n^\varepsilon \|\widetilde{\mathcal{A}}_1 - \widetilde{\mathcal{A}}_2\|_0 \\
&\qquad \leq K\eta (m/n)^a n^\varepsilon \sum_{k=n}^{m-1} (k+1)^{\varepsilon-a} k^{a-\rho} \\
&\qquad \leq K\eta 2^{\varepsilon-a} \zeta(\rho-\varepsilon)(m/n)^a n^\varepsilon \|\widetilde{\mathcal{A}}_1 - \widetilde{\mathcal{A}}_2\|_0,
\end{aligned}$$

and hence

$$\|T_0\widetilde{\mathcal{A}}_1 - T_0\widetilde{\mathcal{A}}_2\|_0 \leq K\eta 2^{\varepsilon-a}\zeta(\rho-\varepsilon)\|\widetilde{\mathcal{A}}_1 - \widetilde{\mathcal{A}}_2\|_0.$$

Provided that η is sufficiently small, the operator T_0 is a contraction. Therefore, there exists a unique $\widetilde{\mathcal{A}} \in \Omega_0$ such that $T_0\widetilde{\mathcal{A}} = \widetilde{\mathcal{A}}$, and one can easily verify that it is a solution of (2.2). Identity (2.3) follows readily from (2.4). This completes the proof of the theorem. □

3. Robustness of polynomial dichotomies

We consider in this section the more general case of nonuniform polynomial dichotomies, and the related robustness problem. In particular, we establish the continuous dependence with the perturbation of the constants in the notion of nonuniform polynomial dichotomy.

Given a sequence $(A_m)_{m\in\mathbb{N}} \subset B(X)$ of invertible operators, we define

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\dots A_n & \text{if } m > n, \\ \mathrm{id} & \text{if } m = n, \\ A_m^{-1}\dots A_{n-1}^{-1} & \text{if } m < n. \end{cases}$$

We say that the sequence $(A_m)_{m\in\mathbb{N}}$ admits a *nonuniform polynomial dichotomy* if there exist projections $P_n\colon X \to X$ for $n \in \mathbb{N}$ such that

$$P_m\mathcal{A}(m,n) = \mathcal{A}(m,n)P_n, \quad m,n \in \mathbb{N},$$

and there exist constants $a < 0 < b$, $\varepsilon \geq 0$ and $K > 0$ such that

$$\begin{aligned} \|\mathcal{A}(m,n)P_n\| &\leq K(m/n)^a n^\varepsilon, \quad m \geq n, \\ \|\mathcal{A}(m,n)Q_n\| &\leq K(n/m)^{-b} n^\varepsilon, \quad m \leq n, \end{aligned} \tag{3.1}$$

where $Q_n = \mathrm{id} - P_n$ is the complementary projection of P_n.

We also consider the perturbed dynamics (2.2), and we define

$$\widetilde{\mathcal{A}}(m,n)=\begin{cases}(A_{m-1}+B_{m-1})\dots(A_n+B_n) & \text{if } m>n,\\ \mathrm{id} & \text{if } m=n,\\ (A_m+B_m)^{-1}\dots(A_{n-1}+B_{n-1})^{-1} & \text{if } m<n,\end{cases}$$

whenever the inverses are well-defined.

The following is our main robustness result.

Theorem 3.1. *Assume that $(A_m)_{m\in\mathbb{N}}$ admits a nonuniform polynomial dichotomy and that there exist constants $\eta,\rho>0$ such that $\|B_m\|\le\eta m^{-\rho}$ for $m\in\mathbb{N}$. If*

$$\rho>2\varepsilon+1,\quad \min\{-a,b\}>\varepsilon,$$

and η is sufficiently small, then $(A_m+B_m)_{m\in\mathbb{N}}$ admits a nonuniform polynomial dichotomy. Namely, there exist projections $\widetilde{P}_m$ for $m\in\mathbb{N}$ such that

$$\widetilde{P}_m\widetilde{\mathcal{A}}(m,n)=\widetilde{\mathcal{A}}(m,n)\widetilde{P}_n,\quad m,n\in\mathbb{N},\tag{3.2}$$

and

$$\begin{aligned}\|\widetilde{\mathcal{A}}(m,n)\widetilde{P}_n\| &\le \frac{2K\widetilde{K}_1}{1-2\widehat{K}}(m/n)^a n^{2\varepsilon}, & m\ge n,\\ \|\widetilde{\mathcal{A}}(m,n)\widetilde{Q}_n\| &\le \frac{2K\widetilde{K}_2}{1-2\widehat{K}}(n/m)^{-b} n^{2\varepsilon}, & m\le n,\end{aligned}$$

where $\widetilde{Q}_n=\mathrm{id}-\widetilde{P}_n$ for each $n\in\mathbb{N}$, and where

$$\begin{aligned}\widetilde{K}_1 &= \frac{K}{1-2^\varepsilon K\eta(2^{-a}+1)\zeta(\rho-\varepsilon)},\\ \widetilde{K}_2 &= \frac{K}{1-2^\varepsilon K\eta\zeta(\rho-\varepsilon)},\\ \widehat{K} &= 2^\varepsilon K\eta\zeta(\rho-2\varepsilon)(\widetilde{K}_1+\widetilde{K}_2).\end{aligned}\tag{3.3}$$

Proof. We separate the proof into several steps.

Step 1. Construction of bounded solutions of equation (2.2).

For each $n\in\mathbb{N}$ we consider the spaces

$$\begin{aligned}\Omega_1 &= \{U=U(m,n)_{m\ge n}\subset B(X):\|U\|_1<\infty\},\\ \Omega_2 &= \{V=V(m,n)_{m\le n}\subset B(X):\|V\|_2<\infty\},\end{aligned}$$

respectively with the norms

$$\|U\|_1=\sup\bigg\{\frac{\|U(m,n)\|}{(m/n)^a n^\varepsilon}:m\ge n\bigg\},\quad \|V\|_2=\sup\bigg\{\frac{\|V(m,n)\|}{(n/m)^{-b} n^\varepsilon}:m\le n\bigg\}.$$

One can easily verify that Ω_1 and Ω_2 are Banach spaces.

Lemma 3.2. *For each $n \in \mathbb{N}$, equation* (2.2) *has a unique solution $U \in \Omega_1$ satisfying, for $m \geq n$,*

$$U(m,n) = \mathcal{A}(m,n)P_n + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_k U(k,n) - \sum_{k=m}^{\infty} \mathcal{A}(m,k+1)Q_{k+1}B_k U(k,n). \tag{3.4}$$

Proof. One can easily verify that any sequence $(U(m,n))_{m\geq n}$ satisfying (3.4) is a solution of equation (2.2). We define an operator T_1 in Ω_1 by

$$(T_1U)(m,n) = \mathcal{A}(m,n)P_n + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_k U(k,n) - \sum_{k=m}^{\infty} \mathcal{A}(m,k+1)Q_{k+1}B_k U(k,n).$$

It follows from (3.1) that

$$\begin{aligned}
&\|\mathcal{A}(m,n)P_n\| + \sum_{k=n}^{m-1} \|\mathcal{A}(m,k+1)P_{k+1}\| \cdot \|B_k\| \cdot \|U(k,n)\| \\
&\quad + \sum_{k=m}^{\infty} \|\mathcal{A}(m,k+1)Q_{k+1}\| \cdot \|B_k\| \cdot \|U(k,n)\| \\
&\leq K(m/n)^a n^\varepsilon + K\eta\bigg(\sum_{k=n}^{m-1} (m/(k+1))^a (k+1)^\varepsilon k^{-\rho} (k/n)^a n^\varepsilon \\
&\quad + \sum_{k=m}^{\infty} ((k+1)/m)^{-b} (k+1)^\varepsilon k^{-\rho} (k/n)^a n^\varepsilon\bigg)\|U\|_1 \\
&\leq K(m/n)^a n^\varepsilon + K\eta\bigg(\sum_{k=n}^{m-1} (k/(k+1))^a (k+1)^\varepsilon k^{-\rho} \\
&\quad + \sum_{k=m}^{\infty} ((k+1)/m)^{-b} (k/m)^a (k+1)^\varepsilon k^{-\rho}\bigg)(m/n)^a n^\varepsilon \|U\|_1 \\
&\leq K(m/n)^a n^\varepsilon + K\eta\bigg(\sum_{k=n}^{m-1} (k+1)^{\varepsilon-a} k^{a-\rho} \\
&\quad + \sum_{k=m}^{\infty} (k+1)^\varepsilon k^{-\rho}\bigg)(m/n)^a n^\varepsilon \|U\|_1 \\
&\leq K(m/n)^a n^\varepsilon + 2^\varepsilon K\eta(2^{-a}+1)\zeta(\rho-\varepsilon)(m/n)^a n^\varepsilon \|U\|_1
\end{aligned} \tag{3.5}$$

for each $m \geq n$. This shows that T_1U is well-defined, and that

$$\|T_1U\|_1 \leq K + 2^\varepsilon K\eta(2^{-a}+1)\zeta(\rho-\varepsilon)\|U\|_1 < \infty. \tag{3.6}$$

Therefore, we obtain an operator $T_1\colon \Omega_1 \to \Omega_1$. Moreover, for each $U_1, U_2 \in \Omega_1$ and $m \geq n$, we have

$$\begin{aligned}
\|(T_1U_1)(m,n) &- (T_1U_2)(m,n)\| \\
&\leq \sum_{k=n}^{m-1} \|\mathcal{A}(m,k+1)P_{k+1}\| \cdot \|B_k\| \cdot \|U_1(k,n) - U_2(k,n)\| \\
&\quad + \sum_{k=m}^{\infty} \|\mathcal{A}(m,k+1)Q_{k+1}\| \cdot \|B_k\| \cdot \|U_1(k,n) - U_2(k,n)\| \\
&\leq K\eta\bigg(\sum_{k=n}^{m-1} (m/(k+1))^a (k+1)^\varepsilon k^{-\rho} (k/n)^a n^\varepsilon \\
&\quad + \sum_{k=m}^{\infty} ((k+1)/m)^{-b} (k+1)^\varepsilon k^{-\rho} (k/n)^a n^\varepsilon\bigg) \|U_1 - U_2\|_1 \\
&\leq 2^\varepsilon K\eta(2^{-a}+1)\zeta(\rho-\varepsilon)(m/n)^a n^\varepsilon \|U_1 - U_2\|_1,
\end{aligned}$$

and hence

$$\|T_1U_1 - T_1U_2\|_1 \leq 2^\varepsilon K\eta(2^{-a}+1)\zeta(\rho-\varepsilon)\|U_1 - U_2\|_1. \tag{3.7}$$

Provided that η is sufficiently small, the operator T_1 is a contraction, and there exists a unique $U \in \Omega_1$ such that $T_1U = U$. □

Lemma 3.3. *For each $n \in \mathbb{N}$, equation* (2.2) *has a unique solution $V \in \Omega_2$ satisfying, for $m \leq n$,*

$$\begin{aligned}
V(m,n) = \mathcal{A}(m,n)Q_n &+ \sum_{k=1}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_kV(k,n) \\
&- \sum_{k=m}^{n-1} \mathcal{A}(m,k+1)Q_{k+1}B_kV(k,n).
\end{aligned} \tag{3.8}$$

Proof. Again one can easily verify that any sequence $(V(m,n))_{m\leq n}$ satisfying (3.8) is a solution of equation (2.2). We define an operator T_2 in Ω_2 by

$$\begin{aligned}
(T_2V)(m,n) = \mathcal{A}(m,n)Q_n &+ \sum_{k=1}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_kV(k,n) \\
&- \sum_{k=m}^{n-1} \mathcal{A}(m,k+1)Q_{k+1}B_kV(k,n).
\end{aligned}$$

By (3.1), for each $m \le n$, we have

$$\begin{aligned}
\|\mathcal{A}(m,n)Q_n\| &+ \sum_{k=1}^{m-1} \|\mathcal{A}(m,k+1)P_{k+1}\| \cdot \|B_k\| \cdot \|V(k,n)\| \\
&+ \sum_{k=m}^{n-1} \|\mathcal{A}(m,k+1)Q_{k+1}\| \cdot \|B_k\| \cdot \|V(k,n)\| \\
&\le K(n/m)^{-b} n^{\varepsilon} + K\eta \biggl(\sum_{k=1}^{m-1} (m/(k+1))^a (k+1)^{\varepsilon} k^{-\rho} (n/k)^{-b} n^{\varepsilon} \\
&+ \sum_{k=m}^{n-1} ((k+1)/m)^{-b} (k+1)^{\varepsilon} k^{-\rho} (n/k)^{-b} n^{\varepsilon} \biggr) \|V\|_2 \\
&\le K(n/m)^{-b} n^{\varepsilon} + K\eta \biggl(\sum_{k=1}^{m-1} (m/(k+1))^a (m/k)^{-b} (k+1)^{\varepsilon} k^{-\rho} \\
&+ \sum_{k=m}^{n-1} ((k+1)/k)^{-b} (k+1)^{\varepsilon} k^{-\rho} \biggr) (n/m)^{-b} n^{\varepsilon} \|V\|_2 \\
&\le K(n/m)^{-b} n^{\varepsilon} + K\eta \biggl(\sum_{k=1}^{m-1} (k+1)^{\varepsilon} k^{-\rho} \\
&+ \sum_{k=m}^{n-1} (k+1)^{\varepsilon} k^{-\rho} \biggr) (n/m)^{-b} n^{\varepsilon} \|V\|_2 \\
&\le K(n/m)^{-b} n^{\varepsilon} + 2^{\varepsilon} K\eta \zeta(\rho-\varepsilon)(n/m)^{-b} n^{\varepsilon} \|V\|_2.
\end{aligned}$$

This shows that T_2V is well-defined, and that

$$\|T_2V\|_2 \le K + 2^{\varepsilon} K\eta\zeta(\rho-\varepsilon)\|V\|_2 < \infty. \tag{3.9}$$

Hence, we obtain an operator $T_2 \colon \Omega_2 \to \Omega_2$. For each $V_1, V_2 \in \Omega_2$ and $m \le n$, we have

$$\begin{aligned}
\|(T_2V_1)(m,n) &- (T_2V_2)(m,n)\| \\
&\le \sum_{k=1}^{m-1} \|\mathcal{A}(m,k+1)P_{k+1}\| \cdot \|B_k\| \cdot \|V_1(k,n) - V_2(k,n)\| \\
&\quad + \sum_{k=m}^{n-1} \|\mathcal{A}(m,k+1)Q_{k+1}\| \cdot \|B_k\| \cdot \|V_1(k,n) - V_2(k,n)\| \\
&\le K\eta \biggl(\sum_{k=1}^{m-1} (m/(k+1))^a (k+1)^{\varepsilon} k^{-\rho} (n/k)^{-b} n^{\varepsilon} \\
&\quad + \sum_{k=m}^{n-1} ((k+1)/m)^{-b} (k+1)^{\varepsilon} k^{-\rho} (n/k)^{-b} n^{\varepsilon} \biggr) \|V_1 - V_2\|_2 \\
&\le 2^{\varepsilon} K\eta\zeta(\rho-\varepsilon)(n/m)^{-b} n^{\varepsilon} \|V_1 - V_2\|_2,
\end{aligned}$$

and hence,

$$\|T_2V_1 - T_2V_2\|_2 \le 2^\varepsilon K\eta\zeta(\rho-\varepsilon)\|V_1 - V_2\|_2. \tag{3.10}$$

For η sufficiently small the operator T_2 is a contraction, and there exists a unique $V \in \Omega_2$ such that $T_2V = V$. □

Step 2. Properties of the bounded solutions.

Lemma 3.4. *We have $U(m,l)U(l,n) = U(m,n)$ for each $m \ge l \ge n$.*

Proof. It follows from (3.4) that

$$\begin{aligned} U(m,l)U(l,n) = {} & \mathcal{A}(m,n)P_n + \sum_{k=n}^{l-1} \mathcal{A}(m,k+1)P_{k+1}B_kU(k,n) \\ & + \sum_{k=l}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_kU(k,l)U(l,n) \\ & - \sum_{k=m}^{\infty} \mathcal{A}(m,k+1)Q_{k+1}B_kU(k,l)U(l,n). \end{aligned}$$

For a fixed l, writing $h(m,l) = U(m,l)U(l,n) - U(m,n)$ for $m \ge l$, we obtain $L_1h = h$, where

$$(L_1H)(m,l) = \sum_{k=l}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_kH(k,l) - \sum_{k=m}^{\infty} \mathcal{A}(m,k+1)Q_{k+1}B_kH(k,l)$$

for each $H \in \Omega_1^l$ and $m \ge l$, where Ω_1^l is obtained from Ω_1 replacing n by l. It follows from (3.5) that L_1 is well-defined. Moreover,

$$\begin{aligned} \|(L_1H)(m,l)\| \le {} & \sum_{k=l}^{m-1} \|\mathcal{A}(m,k+1)P_{k+1}B_kH(k,l)\| \\ & + \sum_{k=m}^{\infty} \|\mathcal{A}(m,k+1)Q_{k+1}B_kH(k,l)\| \\ \le {} & K\eta\bigg(\sum_{k=l}^{m-1} (m/(k+1))^a(k+1)^\varepsilon k^{-\rho}(k/l)^a l^\varepsilon \\ & + \sum_{k=m}^{\infty} ((k+1)/m)^{-b}(k+1)^\varepsilon k^{-\rho}(k/l)^a l^\varepsilon\bigg)\|H\|_1 \\ \le {} & 2^\varepsilon K\eta(2^{-a}+1)\zeta(\rho-\varepsilon)(m/l)^a l^\varepsilon\|H\|_1, \end{aligned}$$

that is,

$$\|L_1H\|_1 \le 2^\varepsilon K\eta(2^{-a}+1)\zeta(\rho-\varepsilon)\|H\|_1 < \infty.$$

We thus obtain an operator $L_1\colon \Omega_1^l \to \Omega_1^l$. For each $H_1, H_2 \in \Omega_1^l$ and $m \ge l$, we have

$$\begin{aligned}
\|(L_1H_1)(m,l) &- (L_1H_2)(m,l)\| \\
&\le \sum_{k=l}^{m-1} \|\mathcal{A}(m,k+1)P_{k+1}\| \cdot \|B_k\| \cdot \|H_1(k,l) - H_2(k,l)\| \\
&\quad + \sum_{k=m}^{\infty} \|\mathcal{A}(m,k+1)Q_{k+1}\| \cdot \|B_k\| \cdot \|H_1(k,l) - H_2(k,l)\| \\
&\le K\eta\biggl(\sum_{k=l}^{m-1} (m/(k+1))^a (k+1)^\varepsilon k^{-\rho} (k/l)^a l^\varepsilon \\
&\quad + \sum_{k=m}^{\infty} ((k+1)/m)^{-b} (k+1)^\varepsilon k^{-\rho} (k/l)^a l^\varepsilon\biggr) \|H_1 - H_2\|_1 \\
&\le 2^\varepsilon K\eta(2^{-a}+1)\zeta(\rho-\varepsilon)(m/l)^a l^\varepsilon \|H_1 - H_2\|_1.
\end{aligned}$$

Therefore,

$$\|L_1H_1 - L_1H_2\|_1 \le 2^\varepsilon K\eta(2^{-a}+1)\zeta(\rho-\varepsilon)\|H_1 - H_2\|_1,$$

and for η sufficiently small there exists a unique $H \in \Omega_1^l$ such that $L_1H = H$. Since $0 \in \Omega_1^l$ also satisfies this identity, we have $H = 0$. Moreover, since $h \in \Omega_1^l$ we conclude that $h = 0$. $\square$

LEMMA 3.5. *We have $V(m,l)V(l,n) = V(m,n)$ for each $m \le l \le n$.*

PROOF. The argument is analogous to that in the proof of Lemma 3.4. By (3.8), we have

$$\begin{aligned}
V(m,l)V(l,n) = {}&\mathcal{A}(m,n)Q_n - \sum_{k=l}^{n-1} \mathcal{A}(m,k+1)Q_{k+1}B_kV(k,n) \\
&+ \sum_{k=1}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_kV(k,l)V(l,n) \\
&- \sum_{k=m}^{l-1} \mathcal{A}(m,k+1)Q_{k+1}B_kV(k,l)V(l,n).
\end{aligned}$$

Now set $\overline{h}(m,l) = V(m,l)V(l,n) - V(m,n)$ for each $m \le l$. Then $L_2\overline{h} = \overline{h}$, where

$$(L_2\overline{H})(m,l) = \sum_{k=1}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_k\overline{H}(k,l) - \sum_{k=m}^{l-1} \mathcal{A}(m,k+1)Q_{k+1}\overline{H}(k,l)$$

for each $\overline{H} \in \Omega_2^l$ and $m \le l$, where Ω_2^l is obtained from Ω_2 replacing n by l. Proceeding in a similar manner to that in the proof of Lemma 3.4, one can show that 0 is the unique fixed point of L_2 in Ω_2^l, and since $\overline{h} \in \Omega_2^l$ we conclude that $\overline{h} = 0$. $\square$

Step 3. Construction of the projections $\widetilde{P}_m$ in (3.2).

Given $p \in \mathbb{N}$ we define

$$\overline{P}_m = \widetilde{\mathcal{A}}(m,p)U(p,p)\widetilde{\mathcal{A}}(p,m), \quad \overline{Q}_m = \widetilde{\mathcal{A}}(m,p)V(p,p)\widetilde{\mathcal{A}}(p,m) \tag{3.11}$$

for each $m \in \mathbb{N}$. We emphasize that the operators $\overline{P}_m$ and $\overline{Q}_m$ may depend on p. It follows from Lemmas 3.4 and 3.5 that:

(a) $\overline{P}_m$ and $\overline{Q}_m$ are projections for each $m \in \mathbb{N}$;
(b) $\overline{P}_m\widetilde{\mathcal{A}}(m,n) = \widetilde{\mathcal{A}}(m,n)\overline{P}_n$ and $\overline{Q}_m\widetilde{\mathcal{A}}(m,n) = \widetilde{\mathcal{A}}(m,n)\overline{Q}_n$ for each m,n in $\mathbb{N}$.

Moreover, since

$$\overline{P}_p = U(p,p) = P_p - \sum_{k=p}^{\infty} \mathcal{A}(p,k+1)Q_{k+1}B_kU(k,p) \tag{3.12}$$

$$\overline{Q}_p = V(p,p) = Q_p + \sum_{k=1}^{p-1} \mathcal{A}(p,k+1)P_{k+1}B_kV(k,p), \tag{3.13}$$

we obtain:

(c) $P_p\overline{P}_p = P_p$, $Q_p\overline{Q}_p = Q_p$, $Q_p(\mathrm{id} - \overline{P}_p) = \mathrm{id} - \overline{P}_p$, $P_p(\mathrm{id} - \overline{Q}_p) = \mathrm{id} - \overline{Q}_p$.

We also note that $\widetilde{U}(m,p) = U(m,p)P_p$ satisfies identity (3.4) with $n = p$. Since $\widetilde{U} \in \Omega_1$, it follows from the uniqueness in Lemma 3.2 that $U(m,p)P_p = U(m,p)$. Similarly, $\widetilde{V}(m,p) = V(m,p)Q_p$ satisfies identity (3.8) and the uniqueness in Lemma 3.3 implies that $V(m,p)Q_p = V(m,p)$. Setting $m = p$ we obtain:

(d) $\overline{P}_pP_p = \overline{P}_p$ and $\overline{Q}_pQ_p = \overline{Q}_p$.

Lemma 3.6. *If η is sufficiently small, then the operator $S_p = \overline{P}_p + \overline{Q}_p$ is invertible.*

Proof. It follows from (c) that

$$\overline{P}_p + \overline{Q}_p - \mathrm{id} = Q_p\overline{P}_p + P_p\overline{Q}_p. \tag{3.14}$$

By (3.12) and (3.13), we obtain

$$\begin{aligned} P_p\overline{Q}_p &= P_pV(p,p) = \sum_{k=1}^{p-1} \mathcal{A}(p,k+1)P_{k+1}B_kV(k,p), \\ Q_p\overline{P}_p &= Q_pU(p,p) = -\sum_{k=p}^{\infty} \mathcal{A}(p,k+1)Q_{k+1}B_kU(k,p). \end{aligned}$$

On the other hand, by (3.6) and (3.3), for each $m \geq n$ we have

$$\|U(m,n)\| \leq \widetilde{K}_1(m/n)^a n^{\varepsilon}, \tag{3.15}$$

and by (3.9) and (3.3), for each $m \le n$ we have

$$\|V(m,n)\| \le \widetilde{K}_2 (n/m)^{-b} n^{\varepsilon}. \tag{3.16}$$

It follows from (3.14)–(3.16) that

$$\begin{aligned}
\|\overline{P}_p + \overline{Q}_p - \mathrm{id}\| &\le \sum_{k=p}^{\infty} \|\mathcal{A}(p,k+1)Q_{k+1}\| \cdot \|B_k\| \cdot \|U(k,p)\| \\
&\quad + \sum_{k=1}^{p-1} \|\mathcal{A}(p,k+1)P_{k+1}\| \cdot \|B_k\| \cdot \|V(k,p)\| \\
&\le \widetilde{K}_1 K\eta \sum_{k=p}^{\infty} ((k+1)/p)^{-b}(k+1)^{\varepsilon} k^{-\rho}(k/p)^a p^{\varepsilon} \\
&\quad + \widetilde{K}_2 K\eta \sum_{k=1}^{p-1} (p/(k+1))^a (k+1)^{\varepsilon} k^{-\rho}(p/k)^{-b} p^{\varepsilon} \\
&\le \widetilde{K}_1 K\eta \sum_{k=p}^{\infty} ((k+1)/p)^{-b}(k+1)^{\varepsilon} k^{-\rho}(k/p)^a k^{\varepsilon} \\
&\quad + \widetilde{K}_2 K\eta \sum_{k=1}^{p-1} (p/(k+1))^a (k+1)^{\varepsilon} k^{-\rho}(p/k)^{-b+\varepsilon}(p/k)^{-\varepsilon} p^{\varepsilon} \\
&\le \widetilde{K}_1 K\eta \sum_{k=p}^{\infty} (k+1)^{\varepsilon} k^{\varepsilon-\rho} + \widetilde{K}_2 K\eta \sum_{k=1}^{p-1} (k+1)^{\varepsilon} k^{\varepsilon-\rho} \\
&\le 2^{\varepsilon} K\eta\zeta(\rho - 2\varepsilon)(\widetilde{K}_1 + \widetilde{K}_2) = \widehat{K}.
\end{aligned}$$

This implies that for η sufficiently small, the operator S_p is invertible. $\square$

Now we set

$$\widetilde{P}_m = \widetilde{\mathcal{A}}(m,p) S_p P_p S_p^{-1} \widetilde{\mathcal{A}}(p,m), \quad \widetilde{Q}_m = \widetilde{\mathcal{A}}(m,p) S_p Q_p S_p^{-1} \widetilde{\mathcal{A}}(p,m) \tag{3.17}$$

for each $m \in \mathbb{N}$. It is easy to show that $\widetilde{P}_m$ and $\widetilde{Q}_m$ are projections for each fixed $m \in \mathbb{N}$, and that (3.2) holds. We note that $\widetilde{P}_m + \widetilde{Q}_m = \mathrm{id}$ for each $m \in \mathbb{N}$.

Step 4. Norm bounds for the evolution operators.

Lemma 3.7. *We have* $\|\widetilde{\mathcal{A}}(m,n)|\mathrm{Im}\overline{P}_n\| \le \widetilde{K}_1 (m/n)^a n^{\varepsilon}$ *for* $m \ge n$.

Proof. We first show that if $(z_m)_{m \ge n}$ is a bounded solution of equation (2.2), then

$$\begin{aligned}
z_m = {} & \mathcal{A}(m,n)P_n z_n + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_k z_k \\
& - \sum_{k=m}^{\infty} \mathcal{A}(m,k+1)Q_{k+1}B_k z_k
\end{aligned} \tag{3.18}$$

for each $m \geq n$. Note that $z_m = P_m z_m + Q_m z_m$, where

$$P_m z_m = \mathcal{A}(m,n) P_n z_n + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1) P_{k+1} B_k z_k,$$
$$Q_m z_m = \mathcal{A}(m,n) Q_n z_n + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1) Q_{k+1} B_k z_k.$$

We rewrite the last identity in the form

$$Q_n z_n = \mathcal{A}(n,m) Q_m z_m - \sum_{k=n}^{m-1} \mathcal{A}(n,k+1) Q_{k+1} B_k z_k. \tag{3.19}$$

On the other hand, we have

$$\begin{aligned}\sum_{k=n}^{\infty} \|\mathcal{A}(n,k+1) Q_{k+1} B_k z_k\| &\leq K\eta \sum_{k=n}^{\infty} ((k+1)/n)^{-b} (k+1)^{\varepsilon} k^{-\rho} \|z_k\| \\ &\leq K\eta 2^{\varepsilon} \zeta(\rho - \varepsilon) \sup_{k \geq n} \|z_k\|,\end{aligned}$$

and

$$\|\mathcal{A}(n,m) Q_m z_m\| \leq K (m/n)^{-b} m^{\varepsilon} \|z_m\| = K (m/n)^{-b+\varepsilon} n^{\varepsilon} \|z_m\|.$$

Therefore, letting $m \to \infty$ in (3.19) yields

$$Q_n z_n = -\sum_{k=n}^{\infty} \mathcal{A}(n,k+1) Q_{k+1} B_k z_k.$$

Consequently,

$$\begin{aligned}Q_m z_m &= -\sum_{k=n}^{\infty} \mathcal{A}(m,k+1) Q_{k+1} B_k z_k + \sum_{k=n}^{m-1} \mathcal{A}(m,k+1) Q_{k+1} B_k z_k \\ &= -\sum_{k=m}^{\infty} \mathcal{A}(m,k+1) Q_{k+1} B_k z_k,\end{aligned}$$

which yields (3.18).

Now given $\xi \in X$ we consider the solution $z_m = \widetilde{\mathcal{A}}(m,n) \overline{P}_n \xi$ of equation (2.2) for $m \geq n$. By (3.11) we have

$$z_m = \widetilde{\mathcal{A}}(m,p) U(p,p) \widetilde{\mathcal{A}}(p,n) \xi = U(m,p) \widetilde{\mathcal{A}}(p,n) \xi.$$

The last identity follows from the fact that both $\widetilde{\mathcal{A}}(m,p) U(p,p)$ and $U(m,p)$ are solutions of equation (2.2), which coincide for $m = p$. Since $m \mapsto U(m,p)$ is bounded, this shows that $(z_m)_{m\ge n}$ is a bounded solution of equation (2.2) with $z_n = \overline{P}_n\xi$. By (3.18), for each $m \ge n$ we have

$$\begin{aligned}\overline{P}_m\widetilde{\mathcal{A}}(m,n)\xi =&\, \mathcal{A}(m,n)P_n\overline{P}_n\xi + \sum_{k=n}^{m-1}\mathcal{A}(m,k+1)P_{k+1}B_k\overline{P}_k\widetilde{\mathcal{A}}(k,n)\xi\\ &- \sum_{k=m}^{\infty}\mathcal{A}(m,k+1)Q_{k+1}B_k\overline{P}_k\widetilde{\mathcal{A}}(k,n)\xi.\end{aligned}$$

Then, writing $\mathcal{B} = (\widetilde{\mathcal{A}}(m,n)|\mathrm{Im}\overline{P}_n)_{m\ge n}$ we obtain

$$\begin{aligned}\|\overline{P}_m\widetilde{\mathcal{A}}(m,n)\xi\| \le&\, K(m/n)^a n^\varepsilon\|\overline{P}_n\xi\|\\ &+ K\eta\sum_{k=n}^{m-1}(m/(k+1))^a(k+1)^\varepsilon k^{-\rho}\|\overline{P}_k\widetilde{\mathcal{A}}(k,n)\xi\|\\ &+ K\eta\sum_{k=m}^{\infty}((k+1)/m)^{-b}(k+1)^\varepsilon k^{-\rho}\|\overline{P}_k\widetilde{\mathcal{A}}(k,n)\xi\|\\ =&\, K(m/n)^a n^\varepsilon\|\overline{P}_n\xi\|\\ &+ K\eta\sum_{k=n}^{m-1}(m/(k+1))^a(k+1)^\varepsilon k^{-\rho}\|\overline{P}_k\widetilde{\mathcal{A}}(k,n)\overline{P}_n\xi\|\\ &+ K\eta\sum_{k=m}^{\infty}((k+1)/m)^{-b}(k+1)^\varepsilon k^{-\rho}\|\overline{P}_k\widetilde{\mathcal{A}}(k,n)\overline{P}_n\xi\|\\ \le&\, K(m/n)^a n^\varepsilon\|\overline{P}_n\xi\|\\ &+ K\eta\sum_{k=n}^{m-1}(m/(k+1))^a(k+1)^\varepsilon k^{-\rho}(k/n)^a n^\varepsilon\|\mathcal{B}\|_1\|\overline{P}_n\xi\|\\ &+ K\eta\sum_{k=m}^{\infty}((k+1)/m)^{-b}(k+1)^\varepsilon k^{-\rho}(k/n)^a n^\varepsilon\|\mathcal{B}\|_1\|\overline{P}_n\xi\|\\ \le&\, K(m/n)^a n^\varepsilon\|\overline{P}_n\xi\|\\ &+ 2^\varepsilon K\eta(2^{-a}+1)\zeta(\rho-\varepsilon)(m/n)^a n^\varepsilon\|\mathcal{B}\|_1\|\overline{P}_n\xi\|.\end{aligned}$$

Therefore,

$$\|\mathcal{B}\|_1 \le K + 2^\varepsilon K\eta(2^{-a}+1)\zeta(\rho-\varepsilon)\|\mathcal{B}\|_1,$$

and since η is sufficiently small (see (3.7)),

$$\|\mathcal{B}\|_1 \le \frac{K}{1-2^\varepsilon K\eta(2^{-a}+1)\zeta(\rho-\varepsilon)} = \widetilde{K}_1.$$

This yields the desired inequality. □

LEMMA 3.8. *We have* $\|\widetilde{\mathcal{A}}(m,n)|\mathrm{Im}\overline{Q}_n\| \le \widetilde{K}_2(n/m)^{-b}n^\varepsilon$ *for* $m \le n$.

PROOF. Since $-a > \varepsilon$, proceeding in a similar manner to the proof of Lemma 3.7, we conclude that if $(z_m)_{m\le n}$ is a bounded solution of equation (2.2), then

$$z_m = \mathcal{A}(m,n)Q_n z_n + \sum_{k=1}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_k z_k - \sum_{k=m}^{n-1} \mathcal{A}(m,k+1)Q_{k+1}B_k z_k. \tag{3.20}$$

Now given $\xi \in X$ we have

$$z_m := \widetilde{\mathcal{A}}(m,n)\overline{Q}_n\xi = V(m,p)\widetilde{\mathcal{A}}(p,n)\xi \quad \text{for } m \le n.$$

Therefore, $(z_m)_{m\le n}$ is a bounded solution of equation (2.2) with $z_n = \overline{Q}_n\xi$, and it follows from (3.20) that

$$\overline{Q}_m\widetilde{\mathcal{A}}(m,n)\xi = \mathcal{A}(m,n)Q_n\overline{Q}_n\xi + \sum_{k=1}^{m-1} \mathcal{A}(m,k+1)P_{k+1}B_k\overline{Q}_k\widetilde{\mathcal{A}}(k,n)\xi - \sum_{k=m}^{n-1} \mathcal{A}(m,k+1)Q_{k+1}B_k\overline{Q}_k\widetilde{\mathcal{A}}(k,n)\xi.$$

Therefore, writing $\mathcal{C} = (\widetilde{\mathcal{A}}(m,n)|\mathrm{Im}\overline{Q}_n)_{m\le n}$ we obtain

$$\begin{aligned}
\|\overline{Q}_m\widetilde{\mathcal{A}}(m,n)\xi\| \le{}& K(n/m)^{-b}n^\varepsilon\|\overline{Q}_n\xi\| \\
&+ K\eta\sum_{k=1}^{m-1}(m/(k+1))^a(k+1)^\varepsilon k^{-\rho}\|\overline{Q}_k\widetilde{\mathcal{A}}(k,n)\xi\| \\
&+ K\eta\sum_{k=m}^{n-1}((k+1)/m)^{-b}(k+1)^\varepsilon k^{-\rho}\|\overline{Q}_k\widetilde{\mathcal{A}}(k,n)\xi\| \\
={}& K(n/m)^{-b}n^\varepsilon\|\overline{Q}_n\xi\| \\
&+ K\eta\sum_{k=1}^{m-1}(m/(k+1))^a(k+1)^\varepsilon k^{-\rho}\|\overline{Q}_k\widetilde{\mathcal{A}}(k,n)\overline{Q}_n\xi\| \\
&+ K\eta\sum_{k=m}^{n-1}((k+1)/m)^{-b}(k+1)^\varepsilon k^{-\rho}\|\overline{Q}_k\widetilde{\mathcal{A}}(k,n)\overline{Q}_n\xi\| \\
\le{}& K(n/m)^{-b}n^\varepsilon\|\overline{Q}_n\xi\| \\
&+ K\eta\sum_{k=1}^{m-1}(m/(k+1))^a(k+1)^\varepsilon k^{-\rho}(n/k)^{-b}n^\varepsilon\|\mathcal{C}\|_2\|\overline{Q}_n\xi\| \\
&+ K\eta\sum_{k=m}^{n-1}((k+1)/m)^{-b}(k+1)^\varepsilon k^{-\rho}(n/k)^{-b}n^\varepsilon\|\mathcal{C}\|_2\|\overline{Q}_n\xi\| \\
\le{}& K(n/m)^{-b}n^\varepsilon\|\overline{Q}_n\xi\| + 2^\varepsilon K\eta\zeta(\rho-\varepsilon)(n/m)^{-b}n^\varepsilon\|\mathcal{C}\|_2\|\overline{Q}_n\xi\|,
\end{aligned}$$

and hence,

$$\|\mathcal{C}\|_2 \le K + 2^\varepsilon K\eta\zeta(\rho-\varepsilon)\|\mathcal{C}\|_2.$$

Since η is sufficiently small (see (3.10)), it follows that

$$\|\mathcal{C}\|_2 \le \frac{K}{1-2^\varepsilon K\eta\zeta(\rho-\varepsilon)} = \widetilde{K}_2.$$

This yields the desired inequality. $\square$

LEMMA 3.9. *We have*

$$\|\widetilde{\mathcal{A}}(m,n)\widetilde{P}_n\| \le \widetilde{K}_1(m/n)^a n^\varepsilon\|\widetilde{P}_n\|, \quad m\ge n, \tag{3.21}$$

$$\|\widetilde{\mathcal{A}}(m,n)\widetilde{Q}_n\| \le \widetilde{K}_2(n/m)^{-b} n^\varepsilon\|\widetilde{Q}_n\|, \quad m\le n. \tag{3.22}$$

PROOF. By property (d), we have

$$S_pP_p = (\overline{P}_p+\overline{Q}_p)P_p = \overline{P}_p, \quad S_pQ_p = (\overline{P}_p+\overline{Q}_p)Q_p = \overline{Q}_p.$$

Setting $S_m = \widetilde{\mathcal{A}}(m,p)S_p\widetilde{\mathcal{A}}(p,m)$ for $m\in\mathbb{N}$, we can show that

$$\widetilde{P}_mS_m = \overline{P}_m \quad\text{and}\quad \widetilde{Q}_mS_m = \overline{Q}_m.$$

Indeed, by (3.17) we have

$$\widetilde{P}_mS_m = \widetilde{\mathcal{A}}(m,p)S_pP_p\widetilde{\mathcal{A}}(p,m) = \widetilde{\mathcal{A}}(m,p)\overline{P}_p\widetilde{\mathcal{A}}(p,m) = \overline{P}_m,$$

since $\overline{P}_p = U(p,p)$. The other identity can be obtained in a similar manner. Since S_m is invertible, we conclude that

$$\mathrm{Im}\widetilde{P}_m = \mathrm{Im}\overline{P}_m \quad\text{and}\quad \mathrm{Im}\widetilde{Q}_m = \mathrm{Im}\overline{Q}_m.$$

Thus, by Lemmas 3.7 and 3.8, we obtain

$$\begin{aligned}\|\widetilde{\mathcal{A}}(m,n)\widetilde{P}_n\| &\le \|\widetilde{\mathcal{A}}(m,n)|\mathrm{Im}\widetilde{P}_n\|\cdot\|\widetilde{P}_n\|\\ &= \|\widetilde{\mathcal{A}}(m,n)|\mathrm{Im}\overline{P}_n\|\cdot\|\widetilde{P}_n\| \le \widetilde{K}_1(m/n)^a n^\varepsilon\|\widetilde{P}_n\|\end{aligned}$$

for $m\ge n$, and

$$\begin{aligned}\|\widetilde{\mathcal{A}}(m,n)\widetilde{Q}_n\| &\le \|\widetilde{\mathcal{A}}(m,n)|\mathrm{Im}\widetilde{Q}_n\|\cdot\|\widetilde{Q}_n\|\\ &= \|\widetilde{\mathcal{A}}(m,n)|\mathrm{Im}\overline{Q}_n\|\cdot\|\widetilde{Q}_n\| \le \widetilde{K}_2(n/m)^{-b} n^\varepsilon\|\widetilde{Q}_n\|\end{aligned}$$

for $m\le n$. $\square$

LEMMA 3.10. *Provided that η is sufficiently small, for each $m\in\mathbb{N}$ we have*

$$\|\widetilde{P}_m\| \le \frac{2K}{1-2\widehat{K}}m^\varepsilon \quad\text{and}\quad \|\widetilde{Q}_m\| \le \frac{2K}{1-2\widehat{K}}m^\varepsilon.$$

PROOF. Given $\xi\in X$, we set

$$\begin{aligned} z_m^1 &= \widetilde{\mathcal{A}}(m,n)\widetilde{P}_n\xi \quad\text{for } m\ge n,\\ z_m^2 &= \widetilde{\mathcal{A}}(m,n)\widetilde{Q}_n\xi \quad\text{for } m\le n.\end{aligned}$$

By Lemma 3.9, $(z_m^1)_{m\geq n}$ and $(z_m^2)_{m\leq n}$ are bounded solutions of equation (2.2). It thus follows from (3.18) and (3.20) that

$$\begin{aligned}\widetilde{P}_m\widetilde{\mathcal{A}}(m,n)\xi = {}&\mathcal{A}(m,n)P_n\widetilde{P}_n\xi + \sum_{k=n}^{m-1}\mathcal{A}(m,k+1)P_{k+1}B_k\widetilde{P}_k\widetilde{\mathcal{A}}(k,n)\xi \\ &- \sum_{k=m}^{\infty}\mathcal{A}(m,k+1)Q_{k+1}B_k\widetilde{P}_k\widetilde{\mathcal{A}}(k,n)\xi,\end{aligned}$$

and

$$\begin{aligned}\widetilde{Q}_m\widetilde{\mathcal{A}}(m,n)\xi = {}&\mathcal{A}(m,n)Q_n\widetilde{Q}_n\xi + \sum_{k=1}^{m-1}\mathcal{A}(m,k+1)P_{k+1}B_k\widetilde{Q}_k\widetilde{\mathcal{A}}(k,n)\xi \\ &- \sum_{k=m}^{n-1}\mathcal{A}(m,k+1)Q_{k+1}B_k\widetilde{Q}_k\widetilde{\mathcal{A}}(k,n)\xi.\end{aligned}$$

Taking $m=n$, we obtain

$$\begin{aligned}Q_m\widetilde{P}_m\xi &= -\sum_{k=m}^{\infty}\mathcal{A}(m,k+1)Q_{k+1}B_k\widetilde{P}_k\widetilde{\mathcal{A}}(k,m)\xi, \\ P_m\widetilde{Q}_m\xi &= \sum_{k=1}^{m-1}\mathcal{A}(m,k+1)P_{k+1}B_k\widetilde{Q}_k\widetilde{\mathcal{A}}(k,m)\xi.\end{aligned}$$

By (3.21) and (3.22), we have

$$\begin{aligned}\|Q_m\widetilde{P}_m\| &\leq \widetilde{K}_1K\eta\sum_{k=m}^{\infty}((k+1)/m)^{-b}(k+1)^{\varepsilon}k^{-\rho}(k/m)^a m^{\varepsilon}\|\widetilde{P}_m\| \\ &\leq \widetilde{K}_1K\eta 2^{\varepsilon}\zeta(\rho-2\varepsilon)\|\widetilde{P}_m\|\end{aligned}$$

and

$$\begin{aligned}\|P_m\widetilde{Q}_m\| &\leq \widetilde{K}_2K\eta\sum_{k=1}^{m-1}(m/(k+1))^{a}(k+1)^{\varepsilon}k^{-\rho}(m/k)^{-b} m^{\varepsilon}\|\widetilde{Q}_m\| \\ &\leq \widetilde{K}_2K\eta 2^{\varepsilon}\zeta(\rho-2\varepsilon)\|\widetilde{Q}_m\|.\end{aligned}$$

Moreover, since $\|P_m\|\leq Km^{\varepsilon}$ and $\|Q_m\|\leq Km^{\varepsilon}$, we obtain

$$\begin{aligned}\|\widetilde{P}_m\| &\leq \|\widetilde{P}_m - P_m\| + \|P_m\| \\ &= \|\widetilde{P}_m - P_m\widetilde{P}_m - P_m + P_m\widetilde{P}_m\| + \|P_m\| \\ &= \|Q_m\widetilde{P}_m - P_m\widetilde{Q}_m\| + \|P_m\| \\ &\leq \|Q_m\widetilde{P}_m\| + \|P_m\widetilde{Q}_m\| + \|P_m\| \\ &\leq \widetilde{K}_1K\eta 2^{\varepsilon}\zeta(\rho-2\varepsilon)\|\widetilde{P}_m\| + \widetilde{K}_2K\eta 2^{\varepsilon}\zeta(\rho-2\varepsilon)\|\widetilde{Q}_m\| + Km^{\varepsilon} \\ &\leq \widehat{K}(\|\widetilde{P}_m\| + \|\widetilde{Q}_m\|) + Km^{\varepsilon},\end{aligned}$$

and

$$\begin{aligned}\|\widetilde{Q}_m\| &\leq \|\widetilde{Q}_m - Q_m\| + \|Q_m\| \\ &= \|\widetilde{P}_m - P_m\| + \|Q_m\| \leq \widehat{K}(\|\widetilde{P}_m\| + \|\widetilde{Q}_m\|) + Km^{\varepsilon}.\end{aligned}$$

Summing the two yields

$$\|\widetilde{P}_m\| + \|\widetilde{Q}_m\| \leq 2\widehat{K}(\|\widetilde{P}_m\| + \|\widetilde{Q}_m\|) + 2Km^{\varepsilon}.$$

This completes the proof of the lemma. □

The statement of Theorem 3.1 follows now readily from the above lemmas. □

References

[1] L. Barreira and Ya. Pesin, *Nonuniform Hyperbolicity*, Encyclopedia of Mathematics and Its Applications, vol. 115, Cambridge University Press, 2007.

[2] L. Barreira and C. Valls, *Robustness of nonuniform exponential dichotomies in Banach spaces*, J. Differential Equations **244** (2008), 2407–2447.

[3] ———, *Stability of Nonautonomous Differential Equations*, Lecture Notes in Mathematics, vol. 1926, Springer, 2008.

[4] ———, *Polynomial contractions and Lyapunov regularity*, Nonlinear Anal. **73** (2010), 202–210.

[5] A. Bento and C. Silva, *Stable manifolds for nonuniform polynomial dichotomies*, J. Funct. Anal. **257** (2009), 122–148.

[6] S.-N. Chow and H. Leiva, *Existence and roughness of the exponential dichotomy for skew-product semiflow in Banach spaces*, J. Differential Equations **120** (1995), 429–477.

[7] W. Coppel, *Dichotomies and reducibility*, J. Differential Equations **3** (1967), 500–521.

[8] Ju. Dalec'kiĭ and M. Kreĭn, *Stability of Solutions of Differential Equations in Banach Space*, Transl. Math. Monogr., vol. 43, Amer. Math. Soc., 1974.

[9] J. Massera and J. Schäffer, *Linear differential equations and functional analysis* I, Ann. of Math. (2) **67** (1958), 517–573.

[10] ———, *Linear Differential Equations and Function Spaces*, Pure Appl. Math., vol. 21, Academic Press, 1966.

[11] R. Naulin and M. Pinto, *Admissible perturbations of exponential dichotomy roughness*, Nonlinear Anal. **31** (1998), 559–571.

[12] V. Oseledets, *A multiplicative ergodic theorem. Liapunov characteristic numbers for dynamical systems*, Trans. Moscow Math. Soc. **19** (1968), 197–221.

[13] O. Perron, *Die Stabilitätsfrage bei Differentialgleichungen*, Math. Z. **32** (1930), 703–728.

[14] Ya. Pesin, *Families of invariant manifolds that corresponding to nonzero characteristic exponents*, Math. USSR-Izv. **10** (1976), 1261–1305.

[15] V. Pliss and G. Sell, *Robustness of exponential dichotomies in infinite-dimensional dynamical systems*, J. Dynam. Differential Equations **11** (1999), 471–513.

[16] L. Popescu, *Exponential dichotomy roughness on Banach spaces*, J. Math. Anal. Appl. **314** (2006), 436–454.

[17] C. Pugh and M. Shub, *Ergodic attractors*, Trans. Amer. Math. Soc. **312** (1989), 1–54.

[18] D. Ruelle, *Characteristic exponents and invariant manifolds in Hilbert space*, Ann. of Math. (2) **115** (1982), 243–290.

Manuscript received July 2, 2010

Luis Barreira and Claudia Valls
Departamento de Matemática
Instituto Superior Técnico
1049-001 Lisboa, PORTUGAL

E-mail address: barreira@math.ist.utl.pt, cvalls@math.ist.utl.pt

Meng Fan
School of Mathematics and Statistics
Northeast Normal University
5268 Renmin Street
Changchun, Jilin, 130024, P. R. CHINA

E-mail address: mfan@nenu.edu.cn

Jimin Zhang
School of Mathematics and Statistics
Northeast Normal University
5268 Renmin Street, Changchun
Jilin, 130024, P. R. CHINA
and
School of Mathematical Sciences
Heilongjiang University
74 Xuefu Street
Harbin, Heilongjiang, 150080, P. R. CHINA

E-mail address: zhangjm1978@hotmail.com